Effort complementarity and role assignments in group contests

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Abstract

This study characterizes role assignments in maximizing a group's winning probability under the influence of the complementarity of group members' efforts in a group contest, in contrast to prize and multiple resource allocations. We use a CES effort aggregator function to parameterize the complementarity. While the prize and resource allocation rules depend on the complementarity, the assignment rule does not when multiple roles are assignable to a single group member: All roles are assigned only to the most productive group member. However, when only a single role per group member is assignable, the assignment rule depends on the complementarity: Roles from greater to less importance are assigned to group members in descending order of their productivity under strong complementarity; only the most important role is assigned to the most productive group member and the others have no effect under weak complementarity.

Keywords: Group contest; role assignment; CES effort aggregator function; resource allocation

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1 Introduction

Group contests, in which individuals join a group to compete for economic rents called a prize, are common in society, such as R&D races, lobbying, military conflicts, and pro-sport games. Usually, group members play roles in their group and exert effort to win a contest. In a soccer team, each team member plays the role of forward, midfielder, defender, or goalkeeper. In a military unit, soldiers play the roles of battle, communication, transportation, scout, etc. In an R&D division, division members play roles of searching for compounds, identifying substances, generating knock-out mice, etc. In all these examples, there is also a complementarity among the group members' efforts. A group manager who wants to maximize the group's winning probability assigns these roles to group members with consideration of their productivity and the complementarity among their efforts. In addition to the role assignment, the group manager allocates resources such as equipment and computers for their roles and promises rewards that are distributed to the group members after they win the competition. While previous studies in the group contest literature have focused on prize allocation, they did not focus on role assignment and resource allocation issues, despite their significant effect on groups' winning probability. In this study, we theoretically clarify the influence of effort complementarity on the group manager's role assignment rule, in contrast to their prize and resource allocation rules,[1] using a stochastic group-contest model.

Regarding complementarity among group members' efforts in assigned roles, each effort may be essential in that an entire collaboration fails if a group member does not contribute. Tasuku Honjo, who won the Nobel Prize in Physiology and Medicine in 2018, would not have identified the function of the gene "PD-1" (which they had already discovered) if his group member had not exerted additional effort to generate targeted knockout mice. The discovery and identification of Honjo's group paved the way for a breakthrough in cancer medicine through the inhibition of

[1]In organizational literature, Bendor and Page (2019), whose model is not a contest model, focus on resource allocations and role assignments in a team.

negative immune regulation, which led him to win the Nobel Prize.[2] The efforts in the roles of discovery and identification were essential for winning the Nobel Prize. We use a CES effort aggregator function to introduce such complementarity among group members' efforts in their roles, following Kolmar and Rommeswinkel (2013). They showed the relationship between effort complementarity and aggregated effort level by first using a CES production function as an aggregator of group members' efforts in group contest literature.

Regarding the assignment of roles in contests, there are three realistic options: (1) Each role is tied up with each group member; (2) the group manager can assign multiple roles to a single group member; and (3) the group manager can assign only a single role per group member. In (1), each group member has a specialized, licensed, or qualified skill related to a role and the group manager cannot change their roles. The group manager only allocates the prizes and resources. In (2), role assignment is unlimited. In (3), each group member has a certain capability and cannot take multiple roles. We considered these options during role assignments to cover realistic variations.

By using the share function approach of Cornes and Hartley (2005), we concisely show the existence and uniqueness of the Nash equilibrium group members' effort profile. Based on this, we characterize the rules of role assignments and prize and resource allocations as follows: First, for non-assignable roles, high productivity gives a high share of prize and resource to the group member with it under weak complementarity and a low share under strong complementarity. Multiple allocatable resources promote this characteristic. Second, once multiple roles are assignable to a single group member, all roles are assigned and full prizes and resources are allocated only to the most productive group member in any effort complementarity, even in effort essentialness. This property is different from that of the prize and resource allocation rules. Third, the assignment rule depends on complementarity when only a single role per group member is assigned. The more important roles are assigned to group members with higher productivity and the less important

[2]See "Facts" and "Biographical" of Tasuku Honjo on the website of Nobel Prize organization (2018).

roles are assigned to those with less productivity under strong complementarity; only the most important role is assigned to the group member with the highest productivity and the other roles have no effect under weak complementarity. This means that the group manager prioritizes the concentration of prizes and resources on the most productive group members by giving up less important and substitutable roles. These results were attained by clarifying the rules of role assignment and prize and multiple resource allocations, contrary to previous studies that focused only on prize allocation (Nitzan and Ueda 2014; Trevisan 2020; Kobayashi and Konishi 2021).

The remainder of this paper is organized as follows. Section 2 discusses related literature. Section 3 presents the proposed model. Section 4 presents the allocation rules of the prize and resources to maximize the group's winning probability without role assignments. Section 5 demonstrates the role assignment rules in two cases: A group manager can assign multiple roles, and only a single role per group member. Section 6 concludes the study and discusses future research possibilities. All proofs are provided in the Appendix.

2 Literature

Previous studies of group contests have analyzed the prize allocation issue. A prize allocation rule comprises the weighted mean of a group member's effort share and equal allocation among group members (Nitzan 1991; Nitzan and Ueda 2011; Gupta 2021). These studies focused on the influence of changing the weight before the contest on each group member's motivation. Under such a prize allocation rule, effort needs to be verifiable, owing to its dependence on each group member's effort share. This is applicable to cases such as group members' pecuniary donations to politicians in rent-seeking competitions among interest groups. However, it is not applicable to cases in which group members' efforts are unverifiable in contests, such as R&D races, pro-sport games, or large litigation among firms, sports teams, or lawyer teams. In such cases, a fixed-prize allocation rule is promised to each group member before the contest and allocated in accordance with the promise after winning the contest. Nitzan and Ueda (2014) clarify whether the expanding disparity in prize allocation to group members improves a group's winning probability. Trevisan (2020) shows a prize allocation rule to maximize a group's probability of winning. In these two studies, when the group members' marginal effort cost is convex, allocating a fixed prize to all group members improves the winning probability. When the marginal cost is concave, allocating the entire prize to a single member improves the probability of winning. This condition is identical to Olson's (1965) group-size paradox. The convexity (concavity) of the marginal effort cost is replaceable with the elasticity of the effort cost presented by Esteban and Ray (2001). These studies used a linear sum of group members' efforts as the group's effort aggregator. Kobayashi and Konishi (2021) extend this condition to the relative sizes between group members' effort complementarity and elasticity of effort cost with a CES function, and show a conflict of interest between the group manager and the group member. Kobayashi, Konishi, and Ueda (2022) endogenize the prize depending on the amount of total effort in the contest and characterize the prize allocation with a homothetic effort-aggregator function following Hartley's (2017) cost minimization approach. Our study aims to clarify the rules of role assignment and prize and resource allocation, characterized by complementarity among group members' efforts in a CES function.

A CES function was used as an effort aggregator[3] in the group contest literature (Kolmar and Rommeswinkel 2013; Brookins, Lightle, and Ryvkin 2015; Choi, Chowdhury, and Kim 2016; Cheikbossian and Fayat 2018; Konishi and Pan 2020; Crutzen, Flamand, and Sahuguet 2020; Kobayashi and Konishi 2021). Kolmar and Rommeswinkel (2013) examined several variables using the CES function $(\sum_{k=1}^{n} a_k e_k^{\sigma})^{\frac{1}{\sigma}}$, where e_k is a group member's effort, σ is a complementarity parameter, a_k is a weight, and n is the population of the group. Other studies use a symmetric CES function, such as $a_k = 1$ or $1/n$ in Kolmar and Rommeswinkel's (2013). Epstein and Mealem (2009)

[3]In addition to the linear sum and CES functions, a discrete aggregator function is also available as an effort aggregator in group contests (Kobayashi 2019).

used a similar form, $\sum_{k=1}^{n} e_k^{\sigma}$. Our study uses an asymmetric CES function, in which each group member's skill parameter is added to Kolmar and Rommeswinkel's (2013). We show how the skill works in the rules for prize and resource allocations and role assignment on every complementarity level. In particular, our study demonstrates a role assignment rule that has not been discussed in previous studies, with the CES function covering a wide range of complementarities.

3 The model

We consider a contest in which $m \geq 2$ groups compete for a prize, focusing on a representative group $i = 1, 2, ..., m$. The population of group i is denoted as $n_i \geq 2$. Group members choose their effort levels e_{ik}, $k = 1, 2, \ldots, n_i$, which contribute to their group's probability of winning simultaneously and non-cooperatively. Group members' efforts $\boldsymbol{e_i} = (e_{i1}, \ldots, e_{in_i})$ are aggregated using the CES function $X_i = [\sum_{k=1}^{n_i} a_{ik}(s_{ik}e_{ik})^{\sigma_i}]^{\frac{1}{\sigma_i}}$, where $-\infty < \sigma_i \leq 1$ indicates the degree of effort complementarity.[4] $s_{ik} > 0$ is group member k's skill level, which converts their efforts into contributions to group i. $a_{ik} \in [0, 1]$ is the weight of k's contribution, which is viewed as either a role assigned to k or inherently tied up with k within the group, and we assume $\sum_{k=1}^{n_i} a_{ik} = 1$, so that the higher the a_{ik}, the more important the role. The CES aggregator becomes a linear function when $\sigma_i = 1$, a Cobb-Douglas function when $\sigma_i = 0$ in the limit, and a function with greater effort complementarity among group members than the Cobb-Douglas function when $\sigma_i < 0$. [5] In the latter two cases, each group member's effort is essential, in that if a group member contributes no effort, the aggregate effort X_i is zero.

[4]Kolmar and Rommeswinkel (2013) call this effort aggregator the impact function.

[5]This CES function converges to the Leontief function of $\min\{s_{i1}e_{i1}, \ldots, s_{in_i}e_{n_i}\}$ with perfect complementarity as $\sigma_i \to -\infty$. We did not consider this form in this study because it required a different analysis. The perfect complementarity of efforts creates multiple Nash equilibria with positive effort levels and coordination problem among them, as Lee (2012) shows. However, our analysis is valid for X_i with sufficiently small $\sigma_i < 0$, where X_i is close to the Leontief form.

The winning probability of group i is described by the Tullock-form contest success function $P_i = X_i/X$ where $X = \sum_{f=1}^{m} X_f$ (Tullock 1980). We assume $P_i = 0$ (or $P_i = 1/m$) when $X = 0$. The prize comprises divisible private goods that are shared among members of the winning group, and the value of the prize is normalized to 1. We denote the prize share of member k of group i by $v_{ik} \in [0,1]$. The share is allocated to each member such that $\sum_{k=1}^{n_i} v_{ik} = 1$ in the winning group after the contest. Members of the losing group obtain nothing. The effort cost function of group member k has constant elasticity $\beta_i > 1$, that is, $e_{ik}^{\beta_i}/(\beta_i c_{ik})$. $c_{ik} = \Pi_{h=1}^{t-1} c_{hk}^i$ when $t \geq 2$ is a composite of the limited resources allocated to group member k to reduce k's effort cost, such as IT equipment. When $t = 1$, only the prize is allocated; there is no allocatable resource for cost reduction, and we assume that each group member has $c_{ik} = 1$ inherently. t is a common value among all the groups.[6] We assume $c_{hk}^i \in [0,1]$ and $\sum_{k=1}^{n_i} c_{hk}^i = 1$ for $h = 1, \ldots, t-1$. If at least one type of $c_{hk}^i = 0$, k's marginal effort cost becomes infinite and they do not contribute to their group. Thus, each limited resource c_{hk}^i is essentially complementary for each group member. The expected payoff for member k in group i is $U_{ik} = P_i v_{ik} - e_{ik}^{\beta_i}/(\beta_i c_{ik})$. We assume that there is a group manager who wants to maximize their group's winning probability P_i in each group. The group manager allocates the prize $\boldsymbol{v_i} = (v_{i1}, \ldots, v_{in_i})$ and cost reduction resources $\boldsymbol{c_h^i} = (c_{h1}^i, \ldots, c_{hn_i}^i)$ for all $1 \leq h \leq t-1$ (when $t \geq 2$) in $c_{ik} = \Pi_{h=1}^{t-1} c_{hk}^i$ and assigns the role $\boldsymbol{a_i} = (a_{i1}, \ldots, a_{in_i})$ to each group member. We consider three variations of allocation and assignment, which are explained in the following sections. We assume that each group member k regards all variables except for e_{ik} as given and that the above information is common knowledge among all players.

We consider a two-stage game played by group managers and group members. In Stage 1, each group manager chooses the rules of prize and resource allocations ($\boldsymbol{v_i}$ and $\boldsymbol{c_h^i}$ for all h) and role assignment ($\boldsymbol{a_i}$) simultaneously (we analyze these rules in the following sections). The group manager allocates resources and assigns roles to group members according to their rules. In Stage 2, each group member determines their effort levels simultaneously, and a contest is conducted.

[6] Although each group may have a different t, we assume this for simplicity.

Only one group becomes the winner in the contest. The manager in the winning group allocates the prize to their group members following the prize-allocation rule. We employ a subgame perfect Nash equilibrium.

In Stage 2, group member k determines their effort level $e_{ik} \geq 0$ to maximize their expected payoff U_{ik} given the other group members' efforts e_{il}, $l \neq k$, and the aggregate effort of the other groups $X - X_i$. The first-order condition of any member k of group i, using $P_i = X_i/X$ and $\partial X/\partial e_{ik} = \partial X_i/\partial e_{ik} = [\sum_{k=1}^{n_i} a_{ik}(s_{ik}e_{ik})^{\sigma_i}]^{\frac{1}{\sigma_i}-1} a_{ik} s_{ik}^{\sigma_i} e_{ik}^{\sigma_i - 1} = X_i^{1-\sigma_i} a_{ik} s_{ik}^{\sigma_i} e_{ik}^{\sigma_i-1}$, is as follows:

$$\begin{aligned} \frac{\partial U_{ik}}{\partial e_{ik}} &= \frac{1}{X^2}\left(\frac{\partial X_i}{\partial e_{ik}} X - X_i \frac{\partial X}{\partial e_{ik}}\right) v_{ik} - \frac{e_{ik}^{\beta_i - 1}}{c_{ik}} \\ &= \frac{1}{X^2}\left(X - X_i\right) X_i^{1-\sigma_i} a_{ik} s_{ik}^{\sigma_i} e_{ik}^{\sigma_i-1} v_{ik} - \frac{e_{ik}^{\beta_i-1}}{c_{ik}} \\ &= \frac{P_i(1-P_i)}{X_i^{\sigma_i}} a_{ik} s_{ik}^{\sigma_i} e_{ik}^{\sigma_i - 1} v_{ik} - \frac{e_{ik}^{\beta_i-1}}{c_{ik}} = 0. \end{aligned} \tag{1}$$

We transpose $e_{ik}^{\beta_i-1}/c_{ik}$ to the other side in (1), multiply both sides of this expression by $c_{ik}/e_{ik}^{\sigma_i-1}$, raise it to the power of $\sigma_i/(\beta_i - \sigma_i)$, and multiply it by $a_{ik}s_{ik}^{\sigma_i}$. We have

$$\left(\frac{P_i(1-P_i)}{X_i^{\sigma_i}}\right)^{\frac{\sigma_i}{\beta_i-\sigma_i}} \left(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i} c_{ik} v_{ik}\right)^{\frac{\sigma_i}{\beta_i - \sigma_i}} = a_{ik}(s_{ik}e_{ik})^{\sigma_i}. \tag{2}$$

We add all the group members' (2) to aggregate all first-order conditions in group i:

$$\left(\frac{P_i(1-P_i)}{X_i^{\sigma_i}}\right)^{\frac{\sigma_i}{\beta_i-\sigma_i}} \sum_{k=1}^{n_i} \left(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i} c_{ik} v_{ik}\right)^{\frac{\sigma_i}{\beta_i - \sigma_i}} = \sum_{k=1}^{n_i} a_{ik}(s_{ik}e_{ik})^{\sigma_i} \quad (= X_i^{\sigma_i}).$$

We raise both sides of the expression to the power of $1/\sigma_i$, multiply it by $X_i^{\frac{\sigma_i}{\beta_i-\sigma_i}}$, and finally raise it to the power of $\beta_i - \sigma_i$. Using $X_i = P_i X$, the aggregate effort X_i of any group i at the Nash equilibrium in Stage 2 is described implicitly as follows:

$$P_i^{1-\beta_i}(1-P_i)A_i = X^{\beta_i}, \tag{3}$$

where

$$A_i = \left[\sum_{k=1}^{n_i} \left(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i} c_{ik} v_{ik}\right)^{\frac{\sigma_i}{\beta_i-\sigma_i}}\right]^{\frac{\beta_i - \sigma_i}{\sigma_i}} \geq 0. \tag{4}$$

A_i comprises the exogenous variables for group member k. All variables that group i's manager determined in Stage 1 are aggregated into A_i, so that those variables affect the Nash equilibrium in Stage 2 via A_i. A_i is a CES function form[7], which converges to a Cobb-Douglass form as $\sigma_i \to 0$[8]: $A_i \to \Pi_{k=1}^{n_i}(a_{ik}s_{ik}^{\beta_i}c_{ik}v_{ik})^{a_{ik}}$. Group i's winning probability P_i is the share function of group i. We base the existence of a unique profile of aggregate effort $(X_1, \ldots, X_m)$ in a Nash equilibrium in Stage 2 of this group contest game on the share function approach of Cornes and Hartley (2005). By totally differentiating (3) with regard to P_i and X and substituting A_i of (3) into it, we obtain, recalling $\beta_i > 1$,

$$\frac{dP_i}{dX} = \frac{\beta_i P_i(1 - P_i)}{[(1 - \beta_i)(1 - P_i) - P_i]\, X} < 0 \tag{5}$$

when $0 < P_i < 1$ and $X > 0$. P_i is then monotonically decreasing in X. In (3), $\lim_{X\to 0} P_i(X) = 1$ and $\lim_{X\to +\infty} P_i(X) = 0$ when $A_i > 0$ in addition to (5). Thus, $\lim_{X\to 0}\sum_{f=1}^{m} P_f(X) = m$ and $\lim_{X\to +\infty}\sum_{f=1}^{m} P_f(X) = 0$. There is then a unique X^*, where $\sum_{f=1}^{m} P_f(X^*) = 1$, implying the existence of a unique $(X_1^*, \ldots, X_m^*)$ with all $X_i^* > 0$ in a Nash equilibrium in Stage 2 from the share function approach. Therefore, we obtain the following lemma immediately.

Lemma 1. *There is a unique profile of aggregate effort $(X_1^*, \ldots, X_m^*)$ with all $X_i^* > 0$ in a Nash equilibrium when $A_i > 0$ for any group i in Stage 2 of the group contest game.*

From Lemma 1, we have the following lemma.

Lemma 2. *For any group i, there is a unique effort profile $\mathbf{e}_i^* \neq \mathbf{0}$ that supports $X_i^* > 0$ in a Nash equilibrium when $A_i > 0$ in Stage 2 of the group contest game.*

All proofs are provided in the Appendix. Even when all the groups' σ_i differ, Lemmas 1 and 2 hold because (1) holds. Lemma 1 does not guarantee the existence of a unique Nash equilibrium. Let us consider the case of $\sigma_i \leq 0$, including $\sigma_i = 0$ in the limit (a Cobb-Douglass function), in

[7]This is a form of the CES function in McFadden (1963). See also the section 3 in Blackorby and Russell (1989).

[8]$(a_{ik}^{\frac{\beta_i}{\sigma_i}})^{\frac{\sigma_i}{\beta_i - \sigma_i}} = a_{ik}^{\frac{\beta_i}{\beta_i - \sigma_i}} \to a_{ik}$ as $\sigma_i \to 0$ in A_i, and recall $\sum_{k=1}^{n_i} a_{ik} = 1$. Therefore, A_i converges to a Cobb-Douglass function.

X_i. When the other group members' efforts are $e_{il} = 0$, any group member k's best response is $e_{ik} = 0$ because $X_i = 0$ for any $e_{ik} > 0$. Then, $\boldsymbol{e_i} = \mathbf{0}$ is also a Nash equilibrium effort profile, in addition to $\boldsymbol{e_i^*} \neq \mathbf{0}$ supporting $X_i^* > 0$ in Lemmas 1 and 2. Hereafter, we focus only on the effort profile $\boldsymbol{e_i^*} \neq \mathbf{0}$.[9]

We must also consider the subgame at $A_i = 0$ [10] in Stage 2. Although (3) does not hold at $A_i = 0$ when $X > 0$, it holds in $P_i \to 0$ as $A_i \to 0$. Instead, we consider the subgame in $P_i \to 0$ as $A_i \to 0$ in (3). $P_i \to 0$ is $X_i \to 0$ when $X > 0$. Then $\boldsymbol{e_i^*} \to \mathbf{0}$.

Before considering Stage 1, we characterize P_i with respect to A_i in (3). By totally differentiating (3) with regard to P_i and A_i for a given X, we obtain, recalling $\beta_i > 1$,

$$\frac{dP_i}{dA_i} = \frac{-P_i(1 - P_i)}{\left[(1 - \beta_i)(1 - P_i) - P_i\right] A_i} > 0, \tag{6}$$

when $0 < P_i < 1$ and $A_i > 0$. For any $A_i < \hat{A}_i$ and X^* with the profile in Lemma 1, $P_i(X^*; A_i) < P_i(X^*; \hat{A}_i)$ from (6). From this, $1 = \sum_{f \neq i} P_f(X^*) + P_i(X^*; A_i) < \sum_{f \neq i} P_f(X^*) + P_i(X^*; \hat{A}_i)$, and $1 = \sum_{f \neq i} P_f(X^{**}) + P_i(X^{**}; \hat{A}_i)$. Then, $X^* < X^{**}$ from (5) and X^{**} comprises the profile of aggregate effort in the Nash equilibrium when $\hat{A}_i$. Therefore, $P_i(X^{**}; \hat{A}_i) = 1 - \sum_{f \neq i} P_f(X^{**}) > 1 - \sum_{f \neq i} P_f(X^*) = P_i(X^*; A_i)$. The last inequality implies that an increase in A_i determined in Stage 1 increases the share P_i (group i's winning probability) in the Nash equilibrium in Stage 2 for any group i.

We consider Stage 1. Any change in the other groups' A_l, $l \neq i$, affects P_i via $X - X_i$ in Stage 2 and does not affect P_i directly in Stage 1. Recall that all the variables that group i's manager determines in Stage 1 are aggregated into A_i without any variables that the other group managers determine. From the characterization of P_i with respect to A_i, an increase in A_i increases P_i in

[9] If we allow $\beta_i = 1$, we must consider the possibility of inactive groups: $P_i = 0$ when $X \geq A_i$ in (3). In addition, we must consider the possibility of multiple Nash equilibria with $X_i > 0$ and the coexistence of active and inactive group members in Stage 2. We assume $\beta_i > 1$ to exclude these complexities. Our main focus is on the rules of prize and resource allocations and role assignment.

[10] For example, when $\sigma_i = 0$ in the limit, one $v_{ik} = 0$ causes $A_i = 0$.

the Nash equilibrium in Stage 2. Therefore, the best response of group i's manager, who wants to maximize their winning probability P_i, to the other group managers' A_l is to maximize their own A_i. In other words, the maximization of A_i is the group manager's dominant strategy in Stage 1. We summarize this as a lemma.

Lemma 3. *The strategy of group i's manager in Stage 1 of the subgame perfect equilibrium is to maximize* A_i.

While the group managers' equilibrium strategy is to maximize A_i, we have not yet shown what rules of prize and resource allocations and role assignment maximize P_i. These rules depend on the type of controllable variables of the group manager. We consider three cases about group managers' control variables. In particular, we show that role assignments have different characteristics from prize and resource allocations.

4 Prize and cost reduction resource allocation

In this section, we consider a case in which the group manager can choose the rules of prize allocation[11] $\boldsymbol{v_i}$ and cost-reduction resource allocation $\boldsymbol{c_h^i}$ for all h in Stage 1 but cannot choose a rule of role assignment $\boldsymbol{a_i}$. Each role is tied to each group member, and $a_{ik} > 0$ for any k. In this case, the group manager has $t \geq 1$ types of allocatable variables v_{ik} and $c^i_{1k}, \ldots, c^i_{t-1k}$ in $c_{ik} = \Pi_{h=1}^{t-1} c^i_{hk}$ for each group member k. When $t = 1$, the group manager only has an allocatable prize, and $c_{ik} = 1$. From Lemma 3 in the previous section, group i's manager chooses the rules

[11] This prize allocation rule is a promise (contract) between the group manager and each group member and is implemented after their win. We assume that each group manager cannot form a contract with each group member depending on either each group member's effort level or the aggregated effort level. If such a contract is feasible, the group manager can achieve their optimal effort level by offering a forcing contract to group members, as shown by Holmstrom (1982); that is, a group member does not obtain anything if the aggregate effort level does not achieve the optimal level.

to maximize (4) with $v_{ik}, c^i_{1k}, \ldots, c^i_{t-1k}$ for $k = 1, \ldots, n_i$ such that $\sum_{k=1}^{n_i} v_{ik} = 1$ and $\sum_{k=1}^{n_i} c^i_{hk} = 1$ for $h = 1, \ldots, t-1$. The allocation rules in the following proposition then hold.

Proposition 1. *In the group contest among m groups, when the group manager can allocate the share of prize v_{ik} and cost reduction resources $c^i_{1k}, \ldots, c^i_{t-1k}$ for $k = 1, \ldots, n_i$ such that $\sum_{k=1}^{n_i} v_{ik} = 1$ and $\sum_{k=1}^{n_i} c_{hk} = 1$ for $h = 1, \ldots, t-1$ to their group members, the allocation rule in the subgame perfect equilibrium (equilibrium strategy) is as follows:*

1. *When $(t+1)\sigma_i < \beta_i$,*

$$v_{ij} = c^i_{1j} = \ldots = c^i_{t-1j} = \frac{(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i})^{\frac{\sigma_i}{\beta_i - (t+1)\sigma_i}}}{\sum_{k=1}^{n_i} (a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i - (t+1)\sigma_i}}} \tag{7}$$

   for $j = 1, \ldots, n_i$. Then, the maximized A_i, described as A^*_i, is

$$A^*_i = \left[\sum_{k=1}^{n_i} (a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i - (t+1)\sigma_i}} \right]^{\frac{\beta_i - (t+1)\sigma_i}{\sigma_i}}. \tag{8}$$

   As $\sigma_i \to 0$, $v_{ij} = c^i_{1j} = \ldots = c^i_{t-1j} \to a_{ij}$ for $j = 1, \ldots, n_i$ and

$$A^*_i \to \Pi_{k=1}^{n_i} (a_{ik}^{t+1} s_{ik}^{\beta_i})^{a_{ik}}. \tag{9}$$

   Then, all group members exert efforts.

2. *When $(t+1)\sigma_i \geq \beta_i$, $v_{ij} = c^i_{1j} = \ldots = c^i_{t-1j} = 1$ is allocated to the group member j with the highest $a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$. When $(t+1)\sigma_i = \beta_i$ and there are multiple js with the highest $a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$, any allocation such that $v_{ij} = c^i_{1j} = \ldots = c^i_{tj} \in [0, 1]$ for each j and $\sum_j v_{ij} = 1$ and $\sum_j c^i_{hj} = 1$ is also allowed. Nothing is allocated to the other group members, that is, $v_{il} = c^i_{1l} = \ldots = c^i_{t-1l} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n_i$. $A^*_i = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$. Only the group member j then exerts effort.*

Let us consider the indications of the allocation rules. In case 1 of Proposition 1, σ_i is small, that is, the group members' efforts have strong complementarity. The allocation rule (7) becomes the share of $(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}} = (a_{ik} s_{ik}^{\sigma_i})^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}$, which depends on each group member's skill weighted by their own role and represents their productivity. (8) is their aggregated productivity, which represents group i's competitiveness in the group contest (recall that a large A_i^* gives a high winning probability to group i) and is a CES function form.

$s_{ik}^{\sigma_i}$ depends on the complementarity σ_i in (7). As $\sigma_i > 0$ becomes weak and close to $\beta_i/(t+1)$, the share of the most productive group member goes to 1[12] and (8) goes to A_i^* in Case 2 of Proposition 1. A high skill level gives a high share to the group member possessing it. On the other hand, as complementarity becomes strong ($\sigma_i = 0$ in the limit), each share in (7) goes to each role a_{ik}, and each skill s_{ik} has no effect on the allocation rule. When complementarity is stronger ($\sigma_i < 0$), each $s_{ik}^{\sigma_i}$ becomes the reciprocal in (7). A high skill level gives a low share to the group member possessing it. This means that the group manager motivates those with low skills more than those with high skills. $\sigma_i = 0$ is the threshold for whether each group member's skill increases or decreases each share. An important role (a high a_{ik}) provides a high share to the group member possessing it under any σ_i. Naturally, group managers motivate members possessing important roles. However, each role's impact in (7) and (8) fades as $\sigma_i < 0$ decreases. In fact, $v_{ij} = c_{1j}^i = \ldots = c_{t-1j}^i \to \frac{(1/s_{ij}^{\beta_i})^{\frac{1}{t+1}}}{\sum_{k=1}^{n_i}(1/s_{ik}^{\beta_i})^{\frac{1}{t+1}}}$ and $A_i^* \to \frac{1}{\left[\sum_{k=1}^{n_i}(1/s_{ik}^{\beta_i})^{\frac{1}{t+1}}\right]^{t+1}}$ as $\sigma_i \to -\infty$.[13] The number t of each group manager's allocatable variable promotes these effects in (7): When $\sigma_i > 0$,

[12]By letting $z_k(\sigma_i) \equiv (a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})/(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i})$, the allocation rule (7) is $\frac{1}{\sum_{k=1}^{n_i} z_k(\sigma_i)^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}}$. All terms of $z_k(\beta_i/(t+1)) < 1$ go to 0 and those of $z_k(\beta_i/(t+1)) > 1$ go to $+\infty$ as $\sigma_i \to \beta_i/(t+1)$ because the power $\sigma_i/(\beta_i - (t+1)\sigma_i) \to +\infty$. In the share of the group member with the highest $a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$ at $\sigma_i = \beta_i/(t+1)$, any $z_k(\beta_i/(t+1)) \leq 1$, so its share goes to 1. The shares of other group members include $z_k(\beta_i/(t+1)) > 1$. Thus, their shares go to 0. If multiple group members have the highest productivity, their shares are equal.

[13]This convergent value of A_i^* is not always the maximized A_i when $X_i = \min\{s_{i1}e_{i1}, \ldots, s_{in_i}e_{in_i}\}$. By substituting (7) into A_i led from only (1) of the group member with the lowest productivity, we have a larger A_i than this A_i^*. Besides, we face the coordination problem of multiple Nash equilibria with positive effort levels as mentioned in footnote 5. However, the allocation rule in Proposition 1 is also valid for sufficiently small $\sigma_i < 0$, where X_i is close to the Leontief form.

the increase in t increases $\sigma_i/(\beta_i - (t+1)\sigma_i)$ and the disparity in allocations between the most productive group member and the least productive group member; when $\sigma_i < 0$, it reduces them.

In Case 2 of Proposition 1, namely, weak complementarity, the group manager allocates the full prize and cost reduction resource to the most productive group member.[14] When $(t+1)\sigma_i = \beta$ and there are multiple most productive group members, any allocation such that $v_{ij} = c^i_{hj}$ per group member and $\sum_j v_{ij} = 1$ and $\sum_j c^i_{hj} = 1$ is also allowed. For example, $v_{i1} = c^i_{h1} = 1/3$ and $v_{i2} = c^i_{h2} = 2/3$ for all $h = 1, \ldots, t-1$ between the two most productive group members 1 and 2. This allocation means that the work done alone by the most productive group member may be shared among those group members and is identical to their solo work.

The threshold condition $(t+1)\sigma_i < (\geq)\beta_i$[15] depends on the number t of each group manager's allocatable variables, while only the case $t = 1$ has been analyzed in previous studies, and its effect on the allocation has not been clarified.[16] Consider an example: When only the prize is allocatable ($t = 1$), allocating it to all group members is the group manager's equilibrium strategy if $2\sigma_i < \beta_i$; when the prize and a single kind of c^i_{1k} are allocatable ($t = 2$), allocating the full prize and cost reduction resource to the most productive group member is the group manager's equilibrium strategy if $2\sigma_i$ is close to β_i and $3\sigma_i \geq \beta_i$. As the group manager has more allocatable cost reduction resources, solo work by the most productive group member tends to occur in the equilibrium.

[14] When the large elasticity of effort cost $\beta_i > t+1$, Case 2 vanishes. If there are multiple most productive group members and $(t+1)\sigma_i > \beta_i$, the group manager allocates all to one of them.

[15] The condition $(t+1)\sigma_i \geq \beta_i$, which brings the strict convexity of $A_i^{\frac{\sigma_i}{\beta_i - \sigma_i}}$, is identical to the condition of Lemma 2.2 in MacFadden (1963).

[16] See Esteban and Ray (2001); Epstein and Mealem (2009); Cheikbossian and Fayat (2018); Trevisan (2020); Kobayashi and Konishi (2021). Technically, this threshold condition is the extension of the condition regarding the elasticity of effort cost in Esteban and Ray (2001) and Trevisan (2020).

5 Role assignment

We consider the case in which any role a_{ik} is assignable in addition to the prize v_{ik} and cost-reduction resources c^i_{hk}. Each role is not tied to a group member. We consider the assignment rules of multiple roles to a single group member and a single role per group member.

5.1 Assignment of multiple roles to a single group member

We assume that the group manager can assign $a_{ik} \in [0, 1]$ continuously to each group member k such that $\sum_{k=1}^{n_i} a_{ik} = 1$. They may assign all the roles to a single group member k such as $a_{ik} = 1$. a_{ik} is inside the CES aggregator X_i, whereas v_{ik} and c^i_{hk} are outside it. This difference gives different powers to the variables in (4). The group manager maximizes (4) by using $t + 1$ types of controllable variables $v_{ik}, c^i_{1k}, \dots, c^i_{t-1k}$, a_{ik} for $k = 1, \dots, n_i$ such that $\sum_{k=1}^{n_i} v_{ik} = 1$, $\sum_{k=1}^{n_i} c^i_{hk} = 1$ for $h = 1, \dots, t-1$, and $\sum_{k=1}^{n_i} a_{ik} = 1$. Recall that (4) converges to the Cobb-Douglass function as $\sigma_i \to 0$. It is impossible to maximize (4) with these $t + 1$ kinds of variables simultaneously for the Cobb-Douglass function when we have the corner solution because it includes the indeterminate form 0^0. Instead, we consider the maximization of A^*_i derived in Proposition 1 with a_{ik}; that is, the group manager determines the role assignment rule, expecting the allocation rule that they determine.[17] We also allow the limit value as the solution. The assignment rule is described as follows:

Proposition 2. *In Proposition 1, suppose that role $a_{ik} \in [0, 1]$ is continuously assignable for any $k = 1, \dots, n_i$ such that $\sum_{k=1}^{n_i} a_{ik} = 1$. The rules of assignment and allocation in the subgame perfect equilibrium (equilibrium strategy) are then as follows: $a_{ij} = 1$ is assigned, $v_{ij} = c^i_{1j} = \dots = c^i_{t-1j} = 1$ is allocated to group member j with the highest $s^{\beta_i}_{ij}$, and $a_{il} = 0$ is assigned, $v_{il} = c^i_{1l} = \dots = c^i_{t-1l} = 0$ is allocated to group members $l = 1, \dots, j-1, j+1, \dots, n_i$. When $(t + 1)\sigma_i = \beta_i$ and there are multiple js with the highest $s^{\beta_i}_{ij}$, equal a_{ij} among js is also allowed*

[17] The group manager determines the assignment rule first and the allocation rule second in Stage 1.

and $v_{ij} = c_{1j} = \ldots = c^i_{t-1j} \in [0,1]$ *such that* $\sum_j v_{ij} = 1$ *and* $\sum_j c^i_{hj} = 1$ *is allocated. Nothing is assigned and allocated to the other group members. The maximized* A^*_i*, described as* A^{**}_i*, is* $A^{**}_i = s^{\beta_i}_{ij}$.

Once any role a_{ik} is assignable, for any effort complementarity, the group manager assigns all roles and allocates the full prize and cost reduction resources to the group member j with the highest skill $s^{\beta_i}_j$ in the subgame perfect equilibrium, while, under the strong complementarity of group members' effort, the prize and cost reduction resources are allocated to all group members when roles are not assignable, as shown in Proposition 1. Even if each group member's effort is essential ($\sigma_i \leq 0$), the group manager does so in the equilibrium. What makes such an assignment and allocation possible is that the group manager gets the other group members with lower skills not to participate in the contest by assigning no role to them. Subsequently, the CES aggregator becomes $X_i = s_{ij}e_{ij}$ for any $\sigma_i > -\infty$. The role assignment brings about more powerful productivity by the group member with the highest skill than that aggregated among all group members, even under the strong complementarity of their efforts.[18]

In Proposition 1, the group manager allocates the prize and cost reduction resources to all group members even under slightly weak complementarity if t is small. Suppose $t = 0$. Does the group manager assign roles to all the group members? Even when the group manager can assign only the roles and cannot allocate any prize or any cost reduction resources, that is, each group member maintains its inherent prize share and cost reduction resource ($c_{ij} = 1$), the result of Proposition 2 holds. The group manager then maximizes (4) for each a_{ik}.

Corollary 1. *When only the roles are assignable (*$t = 0$*),* $a_{ij} = 1$ *is assigned to the group member* j *with the highest* $s^{\beta_i}_{ij}v_{ij}$*, and the other* $a_{il} = 0$ *for* $l = 1, \ldots, j-1, j+1, \ldots, n_i$*.* $A^{**}_i = s^{\beta_i}_{ij}v_{ij}$*.*

Even in this case, the group manager assigns all the roles to the group member with the

[18]However, the role assignment is invalid under perfect complementarity because all assignable roles disappear in the Leontief function, as shown in the footnote 5.

highest $s_{ij}^{\beta_i} v_{ij}$ for any effort complementarity in the subgame perfect equilibrium. The other group members then completely free-ride this group member and receive their initial shares of the prize. For the group manager, their prize shares are unavailable and become wasteful. Despite such waste, it is best for the group manager to concentrate all roles on the most productive group member and not assign roles to the other group members. The results of Proposition 2 and Corollary 1 show the robustness of the concentration of all roles on the group member, regardless of the strength of effort complementarity and the number of types of group managers' allocatable variables.

5.2 Assignment of a single role per group member

Thus far, we have considered the case in which the group manager could assign multiple roles to a single group member. In reality, a case exists in which each group member takes only a single role. We consider this situation as follows: We assume that there are n_i different and discrete roles $a_{i1}, a_{i2}, \ldots, a_{in_i}$, which are $a_{i1} > a_{i2} > \ldots > a_{in_i} > 0$ and $\sum_{k=1}^{n_i} a_{ik} = 1$, instead of continuously assignable roles. a_{i1} is the most important role and a_{in_i} is the least important. The group manager can assign one role a_{ik} per group member. How does the group manager then assign roles to group members in a subgame perfect equilibrium? Proposition 1 is useful in this case. We have the following proposition:

Proposition 3. *In Proposition 1, suppose that roles $a_{i1}, a_{i2}, \ldots, a_{in_i}$ in A_i^* are $a_{i1} > \ldots > a_{in_i} > 0$ and assignable by the group manager, and suppose that the group manager can assign only a single role per group member. The role assignment rule in the subgame perfect equilibrium is then as follows:*

1. *When $(t+1)\sigma_i < \beta_i$ or $\sigma_i = 0$ in the limit, the group manager assigns the roles from a_{i1} to a_{in_i} to the group members in descending order of their skill level $s_{ik}^{\beta_i}$ for $k = 1, \ldots, n_i$.*

2. When $(t+1)\sigma_i \geq \beta_i$, the group manager assigns to the highest role a_{i1} the group member j with the highest $s_j^{\beta_i}$ and assigns any other roles to the other group members.

In Case 1 of Proposition 3, the roles $a_{i1}, \ldots, a_{in_i}$ are assigned to the group members in descending order of their skills in equilibrium. This assignment rule has the properties of both the allocation rule in Case 1 of Proposition 1 and the assignment rule in Proposition 2. While the group manager assigns a role to each group member owing to strong complementarity, the group manager wants to make the roles of the group members with low productivity as small as possible. However, in Case 2 of Proposition 3, the group manager assigns the most important role to the most productive group member and the other roles to any group member in the equilibrium because cost reduction resources and prizes are not allocated to the other group members owing to weak complementarity. This means that the group manager prioritizes the concentration of prizes and resources on the most productive group member by abandoning less important and substitutable roles. The results of Proposition 3 come from the allocation rule depending not on the assigned roles but on the condition of $(t+1)\sigma_i < (\geq)\beta_i$ as shown in Proposition 1.

6 Concluding remarks

We characterizes role assignments in maximizing a group's winning probability under the influence of the complementarity of group members' efforts in a group contest, comparing it with prize and resource allocations. Previous studies in the group contest literature have not analyzed this issue despite the significant effect of role assignment on groups' winning probability. We clarified the different properties of the role assignment rules from those of the prize and multiple resource allocation rules. When the group manager can assign roles to their group members without limitations, the assignment rule does not depend on complementarity: All roles are assigned to the most productive group member. When the group manager needs to assign a single role per group member, the assignment rule depends on complementarity: Roles from more to less importance are assigned to group members in descending order of their productivity under strong complementarity; only the most important role is assigned to the most productive group member and the others have no effect under weak complementarity.

If a group needs to conduct large-scale work in a contest, it is natural that its roles be shared among all group members. Nevertheless, the rule under weak complementarity in the latter assignment case appears strange. This is due to the fact that the prize and resources can be concentrated on a single group member. We need to consider the allocation rules including consideration of some types of capacity, which is a topic for future research.

Appendix: Proofs

Proof of Lemma 2. In (1), $\partial X_i/\partial e_{ik} > 0$ and $e_{ik}^{\beta_i - 1}/c_{ik} = 0$ when $e_{ik} = 0$ and $e_{il} > 0$, $l \neq k$. $0 \leq \partial X_i/\partial e_{ik} < +\infty$ and $e_{ik}^{\beta_i - 1}/c_{ik} = +\infty$ when $e_{ik} = +\infty$ and $e_{il} > 0$. The first-order condition (1) for any k is continuous with regard to e_{ik}. U_{ik} satisfies the second-order condition because X_i is a CES function. Therefore, there is a unique and continuous best-response function, and there is at least one fixed point $\boldsymbol{e_i^*} \neq \boldsymbol{0}$ for a given $X - X_i$.

Suppose that there are different $e_{ik}^* > 0$ and $\hat{e}_{ik} > 0$ that support $X_i^* > 0$ in $(X_1^*, \ldots, X_m^*)$ in a Nash equilibrium. By Lemma 1, such X_i^* is unique. e_{ik}^* and $\hat{e}_{ik}$ must satisfy the first-order condition (1). Noting that $P_i(1 - P_i)/X_i^{\sigma_i}$ in (1) is common for e_{ik}^* and $\hat{e}_{ik}$, (1) does not have multiple solutions. This result contradicts the existence of e_{ik}^* and $\hat{e}_{ik}$. Therefore, $\boldsymbol{e_i^*} \neq \boldsymbol{0}$ is unique. ■

Proof of Proposition 1.

We find the solution to maximize (4) by comparing the interior solution with the corner solution, noting that (4) is a CES function form. First we find the interior solution. Let $c_{hk}^i = y_{hk}$ for $h = 1, \ldots, t-1$, $v_{ik} = y_{tk}$, and $\frac{\sigma_i}{\beta_i - \sigma_i} = \rho$ in (4) for simplicity of notation. Then $A_i =$

$\left[\sum_{k=1}^{n_i}\left(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i}\Pi_{h=1}^{t} y_{hk}\right)^{\rho}\right]^{\frac{1}{\rho}}$. The Lagrange function is defined as

$$L = \left[\sum_{k=1}^{n_i}\left(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i}\Pi_{h=1}^{t} y_{hk}\right)^{\rho}\right]^{\frac{1}{\rho}} + \sum_{h=1}^{t}\lambda_h\left[1-\sum_{k=1}^{n_i} y_{hk}\right].$$

The first-order conditions are

$$\frac{\partial L}{\partial y_{sj}} = \left[\sum_{k=1}^{n_i}\left(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i}\Pi_{h=1}^{t} y_{hk}\right)^{\rho}\right]^{\frac{1}{\rho}-1} (a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}\Pi_{h\neq s} y_{hj})^{\rho} y_{sj}^{\rho-1} - \lambda_s = 0$$

$$\frac{\partial L}{\partial \lambda_s} = 1 - \sum_{k=1}^{n_i} y_{sk} = 0$$

for $s = 1, \ldots, t$ and $j = 1, \ldots, n_i$. From any $\partial L/\partial y_{sj}$ and $\partial L/\partial y_{sl}$, we have

$$\begin{aligned}
\frac{(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}\Pi_{h\neq s} y_{hj})^{\rho} y_{sj}^{\rho-1}}{(a_{il}^{\frac{\beta_i}{\sigma_i}} s_{il}^{\beta_i}\Pi_{h\neq s} y_{hl})^{\rho} y_{sl}^{\rho-1}} &= 1 \\
\iff \quad \frac{y_{sj}}{y_{sl}} &= \left(\frac{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}\Pi_{h\neq s} y_{hj}}{a_{il}^{\frac{\beta_i}{\sigma_i}} s_{il}^{\beta_i}\Pi_{h\neq s} y_{hl}}\right)^{\frac{\rho}{1-\rho}} = \left(\frac{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}(\Pi_{h\neq s,u} y_{hj}) y_{uj}}{a_{il}^{\frac{\beta_i}{\sigma_i}} s_{il}^{\beta_i}(\Pi_{h\neq s,u} y_{hl}) y_{ul}}\right)^{\frac{\rho}{1-\rho}}.
\end{aligned}$$

By choosing any u among $s = 1, \ldots, t$ and substituting y_{uj}/y_{ul} from $\partial L/\partial y_{uj}$ and $\partial L/\partial y_{ul}$ into y_{sj}/y_{sl}, we have

$$\frac{y_{sj}}{y_{sl}} = \left(\frac{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}\Pi_{h\neq s,u} y_{hj}}{a_{il}^{\frac{\beta_i}{\sigma_i}} s_{il}^{\beta_i}\Pi_{h\neq s,u} y_{hl}}\right)^{\frac{\rho}{1-\rho-\rho}}.$$

By repeating these choices and substitutions for all $1, \ldots, t$, we have

$$\frac{y_{sj}}{y_{sl}} = \left(\frac{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}}{a_{il}^{\frac{\beta_i}{\sigma_i}} s_{il}^{\beta_i}}\right)^{\frac{\rho}{1-t\rho}}.$$

This expression is

$$y_{sl} = \left(\frac{a_{il}^{\frac{\beta_i}{\sigma_i}} s_{il}^{\beta_i}}{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}}\right)^{\frac{\rho}{1-t\rho}} y_{sj}.$$

By substituting y_{sl} for all $l = 1, \ldots, n_i$ into $\partial L/\partial\lambda_s$ and solving for y_{sj}, we obtain the allocation rule of (7):

$$
\begin{aligned}
\sum_{k\neq j}\left(\frac{a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i}}{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}}\right)^{\frac{\rho}{1-t\rho}} y_{sj} + y_{sj} &= 1 \\
\frac{y_{sj}}{(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i})^{\frac{\rho}{1-t\rho}}} \sum_{k=1}^{n_i} (a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}} &= 1 \\
y_{sj} &= \frac{(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i})^{\frac{\rho}{1-t\rho}}}{\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}}} = \frac{(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}}{\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}}
\end{aligned}
$$

for $s = 1, \ldots, t$ and $j = 1, \ldots, n_i$.

Furthermore, by substituting all y_{sj} into A_i, we obtain:

$$
\begin{aligned}
A_i &= \left[\sum_{k=1}^{n_i}\left(\left(\frac{(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}}}{\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}}}\right)^t a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i}\right)^\rho\right]^{\frac{1}{\rho}} \\
&= \frac{1}{\left(\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}}\right)^t} \left[\sum_{k=1}^{n_i}\left((a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{t\rho}{1-t\rho}} a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i}\right)^\rho\right]^{\frac{1}{\rho}} \\
&= \frac{1}{\left(\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}}\right)^t} \left[\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}}\right]^{\frac{1}{\rho}} \\
&= \left[\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\rho}{1-t\rho}}\right]^{\frac{1-t\rho}{\rho}} = \left[\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}\right]^{\frac{\beta_i-(t+1)\sigma_i}{\sigma_i}}. \qquad (10)
\end{aligned}
$$

Second, noting that (4) is a CES function form, the corner solution is $c_{1j}^i = \ldots = c_{t-1j}^i = v_{ij} = 1$ for j with the highest $a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$ and the other $c_{1l}^i = \ldots = c_{t-1l}^i = v_{il} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n_i$. When $\sigma_i > 0$, the corner solution gives $A_i = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$. Then, (10) becomes

$$
A_i = \frac{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}}{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}} \left[\sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}\right]^{\frac{\beta_i-(t+1)\sigma_i}{\sigma_i}} = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i} \left[\sum_{k=1}^{n_i}\left(\frac{a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i}}{a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}}\right)^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}\right]^{\frac{\beta_i-(t+1)\sigma_i}{\sigma_i}}. \qquad (11)
$$

The jth term in the summation in (11) is 1. When $\beta_i - (t+1)\sigma_i > 0$, the brackets in (11) are greater than 1 and the interior solution yields a larger A_i than the corner solution. The corner solution gives $\lim_{\sigma_i \to 0} A_i = \Pi_{k=1}^{n_i}(a_{ik}s_{ik}^{\beta_i}c_{ik}v_{ik})^{a_{ik}} = (a_{ij}s_{ij}^{\beta_i})^{a_{ij}}\Pi_{k\neq j}(a_{ik}s_{ik}^{\beta_i}0)^{a_{ik}} = 0$ in (4) and when $\sigma_i < 0$ it gives

$$A_i = \frac{1}{\left[\sum_{k=1}^{n_i} a_{ik}^{\frac{\beta_i}{\beta_i-\sigma_i}}/(s_{ik}^{\beta_i}c_{ik}v_{ik})^{\frac{-\sigma_i}{\beta_i-\sigma_i}}\right]^{\frac{\beta_i-\sigma_i}{-\sigma_i}}} = \frac{1}{\left[a_{ij}^{\frac{\beta_i}{\beta_i-\sigma_i}}/(s_{ij}^{\beta_i})^{\frac{-\sigma_i}{\beta_i-\sigma_i}} + \sum_{k\neq j} a_{ik}^{\frac{\beta_i}{\beta_i-\sigma_i}}/(s_{ik}^{\beta_i}0)^{\frac{-\sigma_i}{\beta_i-\sigma_i}}\right]^{\frac{\beta_i-\sigma_i}{-\sigma_i}}} = 0.$$

When $\sigma_i \leq 0$, the interior solution yields a larger A_i than does the corner solution. Therefore, we obtain (7) and (8) in Case 1 in the proposition.

When $\beta_i - (t+1)\sigma_i < 0$ (then $\sigma_i > 0$) in (11), the inside of the brackets is greater than 1 and the brackets are the denominator, so the brackets are less than 1. The corner solution then yields a larger A_i than the interior solution.

When $\beta_i - (t+1)\sigma_i = 0$ (then $\sigma_i > 0$), that is, $\sigma_i/(\beta_i - (t+1)\sigma_i) = +\infty$, in (11), the inside of brackets in (11) becomes 1 because all terms without the jth are 0 from $(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})/(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}) < 1$ for any $k \neq j$. The brackets become 1 and the interior solution yields the same A_i as the corner solution. When $\beta_i - (t+1)\sigma_i = 0$ and there are multiple group members j with the highest $a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$ in (4), any allocation such that $c_{1j}^i = \ldots = c_{t-1j}^i = v_{ij} \in [0,1]$ for each j and $\sum_j c_{hj}^i = \sum_j v_{ij} = 1$ also becomes a solution because

$$A_i = \left[\sum_j \left(a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}(\Pi_{h=1}^{t-1}c_{hj}^i)v_{ij}\right)^{\frac{\sigma_i}{\beta_i-\sigma_i}}\right]^{\frac{\beta_i-\sigma_i}{\sigma_i}} = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}\left[\sum_j v_{ij}^{\frac{t\sigma_i}{\beta_i-\sigma_i}}\right]^{\frac{\beta_i-\sigma_i}{\sigma_i}} = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$$

where $t\sigma_i/(\beta_i - \sigma_i) = 1$ from $\beta_i = (t+1)\sigma_i$. Therefore, Case 2 in the proposition is obtained.

Finally, we calculate (7) and (8) at $\sigma_i = 0$ in the limit. In (7), we immediately have $v_{ij} = c_{1j}^i = \ldots = c_{t-1j}^i \to \frac{a_{ij}}{\sum_{k=1}^{n_i} a_{ik}} = a_{ij}$ as $\sigma_i \to 0$. In (8), $\log A_i^* = \frac{1}{\sigma_i/(\beta_i-(t+1)\sigma_i)} \log \sum_{k=1}^{n_i}(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}$.

$$\begin{aligned}\lim_{\sigma_i\to 0}\log A_i^* &= \lim_{\sigma_i\to 0}\frac{1}{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}}\log\sum_{k=1}^{n_i} a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} \\ &= \lim_{\sigma_i\to 0}\frac{1}{\frac{\beta_i}{(\beta_i-(t+1)\sigma_i)^2}}\frac{\sum_{k=1}^{n_i}\left[\frac{da_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}}{d\sigma_i} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} + a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}\frac{ds_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}}}{d\sigma_i}\right]}{\sum_{k=1}^{n_i} a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}}}.\end{aligned}$$

Let $Z = a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}$ and $T = s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}}$. Then, $\log Z = \frac{\beta_i}{\beta_i-(t+1)\sigma_i}\log a_{ik} \Rightarrow Z'/Z = \frac{\beta_i(t+1)}{(\beta_i-(t+1)\sigma_i)^2}\log a_{ik} \Leftrightarrow Z' = a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}\frac{\beta_i(t+1)}{(\beta_i-(t+1)\sigma_i)^2}\log a_{ik}$, and $\log T = \frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}\log s_{ik} \Rightarrow T'/T = \frac{\beta_i^2}{(\beta_i-(t+1)\sigma_i)^2}\log s_{ik}$

$\Leftrightarrow T' = s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} \frac{\beta_i^2}{(\beta_i-(t+1)\sigma_i)^2} \log s_{ik}$. Substituting these into the above limit expression, we obtain:

$$\lim_{\sigma_i \to 0} \frac{1}{\frac{\beta_i}{(\beta_i-(t+1)\sigma_i)^2}} \frac{\sum_{k=1}^{n_i} \left[a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} \frac{\beta_i(t+1)}{(\beta_i-(t+1)\sigma_i)^2} (\log a_{ik}) s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} + a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} \frac{\beta_i^2}{(\beta_i-(t+1)\sigma_i)^2} \log s_{ik} \right]}{\sum_{k=1}^{n_i} a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}}}$$

$$= \frac{1}{\frac{1}{\beta_i}} \frac{\sum_{k=1}^{n_i} \left[a_{ik} \frac{t+1}{\beta_i} \log a_{ik} + a_{ik} \log s_{ik} \right]}{\sum_{k=1}^{n_i} a_{ik}}$$

$$= \sum_{k=1}^{n_i} \left[\log a_{ik}^{(t+1)a_{ik}} s_{ik}^{\beta_i a_{ik}} \right] = \log \Pi_{k=1}^{n_i} (a_{ik}^{t+1} s_{ik}^{\beta_i})^{a_{ik}}.$$

Therefore, we obtain (9) in Case 1 of the proposition. ■

Proof of Proposition 2.

Let $a_{ik} < 1$ for any k, which is a component of any interior point $\boldsymbol{a_i}$ in each A_i^* in Proposition 1, and let $s_{ij} = \max\{s_{i1}, \ldots, s_{in_i}\}$. When $\frac{\sigma_i}{\beta_i-(t+1)\sigma_i} > 0$ in (8), $\sigma_i > 0$ and $\beta_i - (t+1)\sigma_i > 0$. Then $\frac{\beta_i}{\beta_i-(t+1)\sigma_i} > 1$. Case 1 in Proposition 1 applies to this case. Noting that $s_{ik}^{\beta_i}/s_{ij}^{\beta_i} \leq 1$ and $a_{ik} < 1$, (8) is

$$A_i^* = \frac{s_{ij}^{\beta_i}}{s_{ij}^{\beta_i}} \left[\sum_{k=1}^{n_i} (a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}} \right]^{\frac{\beta_i-(t+1)\sigma_i}{\sigma_i}} = s_{ij}^{\beta_i} \left[\sum_{k=1}^{n_i} \left(a_{ik}^{\frac{\beta_i}{\sigma_i}} \frac{s_{ik}^{\beta_i}}{s_{ij}^{\beta_i}} \right)^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}} \right]^{\frac{\beta_i-(t+1)\sigma_i}{\sigma_i}}$$

$$\leq s_{ij}^{\beta_i} \left[\sum_{k=1}^{n_i} a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} \right]^{\frac{\beta_i-(t+1)\sigma_i}{\sigma_i}} < \beta_{ij}^{\beta_i}.$$

Therefore, we obtain a corner solution of $a_{ij} = 1$ with the highest $s_{ij}^{\beta_i}$ and the other $a_{il} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n_i$.

When $\frac{\sigma_i}{\beta_i-(t+1)\sigma_i} < 0$ and $\sigma_i > 0$, $\beta_i - (t+1)\sigma_i < 0$. Case 2 of Proposition 1 is applicable to this case. The solution for maximizing $A_i^* = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$ is identical to the corner solution when $\frac{\sigma_i}{\beta_i-(t+1)\sigma_i} > 0$.

When $\frac{\sigma_i}{\beta_i-(t+1)\sigma_i} < 0$ and $\sigma_i < 0$, $\beta_i - (t+1)\sigma_i > 0$. Then $\frac{\beta_i}{\beta_i-(t+1)\sigma_i} < 1$. Case 1 of Proposition

1 is applicable to this case. From (8) and $s_{ij} \geq s_{ik}$, we have

$$\begin{aligned} A_i^* &= \frac{1}{\left[\sum_{k=1}^{n_i} (a_{ik}/s_{ik}^{-\sigma_i})^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}\right]^{\frac{\beta_i-(t+1)\sigma_i}{-\sigma_i}}} \\ &\leq \frac{1}{\left[\sum_{k=1}^{n_i} (a_{ik}/s_{ij}^{-\sigma_i})^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}\right]^{\frac{\beta_i-(t+1)\sigma_i}{-\sigma_i}}} = \frac{s_{ij}^{\beta_i}}{\left[\sum_{k=1}^{n_i} a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}}\right]^{\frac{\beta_i-(t+1)\sigma_i}{-\sigma_i}}} < s_{ij}^{\beta_i}. \end{aligned}$$

The last inequality comes from the brackets larger than 1 because $\frac{\beta_i}{\beta_i-(t+1)\sigma_i} < 1$. Therefore, we have the corner solution.

When $\frac{\sigma_i}{\beta_i-(t+1)\sigma_i} = 0$, $\sigma_i = 0$ and $\beta_i - (t+1)\sigma_i = \beta_i > 0$. Case 1 of Proposition 1 is applicable to this case. The corner solution gives an indeterminate form for (9), 0^0. Instead, suppose that we have the limit value as the corner solution of maximization: $a_{ij} \to 1$ with the highest $s_{ij}^{\beta_i}$ and the other $a_{il} \to 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n_i$. Then, for every factor of $l \neq j$ in (9), $\lim_{a_{il}\to 0}(a_{il}^{t+1} s_{il}^{\beta_i})^{a_{il}} = 1$ because

$$\begin{aligned} \lim_{a_{il}\to 0} \log(a_{il}^{t+1} s_{il}^{\beta_i})^{a_{il}} &= \lim_{a_{il}\to 0} \frac{(t+1)\log a_{il} + \beta_i \log s_{il}}{1/a_{il}} \\ &= \lim_{a_{il}\to 0} \frac{(t+1)/a_{il}}{-1/a_{il}^2} \\ &= \lim_{a_{il}\to 0} [-(t+1)a_{il}] = 0. \end{aligned}$$

Then $A_i^* \to s_{ij}^{\beta_i} (= A_i^{**})$. By contrast, for any interior point with $0 < a_{ik} < 1$ for any k such that $\sum_{k=1}^{n_i} a_{ik} = 1$, if we let $a_{il} = \max\{a_{i1}, \ldots, a_{in_i}\}$, we have

$$\begin{aligned} A_i^* &= \Pi_{k=1}^{n_i} (a_{ik}^{t+1} s_{ik}^{\beta_i})^{a_{ik}} \\ &\leq \Pi_{k=1}^{n_i} (a_{il}^{(t+1)a_{ik}} s_{ij}^{\beta_i a_{ik}}) = a_{il}^{t+1} s_{ij}^{\beta_i} < s_{ij}^{\beta_i}. \end{aligned}$$

Therefore, we have the corner solution as the limit value.

When $\frac{\sigma_i}{\beta_i-(t+1)\sigma_i} = +\infty$, $\sigma_i > 0$ and $\beta_i = (t+1)\sigma_i$. Case 2 of Proposition 1 is applicable to this case. The solution maximizing $A_i^* = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$ is identical to the above corner solution. If there are multiple group members j with the highest $s_{ij}^{\beta_i}$, we have $A^{**} = s_{ij}^{\beta_i}$ by assigning equal a_{ij} to those

js and assigning nothing to the others because $A^* = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$ is obtained from Proposition 1 by this assignment.

By substituting the corner solution into (7) and (8), we have $c_{1j}^i = \ldots = c_{t-1j}^i = v_{ij} = 1$, $c_{1l}^i = \ldots = c_{t-1l}^i = v_{il} = 0$ and $A_i^{**} = s_{ij}^{\beta_i}$. Therefore, the message of the proposition holds true. ■

Proof of Corollary 1. Because only roles are assignable, by setting $t = 0$, replacing A_i^* with A_i, and replacing $s_{ik}^{\beta_i}$ with $s_{ik}^{\beta_i} v_{ik}$ in the proof of Proposition 2, we have this corollary. ■

Proof of Proposition 3.

In Case 1 of Proposition 1, in which $(t+1)\sigma_i < \beta_i$, the parentheses in the brackets in A_i^* of (8) are rewritten as $(a_{ik}^{\frac{\beta_i}{\sigma_i}} s_{ik}^{\beta_i})^{\frac{\sigma_i}{\beta_i-(t+1)\sigma_i}} = a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}}$. For any $s_{ik}^{\beta_i} > s_{il}^{\beta_i}$ and any $a_{ik} > a_{il}$, when $\sigma_i > 0$,

$$
\begin{aligned}
& \left(a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} + a_{il}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{il}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} \right) \\
& - \left(a_{il}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} + a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} s_{il}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} \right) \\
= \; & \left[a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} - a_{il}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} \right] \left[s_{ik}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}} - s_{il}^{\frac{\beta_i\sigma_i}{\beta_i-(t+1)\sigma_i}}) \right] > 0.
\end{aligned} \tag{12}
$$

From this, by assigning the higher a_{ik} to the group member with the higher $s_{ik}^{\beta_i}$, A_i^* becomes larger than any other assignment because of the additive form in the brackets of A_i^*. Therefore, the role assignment rule in $0 < (t+1)\sigma_i < \beta_i$ is to assign roles from a_{i1} to a_{in} to group members in descending order of their productivity $s_{ik}^{\beta_i}$.

When $\sigma_i = 0$ in the limit, A_i^* in (9) is $\Pi_{k=1}^{n_i}(a_{ik}^{1+t} s_{ik}^{\beta_i})^{a_{ik}} = \Pi_{k=1}^{n_i}(a_{ik}^{(1+t)a_{ik}} s_{ik}^{\beta_i a_{ik}})$ in Proposition 1. For any $s_{ik}^{\beta_i} > s_{il}^{\beta_i}$ and any $a_{ik} > a_{il}$,

$$
\begin{aligned}
& \left(a_{ik}^{(t+1)a_{ik}} s_{ik}^{\beta_i a_{ik}} \right) \left(a_{il}^{(t+1)a_{il}} s_{il}^{\beta_i a_{il}} \right) - \left(a_{il}^{(t+1)a_{il}} s_{ik}^{\beta_i a_{il}} \right) \left(a_{ik}^{(t+1)a_{ik}} s_{il}^{\beta_i a_{ik}} \right) \\
= \; & a_{ik}^{(t+1)a_{ik}} a_{il}^{(t+1)a_{il}} s_{ik}^{\beta_i a_{il}} s_{il}^{\beta_i a_{ik}} \left[\left(\frac{s_{ik}^{\beta_i}}{s_{il}^{\beta_i}} \right)^{a_{ik}-a_{il}} - 1 \right] > 0.
\end{aligned}
$$

From this, by assigning the higher a_{ik} to the group member with the higher $s_{ik}^{\beta_i}$, A_i^* becomes larger than any other assignment. Therefore, the role assignment rule when $\sigma_i = 0$ in the limit is identical to that when $0 < (t+1)\sigma_i < \beta_i$.

When $\sigma_i < 0$, the brackets in (8) are in its denominator. The inside of the brackets should be minimized. Noting that $-\sigma_i > 0$ and $-(t+1)\sigma_i > 0$, the inside of the parentheses in the brackets in (8) is $a_{ik}^{\frac{\beta_i}{\beta_i-(t+1)\sigma_i}} / s_{ik}^{\frac{-\sigma_i\beta_i}{\beta_i-(t+1)\sigma_i}}$. The rule identical to that in $0 < (t+1)\sigma_i < \beta_i$ and $\sigma_i = 0$ in the limit minimizes the inside of the brackets in A_i^*, that is, maximizes A_i^*, from the same calculation as (12).

When $(t+1)\sigma_i \geq \beta_i$, in Case 2 of Proposition 1, $A_i^* = a_{ij}^{\frac{\beta_i}{\sigma_i}} s_{ij}^{\beta_i}$. To maximize A_i^*, a_{i1} should be allocated to that group member j who has the highest $s_{ik}^{\beta_i}$, and the other roles $a_{i2}, \ldots, a_{in_i}$ are assigned to any other group members because there are no terms for them in A_i^*. In this case, this maximization is possible because the complementarity is weak. ■

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