Minimum price equilibrium in the assignment market: The Serial Vickrey mechanism[1]

Yu Zhou[2] Shigehiro Serizawa[3]

This version: April 25, 2023

Abstract

We study an assignment market where multiple heterogenous objects are sold to unit demand agents who have general preferences that accommodate income effects and market frictions. The minimum price equilibrium (MPE) is one of the most important equilibrium notions in such settings. Nevertheless, none of the well-known mechanisms that find the MPEs in the quasi-linear environment can identify or even approximate the MPEs for general preferences. We establish novel structural characterizations of MPEs and design the "Serial Vickrey (SV) mechanism" based on the characterizations. The SV mechanism finds an MPE for general preferences in a finite number of steps. Moreover, the SV mechanism only requires agents to report finite-dimensional prices in finitely many times, and also has nice dynamic incentive properties.

Keywords: The assignment market, minimum price equilibrium, general preferences, structural characterizations, Serial Vickrey mechanism, dynamic incentive compatibility

JEL Classification: C63, C70, D44

[1]The preliminary version of this article is presented at 2017, 2019 Economic Design Conferences, 2018 SPMID conference, PET 2018, the 14th Meeting of the Society for Social Choice, the 19th SAET, 2019 APIOC, 2019 Nanjing International Conference on Theory of Games and Economic Behavior, European Meeting on Game Theory (SING 13, 16), 2021 Asian Metting of Econometric Society, 6th World Congress of the Game Theory Society, EEA-ESEM 2021, and seminars at City University of Hong Kong, ITAM, Keio, and Maastricht. We thank Tommy Andersson, Itai Ashlagi, Lawrence Ausubel, Vincent Crawford, Chuangyin Dang, Jean-Jacques Herings, Fuhito Kojima, Scott Duke Kominers, Andy Mackenzie, Debasis Mishra, Toyotaka Sakai, James Schummer, Ning Sun, Dolf Talman, Alex Teytelboym, Ryan Tierney, William Thomson, and Rakesh Vohra for their helpful comments. We gratefully acknowledge financial support from the Joint Usage/Research Center at ISER, International Joint Research Promotion Program (Osaka University), and Grant-in-aid for Research Activity, Japan Society for the Promotion of Science (15J01287, 15H03328, 15H05728, 19K13653, 20H05631, 20KK0027).

[2]Graduate School of Economics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-8501, Japan. E-mail: zhouyu_0105@hotmail.com

[3]Faculty of Economics, Osaka University of Economics, 2-2-8, Osumi, Higashiyodogawa-ku, Osaka, 533-8533, Japan. E-mail: serizawa@iser.osaka-u.ac.jp; serizawa@osaka-ue.ac.jp

1 Introduction

1.1 Motivation and main results

In auction and matching theory, quasi-linearity of preferences is commonly assumed. It means that the payment for a good has no limit and exhibits no income effect for its demand, or utilities are perfectly transferable among agents. Quasi-linearity simplifies analysis and makes linear programming techniques applicable to auction and matching problems (Vohra, 2011).[1]

Nevertheless, there are many important practical applications that lead to the non-quasi-linearity of preferences. In the spectrum license auctions in OECD countries, private firms often borrow to pay for the huge winning bids and face non-linear borrowing costs (Klemperer, 2004). Thus their utility derived from spectrum licenses changes non-linearly with respect to their actual payments. Distortional taxes like transaction taxes are widely seen in practical transactions. Even if agents have quasi-linear preferences over objects, introducing transaction taxes that proportionally charge the payments makes agents' utility change non-linearly with respect to the actual transfers (Fleiner et al. 2019).

The properties of Walrasian equilibria beyond the quasi-linear environment such as its equivalence to stability notions, lattice properties, and rural hospital theorems gradually attract more and more attention, see. e.g., Crawford and Knoer (1981), Quinzi (1984), Demange and Gale (1985), Noldeke and Samuelson (2018), Fleiner et al. (2019), and Schlegel (2021). However, little progress has been made of how to obtain some desirable Walrasian equilibrium via mechanisms with nice properties of information revelation and incentives for non-quasi-linear preferences. We are motivated to fill such a research gap.

This paper builds on the assignment market model where multiple heterogenous objects are sold to unit demand agents whose preferences are not necessarily quasi-linear, i.e., they have *general preferences*. In this setting, a Walrasian equilibrium always exists and there is a minimum price equilibrium (MPE) whose price (vector) is coordinate-wise minimum among all equilibrium prices. The MPE plays a central role in the efficient and incentive-compatible mechanism design, and the MPE rule, which maps to each preference profile an MPE, is characterized by efficiency, strategy-proofness, and fairness.[2] However, the definition of MPE contains no formula to identify its price or allocation. Moreover, the previous characterizations of MPE provide no clue to find MPEs for general preferences. *We establish novel structural characterizations of MPEs that provide bases to construct a mechanism that finds an MPE for general preferences. The mechanism has nice incentive properties and has agents reveal finite-dimensional information on their preferences in a finite number of times.*

When agents have quasi-linear preferences, the MPE is known to be equivalent to the

[1] The recent development of (discrete) convex analysis and tropical geometry techniques relies on quasi-linearity, e.g., Murota (2003) and Baldwin and Klemperer (2019).

[2] See, e.g., Morimoto and Serizawa (2015), Zhou and Serizawa (2018), and Kazumura et al. (2020).

Vickrey outcomes (Leonard, 1983). The sealed-bid Vickrey auction (Leonard, 1983), the exact auction of Demange et al. (1986), and the approximate auction of Demange et al. (1986), originating from the salary adjustment process in Crawford and Knoer (1981), are the well-known mechanisms to find the MPEs. However, Zhou and Serizawa (2021) exemplify that none of these mechanisms identify or even approximate the MPEs and fail to be efficient and incentive-compatible when agents have general preferences (See also Section 3.2), i.e.:

- The sealed-bid generalized Vickrey auction fails to identify the MPE.
- The exact auction of Demange et al. (1986) can overshoot the MPE price arbitrarily far.
- The approximate auction of Demange et al. (1986) may overshoot or undershoot the MPE price arbitrarily far.
- The price paths of the continuous versions of the above exact and approximate auctions may not be well defined.

Zhou and Serizawa (2021) further point out that other mechanisms that find MPEs in the quasi-linear environment proposed by Mishra and Parkes (2010), Andersson and Erlanson (2013), and Liu and Bagh (2019), face the similar shortages as the auctions in Demange et al. (1986). Gul and Stacchett (2000), Ausubel (2006), and Sun and Yang (2006) also propose auctions that identify the MPEs when agents have multi-unit demand quasi-linear preferences. However, when applied to the unit-demand settings, their auctions coincide with the exact auction of Demange et al. (1986), and so face the same problem as pointed above. The salary adjustment process of Kelso and Crawford (1982) and the cumulative offers process proposed by Hatfield and Milgrom (2005) coincide with the approximate auction when applied to our models. Therefore their processes face the same shortage as pointed above.

The above discussion indicates the challenges of our research. Therefore, designing desirable mechanisms that implement the MPE for general preferences requires novel analysis of the properties of Walrasian equilibrium and the MPE.

Our main results consist of two parts, which is illustrated in Section 3.3.

The first part analyzes the properties of Walrasisan equilibria and MPE, stated as Proposition 1, Theorems 1 and 2.

Given an arbitrary Walrasian equilibrium, agents and objects can be partitioned via the "connectedness" property. An object is *connected* if either it is unassigned or is connected to an object with zero price via a sequence of distinct agents' demands, each demanding both her assigned object and the object assigned to the successive agent. An agent is *connected* if she either gets a connected object or nothing.

Proposition 1 shows that a Walrasian equilibrium is an MPE if and only if each object is connected/each agent is connected. Thus, the difference between an MPE and an arbitrary Walrasian equilibrium is whether all the agents and objects are connected.

We further argue that we can always find an MPE from an arbitrary Walrasian equilibrium, by employing an adjustment process that makes the unconnected agents and objects at the given Walrasian equilibrium eventually become connected. This process is called the "I pay others' indifference prices process." We characterize the link between an arbitrary Walrasian equilibrium to an MPE through this process in Theorems 1 and 2. Theorems 1 and 2 show the novel and robust structural properties of Walrasian equilibria beyond the quasi-linear settings.

The second part is to use above characterization results (See Table 1 below) to design the "Serial Vickrey (SV) mechanism." The SV mechanism obtains an MPE via finite-dimensional revelation in finitely many times and moreover is dynamically incentive compatible, as stated in Theorems 3 and 4.

The SV mechanism elicits agents' preference information by asking agents to report "indifference prices" in finitely many times. An agent's indifference price of an object is her willingness to pay for the object, evaluated from her provisionally assigned bundle (object-payment pair). In the quasi-linear environment, an agent's preference is simply presented by a finite-dimensional vector of valuations. However, revealing the general preference is infinite-dimensional information revelation.

The SV mechanism introduces objects one by one, and based on an MPE for k objects, it employs the "SV sub-mechanism," to find an MPE for $k+1$ objects. When introducing the first object, the SV sub-mechanism coincides with the second-price auction. In general, the SV sub-mechanism contains two stages. Given an MPE for k objects, Stage 1 constructs a Walrasian equilibrium for $k+1$ via the "Equilibrium-generating mechanism." If there are agents and objects that are not connected, we further proceed to Stage 2. Stage 2 conducts reassignment and price adjustment of unconnected objects among unconnected agents via "MPE-assignment mechanism." *Stage 2 is not a replication of Theorems 1 and 2, and it is a process drawn on these results that iteratively finds an MPE.*

Table 1: The connection between characterization results and SV sub-mechanism

Characterization	**Implication in the SV sub-mechanism**
Prop. 1	Stage 1: Equilibrium-generating mechanism (Def. 5)
Thms 1 & 2	Stage 2: The MPE-assignment mechanism (Def. 7)

Finally, we show that no agent can benefit from misreporting within each SV sub-mechanism that obtains the MPE with $k+1$ objects from the MPE with k objects. Besides, no agent can benefit from misreporting even in each stage within a given SV sub-mechanism.

1.2 Related literature

The practice-driven and technique-driven investigation of auction and matching theory for general preferences attract more and more attention. It roughly goes into the following directions.

The first direction is to study the optimal mechanism design of Myerson (1980)'s framework, by allowing agents to have non-quasi-linear preferences over alternatives, see, e.g., Baisa (2016,

2017) and Gershkov et al. (2021). They develop new techniques that substantially improve the Myersionian approach of analyzing the optimal mechanism.

The second direction is to study efficiency, strategy-proofness, and fairness properties of rules such as the generalized Vickrey rules, the MPE rules, the minimum rationing price equilibrium rule, and the cumulative offer rules when agents have general preferences, see, e.g., Andersson and Svensson (2014), Afacan (2017), Morimoto and Serizawa (2015), Baisa (2020), Malik and Mishra (2021), and Hatfield et al. (2021). Although rules are functions/correspondences, these characterizations help justify the importance of some particular solution concepts like the MPE and show the possible range/domains that might admit mechanisms with desirable properties. For example, if we extend our model by allowing agents to have multi-unit demand, there is no efficient and strategy-proof rule (Baisa, 2020). Thus, to design efficient and incentive-compatible mechanisms in the non-quasi-linear environment, we generally need to focus on the case where agents have unit demand.

The third direction is to study the properties of Walrasian equilibria and some other solution concepts from the cooperative game theory like stable outcomes and core outcomes. The main focus is to establish the equivalence result between Walrasian equilibrium and cooperative game theoretical solution concepts, the lattice property of Walrasian equilibrium, and rural hospital theorems for Walrasian equilibrium, see, e.g., Quinzi (1984), Demange and Gale (1985), Fleiner et al. (2019), and Schlegel (2021). There are also works that study the above properties of Walraisan equilibria in the matching models with various types of constraints such as price controls and employment constraints, see, e.g., Herings (2018) and Kojima et al. (2020). Noldeke and Samuelson (2018) study the duality relationship without quasi-linearity in the one-to-one two-sided matching model, and characterize the duality of implementable profiles and assignments via the Galois connection. The Galois connection is a pair of mappings without any detailed process to identify the stable outcomes.

On the other hand, how to compute or implement some particular equilibrium notion via mechanisms with socially desirable properties is also of great importance. The above results are important in understanding the structure and properties of certain equilibrium concepts, but not very helpful in implementing particular solution concepts.

To proceed the study of equilibrium computation or equilibrium implementation via desirable mechanisms, the starting point is to analyze some structural properties of equilibria. Some attempts have been made in various models with general preferences.

In the same model as ours, Caplin and Leahy (2014) characterize the MPE price via the graph-allocation structure, which initiates Caplin and Leahy (2020) to employ the homotopy method to conduct the comparative static analysis of the MPE. They conjecture the possibility of designing mechanisms for the MPE based on their graph-allocation structure. The graph-allocation characterization exhausts all connections between all agents and objects via the forest formation, which is different from our results.[3] This paper instead provides a dynamically incentive-compatible mechanism that iteratively obtains an MPE.

In the model of Demange and Gale (1985), Alaei et al. (2016) show that one-sided optimal outcome can be obtained recursively via the optimal outcomes of two sides in all possible subeconomies. Their result indicates that in our model, the MPE can be characterized recursively via the MPE and maximum price equilibrium for all possible subeconomies. In contrast, the SV mechanism only requires the MPE for the subeconomies with fewer object.[4] Besides, the SV mechanism iteratively finds an MPE and has the dynamic incentive property. Section 7 also details some applications of our results that are not covered by theirs.

In the assignment market with price controls, Andersson and Svensson (2018) construct a finite ascending-price sequence that finds a "minimum rationing price equilibrium." This sequence terminates at an MPE price in our model. It is not clear how to identify two adjacent prices in the sequence in finitely many steps so their construction is different from the SV mechanism. Galichon et al. (2019) provide the theoretical model with potential application to structural estimation of imperfectly transferable utilities in the one-to-one two-sided matching model. They define an aggregate equilibrium that can be characterized by a system of equations in terms of matching pairs. Their proposed equilibrium is different from the MPE, and the corresponding characterization does not hold for the MPE.

Notice that our SV mechanism is also essentially different from the equilibrium existence proof via the Scarf lemma (Quinzi, 1984) or the tatonnement process in the spirit of Kelso and Crawford (1982). These approaches in general obtain an approximate equilibrium in the economies by using the piece-wise utility functions or with discretized payments, and then take the limit argument. Thus the finite information revelation and incentive property are lost.

Overall, **our main contribution** is placed on the third direction of existing literature that develops the auction and matching theory with non-quasi-linear preferences. We provide results that characterize new structural properties of Walrasian equilibria. These properties provide novel insights in implementing equilibrium notions via mechanisms with desirable informational and incentive properties for general preferences.

The remainder is organized as follows: Section 2 defines the model and MPEs. Section 3 illustrates the SV mechanism. Section 4 gives the structural characterizations. Sections 5

[3]In Caplin and Leahy (2014), objects are nodes and agents' indifference prices between two objects form branches. In our language, see, Theorem 1, to generate a price for an arbitrary assignment of m unconnected agents, the IPOIP process operates at most m rounds. However, they first obtain a price for each forest and then choose the coordinate-wise price as the price for the given assignment. Notice that the number of forests that supports the given assignment is $\sum_{k=1}^{m} C_m^k \cdot k \cdot m^{m-1-k}$.

[4]Suppose that there are more agents than objects, i.e., $n > m$. In the worst-case scenario, the number of subeconomies that the SV mechanism deals with is $m + 2! + \cdots + m!$. It is independent of n. The number of subeconomies that Alaei et al. (2016) deal with is $m \cdot n + \sum_{k=1}^{m-1} m! \cdot n! \backslash (m-k)! \cdot (n-k)!$.

presents SV mechanism and shows its convergence property. Section 6 shows the incentive property of the SV mechanism. Section 7 concludes by providing further discussion.

2 The model and minimum price equilibrium

There are finite sets of agents $N \equiv \{1, \ldots, n\}$ and objects $M \equiv \{1, \ldots, m\}$. Objects are allowed to be heterogenous or identical. Receiving nothing is called receiving object 0. Let $L \equiv M \cup \{0\}$. Each agent has **unit demand**: She receives at most one object. The **bundle** for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$, consisting of an object $x_i \in L$ and a payment $t_i \in \mathbb{R}$. Each agent i has a complete and transitive preference R_i over $L \times \mathbb{R}$. Let P_i and I_i be the associated strict and indifference relations.

We focus on the **general preference** R_i that satisfies the following two standard properties:

Money monotonicity: For each $x_i \in L$, each pair $t_i, t_j \in \mathbb{R}$, if $t_i < t_i'$, $(x_i, t_i)\, P_i\, (x_i, t_i')$.

Finiteness: For each $t_i \in \mathbb{R}$, each pair $x_i, x_j \in L$, there is $t_j \in \mathbb{R}$ such that $(x_i, t_i)\, I_i\, (x_j, t_j)$.

Money monotonicity states that for a given object, a lower payment makes the agent better off. Finiteness says that no object is infinitely good or bad. Let $\mathcal{R}$ be the set of general preferences and $R \equiv (R_i)_{i \in N} \in \mathcal{R}^n$ be a preference profile.

By money monotonicity and finiteness, for each $R_i \in \mathcal{R}$, each $z_i \in L \times \mathbb{R}$ and each $y \in L$, there is a unique amount $V_i(y; z_i) \in \mathbb{R}$ such that $(y, V_i(y; z_i))\, I_i\, z_i$. We call $V_i(y; z_i)$ the **indifference price of y at z_i for R_i.**

An assignment market is summarized by (N, M, R). We fix an assignment market till we come to analyze the incentives in Section 6.

Let $x \equiv (x_1, \ldots, x_n) \in L^n$ be an (**object**) **assignment** such that except for object 0, no two agents get the same object. Let X be the set of assignments. Given $x \in X$, let $t \equiv (t_1, \ldots, t_n) \in \mathbb{R}^n$ denote the associated payment. An **allocation** is a list of bundles $z \equiv (x, t) \equiv (z_1, \ldots, z_n) \in [L \times \mathbb{R}]^n$ such that $(x_1, \ldots, x_n) \in X$. We denote the set of allocations by Z. Given $z \in Z$ and $N' \subseteq N$, let $z_{N'} \equiv (z_i)_{i \in N'}$ and $z_{N \setminus N'} \equiv (z_i)_{i \in N \setminus N'}$.

Let $p \equiv (p_x)_{x \in M} \in \mathbb{R}^m_+$ be a price (vector). Without loss of generality, assume the price of object 0 and all reserve prices of objects are zero. Agent $i's$ **demand set at price** p is defined as $D_i(p) \equiv \{x \in L : (x, p_x)\, R_i\, (y, p_y),\ \forall y \in L\}$, i.e., the set of objects that maximizes the agent's welfare at the given price.

The solution concept that we use is the standard Walrasian equilibrium.

Definition 1: A pair $(z, p) \in Z \times \mathbb{R}^m_+$ is a **Walrasian equilibrium** if

$$\text{for each } i \in N,\ x_i \in D_i(p) \text{ and } t_i = p_{x_i}, \tag{E-i}$$

$$\text{for each } y \in M, \text{ if for each } i \in N,\ x_i \neq y, \text{ then } p_y = 0. \tag{E-ii}$$

(E-i) says that each agent i receives a bundle z_i consisting of a demanded object x_i and a payment t_i equal to the price of x_i. (E-ii) says that prices of unassigned objects are zero.

When agents have general preferences as defined above, there is a Walrasian equilibrium (Alkan and Gale, 1990). In particular, Demange and Gale (1985) show the following result.

Fact 1: The set of Walrasian equilibrium prices is a complete lattice.

Therefore, among all Walrasian equilibrium prices, there is a unique coordinate-wise minimum price $p^{\min}$. A **minimum price equilibrium** (**MPE**) is a Walrasian equilibrium supported by $p^{\min}$. Since indifferences in preferences are admitted, MPE allocations may not be unique, but they are welfare-equivalent: each agent i has the same welfare across all the MPEs.

3 An illustrative example

In this section, we illustrate an MPE (Section 3.1), differences between the quasi-linear preferences and general preferences (Section 3.2), and the sketch idea of how the SV mechanism finds an MPE (Section 3.3).

3.1 The assignment market

There are two objects $M = \{a, b\}$, and three agents $N = \{1, 2, 3\}$. We use utility functions to represent the preference profile R.

The utility that an agent obtains by getting object 0 and paying 0 is zero, i.e., for each $i \in N$, $u_i(0, 0) = 0$. Agent 1 has the quasi-linear utility function:

$$u_1(x, p_x) = \begin{cases} -1 - p_a & \text{if } x = a \\ 6.1 - p_b & \text{if } x = b \end{cases}$$

where agent 1's valuation of object a is -1 and that of object b is 6.1.

Agent 2's utility function is not quasi-linear in the sense that the marginal utility of payment is non-identical between objects a and b. Her utility function is given by

$$u_2(x, p_x) = \begin{cases} 5.8 - p_a & \text{if } x = a \\ 84.5 - 13p_b & \text{if } x = b \end{cases}$$

where agent 2's valuation of object a is 5.8 and that of object b is 84.5.

Agent 3's utility function is in the same manner as agent 2, which is given by

$$u_3(x, p_x) = \begin{cases} 5.85 - p_a & \text{if } x = a \\ 41.7 - 6p_b & \text{if } x = b \end{cases}$$

where agent 2's valuation of object a is 5.85 and that of object b is 41.7.

We remark that preferences with different price-slopes for different objects are widely used in the online position auctions.

The minimum price equilibrium (MPE)

In the above assignment market, the MPE prices of objects a and b are

$$p^{\min} = (p_a^{\min}, p_b^{\min}) = (0.6, 6.1).$$

There is a *unique* MPE assignment: Agent 1's assigned object is $x_1^{\min} = 0$, agent 2's assigned object is $x_2^{\min} = b$, and agent 3's assigned object is $x_3^{\min} = a$. Thus the MPE allocation is as follows:

$$\begin{aligned} \text{Agent 1} \quad &: \quad z_1^{\min} = (x_1^{\min}, 0) = (0, 0). \\ \text{Agent 2} \quad &: \quad z_2^{\min} = (x_2^{\min}, p_b^{\min}) = (b, 6.1). \\ \text{Agent 3} \quad &: \quad z_3^{\min} = (x_3^{\min}, p_a^{\min}) = (a, 0.6). \end{aligned}$$

In the following, we show that $(z^{\min}, p^{\min})$ is an MPE.

First, we show that $(z^{\min}, p^{\min})$ is a Walrasian equilibrium. For each agent i, recall that $D_i(p) = \{x \in \{0, a, b\} : u_i(x, p_x) \geq u_i(y, p_y), \forall y \in \{0, a, b\}\}$. For agent 1, since

$$u_1(0, 0) = 0 = u_1(b, 6.1) = 6.1 - 6.1 > u_1(a, 0.6) = -1 - 0.6,$$

it holds that $D_1(p^{\min}) = \{0, b\}$ and so $x_1^{\min} = 0 \in D_1(p^{\min})$. Similarly, for agents 2 and 3, we have that $x_2^{\min} = b \in D_2(p^{\min}) = \{a, b\}$ and $x_3^{\min} = a \in D_3(p^{\min}) = \{a\}$. Thus (E-i) in Definition 1 holds. (E-ii) in Definition 1 holds vacuously.

Second, we show that $(z^{\min}, p^{\min})$ is an MPE. Let $p' = (p'_a, p'_b)$ be a Walrasian equilibrium price. We show that $p'_a \geq p_a^{\min} = 0.6$ and $p'_b \geq p_b^{\min} = 6.1$. By contradiction, suppose not, i.e., $p'_a < 0.6$ or $p'_b < 6.1$. In case of $p'_a < 0.6$ and $p'_b < 6.1$, no agent demands object 0, and so at least one agent cannot receive a demanded object. In case of $p'_a < 0.6$ and $p'_b \geq 6.1$, $D_2(p') = D_3(p') = \{a\}$, either agent 2 or agent 3 cannot receive a demanded object. In case of $p'_a \geq 0.6$ and $p'_b < 6.1$, $D_1(p') = D_2(p') = \{b\}$, either agent 1 or 2 cannot receive a demanded object. All these three cases lead to the contradiction that p' is an equilibrium price.

3.2 Quasi-linear preferences versus general preferences

It is worth discussing the points on which non-quasi-linearity drastically alters the results obtained for quasi-linear preferences. The first point is on the assignment. Note that a Walrasian assignment for quasi-linear preferences is also an MPE assignment for the same preferences. This is not true for non-quasi-linear preferences. For instance, in the assignment market of Section 3.1, (z, p) where $z_1 = (0, 0)$, $z_2 = (a, 5.8)$, $z_3 = (b, 6.5)$, and $p = (p_a, p_b) = (5.8, 6.5)$ is a Walrasian equilibrium where agent 2 gets a and 3 gets b. However, at the unique MPE assignment, agents 2 and 3 obtain object b and a respectively. Thus, *non-quasi-linear preferences complicate the analysis of the MPE assignment.*

The second point is that standard mechanisms such as the VCG mechanism, the exact and approximate auctions of Demange et al. (1986), which are designed for quasi-linear preferences, fail to work in non-quasi-linear environments (See also the discussion in the introduction). The VCG mechanism uses agents' valuations on objects to compute assignments and payments. However, for non-quasi-linear preferences, since agents' valuations on objects are not well-defined, the VCG mechanism cannot be applied. Morimoto and Serizawa (2015) demonstrate that a natural generalization of the VCG mechanism for non-quasi-linear preferences no longer generates the MPE, and fails to satisfy efficiency and strategy-proofness.

To discuss two auctions of Demange et al. (1986), consider the assignment market in Section 3.1 and assume the unit price increment. The exact auction generates an outcome price $(p'_a, p'_b) = (6, 7)$. The approximate auction generates an outcome price $(p''_a, p''_b) = (5, 6)$.[5]

For the outcome of exact auction, the price $p'_a = 6$ of object a largely deviates from its MPE price $p^{\min}_a = 0.6$. Zhou and Serizawa (2021) demonstrate that *even though an increment is fixed, the outcome price of the exact auction can deviate from the MPE price arbitrarily large, which may cause arbitrarily large inefficiency and incentives to misreport.*

For quasi-linear preferences, Demange et al. (1986) show that the approximate auction generates an outcome price that deviates from the MPE price at most by $\min\{|M|, |N|\}$. However, $p^{\min}_a = 0.6$ and the deviation $(5 - 0.6)$ is larger than the upper bound $\min\{|M|, |N|\} = 2$. Thus, in non-quasi-linear environments, the outcome price of the approximate auction may deviate from the MPE price more than the upper bound. Indeed, Zhou and Serizawa (2021) demonstrate that *in non-quasi-linear environments, even though an increment is fixed, the outcome price of the approximate auction can deviate from the MPE price arbitrarily large, which may cause arbitrarily large incentives to misreport.*

Zhou and Serizawa (2021) further demonstrate that *for some non-quasi-linear preferences, the price paths of continuous versions of above two auctions are not be well defined.* Thus, the problematic implications of above two auctions in non-quasi-linear settings are not due to the discreteness of price increments, but due to the non-quasi-linearity of preferences.

3.3 The Serial Vickrey (SV) mechanism: an illustration

The SV mechanism introduces objects one by one, and consists of m steps, i.e., each step introduces one additional object. Step k finds an MPE for k objects by employing the "SV sub-mechanism," which consists of two stages:

[5] See Online Appendix for computation details of these two outcomes. The bidding order we use in the approximate auction is agents 1, 2, and 3. Given a bidding order, there is some utility profile whose outcome price of the approximate auction is arbitrarily far from its MPE price (Zhou and Serizawa, 2021).

• Stage 1 constructs a Walrasian equilibrium with k objects from the MPE with $k-1$ objects.
• Stage 2 finds an MPE by reassigning objects and adjusting prices of the Walrasian equilibrium constructed in Stage 1.

Finite dimensionality of transmitted information: **indifference price**

In the SV mechanism, agents report their indifference prices defined in Section 2 to transmit information of their preference. For example, agent i's indifference price of object b at bundle (a, p_a) is $V_i(b; (a, p_a))$ such that

$$u_i(b, V_i(b; (a, p_a))) = u_i(a, p_a).$$

Note that since there is only a finite number of objects and agents, the transmitted information in each stage of each step is always finite-dimensional. The SV mechanism transmits such information in a finite number of times. This is an important feature of the SV mechanism.

Illustration of the SV mechanism for the assignment market in Section 3.1

Step 1: Initially we assign the null bundle $(0, 0)$ to each agent. We introduce an object, say, object a, and obtain an MPE for the assignment market with only object a.

Each agent reports her indifference price of object a at $(0, 0)$. That is, agent 1 reports $V_1(a; (0, 0))$. Since

$$0 = u_1(0, 0) = u_1(a, V_1(a; (0, 0))) = -1 - V_1(a; (0, 0)),$$

$V_1(a; (0, 0)) = -1$. Similarly, agent 2 reports $V_2(a; (0, 0)) = 5.8$, and agent 3 reports $V_3(a; (0, 0)) = 5.85$. We treat the reported indifference prices as agents' bids and conduct *the second-price auction.* Since $V_3(a; (0, 0)) > V_2(a; (0, 0)) > V_1(a; (0, 0))$, we assign (a, p_a) to agent 3 with $p_a = V_2(a; (0, 0)) = 5.8$, the second highest indifference price. We assign $(0, 0)$ to agents 1 and 2. The outcome of SV mechanism at Step 1 is $((0, 0), (0, 0), (a, p_a), p_a)$, which is an MPE for the assignment market with only object a.

Step 2: Each agent starts with the bundle assigned in Step 1. We introduce object b and obtain an MPE for the assignment market with objects a and b in two stages. Stage 1 constructs a Walrasian equilibrium for the assignment market with objects a and b. If the constructed equilibrium is not an MPE, we proceed to Stage 2 and reassign objects and adjust prices.

Stage 1: Construction of a Walrasian equilibrium for objects a and b

Each agent reports her indifference price of object b at the bundle assigned in Step 1. For example, agent 3 gets $(a, 5.8)$ in Step 1 and she reports $V_3(b; (a, 5.8))$. Since

$$5.85 - 5.8 = u_3(a, 5.8) = u_3(b, V_3(b; (a, 5.8))) = 41.7 - 6 \times V_3(b; (a, 5.8)),$$

$V_3(b; (a, 5.8)) = 6.94$. Accordingly, agents 1 and 2 are assigned $(0, 0)$ in Step 1, and they report $V_1(b; (0, 0)) = 6.1$ and $V_2(b; (0, 0)) = 6.5$.

We treat the reported indifference prices as agents' bids and conduct a variant of the second-price auction. Since $V_3(b;(a,5.8)) > V_2(b;(0,0)) > V_1(b;(0,0))$, we assign (b, p_b) to agent 3 with $p_b = V_2(b;(0,0)) = 6.5$, the second highest indifference price. We assign to agent 1 the null bundle $(0,0)$, but assign to agent 2 (a, p_a), agent 3's bundle in Step 1. It is easy to confirm that $((0,0),(a,p_a),(b,p_b))$ with $p_a = 5.8$ and $p_b = 6.5$ is a Walrasian equilibrium. This stage generalizes the standard second-price auction.

We examine whether the obtained Walrasian equilibrium is an MPE or not by checking a condition, which we call *demand connectedness*. Note that agent 2 demands both objects a and b at (p_a, p_b). In this sense, objects a and b are "connected" by agent 2's demand. If an object is connected to object 0 or an object whose price is zero by a sequence of agents' demands, we say such an object is "connected." Unless all objects are connected, a Walrasian equilibrium is not an MPE.[6] In the Walrasian equilibrium constructed above, both objects a and b are unconnected, and so it is not an MPE. Therefore, we proceed to Stage 2.

Stage 2: The object reassignment and price adjustment

The price of a connected object in a Walrasian equilibrium is equal to its price at the MPE.[7] Thus, we adjust only the prices of unconnected objects. By definition, the MPE price is less than or equal to a Walrasian equilibrium price. If $p_a^{\min} < V_1(a;(0,0))$ or $p_b^{\min} < V_1(b;(0,0))$, then no agent demands object 0, leading to an infeasible assignment, and so $V_1(a;(0,0)) \leq p_a^{\min} \leq p_a = 5.8$, and $V_1(b;(0,0)) \leq p_b^{\min} \leq p_b = 6.5$. Thus, we start price adjustment process with $p_a^0 = \max\{0, V_1(a;(0,0))\} = 0$ and $p_b^0 = \max\{0, V_1(b;(0,0))\} = 6.1$.

Different from the quasi-linear settings, with non-quasi-linearity, *a Walrasian equilibrium assignment may not be an MPE assignment,* as illustrated in Section 3.2. This fact necessitates reassigning objects to find an MPE in Stage 2, complicating the adjustment process.

If an agent is assigned an connected object or object 0, we say the agent is "connected." In the Walrasian equilibrium constructed in Stage 1, agent 1 is connected while agents 2 and 3 are unconnected. Notice that the numbers of unconnected agents and objects are equal and an unconnected agent in a Walrasian equilibrium is assigned a unconnected object in the MPE.[8] Thus, we only reassign objects a and b among agents 2 and 3, keeping agent 1 assigned $(0,0)$.

There are two candidates of the assignments at the MPE to agents 2 and 3: $\mu^1 \equiv (\mu_2^1, \mu_3^1) = (a,b)$ (assigning a to 2 and b to 3) or $\mu^2 \equiv (\mu_2^2, \mu_3^2) = (b,a)$ (assigning a to 3 and b to 2) In the following, we conduct a process, which we call "IPOIP (I-pay-others'-indifference-prices) process," to examine the two candidates.

IPOIP process

The IPOIP process for μ^1 We initialize the price of object a as $p_a^0 = \max\{0, V_1(a;(0,0))\} = 0$, and that of object b as $p_b^0 = \max\{0, V_1(b;(0,0))\} = 6.1$. Along with μ^1, we assign $(\mu_2^1, p_a^0) =$

[6] See Proposition 1 in Section 4.
[7] See Theorem 1 (i) in Section 4.
[8] See Lemma 1 in Section 4.

$(a, 0)$ and $(\mu_3^1, p_b^0) = (b, 6.1)$ to agent 2 and 3 respectively as their initial bundles for μ^1. In IPOIP process, each agent reports her indifference prices of the objects assigned to others by μ^1 at her own initial bundle.

Agent 2 reports her indifference price $V_2(b; (a, p_a^0)) = 6.05$ of b at $(a, 0)$, and agent 3 reports her indifference price $V_3(a; (b, 6.1)) = 0.75$ of a at $(b, 6.1)$. Then, we update the prices of objects a and b as

$$
\begin{aligned}
p_a^1 &= \max\{p_a^0, V_3(a; (b, 6.1))\} = 0.75, \text{ and} \\
p_b^1 &= \max\{p_b^0, V_2(b; (a, 0))\} = 6.1.
\end{aligned}
$$

We update the prices of the two agents' bundles to $p^1 \equiv (p_a^1, p_b^1)$, that is, we assign the bundles $(\mu_2^1, p_a^1) = (a, 0.75)$ and $(\mu_3^1, p_b^1) = (b, 6.1)$ to agent 2 and 3 respectively. *Notice that only prices are updated and agents' assignments remain same.*

Then, each agent reports her indifference prices of the objects assigned to others by μ^1 at her updated bundle. Agent 2 reports her indifference price $V_2(b; (a, p_a^1)) = V_2(b; (a, 0.75)) = 6.11$ of b at her updated bundle $(a, p_a^1) = (a, 0.75)$, and agent 3 reports her indifference price $V_3(a; (b, p_b^1)) = V_3(a; (b, 6.1)) = 0.75$ of a at her updated bundle $(b, p_b^1) = (b, 6.1)$. Again, we update the prices of objects a and b as

$$
\begin{aligned}
p_a^2 &= \max\{p_a^1, V_3(a; (b, p_b^1))\} = 0.75, \text{ and} \\
p_b^2 &= \max\{p_b^1, V_2(b; (a, p_a^1))\} = 6.11.
\end{aligned}
$$

We stop price adjustment for μ^1 at $p^2 \equiv (p_a^2, p_b^2)$. μ^1 is the MPE assignment if and only if $p^1 = p^2$.[9] Since $p_b^1 < p_b^2$, μ^1 is not an MPE assignment.

The IPOIP process for μ^2 Along with μ^2, we assign $(\mu_2^1, p_a^0) = (b, 0)$ and $(\mu_3^2, p_b^0) = (a, 6.1)$ to agent 2 and 3 respectively as their initial bundles for μ^2.

Agent 2 reports her indifference price $V_2(a; (b, p_b^0)) = 0.6$ of a at $(b, p_b^0) = (b, 6.1)$. Agent 3 reports her indifference price $V_3(b; (a, p_a^0)) = V_3(b; (a, 0)) = 5.975$ of b at $(a, p_a^0) = (a, 0)$. We update the prices of objects a and b as

$$
\begin{aligned}
p_a'^1 &= \max\{p_a^0, V_2(a; (b, 6.1))\} = 0.6, \text{ and} \\
p_b'^1 &= \max\{p_b^0, V_3(b; (a, 0))\} = 6.1.
\end{aligned}
$$

We update the prices of the two agents' bundles to $p'^1 \equiv (p_a'^1, p_b'^1)$, that is, we assign the bundles $(\mu_2^2, p_b'^1) = (b, 6.1)$ and $(\mu_3^2, p_a'^1) = (a, 0.6)$ to agent 2 and 3 respectively.

Then, agent 2 reports her indifference price $V_2(a; (b, p_b'^1)) = V_2(a; (b, 6.1)) = 0.6$ of a at $(b, p_b'^1) = (b, 6.1)$, and agent 3 reports her indifference price $V_3(b; (a, p_a'^1)) = V_3(b; (a, 0.6)) =$

[9] See Theorem 2 in Section 4.

6.075 of a at her updated bundle $(a, p'^1_a) = (a, 0.6)$. Again, we update the prices of objects a and b as

$$\begin{aligned} p'^2_a &= \max\{p'^1_a, V_3(a; (b, p'^1_b))\} = 0.6, \text{ and} \\ p'^2_b &= \max\{p'^1_b, V_2(b; (a, p'^1_a))\} = 6.1. \end{aligned}$$

Since $p'^1 = p'^2$, $\mu^2 \equiv (\mu^2_2, \mu^2_3) = (b, a)$ is an MPE assignment.[10] Indeed, (μ^2, p'^1) coincide with the MPE assignment and price illustrated in Section 3.1.

Thus, Stage 2 finds an MPE. It is worth repeating that the SV mechanism transmits only finite dimensional information in finitely many times.

Table 2 summarizes the main concepts and their usages.

Table 2: The summary of main concepts and usages

Main concept	**Main usage**
Demand connected path (DCP, Def.2)	Def. 5
Connected object (Def. 2): $M_C(= M\backslash M_U)$	Prop. 1 & Thm. 1
Connected agent (Def. 3): $N_C(= N\backslash N_U)$	Prop. 1 & Thm. 1
Unconnected object: $M_U(= M\backslash M_C)$	Def. 4, Thm. 1, & Thm. 2
Unconnected agent: $N_U(= N\backslash M_U)$	Def. 4, Thm. 1, & Thm. 2
Assignments of unconnected agents: $\Omega(\ni \mu)$ (above Def. 4)	Def. 4, Thm.1, & Def. 7
$k - th$ highest indifference prices of x reported by agents in N' at z: $C^k(x; z_{N'})$ (above Ex. 3)	Defs. 4, 6 & Fact 3
I-pay-others'-indifference-prices (IPOIP) process (Def. 4)	Thms. 1, 2, & Defs. 6, 7
$\overline{p}^k(\mu)$: price of IPOIP process for μ at round k (above Fact 2)	Facts 2, 3, & 4

4 The structural characterizations

Before presenting the SV mechanism in Section 5, this section provides a systematic investigation of the structural properties of MPEs. Proposition 1 shows the connected property of an MPE, for the use of Stage 1 in the SV sub-mechanism (Section 5.1). Theorems 1 and 2 disclose a dynamic relation between an arbitrary Walrasian equilibrium and an MPE, for the use of Stage 2 in the SV sub-mechansim (Section 5.2).

4.1 The connectedness characterization

First, we introduce two concepts: "connected object" and "connected agent."

[10] See Theorem 2 in Section 4.

Definition 2: An object $x \in M$ is **connected at** $(z, p) \in Z \times \mathbb{R}^m_+$ if (i) x is unassigned or (ii) there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents ($\Lambda > 1$) that forms a **demand connectedness path (DCP)** such that
(ii-1) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(ii-2) $x_{i_\Lambda} = x$,
(ii-3) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$, and
(ii-4) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

Definition 2 says that an object x is connected in two cases. In Case (i), object x is unassigned. In Case (ii), there is a sequence of distinct agents such that the first agent receives an object with zero price (ii-1); the last agent receives object x (ii-2); each agent who is not the first agent gets an object with a positive price (ii-3); each agent who is not the last agent demands both her assigned object and the object assigned to her successive agent (ii-4). These agents form a demand connected path to object x. Definition 2 is more general than necessary, and in the following, we only require that (z, p) is a Walrasian equilibrium.

Example 1 illustrates a DCP.

Example 1: Consider the MPE $(z^{\min}, p^{\min})$ illustrated in Section 3.1.

Object a is connected by a DCP formed by agents 3, 2, and 1 where $i_1 = 1$, $i_2 = 2$, and $i_3 = i_\Lambda = 3$: (i) $x_1^{\min} = 0$, (ii) $x_2^{\min} = b$, (iii) $x_3^{\min} = a$, $p_a^{\min} = 0.6 > 0$, and $x_2^{\min} = b$, $p_b^{\min} = 6.1 > 0$, and (iv) $\{0, b\} \in D_1(p^{\min})$ and $\{a, b\} \in D_2(p^{\min})$. (i) to (iv) corresponds to (ii-1) to (ii-4) in Definition 2.

Definition 3: An agent $i \in N$ is **connected** at $z \in Z$ if x_i is connected or $x_i = 0$.

Definition 3 says that an agent is connected if she gets either a connected object or object 0. In Example 1, agent 3 gets a connected object a so she is connected. Agent 1 is also connected since she gets object 0.

In the following, we let M_C and $M_U (\equiv M \backslash M_C)$ denote the sets of connected objects and unconnected objects, N_C and $N_U (\equiv N \backslash N_C)$ the sets of connected agents and unconnected objects at some given allocation, respectively

Example 2 illustrates the connected and unconnected agents and objects.

Example 2: Consider the Walrasian equilibrium (z, p) obtained in Stage 1 of the SV mechanism in Section 3.3.

Agent 1 gets $z_1 = (0, 0)$ and so agent 1 is a connected agent. Agent 2 gets $z_2 = (a, p_a)$ and agent 3 gets $z_3 = (b, p_b)$. Since $V_1(a; (0, 0)) < p_a$ and $V_1(b; (0, 0)) < p_b$, agents 2 and 3 are unconnected and objects a and b are unconnected, either.

Remark 1: By Definition 2(ii-1), an agent who gets an object with zero price is connected. Moreover, there is a connected agent if and only if some agent gets an object with zero price.[11]

[11] "Only if" is easy to see. For "if part", by contradiction, suppose that for each $i \in N$, $p_{x_i} > 0$. Since a

Proposition 1 characterizes the MPE by connected agents and objects.

Proposition 1: Let (z, p) be a Walrasian equilibrium. Then the following statements are equivalent:
(i) p is an MPE price,
(ii) all the agents are connected at (z, p), and
(iii) all the objects are connected at (z, p).

The proof of Proposition 1 is relegated to Appendix A.1. Proposition 1 gives three equivalent conditions to judge whether a Walrasian equilibrium is an MPE.

4.2 The I-pay-others'-indifference-prices characterizations

Given an object x, an allocation z and a group N' of agents, let $C^1(x; z_{N'})$ and $C^2(x; z_{N'})$ be the **highest and second highest indifference prices of x reported by agents in N' at** z. Let $C^1_+(x; z_{N'}) \equiv \max\{C^1(x; z_{N'}), 0\}$, and $C^2_+(x; z_{N'}) \equiv \max\{C^2(x; z_{N'}), 0\}$. If $N' = N$, we write z instead of z_N. If $N' = \emptyset$, let $C^1_+(x; z_{N'}) = C^2_+(x; z_{N'}) \equiv 0$. Example 3 illustrates these notations.

Example 3: Consider Step 1 in Section 3.3.

Let $N = N' = \{1, 2, 3\}$, $x = a$, and $z = (z_1, z_2, z_3) = ((0, 0), (0, 0), (0, 0))$. Recall that $V_3(a; (0, 0)) > V_2(a; (0, 0)) > 0 > V_1(a; (0, 0))$. Thus $C^1(a; z) = V_3(a; (0, 0))$ and $C^2(a; z) = C^2_+(a; z) = V_2(b; (0, 0))$.

The following result shows the connection between a Walrasian equilibrium and an MPE.

Lemma 1: Let (z, p) be a Walrasian equilibrium and $(z^{\min}, p^{\min})$ be an MPE. Let N_U and M_U be the set of unconnected agents and unconnected objects at (z, p).
(i) There are equal numbers of unconnected agents and unconnected objects, i.e., $|N_U| = |M_U|$.
(ii-1) The MPE price of a connected object is equal to the given equilibrium price, i.e., for each $x \in M_C$, $p^{\min}_x = p_x$.
(ii-2) The MPE price of an unconnected object is bounded above by the given equilibrium price and below by the highest indifference price reported by the connected agents $(N \backslash N_U)$, i.e., for each $x \in M_U$, $C^1_+(x; z_{N \backslash N_U}) \leq p^{\min}_x < p_x$.
(iii) Each unconnected agent gets an unconnected object at the MPE, i.e., for each $i \in N_U$, $x^{\min}_i \in M_U$.

The proof of Lemma 1 is relegated to Appendix A.2. Lemma 1(i) and 1(iii) say that to obtain an MPE from a given Walrasian equilibrium, we only need to reallocate unconnected

connected agent exists, for each connected agent i, we have that $p_{x_i} > 0$. Thus for an arbitrary connected agent i, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of distinct agents satisfying Definition 2, with $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, contradicting that for each $i \in N$, $p_{x_i} > 0$.

objects among unconnected agents at the given equilibrium. Lemma 1(ii) shows the boundary of the MPE price of an unconnected object. Example 4 illustrates Lemma 1.

Example 4: Consider the Walrasian equilibrium (z, p) obtained in Stage 1 of the SV mechanism in Section 3.3.

As shown in Example 2, $N_U = \{2, 3\}$ and $M_U = \{a, b\}$. Thus Lemma 1(i) holds. Agent 1 is the only connected agent, and so

$$\begin{aligned} C^1_+(a; z_{N\setminus N_U}) &= C^1_+(a; z_{\{1\}}) = \max\{0, V_1(a; (0, 0))\} = 0. \\ C^1_+(b; z_{N\setminus N_U}) &= C^1_+(b; z_{\{1\}}) = \max\{0, V_1(b; (0, 0))\} = 6.1. \end{aligned}$$

It is easy to see that Lemma 1(ii) holds. As shown in Section 3.1, agent 2 gets a at (z, p), but b at the MPE while agent 3 gets b at (z, p), but a at the MPE. Thus Lemma 1(iii) holds.

As shown in Example 4, when agents have general preferences, a Walrasian equilibrium assignment may not be an MPE assignment, which increases the difficulty in obtaining the MPE. We propose the following process that generates a candidate price for each possible assignment among unconnected agents.

Let μ be an assignment or bijection, from the set of unconnected objects M_U to the set of unconnected agents N_U, and μ_i be accordingly the associated object assigned to agent $i \in N_U$. Let Ω be the set of all such assignments.

Definition 4: Let (z, p) be a Walrasian equilibrium and N_U and M_U be the set of unconnected agents and objects at (z, p). Let $\mu \in \Omega$. **The $k-$round I-pay-others'-indifference-prices (IPOIP) process for μ** is defined as follows:

For each $x \in M_U$ and each $i \in N_U$,
(i) $\overline{p}^0_x \equiv C^1_+(x; z_{N\setminus N_U})$ and $\overline{z}^0_i \equiv (\mu_i, p^0_{\mu_i})$, and
(ii) for each $s = 1, \cdots, k$,

$$\overline{p}^s_x \equiv C^1(x; z^{s-1}_{N_U}) \text{ and } \overline{z}^s_i \equiv (\mu_i, \overline{p}^s_{\mu_i}).$$

The $k-$round IPOIP process for a given assignment updates the prices of unconnected objects and unconnected agents' bundles recursively: First, set the starting price of each unconnected object x as $\overline{p}^0_x = C^1_+(x; z_{N\setminus N_U})$, the maximum value of the indifference prices reported by the connected agents $N\setminus N_U$. Notice that connected agents do not participate in the process.

Each unconnected agent i is assigned an unconnected object μ_i and gets the bundle $\overline{z}^0_i = (\mu_i, \overline{p}^0_{\mu_i})$. Each unconnected agent i reports her indifference price of each unconnected object x at $\overline{z}^0_i$, i.e., $V_i(x; \overline{z}^0_i)$. Then the price of x is updated to the maximum value of reported indifference prices reported by the unconnected agents N_U, i.e., $\overline{p}^1_x = C^1_+(x; \overline{z}^0_{N_U})$. Each unconnected agent i keeps the same object μ_i but pays an updated price $\overline{p}^1_{\mu_i}$, i.e., $\overline{z}^1_i = (\mu_i, \overline{p}^1_{\mu_i})$. In the same manner, the price of each unconnected object x is updated to $\overline{p}^2_x = C^1_+(x; \overline{z}^1_{N_U})$ and so forth.

The formation of agent's tentative payment $\overline{p}_x^s$ is in the spirit of the Vickrey payment. Stage 2 in Step 2 of the SV mechanism in Section 3.3 illustrate two 2−round IPOIP process for two candidate assignments of agents 2 and 3, respectively.

Let $\overline{p}^s(\mu)$ be the price generated in round $s(\leq k)$ in a $k-$IPOIP process for μ. It is easy to see that generated prices in the process are non-decreasing. Formally, we have:

Fact 2: The price generated by a $k-$round IPOIP process for a given assignment μ is non-decreasing, i.e., for each $s = 1, \cdots, k$, $\overline{p}^0(\mu) \leq \overline{p}^1(\mu) \leq \cdots \leq \overline{p}^s(\mu)$.

Now we give the characterizations of MPEs in terms of IPOIP processes.

Theorem 1: Let (z, p) be a Walrasian equilibrium, and N_C, M_C, N_U, and M_U be the sets of connected agents, connected objects, unconnected agents and unconnected objects at (z, p). Let $p^{\min}$ be the MPE price. Then the following holds.
(i) For each $x \in M_C$, $p_x^{\min} = p_x$, and for each $i \in N_C$, z_i is an MPE bundle.
(ii) Let $\overline{p}^{|M_U|-1}(\mu')$ be the price generated in round $|M_U| - 1$ in a $|M_U|-$IPOIP process for $\mu' \in \Omega$. There is $\mu \in \Omega$ such that

(ii-1) μ is an MPE assignment among unconnected agents, and

(ii-2) for each $x \in M_U$, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu) = \min\limits_{\mu' \in \Omega} \overline{p}_x^{|M_U|-1}(\mu')$.

Here we only sketch the proof of Theorem 1 and relegate the formal proof to Appendix A.3.

Sketch of Proof of Theorem 1: First, we show Theorem 1(i). We argue that if the prices of connected objects are lower than p, these objects will be "overdemanded" at the MPE price. Thus, their MPE price should be equal to p. Then by Lemma 1(ii), connected agents could keep the same equilibrium bundles as their MPE bundles.

Second, we show Theorem 1(ii). For (ii-1), by Fact 1 and Theorem 1(i), there must exist an MPE assignment that assign the unconnected objects to the unconnected agents. The proof of (ii-2) consists of two steps.

Step 1 shows that if μ is an MPE assignment, then at each round of the IPOIP process, at least one unconnected object's price reaches its MPE price, and it never increases in the later round. Since there are $|M_U|$ unconnected objects, starting from round 0, the IPOIP process for μ finds the MPE price for unconnected objects at most by $|M_U| - 1$ rounds. Thus, for each unconnected object x, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu)$. Step 2 shows that for any non-MPE assignment μ', at each round of the IPOIP process, at least one unconnected object's price exceeds its MPE price and by Fact 2, it never decreases in the later round. Since there are $|M_U|$ unconnected objects, for each unconnected object x, $p_x^{\min} \leq \overline{p}_x^{|M_U|-1}(\mu')$. **Q.E.D.**

We illustrate Theorem 1 by using Stage 2 of Step 2 in Section 3.3. Agent 1 is connected and she simply keeps $(0, 0)$. Thus Theorem 1(i) holds. Agents 2 and 3 are unconnected and objects a and b are unconnected either (See also Example 2). We run the IPOIP process for two assignments of agents 2 and 3. We can also verify $p_a^{\min} = \min\{p_a^1, p_a'^1\} = p_a'^1 = 0.6$, and

$p_b^{\min} = \min\{p_b^1, p_b'^1\} = p_b'^1 = 6.1$. Since $(p_a'^1, p_b'^1)$ is generated by the second candidate assignment μ^2 and so at the MPE, agent 2 gets b and 3 gets a, as desired. Thus Theorem 1(ii) holds.

Theorem 1 goes beyond Proposition by dealing with the case where there are unconnected agents at the given Walrasian equilibrium.

Remark 2: The outcome of $(|M_U| - 1)-$round IPOIP process for a given assignment μ', i.e., $\overline{p}^{|M_U|-1}(\mu')$ is generally not an equilibrium price for unconnected objects. Thus, Theorem 1(ii-2) is different from the meet operation of equilibrium prices that generates a new equilibrium price, by their lattice property (Fact 1).

Note that if we conduct $|M_U|-$IPOIP process for each $\mu \in \Omega$, since $|\Omega| = |M_U|!$, it will take long to obtain an MPE. Fortunately, it is possible to obtains MPEs without conducting $|M_U|-$IPOIP process for all assignments in Ω. Theorem 2 below provide some way to detect an MPE assignment without exhausting all possible candidate assignments.

Intuitively, if μ is an MPE assignment and $\overline{p}^s$ obtained at some round s in the IPOIP process for μ is an MPE price, then in the later rounds, the price remains unchanged, i.e., $\overline{p}^s = \cdots = \overline{p}^k$. Theorem 2 surprisingly shows that for any assignment μ, if the prices in two adjacent rounds $s-1$ and s of the IPOIP process for μ remain unchanged, i.e., $\overline{p}^{s-1} = \overline{p}^s$, then $\overline{p}^{s-1}$ is an MPE price and μ is an MPE assignment of unconnected objects and agents.

Theorem 2: Let N_U and M_U be sets of unconnected agents and objects at some given Walrasian equilibrium (z, p). In the $|M_U|-$round IPOIP process for $\mu \in \Omega$, the following are equivalent:
(i) There is some round $s \leq |M_U|$ in the IPOIP process such that the price in the precedent round remains unchanged, i.e., $\overline{p}^{s-1}(\mu) = \overline{p}^s(\mu)$;
(ii) μ is an MPE assignment and $\overline{p}^{s-1}(\mu)$ is the MPE price of unconnected agents and objects.

Here we only sketch the proof of Theorem 2 and relegate the formal proof to Appendix A.4.

Sketch of Proof of Theorem 2: First (ii) $\Rightarrow$ (i). As argued in Step 1 of the proof Theorem 1(ii-2), when $\overline{p}^{s-1}(\mu)$ is the MPE price of unconnected objects and μ is an MPE assignment of unconnected agents, $\overline{p}^{s-1}$ remain unchanged in the later round of the IPOIP process so $\overline{p}^{s-1}(\mu) = \overline{p}^s(\mu)$.

Next, (i) $\Rightarrow$ (ii). Since the outcome of the IPOIP process may not assign unconnected agents bundles in their demand sets, we define weak connected objects and agents by relaxing (ii-4) in Definition 2: Agents on the path are indifferent between their bundles and the successive agents' bundles. The proof contains five steps. Step 1 shows that all the unconnected objects are weakly connected at $(\mu, \overline{p}^{s-1})$. Steps 2 and 3 show that the price of unconnected objects generated by the IPOIP process is bounded above by their MPE price. Step 4 shows that each unconnected agent i weakly prefer $(\mu_i, \overline{p}_{\mu_i}^{s-1})$ to the bundles consisting of connected objects paired with their MPE price. Step 5 concludes that (ii) holds. **Q.E.D.**

In Stage 2 of Step 2 in Section 3.3, the IPOIP process for μ^2 illustrates Theorem 2(ii).

5 Serial Vickrey mechanism

In this section, we present the "Serial Vickrey (SV) mechanism," that finds an MPE in a finite number of steps via agents' reports of indifference prices in finitely many times.

In short, the SV mechanism works in the following way. Each agent is initially assigned $(0, 0)$. Introduce objects sequentially by its index, $1, 2, \cdots$. In Step $k(\geq 1)$, suppose that the MPE for the market with $k-1$ objects is given. We introduce object k and run the **SV sub-mechanism** consisting of the following two stages.

In Stage 1, we **construct a Walrasian equilibrium** with k objects via "Equilibrium-generating mechanism." If the resulting Walrasian equilibrium is shown to be an MPE, by verifying that all the agents are connected, we stop. Otherwise, we go to Stage 2 and **reassign objects and adjust prices for the unconnected agents and objects** via "MPE-adjustment mechanism," to obtain the MPE.

Section 5.1 defines the Equilibrium-generating mechanism (Definition 5) and studies its property (Proposition 2). Section 5.2 defines the "MPE-adjustment mechanism," (Definition 7) and studies its property (Proposition 3). Section 5.3 gives a formal definition of the Serial Vickrey mechanism (Definition 8) and shows its convergence property (Theorem 3).

5.1 Implication of Proposition 1: Stage 1 of SV sub-mechanism

First, we define the equilibrium-generating mechanism, which constructs a Walrasian equilibrium for $k+1$ objects from an MPE for k objects.

Definition 5: **Equilibrium-generating mechanism** Let $(z^{\min}, p^{\min})$ be an MPE of k objects. Introduce a new object y. Each agent i reports $V_i(y; z_i^{\min})$, and then compute $C^1(y; z^{\min})$.
Phase 1: If $C^1(y; z^{\min}) < 0$, set (z, p) as
(a) $p_y = 0$, and $p = (p^{\min}, p_y)$, and (b) for each $i \in N$, $z_i = z_i^{\min}$.
Otherwise, go to Phase 2.
Phase 2: Arbitrarily select an agent i with the highest indifference price of y at $z_i^{\min}$, i.e., $V_i(y; z_i^{\min}) = C^1(y; z^{\min})$. Identify a demand connectedness path (DCP, Def. 2) for $x_i^{\min}$, i.e., the sequence of agents $\{i_\lambda\}_{\lambda=1}^{\Lambda}$. Set (z, p) as
(a) $p_y = C_+^2(y; z^{\min})$, and $p = (p^{\min}, p_y)$,
(b) $z_{i_\Lambda} = (y, p_y)$,
(c) for each $i_l \in \{i_\lambda\}_1^{\Lambda-1}$, $z_{i_l} = z_{i_{l+1}}^{\min}$, and
(d) for each $j \in N \backslash \{i_\lambda\}_1^{\Lambda}$, $z_j = z_j^{\min}$.

By Proposition 1, at the MPE for k objects, each object is connected and so we can always identify a DCP for each object. This idea is essential in Phase 2 to update agents' bundles. The process of identifying a DCP for the given object can be referred to Corollary 1 in Morimoto and Serizawa (2015). For the completeness, we provide a process that identifies the DCP via agents' reported indifference prices in finitely many times in the Online Appendix.

The essence part of the above mechanism is Phase 2: When object y is introduced, each agent reports her indifference of y at $z_i^{\min}$, i.e., $V_i(y; z_i^{\min})$. We assign the new object to an arbitrary agent i whose has the highest indifference, but ask her to pay the second highest indifference price $C_+^2(y; z^{\min})$. We identify a DCP for agent $i's$ assignment $x_i^{\min}$ at $(z^{\min}, p^{\min})$. After agent i is assigned object y, we alternate the bundles of agents on the identified DCP from the object with zero price. All other agents keep their bundles at $z^{\min}$.

Example 5 illustrates the Phase 2.

Example 5: Consider Stage 1 in Step 2 of the SV mechanism in Section 3.3.

The MPE with only object a is $z^{\min}$ such that that agent 1 gets $(0,0)$, 2 gets $(0,0)$, and 3 gets $(a, 5.8)$. When object b is introduced, agent 3 has the highest indifference price and she gets object b, but pays $p_b = C_+^2(b; z^{\min}) = 6.5$. Thus (a) and (b) of Phase 2 holds. Object a's DCP is formalized by agents 2 and 3, and so agent 2 takes 3's previously assigned bundle, i.e., $(a, 5.8)$. Thus (c) of Phase 2 holds. Agent 1 remains $(0,0)$. Thus (d) of Phase 2 holds.

Proposition 2: Given an MPE with k objects, the equilibrium-generating mechanism finds a Walrasian equilibrium for $k+1$ objects via agents' reports of indifference prices in finitely many times.

The proof of Proposition 2 is relegated to Appendix B.1.

5.2 Implication of Theorems 1 and 2: Stage 2 of SV sub-mechanism

Theorem 1 implies that by conducting an IPOIP process for each possible assignment of unconnected agents, we can find an MPE from an arbitrary Walrasian equilibrium. On the other hand, Theorem 2 implies that if the price $\overline{p}^r(\mu)$ generated in round $r \leq |M_U|$ in IPOIP process for μ stops increasing, i.e., $\overline{p}^r(\mu_s) = \overline{p}^{r-1}(\mu_s)$, then μ is confirmed to be an MPE assignment. Thus, we can obtain an MPE assignment without conducting all possible assignments of unconnected agents except for the worst case. Thus, Theorem 2 enables us to find an MPE assignment faster.

In the following, we argue that it is possible to find an MPE even faster by excluding some undesirable assignments of unconnected agents.

Fact 3: Let (z, p) be a Walrasian equilibrium and N_U and M_U be the sets of unconnected agents and objects at (z, p). For each $x \in M_U$, $\overline{p}_x^0 = C_+^1(x; z_{N_C})$. For each $i \in N_U$, let $\Gamma_i = \{x \in M_U : (x, \overline{p}_x^0)\, R_i\, (y, \overline{p}_y^0), \forall y \in M_U\}$.

(i) Let $\overline{p} = (\overline{p}_{M_U}^0, p_{M \setminus M_U})$. If there is $\mu \in \Omega$ such that for each $i \in N_U$, $\mu_i \in \Gamma_i$, then $(\mu, \overline{p})$ is an MPE.

(ii) Let N_1 and N_2 be a partition of N_U, and M_1 and M_2 be a partition of M_U such that $|N_1| = |M_1|$, $|N_2| = |M_2|$. Assume for each $i \in N_1$, each $x \in M_1$ and each $y \in M_2$, $(x, \overline{p}_x^0)\, P_i\, (y, \overline{p}_y^0)$,

and for each $i \in N_2$, each $x \in M_2$ and each $y \in M_1$, $(x, \overline{p}_x^0)\, P_i\, (y, \overline{p}_y^0)$. Then, at any MPE, agent $i \in N_1$ receives object $x \in M_1$, and agent $i \in N_2$ receives object $x \in M_2$.
(iii) Let $i \in N_U$ and $x \in M_U$. Agent i never gets x at any MPE if $V_i(x; z_i) \leq \overline{p}_x^0$.

The proof of Fact 3 is relegated to Appendix B.2. Fact 3(i) says that at the starting price $\overline{p}_{M_U}^0$, if there is an assignment μ among unconnected agents such that each agent weakly prefers her assigned bundle $(\mu_i, \overline{p}_{\mu_i}^0)$ to any other bundle assigned to other unconnected agents, then $(\mu, \overline{p})$ is an MPE. Fact 3(ii) partitions unconnected agents into two parts, and objects into two parts. It shows that any assignment such that some agent in N_1 gets some object from M_2 will not be the MPE assignment. Fact 3(iii) is a direct outcome of Lemma 1.

Let $DQ_0(ii)$ be the set of assignments in Ω such that the condition of Fact 3(ii) holds for N_1, N_2, M_1 and M_2, but some agents in N_1 are assigned objects in M_2. Let $DQ_0(iii)$ be the set of assignments in Ω such that the condition of Fact 3(ii) holds for some $i \in N_U$ and $x \in M_U$. We exclude the set $DQ_0(ii) \cup DQ_0(iii)$ as the **disqualified** candidates of MPE assignments. In other words, we only need to conduct IPOIP process for $\Omega^{*0} \equiv \Omega \backslash \{DQ_0(ii) \cup DQ_0(iii)\}$.

Fact 4: Let (z, p) be a Walrasian equilibrium and N_U and M_U be the sets of unconnected agents and objects at (z, p). Let $i \in N_U$ and $x \in M_U$, and $\overline{p}_x^0 = C_+^1(x; z_{N_C})$. Agent i never gets x at any MPE if for some $\mu \in \Omega$, $V_i(x; \overline{z}_i(\mu)) < \overline{p}_x^0$ where $\overline{z}_i(\mu) = (\mu_i, \overline{p}_{\mu_i}^{|M_U|-1}(\mu))$.

By using Fact 4, we might be able to exclude assignments even from Ω^{*0}. After operating the $(|M_U| - 1)$-round IPOIP process for an assignment $\mu \in \Omega^{*0}$, we obtain its outcome $\overline{z}(\mu) \equiv (\mu, \overline{p}^{|M_U|-1}(\mu))$. At this point, we ask each $i \in N_U$ to report her indifference prices $V_i(x; \overline{z}_i(\mu)), x \in M_U$, where $\overline{z}_i(\mu) = (\mu_i, \overline{p}_{\mu_i}^{|M_U|-1}(\mu))$. Fact 4 says that if $V_i(x; \overline{z}_i^{|M_U|-1}(\mu)) < \overline{p}_x^0$ for some $i \in N_U$, then any $\mu' \in \Omega$ such that $\mu'_i = x$ cannot be an MPE assignment. Thus, we also exclude the set of such assignments as well. Let

$$DQ(\mu) \equiv \{\mu' \in \Omega^{*0} : \exists i \in N_U \text{ s.t. } V_i(\mu'_i; \overline{z}_i^{|M_U|-1}(\mu)) < \overline{p}_x^0\}.$$

$DQ(\mu)$ is the **set of disqualified assignments after the IPOIP process for** μ. We also remove $DQ(\mu)$ from Ω^{*0}. Whenever we conduct IPOIP process for some $\mu \in \Omega^{*0}$, we remove $DQ(\mu)$ from the current candidate set of MPE assignments.

We also consider the possibilities to disqualify an assignment $\mu \in \Omega^{*0}$ while conducting the IPOIP process for μ itself. An assignment $\mu \in \Omega^{*0}$ **departs at round** $s \leq |M_U|-1$ **(in IPOIP process from** $\overline{p}^0 \equiv (C_+^1(x; z_{N \backslash N_U}))_{x \in M_U}$) if for each $x \in M_U$, $\overline{p}_x^s(\mu) > \overline{p}_x^0$. Since $\overline{p}^s(\mu)$ is non-decreasing in s during the IPOIP process (Fact 2), if μ departs at some round $s \leq |M_U| - 1$, then μ also departs at round $|M_U| - 1$, which implies that no $x \in M_U$ is connected in $\overline{z}(\mu)$. Thus, by Proposition 1, if an assignment $\mu \in \Omega^{*0}$ departs at some round $s \leq |M_U| - 1$, then μ cannot be an MPE assignment. Thus, an assignment $\mu \in \Omega^{*0}$ is disqualified as soon as it departs at some round $s \leq |M_U| - 1$.

By Lemma 1(ii) and Theorems 1 and 2, we have Fact 5 below:

Fact 5: Let (z, p) be a Walrasian equilibrium. Let N_U and M_U be the sets of unconnected agents and unconnected objects at (z, p). Let $\mu \in \Omega^{*0}$. Let $p' = p$ or $p' = \overline{p}^{|M_U|-1}(\mu)$. An assignment μ is not an MPE assignment of N_U if for some $x \in M_U$, $\overline{p}_x^{|M_U|-1}(\mu) > p'_x$.

We refer to μ and p' of Fact 5 as **reference assignment** and **reference price** respectively. Since $\overline{p}^s(\mu)$ is non-decreasing in s during the IPOIP process (Fact 2), Fact 5 implies that an assignment $\mu \in \Omega^{*0}$ is disqualified as soon as its price of some $x \in M_U$ exceeds the reference price at some round $s \leq |M_U| - 1$.

The above argument motivates the following concept.

Definition 6: Let (z, p) be a Walrasian equilibrium. Let N_U and M_U be the sets of unconnected agents and unconnected objects at (z, p). For each $x \in M_U$, let $\overline{p}_x^0 \equiv C_+^1(x; z_{N \setminus N_U})$. Let $p' \in \mathbb{R}^{M_U}$ be a reference price. An assignment μ is **disqualified at round** $s \leq |M_U| - 1$ (in IPOIP process) **by** p' if (i) for some $x \in M_U$, $\overline{p}_x^s(\mu) > p'_x$, or (ii) for each $x \in M_U$, $\overline{p}_x^s(\mu) > \overline{p}_x^0$. An assignment μ **survives (**in IPOIP process) **against** p' if neither of (i) and (ii) holds.

Now we are ready to show the following mechanism.

Definition 7: MPE-adjustment mechanism Let (z, p) be a Walrasian equilibrium, and N_U and M_U be the set of unconnected agents and objects at (z, p). If conditions in Fact 3(i) hold, stop at $(\mu, \overline{p})$ defined in Fact 3(i). Otherwise, let $\Omega^{*1} \equiv \Omega^{*0}$ and conduct the following procedure.

Session 1: Set the Walrasian equilibrium $p^{*1} \equiv (p_x)_{x \in M_U}$ be a reference price and conduct the $|M_U|$ −round IPOIP process for the Walrasian equilibrium assignment $\mu^{*1} \equiv (x_i)_{i \in N_U}$.

If $\overline{p}^r(\mu^{*1}) = \overline{p}^{r-1}(\mu^{*1})$ at some round $r \leq |M_U|$ (*Case 1*), stop the process, and μ^{*1} is an MPE assignment and $p^* \equiv \overline{p}^r(\mu^{*1})$ is the MPE price of M_U.

If not (*Case 1* fails), but μ^{*1} survives against p^{*1} (*Case 2*), set $\mu^{*2} \equiv \mu^{*1}$ and $p^{*2} \equiv \overline{p}^{|M_U|-1}(\mu^{*1})$ as reference assignment and price. Set $\Omega^{*2} \equiv DQ(\mu^{*1}) \cup \{\mu^{*1}\}$. Go to Session 2.

If both Cases 1 and 2 fail, keep reference assignment and price, i.e., $\mu^{*2} \equiv \mu^{*1}$, $p^{*2} \equiv p^{*1}$, and set $\Omega^{*2} \equiv \Omega^{*1} \backslash \{\mu^{*1}\}$. Go to Session 2.

Session $s(\geq 2)$: Choose $\mu_s \in \Omega^{*s} \equiv \Omega^{*s}$ and conduct the $|M_U|$ −round IPOIP process for μ_s.

If $\overline{p}^r(\mu_s) = \overline{p}^{r-1}(\mu_s)$ at some round $r \leq |M_U|$ (*Case 1*), stop the process, and μ_s is an MPE assignment and $p^* \equiv \overline{p}^r(\mu_s)$ is the MPE price of M_U.

If *Case 1* fails, but μ_s survives against p^{*s} (*Case 2*), set $\mu^{*s+1} \equiv \mu_s$ and $p^{*s+1} \equiv \overline{p}^{|M_U|-1}(\mu_s)$ as reference assignment and price. Set $\Omega^{*s+1} \equiv \Omega^{*s} \backslash (DQ(\mu_s) \cup \{\mu_s\})$. Go to Session $s + 1$.

If both *Cases 1 and 2* fail, keep reference assignment and price, i.e., $\mu^{*s+1} \equiv \mu^{*s}$, $p^{*s+1} \equiv p^{*s}$, and set $\Omega^{*s+1} \equiv \Omega^{*s} \backslash \{\mu_s\}$. Go to Session $s + 1$.

In Session 1 of MPE-adjustment mechanism, the given Walrasian equilibrium price $p^{*1} = (p_x)_{x \in M_U}$ plays a reference price to examine the Walrasian equilibrium assignment $\mu^{*1} \equiv (x_i)_{i \in N_U}$. If μ^{*1} is diagnosed as an MPE assignment (Case 1), then the mechanism stops at Session 1.

Note that in Session s, if μ_s survives, then $p^{*s} = \overline{p}^r(\mu_s) \leq p^{*s-1}$. Thus, p^{*s} is non-increasing in s. Moreover, by disqualification conditions, an MPE assignment always survives until an IPOIP process is conducted for it. Thus, the mechanism ends up with finding an MPE assignment and its price.

Example 6 illustrates the MPE-adjustment mechanism.

Example 6: Consider Stage 2 in Step 2 of the SV mechanism in Section 3.3.

The set of initial disqualified assignments is empty, i.e., $\Omega^{*0} = \Omega$. In Session 1, we check the first possible assignment of agents 2 and 3 that coincides with their Walrasian equilibrium assignment constructed at Stage 1. The outcome falls into Case 3. Then we proceed to the second possible assignment of agents 2 and 3, the outcome of which falls into Case 1.

By the above analysis, we can establish the following result.

Proposition 3: The MPE-adjustment mechanism generates a finite sequence $\{(\mu^{*t}, p^{*t}\}_{t=0}^{T}$ via agents' reports of indifference prices in finitely many times such that
(i) $T < +\infty$, and for each $t = 1, \cdots, T$, $p^{*t} \leq p^{*t-1}$.
(ii) μ^{*T} and p^{*T} are MPE assignment and price of unconnected agents and objects.

5.3 Definition and convergence of Serial Vickrey mechanism

We now present the formal definition of **SV mechanism**.

Definition 8: The **SV mechanism** is defined as follows. Each agent is initially assigned $(0, 0)$. Introduce object sequentially by its index, $1, 2, \cdots$.
Step $k(\geq 1)$: Given the MPE for the market with $k-1$ objects, introduces object k and run the following **SV sub-mechanism**:

Stage 1: Construct a Walrasian equilibrium with k objects via Equilibrium-generating mechanism. If all the agents are connected, stop at the constructed equilibrium. Otherwise, identify the unconnected agents and objects. Then go to Stage 2.

Stage 2: Reassign objects and adjust prices for the unconnected agents and objects via MPE-adjustment mechanism, and obtain an MPE with k objects. If $k = m$, stop. Otherwise, go to **Step** $k + 1$.

To identify the set of connected agents for a given Walrasian equilibrium, we could first identify the DCP for each object (Def. 2) if there is any, and then obtain the set of connected agents. In the Online Appendix, we provide a process with the use of DCP to identify the set of connected agents at the constructed Walrasian equilibrium via agents' reports of indifference prices in finitely many times.

By Fact 1, the MPE price for the assignment market with the whole set of objects is unique so the outcome of the SV mechanism is independent of the order that we introduce objects to the market. By Propositions 2 and 3, we can establish the following results:

Proposition 4: Given an MPE for k objects, the SV sub-mechanism finds an MPE for $k+1$ objects via agents' reports of indifference prices in finitely many times.

By Propositions 2, 3, and 4, the convergence of the SV mechanism follows.

Theorem 3: The SV mechanism finds an MPE in a finite number of steps via agents' reports of indifference prices in finitely many times.

6 Incentive properties

In this section, we investigate the incentive properties of the SV mechanism. A *rule* f is a mapping from the set of general preference profiles $\mathcal{R}^n$ to the set of allocations Z. It assigns each agent i with a bundle $f_i(R)$ at each preference profile R. A rule f is *strategy-proof* if no agent can gain from misreporting her preference, i.e., for each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R_i, R_{-i})\, R_i\, f_i(R'_i, R_{-i})$.

In our model, the *MPE rule* that assigns to each general preference profile an MPE allocation is strategy-proof (Demange and Gale, 1985; Morimoto and Serizawa, 2015). Let f^{SV} be the rule that selects the outcome of the SV mechanism. By Theorem 3, f^{SV} coincides with the MPE rule and so we have the following result.

Fact 6: The rule f^{SV} is strategy-proof on the set of general preference profiles.

The SV mechanism is decomposed into m steps, i.e., m SV sub-mechanisms. Each SV sub-mechanism is decomposed into two stages, i.e., the equilibrium-generating and MPE-adjustment mechanisms. We show that these sub-mechanisms are also incentive compatible. In other words, even if agents are not fully rational and their perspectives are limited to step-wise or stage-wise, agents still have no incentives to misreport. *Step-wise and stage-wise incentive compatibilities are remarkable properties of the SV mechanism*, as shown below.

We first study the cases in which agents' perspectives are limited to stages, and analyze the incentive properties of the equilibrium-generating and MPE-adjustment mechanisms. Note that the outcomes of these mechanisms depend both on the revealed preferences and allocations obtained from previous steps or stages. Thus, we need to introduce some notations.

For each $k \in \mathbb{N}_+$, let $Z^k \equiv \{z \in Z : \forall i \in N, x_i \in \{0, 1, \ldots, k\}\}$ be the set of allocations for the assignment market with k objects. If $k = m$, then $Z^k = Z$.

An *augmented rule*, $g^{k \to k'} : \mathcal{R}^n \times Z^k \to Z^{k'}$, associates to each pair of preference profile R and an allocation z with k objects an allocation $g^{k \to k'}(R, z)$ with k' objects, where $g_i^{k \to k'}(\cdot; \cdot)$ denotes the bundle assigned to agent i. Given an allocation $z \in Z^k$, $(g^{k \to k'}(\cdot; z), R)$ forms a (normal-form) revelation game: agents report their preferences R and the outcome of their reports is selected by $g^{k \to k'}(\cdot; z)$.

Definition 9: Let $k, k' \in \mathbb{N}_+$ and $z \in Z^k$. An augmented rule $g^{k \to k'}(\cdot; z)$ is **strategy-proof from** z if for each $i \in N$, each $R'_i \in \mathcal{R}$, and each $R_{-i} \in \mathcal{R}^{n-1}$, $g_i^{k \to k'}(R; z)\, R_i\, g_i^{k \to k'}(R'_i, R_{-i}; z)$.

Now we consider the incentive properties of the following augmented rules:

• **SV sub-rule** $SV^{k\to k+1}(\cdot;z)$: Given a preference profile R, $SV^{k\to k+1}(R;z)$ is the outcome of the SV sub-mechanism for $k+1$ objects based on an allocation $z \in Z^k$.[12]

• **Equilibrium-generating rule** $SV_E^{k\to k+1}(\cdot;z)$: Given a preference profile R, $SV_E^{k\to k+1}(R;z)$ is the outcome of equilibrium-generating mechanism in the SV sub-mechanism for $k+1$ objects based on an allocation $z \in Z^k$.[13]

• **MPE-adjustment rule** $SV_M^{k+1\to k+1}(\cdot;z)$: Given a preference profile R, $SV_M^{k\to k+1}(R;z)$ is the outcome of the SV sub-mechanism for $k+1$ objects based on an allocation $z \in Z^{k+1}$.[14]

Notice that SV sub-rule can be decomposed into Equilibrium-generating rule and MPE-adjustment rule. Now we propose the following result.

Theorem 4: Let $0 \le k < |M|$ and $R \in \mathcal{R}^n$.
(i) (**Stage-wise**) For each $z \in Z^k$, if z is an MPE allocation for R, then $SV_E^{k\to k+1}(\cdot;z)$ is strategy-proof from z.
(ii) (**Stage-wise**) For each $z \in Z^{k+1}$, if z is a Walrasian equilibrium allocation for R, then $SV_M^{k+1\to k+1}(\cdot;z)$ is strategy-proof from z.
(iii) (**Step-wise**) For each $z \in Z^k$, if z is an MPE allocation for R, then $SV^{k\to k+1}(\cdot;z)$ is strategy-proof from z.

The proof of Theorem 4 is relegated to Appendix C. Theorem 4(i) and 4(ii) state "stage-wise strategy-proofness" of the SV sub-mechanism. In Stage 1, when a new object is introduced, no agent can gain by misreporting. In Stage 2, at the Walrasian equilibrium constructed in stage 1, no agent can gain by misreporting. Theorem 4(iii) states "step-wise strategy-proofness" of the SV sub-mechanism. When a new object is introduced, to obtain an MPE for the assignment market with additional one object, no agent has incentive to misreport her preference. Thus, even if agents' perspectives are myopic and limited to the stages where they are currently interacting, agents have incentives to reveal their true preferences.

It is worth noting that Theorem 4(iii) is not implied by Fact 6, since at each step, the SV sub-mechanism involves a different number of objects. It is not implied by Theorem 4(i) and 4(ii) either since there is no guarantee that the aggregation of two incentive compatible mechanisms still preserves the incentive property.

[12] $SV^{k\to k+1}(\cdot;z)$ depends not only on the preferences revealed in the SV sub-mechanism for $k+1$ objects and $z \in Z^k$ but also on the information on DCPs at z that are revealed in the previous step.

[13] $SV_E^{k\to k+1}(\cdot;z)$ depends not only on the preferences revealed in the equilibrium-generating mechanism and $z \in Z^k$ but also on the information on DCPs at z that are revealed in the previous step.

[14] $SV_M^{k+1\to k+1}(\cdot;z)$ depends not only on the preferences revealed in the MPE-adjustment mechanism and $z \in Z^{k+1}$ but also on the information of connected and unconnected agents and objects at z that are revealed in the previous stage.

7 Concluding remarks

We conclude by providing some further discussions of the SV mechanism.

• **The welfare property of the SV mechanism**

First, an agent's welfare is non-decreasing with the number of introduced objects in the SV mechanism. This result follows Proposition 2 and Fact 1. It is in line with the comparative static analysis of the agents' welfare regarding the number of objects in Demange and Gale (1985). Second, agents' welfare is non-decreasing stage-wise within the SV sub-mechanism. By construction, agents' welfare at the outcome of Stage 1 is bounded below by that at the given MPE. By Lemma 1, agents' welfare at the outcome of Stage 2 is bounded below by that at the Walrasian equilibrium constructed in Stage 1.

• **Quasi-linear preferences and assortative-matching preferences**

As illustrated in Section 3.2, there are significant differences between quasi-linear environment and general environment. If we apply the SV mechanism to the quasi-linear environment, then any Walrasian equilibrium assignment can be an MPE assignment. Although this fact cannot simplify the characterizations of Theorems 1 and 2, but it largely simplifies the Stage 2 of the SV sub-mechanism: we only run a one-round IPOIP process for the assignment at the constructed equilibrium in Stage 1 and then obtain an MPE.

The Alonso-type housing market is well-studied in the urban economics where agents have positive-assortative preferences: they have common ranking over houses and preferences exhibits positive income effects. As a consequence, all the Walrasian equilibrium assignments share the same shape, i.e., agents with higher incomes obtain houses closer to the city center.[15] The insight of how quasi-linear preferences simplifies the SV mechanism also carries over here.

• **The assignment market without outside option**

In the assignment market without outside option, there is no object 0 and each agent gets one object ($|N| \leq |M|$). This model is also called the task-assignment model analyzed by, e.g., Sun and Yang (2003), Andersson (2007), and Noldeke and Samuelson (2018).

In such settings, Walrasian equilibria and MPEs are defined as "envy-free allocations," and "fair and optimal allocations" (Sun and Yang, 2003). Theorems 1 and 2, together with MPE-adjustment mechanism, identify a fair and optimal allocation.

We remark that the structural result in Alaei et al. (2016) depends on the existence of object 0, so it fails to characterize the fair and optimal allocation.

• **Other incentive notions**

Li (2017) introduces a notion of "obvious strategy-proofness," which requires that in the game form induced by the mechanism, along the equilibrium path, by comparing the maximum

[15] Houses are identical but different in locations. Agents have the same utility functions and at each payment, agents commonly prefer houses with shorter distance to the city center than those with longer distance. See Zhou and Serizawa (2018) for details.

payoff among all deviations with the minimum payoff by following the truth-telling strategy, no agent has any incentive to deviate. The obvious strategy-proofness is stronger than our incentive notions. It is demanding and some well-know mechanisms fail to be obviously strategy-proof, e.g., the deferred-acceptance mechanism (Ashlagi and Gonczarowski, 2018). The SV mechanism is not obviously strategy-proof, but still incentivizes agents with limited perspectives to reveal their true preferences (step-wise and stage-wise strategy-proofness).

Ausubel (2006) and Sun and Yang (2014) study the multi-object auction mechanisms and show that sincere bidding forms an ex-post perfect equilibria (EXPE). We conjecture that truthfully reporting the indifference prices is an EXPE in the SV mechanism although we have not enabled to prove it since the SV mechanism is extremely complex.

References

[1] Afacan, M.O., 2017. Some further properties of the cumulative offer process. Games and Economic Behavior 104, 656-665.

[2] Alaei, S., Kamal, J., Malekian, A., 2016. Competitive equilibria in two-sided matching markets with general utility functions. Operations Research 64(3), 638-645.

[3] Alkan, A., Gale, D., 1990. The core of the matching game. Games and Economic Behavior 2(3), 203-212.

[4] Andersson, T., Svensson, L.G., 2018. Sequential rules for house allocation with price restrictions. Games and Economic Behavior 107, 41-59.

[5] Andersson, T., Svensson, L.G., 2014. Non-manipulable house allocation with rent control. Econometrica 82(2), 507-539.

[6] Andersson, T., Erlanson, A., 2013. Multi-item Vickrey–English–Dutch auctions. Games and Economic Behavior 81, 116-129.

[7] Andersson, T., 2007. An algorithm for identifying fair and optimal allocations. Economics Letters 96(3), 337-342.

[8] Ashlagi, I., Gonczarowski, Y.A., 2018. Stable matching mechanisms are not obviously strategy-proof. Journal of Economic Theory 177, 405-425.

[9] Ausubel, L.M., 2006. An efficient dynamic auction for heterogeneous commodities. American Economic Review 96(3), 602-629.

[10] Baisa, B.H, 2020. Efficient multiunit auctions for normal goods. Theoretical Economics 15(1), 361-413.

[11] Baisa, B.H., 2017. Auction design without quasilinear preferences. Theoretical Economics 12(1), 53-78.

[12] Baisa, B.H, 2016. Overbidding and inefficiencies in multi-unit Vickrey auctions for normal goods. Games and Economic Behavior 99, 23-35.

[13] Baldwin, E., Klemperer, P., 2019. Understanding preferences:"demand types", and the existence of equilibrium with indivisibilities. Econometrica, 87(3), pp.867-932.

[14] Caplin, A., Leahy, J.V., 2014. A graph theoretic approach to markets for indivisible goods. Journal of Mathematical Economics 52, 112-122.

[15] Caplin, A., Leahy, J.V., 2020. Comparative statics in markets for indivisible goods. Journal of Mathematical Economics 90, 80-94.

[16] Crawford, V.P., Knoer, E.M., 1981. Job matching with heterogeneous firms and workers. Econometrica 49(2), 437-450.

[17] Demange, G., Gale, D., 1985. The strategy structure of two-sided matching markets. Econometrica 53(4), 873-888.

[18] Demange, G., Gale, D., Sotomayor, M., 1986. Multi-item auctions. Journal of Political Economy 94(4), 863-872.

[19] Fleiner, T., Jagadeesan, R., Janko, Z., Teytelboym, A., 2019. Trading networks with frictions. Econometrica 87(5), 1633-1661.

[20] Galichon, A., Kominers, S.D., Weber, S., 2019. Costly concessions: An empirical framework for matching with imperfectly transferable utility. Journal of Political Economy 127(6), 2875-2925.

[21] Gershkov, A., Moldovanu, B., Strack, P., Zhang, M., 2021. A theory of auctions with endogenous valuations. Journal of Political Economy 129(4), 1011-1051.

[22] Gul, F., Stacchetti, E., 2000. The English auction with differentiated commodities. Journal of Economic Theory 92(1), 66-95.

[23] Hatfield, J.W., Kominers, S.D., Westkamp, A., 2021. Stability, strategy-proofness, and cumulative offer mechanisms. The Review of Economic Studies 88(3), 1457-1502.

[24] Hatfield, J.W., Milgrom, P.R., 2005. Matching with contracts. American Economic Review 95(4), 913-935.

[25] Herings, P.J.J., 2018. Equilibrium and matching under price controls. Journal of Economic Theory 177, 222-244.

[26] Kazumura, T., Mishra, D., Serizawa, S., 2020. Strategy-proof multi-object mechanism design: Ex-post revenue maximization with non-quasilinear preferences. Journal of Economic Theory 188, 105036.

[27] Kazumura, T., Mishra, D., Serizawa, S., 2020b. Mechanism design without quasilinearity. Theoretical Economics 15(2), 511–544.

[28] Kelso Jr, A. S., Crawford, V. P., 1982. Job matching, coalition formation, and gross substitutes. Econometrica 50(6), 1483-1504.

[29] Klemperer, P., 2004. Auctions: Theory and practice. Princeton University Press.

[30] Kojima, F., Yu, N., Sun, N., 2020. Job matching under constraints. American Economic Review 110(9), 2935-2947.

[31] Leonard, H.B., 1983. Elicitation of honest preferences for the assignment of individuals to positions. Journal of Political Economy 91(3), 461-479.

[32] Li, S., 2017. Obviously strategy-proof mechanisms. American Economic Review 107(11), 3257-87.

[33] Liu, D., Bagh, A, 2019. Preserving bidder privacy in assignment auctions: design and measurement. Management Science, forthcoming.

[34] Malik, K., Mishra, D., 2021. Pareto efficient combinatorial auctions: dichotomous preferences without quasilinearity. Journal of Economic Theory 191, 105-128.

[35] Mishra, D., Talman A.J.J., 2010. Characterization of the Walrasian equilibria of the assignment model. Journal of Mathematical Economics 46(1), 6-20.

[36] Mishra, D., Parkes, D.C., 2009. Multi-item Vickrey–Dutch auctions. Games and Economic Behavior 66, 326-347.

[37] Morimoto, S., Serizawa, S., 2015. Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule. Theoretical Economics 10(2), 445-487.

[38] Murota, K., 2003. Discrete convex analysis. SIAM.

[39] Myerson, R.B., 1981. Optimal auction design. Mathematics of Operations Research 6(1), 58-73.

[40] Noldeke, G., Samuelson, L., 2018. The implementation duality. Econometrica, 86(4), 1283-1324.

[41] Vohra, R.V., 2011. Mechanism design: a linear programming approach. Cambridge University Press.

[42] Schlegel, J.C., 2021. The structure of equilibria in trading networks with frictions. Theoretical Economics, forthcoming.

[43] Sun, N., Yang, Z., 2014. An efficient and incentive compatible dynamic auction for multiple complements. Journal of Political Economy 122(2), 422-466.

[44] Sun, N., Yang, Z., 2009. A double-track adjustment process for discrete markets with substitutes and complements. Econometrica 77(3), 933-952.

[45] Sun, N., Yang, Z., 2003. A general strategy proof fair allocation mechanism. Economics Letters 81(1), 73-79.

[46] Zhou, Y, Serizawa, S., 2018. Strategy-proofness and efficiency for non-quasi-linear common-tiered-object preferences: Characterization of minimum price rule. Games and Economic Behavior 109, 327-363.

[47] Zhou, Y, Serizawa, S., 2021. Multi-object auction design beyond quasi-linearity: Leading examples. ISER Discussion Paper 1116.

Appendix

Part A gives the proofs of results in Section 4. Part B gives the proofs of results in Section 5. Part C gives the proofs of results in Section 6.

In the following, Let N_C and M_C be the sets of connected agents and objects. Let $N_U \equiv N \backslash N_C$ and $M_U \equiv M \backslash M_C$ be the sets of unconnected agents and objects at the equilibria we investigate. Let W be the set of Walrasian equilibria, and let $W^{\min} \subseteq W$ be the corresponding set of MPEs. Given $p \in \mathbb{R}^m_+$ and $M' \subseteq M$, let $p_{M'} \equiv (p_x)_{x \in M'}$ and $p_{M \backslash M'} \equiv (p_x)_{x \in M \backslash M'}$.

Appendix A: Proofs of Proposition 1, Lemma 1, Theorem 1, and Theorem 2

Definition A.1: (i) A non-empty set $M' \subseteq M$ of objects is **overdemanded at** p if

$$|\{i \in N : D_i(p) \subseteq M'\}| > |M'| .$$

(ii) A non-empty set $M' \subseteq M$ of objects is (**weakly**) **underdemanded at** p if

$$[\forall x \in M', \, p_x > 0] \Rightarrow |\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \, (\leq) < |M'| .$$

Fact A.1 (Mishra and Talman, 2010; Morimoto and Serizawa, 2015). p is an equilibrium price vector if and only if no set is overdemanded and no set is underdemanded at p.

Fact A.2 (Alkan and Gale, 1990; Morimoto and Serizawa, 2015). p is an MPE price if and only if no set is overdemanded and no set is weakly underdemanded at p.

Fact A.3: Let $(z, p) \in W$ and M_C be defined at (z, p). Let $M' \subseteq M_C$ be such that $M' \neq \emptyset$ and for each $x \in M'$, $p_x > 0$. Then, (i) $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| > |M'|$, and (ii) for each $x \in M_C$, $p_x^{\min} \leq p_x$.

Proof of (i): Since $(z, p) \in W$, and for each $x \in M'$, $p_x > 0$, then by Fact A.1, $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \geq |M'|$. To show ">", we proceed by contradiction. Suppose that $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$. Then, by $M' \subseteq M_C$,

$$\text{for each } i \in N \text{ such that } D_i(p) \cap M' \neq \emptyset,\ x_i \in M' \text{ and } i \in N_C. \tag{$*$}$$

Let $i \in N$ such that $x_i \in M'$. Then by $(*)$, $i \in N_C$. By $x_i \in M'$, $p_{x_i} > 0$. By Definition 2, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents such that
(a) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(b) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$,
(c) $x_{i_\Lambda} = x_i$, and
(d) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

Claim: Let $l = 1, \cdots, \Lambda - 1$ and $N(l) \equiv \{i_{\Lambda-1}, \cdots, i_{\Lambda-l}\}$. Then, for each $j \in N(l)$, $x_j \in M'$.

Step 1: The Claim holds for $l = 1$.

By (c), $x_{i_\Lambda} = x_i \in M'$. By (d), $D_{i_{\Lambda-1}}(p) \cap M' \neq \emptyset$. Thus by $(*)$, $x_{i_{\Lambda-1}} \in M'$.

Induction hypothesis: The Claim holds for s such that $1 \leq s < \Lambda - 1$.

Step 2: The Claim holds for $l = s + 1$.

By induction hypothesis, $x_{i_{\Lambda-s}} \in M'$. By (d), $x_{i_{\Lambda-s}} \in D_{i_{\Lambda-(s+1)}}(p)$. Thus $D_{i_{\Lambda-(s+1)}}(p) \cap M' \neq \emptyset$. Thus by $(*)$, $x_{i_{\Lambda-(s+1)}} \in M'$.

Thus Claim holds.

Let $l = \Lambda - 1$. Then, Claim implies that for each $j \in \{i_1, \cdots, i_{\Lambda-1}\}$, $x_j \in M'$. By $\Lambda \geq 1$, $x_{i_1} \in M'$. Thus, $p_{x_{i_1}} > 0$ and (a) is violated. Thus $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$ does not hold.

Proof of (ii): By $(z, p) \in W$, for each $x \in M_C$, $p_x^{\min} \leq p_x$. Let $M' \equiv \{x \in M_C : p_x^{\min} < p_x\}$. We proceed by contradiction. Suppose $M' \neq \varnothing$.

Then, by $M' \subseteq M^C$, $(z, p) \in W$, by Fact A.3, $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| > |M'|$. Note that for each $i \in N$ such that $D_i(p) \cap M' \neq \emptyset$, by $p_{M'}^{\min} < p_{M'}$, $D_i(p^{\min}) \subseteq M'$. Thus

$$\left|\{i \in N : D_i(p^{\min}) \subseteq M'\}\right| > |M'|.$$

Thus M' is overdemanded at $p^{\min}$, violating Fact A.2. Thus $p = p^{\min}$. **Q.E.D.**

A.1 Proof of Proposition 1

Proof: We prove Proposition 1 by showing (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i).

Step 1: (i)$\Longrightarrow$(ii), i.e., $p = p^{\min} \Longrightarrow N = N_C$

Obviously $N_C \subseteq N$. For each $i \in N$, if $p_{x_i} = 0$, by Definition 3 and Remark 1, x_i is connected so $i \in N_C$. If $p_{x_i} > 0$, by Corollary 2 in Morimoto and Serizawa (2015) and Definition 3, $i \in N_C$. Thus $N \subseteq N_C$. Thus $N = N_C$.

Step 2: (ii)$\Longrightarrow$(iii), i.e., $N = N_C \Longrightarrow M = M_C$.

Obviously $M_C \subseteq M$. For each $x \in M$, if x is assigned, by $N = N_C$ and Definition 3, $x \in M_C$. If x is unassigned, by Definition 2, $x \in M_C$. Thus $M \subseteq M_C$. Thus $M = M_C$.

Step 3: (iii)$\Longrightarrow$(i), i.e., $M = M_C \Longrightarrow p = p^{\min}$.

By $M = M_C$, Fact A.3 (ii): implies $p = p^{\min}$. **Q.E.D.**

A.2 Proof of Lemma 1

Proof: **Part (i):** Let $x \in M_U$. Then, by Definition 2, $p_x > 0$. Thus, by $(z, p) \in W$, for each $x \in M_U$, there is $i \in N$ such that $x_i = x$. By Definition 3, $i \in N \backslash N_C = N_U$. Thus, for each $x \in M_U$, there is $i \in N_U$ such that $x_i = x$. This implies $|M_U| \leq |N_U|$.

Let $i \in N_U$. If $x_i = 0$, then by Definition 3 and Remark 1, $i \in N_C$, a contradiction. Thus, $x_i \in M$, and by Definition 2, $x_i \notin M_C$, ie, $x_i \in M \backslash M_C = M_U$. Thus, for $i \in N_U$, $x_i \in M_U$, which implies $|N_U| \leq |M_U|$. Thus $|M_U| = |N_U|$.

Part (ii-1): It follows from Fact A.3 (ii).

Part (ii-2): Step 1: For each $x \in M_U$, $p_x^{\min} < p_x$.

Let $M' \equiv \{x \in M_U : p_x^{\min} = p_x\}$. We proceed by contradiction. Suppose $M' \neq \varnothing$.

By contradiction, suppose that there is $i \in N_C$ such that $D_i(p) \cap M' \neq \emptyset$. Then any $x \in D_i(p) \cap M'$ is connected to $x_i \in M_C$, and so for any $x \in D_i(p) \cap M'$, $x \in M_C$. This contradicts $M' \subseteq M_U$. Thus,

$$\text{for each } i \in N_C,\ D_i(p^{\min}) \cap M' = \emptyset. \qquad (*)$$

Since $p^{\min}_{M_U \backslash M'} < p_{M_U \backslash M'}$ and $p^{\min}_{M'} = p_{M'}$, then

$$\text{for each } i \in N_U \text{ such that } x_i \in M_U \backslash M',\ D_i(p^{\min}) \cap M' = \emptyset. \qquad (**)$$

Thus,

$$\begin{aligned}
& \left|\{i \in N : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| \\
= \; & \left|\{i \in N \backslash N_C : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| && \text{by } (*) \\
= \; & \left|\{i \in N_U : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| \\
\leq \; & |N_U| - |\{i \in N_U : x_i \in M_U \backslash M'\}| && \text{by } (**) \\
= \; & |\{i \in N_U : x_i \in M'\}| = |M'|.
\end{aligned}$$

Thus M' is weakly underdemanded, violating Fact A.2.

Step 2: For each $x \in M_U$, $p_x^{\min} \geq C_+^1(x; z_{N_C})$.

We proceed by contradiction. Suppose that there is a non-empty set $M' \subseteq M_U$ such that for each $x \in M'$, $0 \leq p_x^{\min} < C_+^1(x; z_{N_C})$.

For each $i \in N_U$ and each $x \in M_C \cup \{0\}$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x_i, p_{x_i}^{\min}) \underset{\text{Step 1}}{P_i} z_i \underset{\text{Def of Equilibrium}}{R_i} (x, p_x) = (x, p_x^{\min}).$$

Thus, for each $i \in N_U$, $D_i(p^{\min}) \cap (M_C \cup \{0\}) = \emptyset$ and thus $D_i(p^{\min}) \subseteq M_U$.

Since for each $x \in M'$, $0 \leq p_x^{\min} < C_+^1(x; z_{N_C})$, then there is $i \in N_C$ such that $V_i(x; z_i) = C_+^1(x; z_{N_C}) > 0$, and so by $p_{M_C} = p_{M_C}^{\min}$ (ii-C), $D_i(p^{\min}) \subseteq M_U$. Thus,

$$\left|\{i \in N : D_i(p^{\min}) \subseteq M_U\}\right| \geq 1 + |N_U| \underset{\text{(i)}}{>} |M_U|.$$

Thus M' is overdemanded, violating Fact A.2.

Part (iii): For each $i \in N_U$ and each $x \in M_C \cup \{0\}$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x_i, p_{x_i}^{\min}) \underset{\text{Step 1 in (ii)}}{P_i} z_i \underset{\text{Def of Equilibrium}}{R_i} (x, p_x) \underset{\text{(ii-C)}}{=} (x, p_x^{\min}).$$

Thus, for each $i \in N_U$, $x_i^{\min} \in M_U$. **Q.E.D.**

A.3 Proof of Theorem 1

First, we propose two lemmas.

Lemma A.1: Let $(z, p) \in W$, and $(z^{\min}, p^{\min}) \in W^{\min}$. Let N_C and M_U be defned at (z, p).Then there is $x \in M_U$ such that $p_x^{\min} = C_+^1(x; z_{N_C})$.

Proof: Let M_C and N_U be the sets of connected objects and unconnected agents at (z, p), respectively. We proceed by contradiction. Suppose that for each $x \in M_U$, $p_x^{\min} > C_+^1(R_{N_C}, x; z)$. Then, for each $i \in N_C$, $D_i(p^{\min}) \cap M_U = \emptyset$. Thus,

$$\begin{aligned} & \left|\{i \in N : D_i(p^{\min}) \cap M_U \neq \emptyset\}\right| \\ = & \left|\{i \in N_U : D_i(p^{\min}) \cap M_U \neq \emptyset\}\right| \\ \leq & |N_U| \underset{\text{Lemma 1(i)}}{=} |M_U|. \end{aligned}$$

Thus, M_U is weakly underdemanded, contradicting Fact A.2. Thus there is $x \in M_U$ such that $p_x^{\min} = C_+^1(R_{N_C}, x; z)$. **Q.E.D.**

Lemma A.2: Let μ be an MPE assignment. In the IPOIP process for μ,
(i) for each $x \in M_U$ and each $s = 1, \cdots$, $\overline{p}_x^s(\mu) \leq p_x^{\min}$, and
(ii) for each $x \in M_U$ and each $s = 1, \cdots$, if $\overline{p}_x^{s-1}(\mu) = p_x^{\min}$, then $\overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu)$.

Proof: Part (i) First, we show the following claim:

Claim: For each $s = 1, \cdots$, if for each $x \in M_C$, $\overline{p}_x^s(\mu) \leq p_x^{\min}$, then for each $x \in M_C$, $\overline{p}_x^{s+1}(\mu) \leq p_x^{\min}$.

Let $s = 1, \cdots$, and let $\overline{p}_x^s(\mu) \leq p_x^{\min}$ for each $x \in M_C$. Then, for each $x \in M_U$ and each $j \in N_U$,

$$V_j(x; \overline{z}_j^s(\mu)) \underset{\overline{p}_{\mu(j)}^s \leq p_{\mu(j)}^{\min}}{\leq} V_j(x; (\mu(j), p_{\mu(j)}^{\min})) \underset{\text{Def of Equilibrium}}{\leq} p_x^{\min}.$$

Thus, for each $x \in M_U$, $\overline{p}_x^{s+1}(\mu) = C_+^1(x; \overline{z}_{N_U}^s(\mu)) \leq p_x^{\min}$. Thus, Claim holds.

For each $x \in M_U$, by Lemma 1(ii), $\overline{p}_x^0 \equiv C_+^1(R_{N_C}, x; z^0) \leq p_x^{\min}$. Thus, by Claim, $\overline{p}_x^1(\mu) \leq p_x^{\min}$. Assume that for $s \geq 1$, and for each $x \in M_U$, $\overline{p}_x^{s-1}(\mu) \leq p_x^{\min}$. Then, by Claim, $\overline{p}_x^s(\mu) \leq p_x^{\min}$. Thus Part (i) holds.

(ii) Let $x \in M_U$ and $s \in \mathbb{N}^+$ be such that $\overline{p}_x^{s-1}(\mu) = p_x^{\min}$. By Part (i) and Fact 2, $\overline{p}_x^{s-1}(\mu) \leq \overline{p}_x^s(\mu) = p_x^{\min}$. Thus, $\overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu)$. **Q.E.D.**

Part (i) of Theorem 1: By Lemma 1, we only show that for each $i \in N_C$, z_i is an MPE bundle. Let $i \in N_C$. For each each $x \in M_U$,

$$z_i \, R_i \, (x, C_+^1(x; z_{N_C})) \underset{\text{Lemma 1(ii)}}{R_i} (x, p_x^{\min}),$$

and for each $x \in M_C \cup \{0\}$,

$$z_i \underset{\text{Def of Equilibrium}}{R_i} (x, p_x) = (x, p_x^{\min}).$$

Thus for each $x \in L$, $z_i \, R_i \, (x, p_x^{\min})$ and thus $z_i \, R_i \, z_i^{\min}$. Note that

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x_i, p_{x_i}^{\min}) \underset{\text{Lemma 1(ii)}}{I_i} (x_i, p_{x_i}) = z_i.$$

Thus $z_i \, I_i \, z_i^{\min}$.

Let x' be such that for each $i \in N_U$, $x_i' = x_i^{\min}$, and for each $i \in N_C$, $x_i' = x_i$. By Lemma 1(iii), for each $i \in N_U$, $x_i^{\min} \in M_U$, and for each $i \in N_C$, $x_i^{\min} \in M_C \cup \{0\}$. Thus, x' is an assignment. Let z' be such that for each $i \in N_U$, $z_i' = z_i^{\min}$, and for each $i \in N_C$, $z_i' = z_i$. Then, since x' is an assignment, and since for each $i \in N_C$ and each for each $x \in L$, $z_i \, I_i \, z_i^{\min} \, R_i \, (x, p_x^{\min})$, $(z', p^{\min})$ is an MPE. Thus, for each $i \in N_C$, $z_i = z_i'$ is an MPE bundle.

Part (ii) of Theorem 1: Step 1: Let μ be an MPE assignment. Then $\overline{p}^{|M_U|-1}(\mu) = p_{M_U}^{\min}$.

First, we show Claim A.2.

Claim A.2: For each $s = 0, 1, \cdots$, let $M_s \equiv \{x \in M_U : \overline{p}_x^s(\mu) = p_x^{\min}\}$ and $N_s \equiv \{i \in N_U : \mu_i \in M_s\}$. Then, for each $s = 0, 1, \cdots$,

(a) $|M_s| = |N_s|$, and (b) if $M_U \backslash M_s \neq \emptyset$, then $M_{s+1} \supsetneq M_s$.

By Definition, for each $s = 0, 1, \cdots$, (a) holds. Thus, we show only (b).

Let $M_U \backslash M_s \neq \emptyset$. By Lemma A.2(ii), $M_{s+1} \supseteq M_s$. Suppose that $M_{s+1} = M_s$. By Lemma A.2(i), for each $x \in M_U \backslash M_s$, $p_x^{\min} > \overline{p}_x^s(\mu) \geq \overline{p}_x^0$. Thus, by $M_{s+1} = M_s$, for each $x \in M_U \backslash M_s$, $\overline{p}_x^0 \leq \overline{p}_x^{s+1}(\mu) < p_x^{\min}$. By Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

If $i \in N_s$, then for each $x \in M_U \backslash M_s$,

$$V_i(x, z_i^{\min}) \underset{i \in N_s}{=} V_i(x, \overline{z}_i^s(\mu)) \leq C_+^1(x; \overline{z}_{N_U}^s(\mu)) = \overline{p}_x^{s+1}(\mu) < p_x^{\min}.$$

Thus, for each $i \in N_s$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

Since for each $i \in N_C \cup N_s$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$, then

$$\{i \in N : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} = \{i \in N_U \backslash N_s : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\}.$$

and so

$$\left|\{i \in N : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\}\right| \leq |N_U \backslash N_s| \underset{\text{Lemma 1(i) and (a)}}{=} |M_U \backslash M_s|.$$

Thus $M_U \backslash M_s$ is weakly underdemanded, contradicting Fact A.2.

Thus, Claim A.2 holds.

Now we complete the proof of Step 1. If $M_U \backslash M_0 = \emptyset$, then for each $x \in M_U$, $\overline{p}_x^0(\mu) = p_x^{\min}$, and so Lemma A.2(ii) implies that for each $x \in M_U$, $\overline{p}_x^{|M_U|-1}(\mu) = p_x^{\min}$. Thus, assume $M_U \backslash M_0 \neq \emptyset$. Then, Claim A.2 says that as s increases, M_s expands strictly until $\{x \in M_U : \overline{p}_x^s(\mu) = p_x^{\min}\} = M_U$. Since Lemma A.1 implies $M_0 \neq \emptyset$, M_s expands strictly at most $|M_U| - 1$ times. Thus, $\{x \in M_U : p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu)\} = M_U$, i.e., for $x \in M_U$, $\overline{p}_x^{|M_U|-1}(\mu) = p_x^{\min}$.

Step 2: Let $\mu' \in \Omega$ be a non-MPE assignment. Then, $p_{M_U}^{\min} \leq p^{|M_U|-1}(\mu')$.

Definition A.2: Let $x_{N_U}^{\min}$ be an MPE assignment, $\mu' \in \Omega$, and $i \in N_U$. A sequence $\{\sigma^l(i)\}_{l=1}^d$ of distinct agents ($1 \leq d \leq n$) is called a **trading cycle from** i **in** μ' if (i) $\sigma^1(i) = i$, and (ii) for each $l \in \{1, \cdots, d-1\}$, $\mu'_{\sigma^{l+1}(i)} = x_{\sigma^l(i)}^{\min}$ and $\mu'_{\sigma^1(i)} = x_{\sigma^d(i)}^{\min}$.

Step 2-1: Let $i \in N_U$, and $\{\sigma^l(i)\}_{l=1}^d$ be a trading cycle from i in μ' and $s \geq 0$. If $\overline{p}_{\mu'_i}^s(\mu') \geq p_{\mu'_i}^{\min}$, then for each $j \in \{\sigma^1(i), \cdots, \sigma^d(i)\}$, $\overline{p}_{x_j^{\min}}^{s+d}(\mu') \geq p_{x_j^{\min}}^{\min}$.

If $d = 1$, then since $\{\sigma^l(i)\}_{l=1}^d = \{i\}$, Step 2-1 follows from Fact 2. Thus, let $d \geq 2$. Without loss of generality, we assume that $i = \sigma^1(i) = 1$, $\sigma^2(i) = 2$, $\cdots$, $\sigma^d(i) = d$. Then, $\mu'_2 = x_1^{\min}$, $\mu'_3 = x_2^{\min}$, $\cdots$, $\mu'_d = x_{d-1}^{\min}$, $\mu'_1 = x_d^{\min}$ and $\overline{p}_{\mu'_1}^s(\mu') \geq p_{\mu'_1}^{\min}$.

Let $j \in \{1, \cdots, d\}$ and assume $\overline{p}_{x_{j-1}^{\min}}^{s+j}(\mu') \geq p_{x_{j-1}^{\min}}^{\min}$. Then,

$$\begin{aligned}
\overline{p}_{x_j^{\min}}^{s+j+1}(\mu') &= C_+^1(x_j^{\min}; \overline{z}_{N_U}^{s+j}(\mu')) \\
&\geq V_j(x_j^{\min}, \overline{z}_j^{s+j}(\mu')) && \text{by } j \in N_U \\
&= V_j(x_j^{\min}, (x_{j-1}^{\min}, \overline{p}_{x_{j-1}^{\min}}^{s+j}(\mu'))) && \text{by } \mu'_j = x_{j-1}^{\min} \\
&\geq V_j(x_j^{\min}, (x_{j-1}^{\min}, p_{x_{j-1}^{\min}}^{\min}(\mu'))) && \text{by } \overline{p}_{x_d^{\min}}^s(\mu') \geq \overline{p}_{x_d^{\min}}^{\min}(\mu') \\
&\geq p_{x_j^{\min}}^{\min}. && \text{Def of Equilibrium}
\end{aligned}$$

Thus, it inductively holds that for each $k \in \{1, \cdots, d\}$, $\overline{p}^{s+k}_{x^{\min}_k}(\mu') \geq p^{\min}_{x^{\min}_k}$. Thus, by Fact 2, for each $j \in \{1, \cdots, d\}$, $\overline{p}^{s+d}_{x^{\min}_j}(\mu') \geq p^{\min}_{x^{\min}_j}$.

Step 2-2: Let $x^{\min}_{N_U}$ be an MPE assignment and $\mu' \in \Omega$. Let $\{N_l(\mu')\}_{l \in K}$ be a partition of N_U such that $K \equiv \{1, \cdots, k\}$, and for each $l \in K$, agents in $N_l(\mu')$ form a trading cycle.[16] Let $L_0 \equiv \{l \in K :$ there is $i \in N_l(\mu')$ s.t. $p^{\min}_{x^{\min}_i} = \overline{p}^0_{x^{\min}_i}\}$ and $M_0 \equiv \{x \in M_U :$there is $i \in \underset{r \in L_0}{\cup} N_r(\mu')$ s.t. $\mu'_i = x\}$. For each $s = 1, 2, \cdots$, let $L_s \equiv \{l \in K :$there are $i \in N_l(\mu')$ and $j \in \underset{r \in L_{s-1}}{\cup} N_r(\mu')$ s.t. $z^{\min}_j \, I_j \, z^{\min}_i\}$ and $M_s \equiv \{x \in M_U :$there is $i \in \underset{r \in L_s}{\cup} N_r(\mu')$ s.t. $\mu'_i = x\}$. Then, for each $s = 0, 1, \cdots$,

$$\text{(a)} \left| \underset{r \in L_s}{\cup} N_r(\mu') \right| = |M_s|, \text{ and (b) if } K \backslash L_s \neq \emptyset, \text{ then } L_{s+1} \supsetneq L_s.$$

By definition, for each $s = 0, 1, \cdots$, (a) holds. Thus, we show only (b).

Let $K \backslash L_s \neq \emptyset$. By Definition, $L_{s+1} \supseteq L_s$. Suppose that $L_{s+1} = L_s$. By $K \backslash L_s \neq \emptyset$, and Lemma 1(i), $M_U \backslash M_s \neq \emptyset$. For each $x \in M_U \backslash M_s$, by $L_{s+1} = L_s$, $x \notin M_0$ and so by Lemma 1(ii), $p^{\min}_x > \overline{p}^0_x$. Thus, by Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

Let $i \in \underset{r \in L_s}{\cup} N_r(\mu')$ and $x \in M_U \backslash M_s$. Let $j \in N_U$ be such that $\mu'(j) = x$, i.e., $z^{\min}_j = (x, p^{\min}_x)$. By $x \in M_U \backslash M_s$ and $L_{s+1} = L_s$, $j \notin \underset{r \in L_{s+1}}{\cup} N_r(\mu')$. Thus, by $i \in \underset{r \in L_s}{\cup} N_r(\mu')$, $z^{\min}_i \, I_i \, z^{\min}_j$ does not hold. By the definition of equilibrium, $z^{\min}_i \, R_i \, z^{\min}_j$ and so $z^{\min}_i \, P_i \, z^{\min}_j$, i.e., $x \notin D_i(p^{\min})$. Thus for each $i \in \underset{r \in L_s}{\cup} N_r(\mu')$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

Since for each $i \in \underset{r \in L_s}{\cup} N_r(\mu') \cup N_C$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$, then

$$\{i \in N : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} = \{i \in N_U \backslash \underset{r \in L_s}{\cup} N_r(\mu') : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\},$$

and so

$$\begin{aligned} & \left| \{i \in N_U \backslash \underset{r \in L_s}{\cup} N_r(\mu') : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} \right| \\ \leq & \left| N_U \backslash \underset{r \in L_s}{\cup} N_r(\mu') \right| \underset{\text{Lemma 1(i) and (a)}}{=} |M_U \backslash M_s|. \end{aligned}$$

Thus, $M_U \backslash M_s$ is weakly underdemanded, contradicting Fact A.2. Thus, (b) $L_{s+1} \supsetneq L_s$ holds.

Now we complete the proof of Step 2. By the finiteness of N_U, Step 2-2 implies that there is $q \in \{0, \cdots, k\}$ such that $L_q = K$. Let $d_0 \equiv \max_{l \in L_0} |N_l(\mu')|$ and for each $r = 1, \cdots, q$, $d_r \equiv \max_{l \in L_r \backslash L_{r-1}} |N_l(\mu')|$. By Fact B.4, $L_0 \neq \emptyset$.

[16] Precisely, there is $i \in N_l(\mu')$ such that there is a sequence $\{\sigma^l(i)\}_{l=1}^{|N_l(\mu')|}$ of distinct agents forming a trading cycle from i in μ'

If $q = 0$, then $d_0 \leq |N_U| = |M_U|$. Thus, by Step 2-1 and Fact 2, at round $d_0 - 1$, for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu') \leq \overline{p}_x^{|M_U|-1}(\mu')$. If $q > 0$, by Step 2-1, at round $d_0 - 1$, for each $x \in M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu')$ and there is $y \in M_1 \backslash M_0$ such that $p_y^{\min} \leq \overline{p}_y^{d_0-1}(\mu')$. By Step 2-1, at round $d_0 + d_1 - 1$, for each $x \in M_1 \backslash M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. By Fact 2, for each $x \in M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. Thus for each $x \in M_1$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. By induction argument, at round $D \equiv \sum_{i=0}^{q} d_i - 1$, for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{D}(\mu')$. Since $D \equiv \sum_{i=0}^{q} d_i \leq |N_U| = |M_U|$, then for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu') \leq \overline{p}_x^{|M_U|-1}(\mu')$. Thus Step 2 holds.

Step 3: Completion of the proof

By Fact 1, there is $\mu \in \Omega$ such that μ is an MPE assignments. By Step 1, for each $x \in M_U$, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu)$. By Step 2, for each $\mu' \in \Omega \backslash \{\mu\}$ and each $x \in M_U$, $p_x^{\min} \leq p_x^{|M_U|-1}(\mu')$. Thus Theorem 1(ii-2) holds. **Q.E.D.**

A.4 Proof of Theorem 2

By Lemma A.2(ii), (ii) implies (i). Thus, we only show that (i) implies (ii). Let $\mu \in \Omega$ and $s \leq |M_U|$ be such that $\overline{p}_{M_U}^{s-1}(\mu) = \overline{p}_{M_U}^{s}(\mu)$.

First, we introduce a weak variant of connectedness.

Definition A.3: Let $(z, p) \in Z \times \mathbb{R}^m$. An agent $i \in N$ is **weakly connected** at p if there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents such that
(i) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(ii) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$,
(iii) $x_{i_\Lambda} = x_i$, and
(iv) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $z_{i_\lambda} \, I_{i_\lambda} \, z_{i_{\lambda+1}}$.

Definition A.3 is weaker than Definition 3 since the weak connectedness does not require that for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \subseteq D_{i_\lambda}(p)$, but instead only $z_{i_\lambda} \, I_{i_\lambda} \, z_{i_{\lambda+1}}$.

Definition A.4: Let $(z, p) \in Z \times \mathbb{R}^m$. An object $x \in M$ is **weakly connected** at p if (i) x is assigned to a weakly connected agent or (ii) x is unassigned.

Let $(z, p) \in W$, and N_C and M_C be defined at (z, p). Then agents in N_C and objects in M_C are all weakly connected.

Step 1: For each $x \in M_U$, x is a weakly connected object at $(\overline{p}^{s-1}(\mu), p_{M_C})$.

Let M' be the set of weakly connected objects in M_U at $(\overline{p}^{s-1}(\mu), p_{M_C})$. To prove $M' = M_U$, we proceed by contradiction. Suppose that $M_U \backslash M' \neq \emptyset$. Let $N' \equiv \{i \in N_U : \mu_i \in M'\}$. Then, N' is the set of weakly connected agents in N_U at $(\overline{p}^{s-1}(\mu), p_{M_C})$, and $|M'| = |N'|$. Then by Lemma 1(i) and $|M'| = |N'|$, $N_U \backslash N' \neq \emptyset$.

If there is $x \in M_U \backslash M'$ such that $\overline{p}_x^{s-1}(\mu) = C_+^1(x; z_{N_C})$, then either $\overline{p}_x^{s-1}(\mu) = 0$ or there is some $j \in N_C$ such that $(x, \overline{p}_x^{s-1}(\mu)) \, I_j \, z_j$, contradicting $x \in M_U \backslash M'$. Thus, for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) \neq C_+^1(x; z_{N_C})$. Thus, by Fact 2, for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) \geq C_+^1(x; z_{N_C})$ and so $\overline{p}_x^{s-1}(\mu) > C_+^1(x; z_{N_C}) \geq 0$.

Let $x \in M_U \backslash M'$. Note that $\overline{p}_x^{s-1}(\mu) \equiv C_+^1(x; \overline{z}_{N_U}^{s-1}(\mu)) \geq C^1(x; \overline{z}_{N'}^{s-1}(\mu))$. Suppose $\overline{p}_x^{s-1}(\mu) = C^1(x; \overline{z}_{N'}^{s-1}(\mu))$. Then, there is $i \in N'$ such that $\overline{p}_x^{s-1}(\mu) = V_i(x; \overline{z}_i^{s-1}(\mu))$. By $i \in N'$ and $\overline{p}_x^{s-1}(\mu) = V_i(x; \overline{z}_i^{s-1}(\mu))$, x is a weakly connected object at $(\overline{p}^{s-1}(\mu), p_{M_C})$, contradicting $x \in M_U \backslash M'$. Thus, for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) > C^1(x; \overline{z}_{N'}^{s-1}(\mu))$.

Let s' be the earliest round in the IPOIP process such that there is $x \in M_U \backslash M'$ such that $\overline{p}_x^{s'}(\mu) = \overline{p}_x^{s-1}(\mu)$. Then, by Fact 2 and $s' \leq s-1$,

$$\text{for each } s'' < s' \text{ and each } y \in M_U \backslash M',\ \overline{p}_y^{s''}(\mu) < \overline{p}_y^{s'}(\mu) \leq \overline{p}_y^{s-1}(\mu). \tag{$*$}$$

Since for each $y \in M_U \backslash M'$, $\overline{p}_y^{s-1}(\mu) > C_+^1(x; z_{N_C})$, then $s' \geq 1$.

To derive a contradiction to $(*)$, we first show the following claim.

Claim A.3: Let $i \in N_U$, $x \in M_U$, $s' \leq s-1$, and $V_i(x, \overline{z}_i^{s'-1}(\mu)) = \overline{p}_x^{s-1}(\mu)$. Then $\overline{p}_{\mu_i}^{s'-1}(\mu) = \overline{p}_{\mu_i}^{s-1}(\mu)$.

Note that

$$\overline{p}_x^{s-1}(\mu) = V_i(x, \overline{z}_i^{s'-1}(\mu)) \underset{\text{Fact2 \& } s' \leq s-1}{\leq} V_i(x, \overline{z}_i^{s-1}(\mu)) \underset{i \in N_U}{\leq} C_+^1(x; \overline{z}_{N_U}^{s-1}(\mu)) = \overline{p}_x^{s}(\mu).$$

Thus, by $\overline{p}^{s-1}(\mu) = \overline{p}^s(\mu)$, $V_i(x, \overline{z}_i^{s'-1}(\mu)) = V_i(x, \overline{z}_i^{s-1}(\mu))$. Since $\overline{z}_i^{s'-1}(\mu) = (\mu_i, \overline{p}_{\mu_i}^{s'-1}(\mu))$ and $\overline{z}_i^{s-1}(\mu) = (\mu_i, \overline{p}_{\mu_i}^{s-1}(\mu))$, then $\overline{p}_{\mu_i}^{s'-1}(\mu) = \overline{p}_{\mu_i}^{s-1}(\mu)$.

By the definition of IPOIP process and $s' \geq 1$, there is $i \in N_U$ such that $V_i(x, \overline{z}_i^{s'-1}(\mu)) = \overline{p}_x^{s'}(\mu) = \overline{p}_x^{s-1}(\mu)$. Note that for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) > C^1(x; \overline{z}_{N'}^{s''}(\mu))$. By Fact 2, for each $s'' \leq s-1$, $\overline{p}_x^{s-1}(\mu) > C^1(x; \overline{z}_{N'}^{s''}(\mu))$. Thus, $i \notin N'$ and so $i \in N_U \backslash N'$, and $\mu_i \in M_U \backslash M'$. By Claim A.3, $\overline{p}_{\mu_i}^{s'-1}(\mu) = \overline{p}_{\mu_i}^{s-1}(\mu)$, contradicting $(*)$.

Thus $M_U \backslash M' \neq \emptyset$ fails to hold, i.e., $M_U = M'$.

Step 2: Let $M_0 \equiv \{x \in M_U : \overline{p}_x^{s-1}(\mu) = C_+^1(x; z_{N_C})\}$. Then $M_0 \neq \emptyset$.

By Definitions A.3 and A.4 and Step 1, there is no $x \in M_U \backslash M_0$ that is weakly connected to some $y \in M_C$ at $(\overline{p}^{s-1}(\mu), p_{M_C})$. Thus, $M_0 \neq \emptyset$ just follows Step 1.

Step 3: For each $x \in M_U$, $\overline{p}_x^{s-1}(\mu) \leq p_x^{\min}$.

If $M_U = M_0$, by Lemma 1(ii), Step 3 trivially holds. Thus, let $M_U \backslash M_0 \neq \emptyset$.

Let $M' \equiv \{x \in M_U : \forall\ x \in M', \overline{p}_x^{s-1}(\mu) > p_x^{\min}\}$. To show $M' = \emptyset$, we proceed by contradiction. Suppose that $M' \neq \emptyset$.

Let $N' \equiv \{i \in N_U : \mu_i \in M'\}$. By Definition, $|N'| = |M'|$. By Lemma 1(i) and $|M'| = |N'|$, $|N_U \backslash N'| = |M_U \backslash M'| \neq \emptyset$. By Step 2, $M_U \backslash M' \supseteq M_0 \neq \emptyset$ and so $N_U \backslash N' \neq \emptyset$.

For each $i \in N_U$ and each $x \in L \backslash M_U$,

$$z_i^{\min} \, R_i \, (x_i, p_{x_i}^{\min}) \underset{\text{Lemma 1(ii)\&(iii)}}{P_i} (x_i, p_{x_i}) = z_i \underset{\text{Def of Equilibrium}}{R_i} (x, p_x) \underset{\text{Theorem 1(i)}}{=} (x, p_x^{\min}).$$

and so $D_i(p^{\min}) \subseteq M_U$.

For each $i \in N_U$ and each $y \in M_U \backslash M'$,

$$\begin{aligned} \overline{z}_i^{s-1}(\mu) \quad & R_i\ (y, \overline{p}_y^{s-1}(\mu)) && \text{by } \overline{p}_y^{s-1}(\mu) = \overline{p}_y^{s}(\mu) \geq V_i(y, \overline{z}_i^{s-1}(\mu)) \\ & R_i\ (y, p_y^{\min}). && \text{by } \overline{p}_y^{s-1}(\mu) \leq p_y^{\min} \end{aligned}$$

For each $i \in N'$ and each $y \in M_U \backslash M'$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (\mu_i, p_{\mu_i}^{\min}) \underset{p_{\mu_i}^{\min} < \overline{p}_{\mu_i}^{s-1}(\mu)}{P_i} \overline{z}_i^{s-1}(\mu) \underset{N' \subseteq N_U}{R_i} (y, p_y^{\min}).$$

Thus, for each $i \in N'$, by $D_i(p^{\min}) \subseteq M_U$, $D_i(p^{\min}) \subseteq M'$. Thus,

$$\left|\{i \in N_U : D_i(p^{\min}) \subseteq M'\}\right| \geq |N'| = |M'|.$$

By Step 1, for each $x \in M'$, x is weakly connected at $(\overline{p}^{s-1}(\mu), p_{M_C})$ and $\overline{p}_x^{s-1}(\mu) > p_x^{\min} \geq 0$. Then by $N_U \backslash N' \neq \emptyset$, there is $i \in N_U \backslash N'$ and $x' \in M'$ such that $\overline{z}_i^{s-1}(\mu)\, I_i\, (x', \overline{p}_{x'}^{s-1}(\mu))$. Thus for each $y \in M_U \backslash M'$,

$$z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x', p_{x'}^{\min}) \underset{p_{x'}^{\min} < \overline{p}_{x'}^{s-1}(\mu)}{P_i} (x', \overline{p}_{x'}^{s-1}(\mu))\, I_i\, \overline{z}_i^{s-1}(\mu) \underset{i \in N_U \backslash N'}{R_i} (y, p_y^{\min}).$$

Thus $y \notin D_i(p^{\min})$. By $D_i(p^{\min}) \subseteq M_U$, $D_i(p^{\min}) \subseteq M'$ so

$$|M'| < |N'| + 1 \leq \left|\{i \in N_U : D_i(p^{\min}) \subseteq M'\}\right|,$$

contradicting Fact A.2.

Step 4: For each $i \in N_U$ and each $x \in L \backslash M_U$, $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq p_x^{\min}$.

Let $i \in N_U$ and $x \in L \backslash M_U$. By Lemma 1(iii), $x_i^{\min} \in M_U$. Thus,

$$V_i(x_i^{\min}; \overline{z}_i^{s-1}(\mu)) \leq C_+^1(x_i^{\min}; \overline{z}^{s-1}(\mu)) = \overline{p}_{x_i^{\min}}^{s}(\mu) \underset{\overline{p}^{s-1}(\mu) = \overline{p}^{s}(\mu)}{=} \overline{p}_{x_i^{\min}}^{s-1}(\mu).$$

Thus, $\overline{z}_i^{s-1}(\mu)\, R_i\, (x_i^{\min}, \overline{p}_{x_i^{\min}}^{s-1}(\mu))$. Note

$$(x_i^{\min}, \overline{p}_{x_i^{\min}}^{s-1}(\mu)) \underset{\text{Step 3}}{R_i} (x_i^{\min}, p_{x_i^{\min}}^{\min}) = z_i^{\min} \underset{\text{Def. of Equilibrium}}{R_i} (x, p_x^{\min}).$$

Thus, by $\overline{z}_i^{s-1}(\mu)\, R_i\, (x_i^{\min}, \overline{p}_{x_i^{\min}}^{s-1}(\mu))$, $\overline{z}_i^{s-1}(\mu)\, R_i\, (x, p_x^{\min})$, i.e., $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq p_x^{\min}$.

Step 5: $((\overline{z}^{s-1}(\mu), z_{N_C}), (\overline{p}^{s-1}(\mu), p_{M_C})) \in W^{\min}$

By Lemma 1(ii) and Theorem 1(i), for each $i \in N_C$, (E-i) holds. For each $i \in N_U$ and each $x \in M_U$, $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq C_+^1(x; \overline{z}^{s-1}(\mu)) = \overline{p}_x^{s}(\mu) = \overline{p}_x^{s-1}(\mu)$, and for each $x \in L \backslash M_U$,

$$V_i(x; \overline{z}_i^{s-1}(\mu)) \underset{\text{Step 5}}{\leq} p_x^{\min} \underset{\text{Theorem 1(i)}}{=} p_x.$$

Thus (E-i) holds. (E-ii) holds obviously. Thus $((\overline{z}^{s-1}_{N_U}(\mu), z_{N_C}), (\overline{p}^{s-1}_{M_U}(\mu), p_{M_C})) \in W$. By Theorem 1(i), Step 3, and Fact 1, $p^{\min} = (\overline{p}^{s-1}_{M_U}(\mu), p_{M_C})$. Thus Step 5 holds. **Q.E.D.**

Appendix B: Proofs of Proposition 2 and Fact 3

B.1 Proof of Proposition 2

Proof: Let $(z^{\min}, p^{\min})$ be an MPE for k objects. Let (z, p) be the output of the equilibrium-generating mechanism. We prove that (z, p) is an equilibrium. Let $M(k)$ be the set of k objects and let $L(k) = M(k) \cup \{0\}$.

Mechanism stops at Phase 1: In this case, $C^1(y; z^{\min}) \leq 0$ and $z = z^{\min}$. Since $z = z^*$, then for each $i \in N$ and each $x \in L(k)$,

$$z_i = z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x, p_x^{\min}) = (x, p_x)$$

and for y, by $C^1(y; z^{\min}) \leq 0$ and $p_y = 0$, $z_i = z_i^{\min} \, R_i \, (y, C^1(y; z^{\min})) \, R_i \, (y, p_y)$. Thus, (z, p) satisfies (E-i). It is straightforward that (z, p) satisfies (E-ii).

Mechanism stops at Phase 2: In this case, $C^1(y; z^{\min}) > 0$, and there is $i \in N'$ such that $z_i^* = (y, C_+^2(y; z^{\min}))$. For each $x \in L(k) \cup \{y\}$,

$$z_i \underset{C_+^2(y;z^{\min}) \leq C_+^1(y;z^{\min})}{R_i} z_i^{\min} \underset{\text{Def of Equilibrium}}{R_i} (x, p_x^{\min}) = (x, p_x).$$

For each $j \in N \backslash \{i\}$ and each $x \in L(k)$, by Definition 8

$$z_j \, R_j \, z_j^{\min} \underset{\text{Def of Equilibrium}}{R_j} (x, p_x^{\min}) = (x, p_x),$$

and for y, by $V_j(y; z_j^{\min}) \leq C_+^2(y; z^{\min}) = p_y$, $z_j \, R_j \, z_j^{\min} \, R_j \, (y, p_y)$.

Thus, (z, p) satisfies (E-i). Unassigned objects at $M(k)$ remain unassigned with zero prices, and $p_{x_{i_1}} = p^{\min}_{x_{i_1}} = 0$. Thus (z, p) satisfies (E-ii). **Q.E.D.**

B.2 Proof of Fact 3

(i): Let $\mu \in \Omega$ be such that for each $i \in N_U$, $\mu_i \in \Gamma_i$. Then, for each $i \in N_U$, $D_i(\overline{p})$. Thus, $\overline{p}$ is an equilibrium price. Moreover, by Lemma 1, it is an MPE price. Thus, μ is an MPE assignment.

(ii): Suppose that there is $i \in N_1$ such that $x_i^{\min} \in M_2$. Then by $|N_1| = |M_1|$ and $|N_2| = |M_2|$, there is $j \in N_2$ such that $x_j^{\min} \in M_2$. Note that there is $y \in M_U$ such that $p_y^{\min} = \overline{p}_y^0$. If $y \in M_2$, then

$$(y, p_y^{\min}) = (y, \overline{p}_y^0) \underset{\text{by } j \in N_2}{P_j} (x_j^{\min}, \overline{p}_x^0) \underset{\text{by } \overline{p}^0_{x_j^{\min}} \leq p^{\min}_{x_j^{\min}}}{R_i} (x_j^{\min}, p_y^{\min}),$$

which implies $x_j^{\min} \notin D_j(p^{\min})$, a contradiction. Thus, $y \in M_1$. Thus,

$$(y, p_y^{\min}) = (y, \overline{p}_y^0) \underset{\text{by } i \in N_1}{P_i} (x_i^{\min}, \overline{p}_x^0) \underset{\text{by } \overline{p}^0_{x_i^{\min}} \le p^{\min}_{x_i^{\min}}}{R_i} (x_i^{\min}, p_y^{\min}),$$

which also implies $x_i^{\min} \notin D_i(p^{\min})$, a contradiction. Thus, for each $i \in N_1$, $x_i^{\min} \in M_1$.

Similarly we can show that for each $i \in N_2$, $x_i^{\min} \in M_2$.

(iii): It follows from Lemma 1. **Q.E.D.**

Appendix C: Proof Theorem 4

Let $r \in \mathbb{R}_+^m$ be the reserve price vectors. A pair $(z, p) \in Z \times \mathbb{R}_+^m$ is an **equilibrium with reserve price** r if (i) (E-i) holds and (ii) for each $y \in M$, $p_y \ge r_y$, and if $p_y > r_y$ then there is some agent $i \in N$ such that $y = x_i$.

By Demange and Gale (1985), there is an equilibrium with reserve price r and the set of equilibrium prices with reserve price r is a complete lattice. Thus, there is an MPE with reserve price r. Let the MPE rule with reserve price r be a mapping from each preference profile to an MPE with reserve price r.

Fact C.1 (Demange and Gale, 1985): The MPE rule with reserve price r is strategy-proof on the set of general preference profiles.

For each $R \in \mathcal{R}^n$, let $W(R)$ and $W^{\min}(R)$ be the set of equilibria and that of MPEs for R. For each $0 \le k \le m$, let $W(k, R)$ and $W^{\min}(k, R)$ be the corresponding notions and $M(k)$ be the set of objects in the assignment market with k objects. Fact C.2 follows Definition 6 of equilibrium-generating mechanism.

Fact C.2: Let $R \in \mathcal{R}^n$ and $(z^{\min}, p^{\min}) \in W^{\min}(k, R)$. Let $i \in N$, $R_i' \in \mathcal{R}$, and $R' = (R_i', R_{-i}) \in \mathcal{R}^n$. Let (z', p') be the outcome of equilibrium-generating mechanism for R'. Let $i^* \in N \backslash \{i\}$ be such that $x'_{i^*} = y$. Then

(i) $z'_{i^*} \, R_{i^*} \, z^{\min}_{i^*}$ and for each $j \in N \backslash \{i^*\}$, $z'_j \, I_j \, z_j^{\min}$,

(ii) $p'_{M(k)} = p^{\min}$, and

(iii) for each $x \in M(k)$ such that $p_x^{\min} > 0$, there is $j \in N \backslash \{i^*\}$ such that $x'_j = x$.

Proof of Theorem 4(i)

Let $R \in \mathcal{R}^n$, $(z^{\min}, p^{\min}) \in W^{\min}(k, R)$, and $i \in N$. Let (z, p) and (z', p') be the outcomes of equilibrium-generating mechanism for R and $R' = (R_i', R_{-i})$, i.e, $z = g^{sub1}(R; z^{\min})$ and $z' = g^{sub1}(R'; z^{\min})$. We show $z_i \, R_i \, z_i'$.

By contradiction, suppose that $z_i' \, P_i \, z_i$. By Fact C.2(i), $x_i' = y$ and so $p'_{k+1} = C_+^1(y; z^{\min}_{N\backslash\{i\}})$. In case of $x_i = y$, by $V_i(y; z_i^{\min}) = C_+^1(y; z^{\min})$, $p_{k+1} = C_+^2(y; z^{\min}) = C_+^1(; z^{\min}_{N\backslash\{i\}})$. Thus $z_i' = z_i$, contradicting $z_i' \, P_i \, z_i$. In case of $x_i \in M(k)$, by $V_i(y; z_i^{\min}) \le C_+^2(y; z^{\min}) \le C_+^1(; z^{\min}_{N\backslash\{i\}}) = p'_{k+1}$, $z_i \, I_i \, z_i^{\min} \, R_i \, (y, p'_y) = z_i'$, contradicting $z_i' \, P_i \, z_i$. Thus, $z_i \, R_i \, z_i'$.

Proof of Theorem 4(ii)

Let $R \in \mathcal{R}^n$ and $z \in Z^{k+1}$ be an equilibirum allocation. Let N_C and N_U be defined at z for R. If $i \in N_C$, then agent i does not participate in the MPE-adjustment mechansim and keeps the same allocation as z_i, and so incentive property holds trivially. Thus, let $i \in N_U$.

Let $z_{N_U} = g^{sub2}(R; z)$ and $z'_{N_U} = g^{sub2}(R'; z)$ be outcomes of MPE-adjustment mechanism for R and $R' = (R'_i, R_{-i})$. Since z_{N_U} is an MPE for (N_U, M_U, R_{N_U}) with $r' = (C^1_+(x; z^{\min}_{N_C}))_{x \in M_U}$, and that z'_{N_U} is an MPE for (N_U, M_U, R'_{N_U}) with r', by Fact C.1, $z_i \, R_i \, z'_i$.

Proof of Theorem 4(iii)

The proof contains three steps. Step 1 gives additional four facts. Step 2 establishes three lemmas based on the above facts, together with Facts C.1 and C.2. Step 3 completes the proof.

Step 1: **Construction of Facts C.3 to C.6**

Fact C.3: Let $R \in \mathcal{R}^n$, $(z, p) \in W(R)$, $N' \subseteq N$, and $M' \subseteq M$. Let N_C be defined at (z, p).
(i) Let $(z_{N'}, p_{M'})$ be an MPE in $(N', M', R_{N'})$. Then $N' \subseteq N_C$.
(ii) Let $i \in N$, $R'_i \in \mathcal{R}$, $R' = (R'_i, R_{-i}) \in \mathcal{R}^n$ and $(z, p) \in W^{\min}(R')$. Then $i \in N_C$.

Proof: (i) Let $x \in M'$ be such that $p_x > 0$ for $(N', M', R_{N'})$. Since $(z_{N'}, p_{M'})$ is an MPE in $(N', M', R_{N'})$, there is a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x in $(N', M', R_{N'})$. Note that for each $\lambda = 1, \ldots, \Lambda$,

$$\{y \in M' : \forall y' \in M', (y, p_y) \, R_\lambda \, (y', p_{y'})\} \subseteq \{y \in M : \forall y' \in M, (y, p_y) \, R_\lambda \, (y', p_{y'})\}.$$

Thus, the DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x is also a DCP in (N, M, R). Thus $x \in N_C$. Thus, $M' \subseteq M_C$ and so $N' \subseteq N_C$.
(ii) If $x_i = 0$ or $p_{x_i} = 0$, by definition, $i \in N_C$. Thus, let $p_{x_i} > 0$. By $(z, p) \in W^{\min}(R')$, there is a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x_i at (z, p) in (N, M, R'). By $R_{-i} = R'_{-i}$ and $i \notin \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$, the agents in $\{i_\lambda\}_{\lambda=1}^{\Lambda-1}$ have the same demands at p in (N, M, R) as in (N, M, R'). Thus, $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ is also a DCP to x_i at the same pair (z, p) in (N, M, R). Thus $i \in N_C$. **Q.E.D.**

Fact C.4 follows the definition of DCP (Definition 2).

Fact C.4: For $x \in M_C$ such that $p_x > 0$ and each DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x, $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$.

Given $N' \subseteq N$, $M' \subseteq M$, $R'_{N'} \in \mathcal{R}^n$ and $r \in \mathbb{R}_+^{|M'|}$, let $Z(N', M', R'_{N'}, r)$ and $Z^{\min}(N', M', R'_{N'}, r)$ denote the sets of equilibrium and MPE allocations for $(N', M', R'_{N'})$ with reserve price r, respectively. Let $W(\cdot, \cdot, \cdot, \cdot)$ and $W^{\min}(\cdot, \cdot, \cdot, \cdot)$ be the sets of equilibria and MPEs similarly defined for $(N', M', R'_{N'})$ with reserve price r. When $r = \mathbf{0}$ or $N' = N$ or $M' = M$, we just omit writing r or N' or M'. Recall that $Z(R)$ is the set of equilibrium allocations for (N, M, R) with $r = \mathbf{0}$. Given $N' \subseteq N$, denote

$$Z^{\min}_{N'}(R) \equiv \{z_{N'} : \exists z_{N \setminus N'} \text{ such that } (z_{N'}, z_{N \setminus N'}) \in Z^{\min}(R)\}.$$

The following fact is easy to see.

Fact C.5: Let $R \in \mathcal{R}^n$ and $(z,p) \in W(R)$. Let N_C be defined at (z,p) for R. Let $N' \subseteq N_C$, $N'' = N \backslash N'$ and $M'' = \{x_i : i \in N''\}$. Then

$$Z^{\min}_{N''}(R) = Z^{\min}(N'', M'', R_{N''}, r) \text{ where } r_x = C^1_+(R, x; z_{N'}) \text{ for each } x \in M''.$$

Fact C.6: Let $R \in \mathcal{R}^n$ and $(z,p) \in W(R)$. Let N_U be defined at (z,p). Let $N' \subseteq \{i \in N : x_i \in M \text{ and } p_{x_i} > 0\}$. If for each $j \in N \backslash N'$, $D_j(p) \cap \{x_i : i \in N'\} = \emptyset$, then $N' \subseteq N_U$.

Proof: By contradiction, suppose that there is $k \in N' \cap N_C$. Then, by $x_k \in M_C$, $p_{x_k} > 0$ and Definition 2, there is a DCP of agents $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ to x_k satisfying (a) $p_{x_{i_1}} = 0$, $i_\Lambda = k$, and $x_{i_\Lambda} = x_k$, and (b) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \subseteq D_{i_\lambda}(p)$. By (b), for agent $i_{\Lambda-1}$, $\{x_k, x_{i_{\Lambda-1}}\} \subseteq D_{i_{\Lambda-1}}(p)$. Since $k \in N'$, $x_k \in D_{i_{\Lambda-1}}(p)$ and $D_j(p) \cap \{x_i : i \in N'\} = \emptyset$ for each $j \in N \backslash N'$, we have $i_{\Lambda-1} \notin N \backslash N'$, ie., $i_{\Lambda-1} \in N'$. Repeating the same argument, we can show $\{i_\lambda : \lambda = 1, ..., \Lambda\} \subseteq N'$. By (a), we have $p_{x_{i_1}} = 0$, contradicting that for each $i \in N'$, $x_i \in M$ and $p_{x_i} > 0$. **Q.E.D.**

Step 2: **Construction of Lemmas C.1 to C.3**

Lemma C.1: Let $R \in \mathcal{R}^n$ and $(z^*, p^{\min}) \in W^{\min}(k, R)$. Let (z,p) be the outcome of equilibrium-generating mechanism for R. Let N_C, N_U, M_C, and M_U be defined at (z,p) for R. Let $i^* \in N$ be such that $x_{i^*} = k+1$.

(i) $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$ and $(z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C})$.

(ii) If (a) $C^1_+(R, k+1; z^*) > C^2_+(R, k+1; z^*)$, or

(b) $\left|\{j \in N_C \backslash \{i^*\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| \leq 1$,

then $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$.

Proof: **Part (i)**: By Facts C.2(i) and C.2(ii), (E-i) holds for $(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$. By Fact C.2(iii), (E-ii) holds for $(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$. Thus $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$. By the same reasoning, $(z_{N_C}, p_{M_C}) \in W(N_C, M_C, R_{N_C})$.

Let $x \in M_C$ be such that $p_x > 0$ and $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ be a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x. Then by Fact C.4, $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$. Thus, each $x \in M_C$ such that $p_x > 0$ is also connected in (N_C, M_C, R_{N_C}). Thus by Proposition 1, $(z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C})$.

Part (ii) Let $x \in M_C \backslash \{k+1\}$ be such that $p_x > 0$ and $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ be a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x. Then by Fact C.4, $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$. Note that to establish that $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$, by Proposition 1 and Part (i), we only need to show $i^* \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$, which implies $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C \backslash \{i^*\}$, in each of Case (a) and Case (b). In the following, we show $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C \backslash \{i^*\}$ in Case (a).

Assume $C^1_+(R, k+1; z^*) > C^2_+(R, k+1; z^*)$. Then by $V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R, k+1; z^*)$ and $x_{i^*} = k+1$, we have: $V_{i^*}(k+1; z^*_{i^*}) > C^2_+(R, k+1; z^*) = t_{i^*}$ and $z_{i^*} = (k+1, t_{i^*})\, P_{i^*}\, z^*_{i^*}$. Thus, $D_{i^*}(p) = \{k+1\}$. By Definition 2(ii-4), $i^* \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$.

For Case (b), if $\left|\{j \in N_C \backslash \{i^*\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| = 0$. it is straightforward to see $i^* \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$. For the other case, the same reasoning of Case (a) works. **Q.E.D.**

Lemma C.2: Let $R \in \mathcal{R}^n$ and $(z^*, p^{\min}) \in W^{\min}(k, R)$. Let $i \in N$, $R'_i \in \mathcal{R}$ and $R' = (R'_i, R_{-i}) \in (R^G)^n$. Let (z, p) and (z', p') be the outcomes of equilibrium-generating mechanism for R and R'. Let N_C, N_U, M_C, and M_U be defined at (z, p) for R. Let $i^* \in N$ be such that $x_{i^*} = k + 1$.

(i) For each $j \in N_C \backslash \{i^*\}$, $x'_j \in M_C \cup \{0, k+1\}$.

(ii) For each $x \in M_U \backslash \{k+1\}$, there is $j \in N_U \cup \{i^*\}$ such that $x'_j = x$.

Proof: **Part (i)**: Let $j \in N_C \backslash \{i^*\}$. By contradition, suppose that $x'_j \notin M_C \cup \{0, k+1\}$. By $x'_j \neq k+1$, we have $x'_j \in M_U \backslash \{k+1\}$. By $x'_j \neq k+1$ and Fact C.2(ii), $p_{x'_j} = p^{\min}_{x'_j} = p'_{x'_j}$. Thus, by Lemma 1(ii), and $x'_j \in M_U$, we have $V_j(x'_j; z^*_j) < p_{x'_j} = p'_{x'_j}$. On the other hand, by Fact C.2(i), $(x'_j, p'_{x'_j}) = z'_j \, R_j \, z^*_j$. Thus, $V_j(x'_j; z^*_j) \geq p'_x$. This is a contradiction.

Part (ii): Let $x \in M_U \backslash \{k+1\}$. By Lemma 1(ii) and $x \in M_U$, $p_x > 0$. By Fact C.2(ii) and $x \neq k+1$, $p_x = p^{\min}_x = p'_x$. Thus, $p'_x > 0$. By Fact C.2(iii) and Part (i), there is $j \in N_U \cup \{i^*\}$ such that $x'_j = x$. **Q.E.D.**

Lemma C.3: Let $R \in \mathcal{R}^n$ and $(z^*, p^{\min}) \in W^{\min}(k, R)$. Let $i \in N$, $R'_i \in R^G$ and $R' = (R'_i, R_{-i}) \in \mathcal{R}^n$. Let (z, p) and (z', p') be respectively the outcomes of some equilibrium-generating mechanisms for R and R'. Let N_C, N_U, M_C, and M_U be defined at (z, p) for R. Let N'_C, N'_U, M'_C, and M'_U be defined at (z', p') for R'. Let $i^* \in N$ be such that $x_{i^*} = k + 1$.

(i) Let $i^* \neq i$, $k + 1 \in M_C$ and $p_{k+1} = C^1_+(R, k+1; z^*) > 0$. Then $(z, p) \in W^{\min}(M(k+1), R)$.

(ii) Let $i \in N_C$ and $i^* \neq i$.

(ii-1) Let $V'_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) > 0$. Then

$$V'_i(k+1; z^*_i) = p'_{k+1} = C^1_+(R, k+1; z^*) = C^1_+(R', k+1; z^*) = V_{i^*}(k+1; z^*_{i^*}) > 0.$$

(ii-2) Let $V'_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) > 0$. Then $(z', p') \in W^{\min}(N, M(k+1), R')$.

(ii-3) Let $V'_i(k+1; z^*_i) > C^1_+(R, k+1; z^*) > 0$. Let $R''_i \in R^G$ be such that for each $x \in M(k)$, $V''_i(x; \cdot) = V'_i(x; \cdot)$ and $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*)$. Let $R'' \equiv (R''_i, R_{-i})$. Then $(z', p') \in W^{\min}(N, M(k+1), R'')$.

(iii) Let $i^* \in N_U$ and $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$. Then $N_C = N'_C$, $M_C = M'_C$, $M_U = M'_U$, and $N_U = N'_U$.

Proof: **Part (i-1)** By Proposition 2, $(z, p) \in W(M(k+1), R)$. To establish $(z, p) \in W^{\min}(M(k+1), R)$, by Proposition 1, we only need to show $N_C = N$.

By Fact C.2(ii), we have: (1) $p_{M(k)} = p^{\min}$. Since z is generated by equilibrium-generating mechanism for R and $x_{i^*} = k+1$, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ to $x^*_{i^*}$ such that (2) $i_\Lambda = i^*$, (3) $x^*_{i_1} = 0$ or $p^{\min}_{x^*_{i_1}} = 0$, and (4) for each $\lambda \in \{1, ..., \Lambda - 1\}$, $\{x^*_{i_\lambda}, x^*_{i_{\lambda+1}}\} \in D_{i_\lambda}(p^{\min})$ and $x_{i_\lambda} = x^*_{i_{\lambda+1}}$. By $x_{i^*} = k+1$, $V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R', k+1; z^*) = p_{k+1}$. Thus by (2), we have: (5) $x_{i^*} = x_{i_\Lambda} = k+1$ and (6) $V_{i_\Lambda}(k+1; z^*_{i_\Lambda}) = p_{k+1}$. In the next two paragraphs, we show that $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$.

By $k+1 \in M_C$, $i_\Lambda \in N_C$, and there is a DCP $\{i'_\lambda\}_{\lambda=1}^{\Lambda'}$ to $k+1$ in (z,p). Note

$$(k+1, p_{k+1}) \underset{(6)}{I_{i_\Lambda}} z^*_{i_\Lambda} = (x^*_{i_\Lambda}, p^{\min}_{x^*_{i_\Lambda}}) \underset{(1)\ \&\ (4)}{=} (x_{i_{\Lambda-1}}, p_{x_{i_{\Lambda-1}}}).$$

Thus, by (1), (4) and (5), we have $\{x_{i_{\Lambda-1}}, x_{i_\Lambda}\} \subseteq D_{i_\Lambda}(p)$. Thus, $\{i'_\lambda\}_{\lambda=1}^{\Lambda'} \cup \{i_{\Lambda-1}\}$ is a DCP to $x_{i_{\Lambda-1}} = x^*_{i^*}$ in (z,p). Thus, $i_{\Lambda-1} \in N_C$.

By (4), we have: $\{x^*_{i_{\Lambda-1}}, x^*_{i_\Lambda}\} \subseteq D_{i_{\Lambda-1}}(p^{\min})$, $x^*_{i_{\Lambda-1}} = x_{i_{\Lambda-2}}$ and $x^*_{i_\Lambda} = x_{i_{\Lambda-1}}$. Thus, by (1), $\{x_{i_{\Lambda-2}}, x_{i_{\Lambda-1}}\} \in D_{i_{\Lambda-1}}(p)$. Thus, $\{i'_\lambda\}_{\lambda=1}^{\Lambda'} \cup \{i_{\Lambda-1}, i_{\Lambda-2}\}$ is a DCP to $x_{i_{\Lambda-2}} = x^*_{i_{\Lambda-1}}$ in (z,p). Thus, $i_{\Lambda-2} \in N_C$. Similarly, we have that for each $\lambda = \Lambda-3, \ldots, 1$, $i_\lambda \in N_C$. Thus, $\{i_\lambda\}_{\lambda=1}^{\Lambda} \subseteq N_C$.

Finally, we show that for each $j \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$, $j \in N_C$. Let $j \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$. Note that $x_j = x^*_j$, and so by $(z^*, p^{\min}) \in W^{\min}(k,R)$ and Proposition 1, there is a DCP $\{i''_\lambda\}_{\lambda=1}^{\Lambda''}$ to $x_j = x^*_j$ in $(z^*, p^{\min})$. If $\{i''_\lambda\}_{\lambda=1}^{\Lambda''} \cap \{i_\lambda\}_{\lambda=1}^{\Lambda} = \emptyset$, since for each λ, $x_{i''_\lambda} = x^*_{i''_\lambda} \neq k+1$, by (1), $\{i''_\lambda\}_{\lambda=1}^{\Lambda''}$ is also a DCP to $x_j = x^*_j$ in (z,p) and so $j \in N_C$. Thus, assume $\{i''_\lambda\}_{\lambda=1}^{\Lambda''} \cap \{i_\lambda\}_{\lambda=1}^{\Lambda} \neq \emptyset$. Then, there is $\lambda' \in \{1, \ldots, \Lambda''\}$ such that $i''_{\lambda'} \in \{i_\lambda\}_{\lambda=1}^{\Lambda}$ and for any $\lambda'' > \lambda'$, $i''_{\lambda''} \notin \{i_\lambda\}_{\lambda=1}^{\Lambda}$. Let λ'' be such that $i_{\lambda''} = i''_{\lambda'}$. Note that for any $\lambda''' > \lambda'$, $x_{i''_{\lambda'''}} = x^*_{i''_{\lambda'''}}$, and that $x_{i_{\lambda''}} = x_{i''_{\lambda'}} \, I_{i''_{\lambda'}} \, x^*_{i''_{\lambda'}} \, I_{i''_{\lambda'}} \, x^*_{i''_{\lambda'+1}} = x_{i''_{\lambda'+1}}$, and $\{x_{i''_{\lambda'+1}}, x_{i_{\lambda''}}\} \subseteq D_{i'_{\lambda'}}(p)$. Thus, the sequence $\{i'_\lambda\}_{\lambda=1}^{\Lambda'} \cup \{i_\lambda\}_{\lambda=\lambda''}^{\Lambda} \cup \{i''_\lambda\}_{\lambda=\lambda'+1}^{\Lambda'}$ is a DCP to x_j in (z,p) and so $j \in N_C$.

Part (ii-1): By $x_{i^*} = k+1$, $V_{i^*}(k+1; z^*_{i^*}) = C^1_+(R, k+1; z^*)$. By $V'_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) > 0$,

$$V'_i(k+1; z^*_i) = C^1_+(R', k+1; z^*) = C^1_+(R, k+1; z^*) = V_{i^*}(k+1; z^*_{i^*}) > 0,$$

and so by $i \neq i^*$, $p'_{k+1} = C^2_+(R', k+1; z^*) = C^1_+(R, k+1; z^*)$. Thus, we have:

$$V'_i(k+1; z^*_i) = p'_{k+1} = C^1_+(R, k+1; z^*) = C^1_+(R', k+1; z^*) = V_{i^*}(k+1; z^*_{i^*}) > 0.$$

Part (ii-2): By Proposition 2, $(z', p') \in Z(N, M(k+1), R')$. Thus, if p' is an MPE price for $(N, M(k+1), R')$, $(z', p') \in W^{\min}(N, M(k+1), R')$. Let z'' be such that $z''_{i^*} = (k+1, p'_{k+1})$ and for each $j \in N \backslash \{i^*\}$, $z''_j = z_j$. We show $(z'', p') \in W^{\min}(N, M(k+1), R')$.

First, we show $(z'', p') \in W(N, M(k+1), R')$. By Part (ii-1) and Fact C.2(ii), we have: (1) $p'_{M(k)} = p^{\min} = p_{M(k)}$ and $p'_{k+1} \geq p_{k+1}$. By construction, unassigned objects at (z,p) remain unassigned at (z'', p'). Thus (E-ii) holds. By (1), for each $j \in N \backslash \{i^*\}$, $x''_j = x_j \in D_j(p')$. By Part (ii-1) and $z''_{i^*} = (k+1, p'_{k+1})$, $z''_{i^*} \, I_{i^*} \, z^*_{i^*}$. Thus $x''_{i^*} = k+1 \in D_{i^*}(p')$. Thus (E-i) holds. Thus, $(z'', p') \in W(N, M(k+1), R')$.

Next, we show that $(z'', p') \in W^{\min}(N, M(k+1), R')$. Let N''_C, N''_U, M''_C, and M''_U be defined at (z'', p') for R'. Note that we only need to show $x''_{i^*} = k+1 \in M''_C$, which, by Part (i) and Part (ii-1), implies $(z'', p') \in W^{\min}(N, M(k+1), R')$.

By $i \in N_C$, there is a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of agents to x_i at (z,p) for R. By $i^* \in N \backslash \{i\}$ and $i_\Lambda = i$, $i^* \neq i_\Lambda$. Thus, there are the two cases below.

Case 1: $i^* \notin \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$ Since $R'_{-i} = R_{-i}$, $i_\Lambda = i \neq i^*$, $i^* \notin \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$, and $z''_j = z_j$ for each $j \in N\backslash\{i^*\}$, by (1), $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ is also a DCP to x_i at (z'', p') for R'. Thus $i \in M'_C$. By $i \neq i^*$, $z_i \, I'_i \, z^*_i$. Thus, by Part (ii-1), $z''_i = z_i \, I'_i \, z^*_i \, I'_i \, (k+1, p'_{k+1}) = z''_{i^*}$. Thus, $\{x''_i, x''_{i^*}\} \in D_{i_\Lambda}(p')$, and $\{i_\lambda\}_{\lambda=1}^{\Lambda} \cup \{i^*\}$ is a DCP to $k+1$ at (z'', p') for R'. Thus $k+1 \in M''_C$.
Case 2: $i^* \in \{i_\lambda\}_{\lambda=1}^{\Lambda-1}$ Let $\Lambda' \leq \Lambda - 1$ be such that $i^* = i_{\Lambda'}$. Then, the subsequence $\{i_\lambda\}_{\lambda=1}^{\Lambda'}$ of $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ is a DCP to $x_{i^*} = k+1$ at (z, p) for R. For each $\lambda \in \{1, \ldots, \Lambda' - 1\}$, by $i_\lambda \neq i^*$, $z''_{i_\lambda} = z_{i_\lambda}$. Thus by (1), $\{i_\lambda\}_{\lambda=1}^{\Lambda'}$ is also a DCP to $x''_{i^*} = x_{i^*} = k+1$ at (z'', p') for R'. Thus $k+1 \in M''_C$.

Part (ii-3): By $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*)$, there is an outcome (z'', p'') of equilibrium-generating mechanism for R'' such that $x''_i = k+1$. By $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) > 0$ and Part (ii-2), $(z'', p'') \in W^{\min}(N, M(k+1), R'')$. To show $(z', p') \in W^{\min}(N, M(k+1), R'')$, we need to show that $p' = p''$, $z''_i = z'_i$, and for each $j \in N\backslash\{i\}$, $z'_j \, I_j \, z''_j$.

By $i \neq i^*$ and $V'_i(k+1; z^*_i) > C^1_+(R, k+1; z^*) > 0$,

$$p'_{k+1} = C^2_+(R', k+1; z^*) = C^1_+(R, k+1; z^*).$$

By Part (ii-1) and $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*)$, $p''_{k+1} = C^1_+(R, k+1; z^*)$. Thus, $p'_{k+1} = p''_{k+1}$. By Fact C.2(ii), $p'_{M(k)} = p^{\min} = p''_{M(k)}$. Thus, $p' = p''$.

By $V'_i(k+1; z^*_i) > C^1_+(R, k+1; z^*) = C^2_+(R', k+1; z^*)$, $x'_i = k+1 = x''_i$. Thus, $z'_i = z''_i$. By Fact C.2(i), for each $j \in N\backslash\{i\}$, $z'_j \, I_j \, z^*_j \, I_j \, z''_j$.

Part (iii): By Fact C.2(ii), $p_{M(k)} = p^{\min} = p'_{M(k)}$. Since $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$, we have: (1) there is some $i' \in N_U$ such that $x'_{i'} = k+1$.

By Lemma C.2(i), $i^* \in N_U$, and (1), we have: (2) for each $j \in N_C$, $x'_j \in M_C \cup \{0\}$.

By Lemma C.1(i), $i^* \in N_U$, and $x_{i^*} = k+1$, $(z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C})$. By $p_{M_C} = p'_{M_C}$, (2), Fact C.2(i) and C.2 (iii), $(z'_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C})$. Thus, by Proposition 1 and Fact C.3, we have: (3) $N_C \subseteq N'_C$ and $M_C \subseteq M'_C$.

By Fact C.2(ii), Lemma 1(ii), and Lemma C.2(ii), for each $x \in M_U$, $p'_x > 0$ and x is assigned to some $j \in N_U$ at (z', p') for R'. Since $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$ and $p'_{M(k)} = p_{M(k)}$, $\{i \in N : D_i(p') \cap M_U \neq \emptyset)\} = N_U$. Thus, $N_U \subseteq N'_U$ and $M_U \subseteq M'_U$. By (3), $N_C = N'_C$, $M_C = M'_C$, $M_U = M'_U$, and $N_U = N'_U$. **Q.E.D.**

Step 3: Completion of the proof

Let $R \in \mathcal{R}^n$, $(z^*, p^{\min}) \in W^{\min}(k, R)$, and $i \in N$. Let (z, p) and (z', p') be the outcomes of equilibrium-generating mechanism for R and $R' \equiv (R'_i, R_{-i})$. Let N_C, N_U, M_C, and M_U be defined at (z, p) for R, and N'_C, N'_U, M'_C, and M'_U be defined at (z', p') for R'. Let $(\widehat{z}, \widehat{p})$ and $(\widetilde{z}, \widetilde{p})$ be the outcomes of MPE-adjustment mechanism for R and R' from (z, p) and (z', p'), respectively. Thus $\widehat{z} = f_{SV}(R; z; k+1)$ and $\widetilde{z} = f_{SV}(R'; z; k+1)$.

Assume for each $j \in N$, $V_j(k+1; z^*_j) \leq 0$. Then $p_{k+1} = 0$ and $(z^*, p) \in W^{\min}(M(k+1), R)$ where $p = (p^{\min}, 0)$. In case $V'_i(k+1; z^*_i) \leq 0$, $z'_i = z^*_i = z_i$ holds. By Theorem 1(i) and

Definition 11, $\widetilde{z}_i = z'_i = z_i = \widehat{z}_i$. In case $V'_i(k+1; z^*_i) > 0$, $z'_i = (k+1, 0)$ and $i \in N'_C$ hold. Thus, by $V_i(k+1; z^*_i) \leq 0$, $\widehat{z}_i = z_i \, R_i \, z'_i = \widetilde{z}_i$. In the following, assume $C^1_+(R, k+1; z^*) > 0$. Then there is $i^* \in N$ such that $x_{i^*} = k+1$. We show $\widehat{z}_i \, R_i \, \widetilde{z}_i$ by considering two cases where $i \in N_C$ in Case I and $i \in N_U$ in Case II.

Case I: $i \in N_C$

By Theorem 1(i) and Definition 11, for agent i, $\widehat{z}_i = z_i$. We conclude that $i \in N'_C$ at (z', p') for R' for each subcase. This implies that i does not participate the MPE-adjustment mechanism from (z', p') so that by $\widehat{z}_i = z_i$ and Fact C.2(i), $\widehat{z}_i = z_i \, R_i \, z^*_i \, I_i \, z'_i = \widetilde{z}_i$.

Case I-1: $i \neq i^*$.

Case I-1-1: $V'_i(k+1; z^*_i) < C^1_+(R, k+1; z^*)$

Case I-1-1-1-1: $C^1_+(R, k+1; z^*) > C^2_+(R, k+1; z^*)$ or $\left|\{j \in N_C \backslash \{i^*, i\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| = 0$.

Since $V'_i(k+1; z^*_i) < C^1_+(R, k+1; z^*)$ and since $C^1_+(R, k+1; z^*) > C^2_+(R, k+1; z^*)$ or $\left|\{j \in N \backslash \{i, i^*\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| = 0$, we have: (1) for each $j \in N_C \backslash \{i^*\}$, $x'_j \neq k+1$. By Lemma C.1(ii), we have: (2) $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$. By Lemma C.2(ii) and (1), we have: (3) for each $j \in N_C \backslash \{i^*\}$, $x'_j \in (M_C \backslash \{k+1\}) \cup \{0\}$.

By Fact C.2(i) and (ii), $p_{M_C \backslash \{k+1\}} = p'_{M_C \backslash \{k+1\}}$ and for each $j \in N_C \backslash \{i^*\}$, $z'_j \, I_j \, z_j$. By $R_{N_C \backslash \{i^*\}} = R'_{N_C \backslash \{i^*\}}$, (2), (3), and Fact C.2(iii), $(z'_{N_C \backslash \{i^*\}}, p'_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R'_{N_C \backslash \{i^*\}})$. Thus, by $i \neq i^*$, Proposition 1 and Fact C.3, $i \in N'_C$ at (z', p') for R'.

Case I-1-1-2: $C^1_+(R, k+1; z^*) = C^2_+(R, k+1; z^*)$ and $\left|\{j \in N_C \backslash \{i^*, i\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| \geq 1$.

Since $p_{k+1} = C^2_+(R, k+1; z^*) = C^1_+(R, k+1; z^*) > 0$, $i^* \in N_C$, and $x_{i^*} = k+1$, by Lemma C.3(i), $(z, p) \in W^{\min}(M(k+1), R)$. Thus, by Proposition 1, $N = N_C$.

By $\left|\{j \in N \backslash \{i, i^*\} : V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)\}\right| \geq 1$, there is $j \in N_C = N \backslash \{i, i^*\}$ such that $V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)$. Thus, $p'_{k+1} = C^2_+(R', k+1; z^*) = C^2_+(R, k+1; z^*)$. By Fact C.2(i) and (ii), $p'_{M(k)} = p_{M(k)}$. Thus $p = p'$. By $V'_i(k+1; z^*_i) < C^1_+(R, k+1; z^*)$, $x'_i \neq k+1$. Since for each $j \in N \backslash \{i\}$, $z'_j \, I_j \, z_j$, by $p' = p$, $(z', p') \in W^{\min}(M(k+1), R)$. Since $(z', p') \in W(M(k+1), R)$, by Fact C.3, $i \in N'_C$ at (z', p') for R'.

Case I-1-2: $V'_i(k+1; z^*_i) = C^1_+(R, k+1; z^*)$

Since $i \neq i^*$, $i \in N_C$, and $V'_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) > 0$, by Lemma C.3(ii-2), $(z', p') \in W^{\min}(N, M(k+1), R')$. Thus, $i \in N'_C$ at (z', p') for R'.

Case I-1-3: $V'_i(k+1; z^*_i) > C^1_+(R, k+1; z^*)$

Let R''_i be such that for each $x \in M(k)$, $V''_i(x; \cdot) = V'_i(x; \cdot)$ and $V''_i(k+1; z^*_i) = C^1_+(R, k+1; z^*)$. Let $R'' \equiv (R''_i, R_{-i})$. Since $i \neq i^*$, $i \in N_C$, and $V'_i(k+1; z^*_i) > C^1_+(R, k+1; z^*) > 0$, by Lemma C.3(ii-3), $(z', p') \in W^{\min}(N, M(k+1), R'')$. Thus, by $(z', p') \in Z(R')$, and Fact C.3(ii), $i \in N'_C$ at (z', p') for R'.

Case I-2: $i = i^* \in N_C$.

Assume $C^2_+(R, k+1; z^*) = 0$. If $V'_i(k+1; z^*_i) \leq 0$, by Definition 8, $z' = z^* = \widetilde{z}$. Since $V_i(k+1; z^*_i) > 0$, then $\widehat{z}_i = z_i \, P_i \, z'_i = \widetilde{z}_i$. If $V'_i(k+1; z^*_i) > 0$, then $z'_i = z_i = (k+1, 0)$. Thus $i \in N'_C$ at (z', p') for R'.

In the following, assume $C^2_+(R, k+1; z^*) > 0$. Then there is $j \in N_C \backslash \{i\}$ such that $V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)$.

Case I-2-1: $V'_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$

Case I-2-1-1: $V_i(k+1; z^*_i) = C^2_+(R, k+1; z^*)$

Since $x_i = k+1$, $V_i(k+1; z^*_i) = C^1_+(R, k+1; z^*) = C^2_+(R, k+1; z^*) > 0$. Since $j \in N_C$, by Lemma 3(ii-2), $(z, p) \in W^{\min}(R)$.

Next we show $(z', p') \in W^{\min}(R)$. By Proposition 2, $(z', p') \in W(R')$. Since $V'_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$, $p'_{k+1} = C^2_+(R, k+1; z^*) = p_{k+1}$. By Fact C.2(ii), $p'_{M(k)} = p^{\min} = p_{M(k)}$. Thus $p' = p$. By Fact C.2(i), for each $j \in N \backslash \{i\}$, $z_j \, I_j \, z'_j$. Together with $z'_i = z_i$, we have $(z', p') \in W(R)$. Since $p = p'$ is an MPE price for R, $(z', p') \in W^{\min}(R)$.

By $(z', p') \in W^{\min}(R)$, $(z', p') \in W(R)$, and Fact C.3(ii), $i \in N'_C$ at (z', p') for R'.

Case I-2-1-2: $V_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$

Since $V'_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$, we have: (1) $x'_i = k+1$. By $x_i = k+1$, $j \in N_C \backslash \{i\}$, and Lemma C.1(i), we have: (2) $(z_{N_C \backslash \{i^*\}}, p_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i^*\}, M_C \backslash \{k+1\}, R_{N_C \backslash \{i^*\}})$. By Lemma C.2(i) and (1), we have: (3) for each $j \in N_C \backslash \{i\}$, $x'_j \in (M_C \backslash \{k+1\}) \cup \{0\}$.

By Fact C.2(i) and (ii), $p_{M_C \backslash \{k+1\}} = p'_{M_C \backslash \{k+1\}}$ and for each $j \in N_C \backslash \{i\}$, $z'_j \, I_j \, z^*_j \, I_j \, z_j$. By $R_{N_C \backslash \{i\}} = R'_{N_C \backslash \{i\}}$, (2), (3), Fact C.2(iii), $(z'_{N_C \backslash \{i\}}, p'_{M_C \backslash \{k+1\}}) \in W^{\min}(N_C \backslash \{i\}, M_C \backslash \{k+1\}, R'_{N_C \backslash \{i\}})$. By Fact C.3(i), $N_C \backslash \{i\} \subseteq N'_C$. Thus $j \in N'_C$. By $j \in N'_C$, $V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)$, and $z'_j \, I_j \, z^*_j$, $i \in N'_C$ at (z', p') for R'.

Case I-2-2: $V'_i(k+1; z^*_i) = C^2_+(R, k+1; z^*)$

In this case, $V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*) = C^1_+(R', k+1; z^*)$. Since $j \neq i$, $j \in N_C$, and $x_i = k+1$ at (z, p) for R, by Lemma C.3(ii-2), $(z', p') \in W^{\min}(N, M(k+1), R')$. Thus $i \in N'_C$ at (z', p') for R'.

Case I-2-3: $V'_i(k+1; z^*_i) < C^2_+(R, k+1; z^*)$

Let R''_i be such that for each $x \in M(k)$, $V''_i(x; \cdot) = V_i(x; \cdot)$ and $V''_i(k+1; z^*_i) = C^2_+(R, k+1; z^*)$. Let $R'' \equiv (R''_i, R_{-i})$.

First we show that $(z, p'') \in Z(R'')$ where $p''_{k+1} = C^2_+(R'', k+1; z^*)$ and $p''_{M(k)} = p_{M(k)}$. Since $V''_i(k+1; z^*_i) = V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)$, we have $p''_{k+1} = C^2_+(R'', k+1; z^*) = C^2_+(R, k+1; z^*) = p_{k+1}$. Thus $p'' = p$. Since $R_{-i} = R''_{-i}$ and $j \in N_C$, then j is connected at (z, p'') for R''. By $V_j(k+1; z^*_j) = C^2_+(R, k+1; z^*)$ and $x_i = k+1$, i and $k+1$ are also connected at (z, p'') for R''. Since $V''_i(k+1; z^*_i) = C^1_+(R'', k+1; z^*)$, by Lemma C.3(ii-3), $(z, p'') \in W^{\min}(R'')$.

Then we consider the following two scenarios.

Case I-2-3-1: $\left|\{k \in N \backslash \{i, j\} : V_k(k+1; z^*_k) \geq C^2_+(R, k+1; z^*)\}\right| \geq 1$

We show $(z', p') \in W^{\min}(R'')$. By $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$, $V_i'(k+1; z_i^*) < C_+^2(R, k+1; z^*)$, and $\left|\{k \in N\backslash\{i,j\} : V_k(k+1; z_k^*) \geq C_+^2(R, k+1; z^*)\}\right| \geq 1$, $p'_{k+1} = C_+^2(R', k+1; z^*) = C_+^2(R, k+1; z^*)$. Thus $p'_{k+1} = p_{k+1}$. By Fact C.2(ii) and $p = p''$, $p'_{M(k)} = p_{M(k)} = p''_{M(k)}$. Thus, $p' = p''$. By Proposition 2, $(z', p') \in W(R')$. Thus for each $k \in N\backslash\{i\}$, $x'_k \in D_k(p')$. Since $V_i'(k+1; z_i^*) < C_+^2(R, k+1; z^*)$, $x'_i \neq k+1$ and by Fact C.2(i), $z'_i \, I_i \, z_i^*$. By the construction of R''_i, $x'_i \in D''_i(p')$. Thus, $(z', p') \in W(R'')$. Since p'' is an MPE price for R'' and $p'' = p'$, $(z', p') \in W^{\min}(R'')$. Since $(z', p') \in W(R')$, by Fact C.3(ii), $i \in N'_C$ at (z', p') for R'.

Case I-2-3-2: $\left|\{k \in N\backslash\{i,j\} : V_k(k+1; z_k^*) \geq C_+^2(R, k+1; z^*)\}\right| = 0$

Let $\widehat{z}''$ such that $\widehat{z}''_j = (k+1, p''_{k+1})$ and for each $i' \in N\backslash\{j\}$, $\widehat{z}''_{i'} = z'_{i'}$. By Fact C.2(i), for each $i' \in N\backslash\{j\}$, $z_i^* \, I_i \, z'_{i'} = \widehat{z}''_{i'}$. Since $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*) = p''_{k+1}$, $(\widehat{z}'', p'') \in W(R'')$. Thus, $(\widehat{z}'', p'') \in W^{\min}(R'')$. Note that for each $i' \in N\backslash\{i,j\}$, $V_{i'}(k+1; z_{i'}^*) < C_+^2(R, k+1; z^*) = p''_{k+1}$. By Lemma C.1(ii) and (1), $(\widehat{z}''_{N\backslash\{j\}}, p''_{M(k)}) = (z'_{N\backslash\{j\}}, p'_{M(k)}) \in W^{\min}(N\backslash\{j\}, M(k), R''_{N\backslash\{j\}})$. By the construction of R''_i, $(z'_{N\backslash\{j\}}, p'_{M(k)}) \in W^{\min}(N\backslash\{j\}, M(k), R'_{N\backslash\{j\}})$. By Fact C.3(ii), $i \in N'_C$ at (z', p') for R'.

Case II: $i \in N_U$

Case II-1: $i^* \in N_C$

By $i \in N_U$ and $i^* \in N_C$, $i \neq i^*$. Thus, by $V_{i^*}(k+1; z_i^*) = C_+^1(R, k+1; z^*)$, we have: $(*1)$ $V_i(k+1; z_i^*) \leq C_+^2(R, k+1; z^*)$. By $i^* \in N_C$ and $z_{i^*}^* = (k+1, C_+^2(R, k+1; z^*))$, we have: $(*2)$ there is $j \in N_C$ such that $V_j(k+1; z_j^*) = C_+^2(R, k+1; z^*)$. By $i \in N_U$ and $R_{-i} = R'_{-i}$, we have: $R_{N_C} = R'_{N_C}$ and $R'_{N_U} = (R'_i, R_{N_U\backslash\{i\}})$.

By contradiction, suppose $C_+^2(R, k+1; z^*) = C_+^1(R, k+1; z^*)$. Then $p_{k+1} = C_+^1(R, k+1; z^*) > 0$. By $i^* \in N_C$, $k+1 \in M_C$. Thus, by $i \neq i^*$ and Lemma C.3(i), $(z, p) \in W^{\min}(M(k+1), R)$ and so by Proposition 1, $N = N_C$, contradicting $i \in N_U$. Thus we have: $(*3)$ $C_+^2(R, k+1; z^*) < C_+^1(R, k+1; z^*) = V_{i^*}(k+1; z_{i^*}^*)$.

Case II-1-1: $V_i'(k+1; z_i^*) \leq C_+^2(R, k+1; z^*)$

By Fact C.5 and Theorem 1, we have $\widehat{z}_{N_U} \in Z^{\min}(N_U, M_U, R_{N_U}, \widehat{r})$ where for each $x \in M_U$, $\widehat{r}_x = C_+^1(R_{N_C}, x; z^*)$, and $\widetilde{z}_{N'_U} \in Z^{\min}(N'_U, M'_U, R'_{N_U}, \widetilde{r})$ where for each $x \in M'_U$, $\widetilde{r}_x = C_+^1(R_{N'_C}, x; z^*)$.

Thus if $N_U = N'_U$ and $M_U = M'_U$, then $N_C = N'_C$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U = N'_U$, $R'_{N_U} = (R'_i, R_{N_U\backslash\{i\}})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus we need to show $N_U = N'_U$ and $M_U = M'_U$.

By $i \in N_U$, $(*1)$, $(*2)$ and $V_i'(k+1; z_i^*) \leq C_+^2(R, k+1; z^*)$,

$$p'_{k+1} = C_+^2(R', k+1; z^*) = C_+^2(R, k+1; z^*) = p_{k+1}.$$

Thus, by $(*3)$, $x'_{i^*} = x_{i^*} = k+1 \in M_C$ and $z'_{i^*} = z_{i^*}$. By Fact C.2(ii), $p'_{M(k)} = p^{\min}_{M(k)} = p_{M(k)}$. Thus, by $p'_{k+1} = p_{k+1}$, we have: (1) $p = p'$.

By Lemma C.2(i) and $x'_{i^*} = k+1 \in M_C$, we have: (2) $\{x'_j\}_{j \in N_C} \subseteq M_C \cup \{0\}$. By $z_{i^*} = z'_{i^*}$ and Fact C.2(i), we have: (3) for each $j \in N_C$, $z'_j \, I_j \, z_j^* \, I_j \, z_j$. Thus, by (1), we have: (4) for each

$j \in N_C$, $\{x_j, x'_j\} \subseteq D_j(p) = D_j(p')$. Thus,

$$\begin{aligned} \text{Lemma C.1(i)} \;&\Rightarrow\; (z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C}) \\ &\Rightarrow\; (z_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \text{ by (1), } R_{N_C} = R'_{N_C} \\ &\Rightarrow\; (z'_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \text{ by (2), (4)} \end{aligned}$$

Thus, by $N_C \subseteq N$, $M_C \subseteq M$ and Fact C.3(i), we have: $N_C \subseteq N'_C$, which implies (5) $N_U \supseteq N'_U$.

In the following, we show $N_U \subseteq N'_U$.

By Lemma C.2(ii) and $x_{i^*} = x'_{i^*} = k+1 \in M_C$, we have $M_U \subseteq \{x'_j\}_{j \in N_U}$. Thus, by $|M_U| = |N_U|$ (Lemma 1(i)), we have: (6) $M_U = \{x'_j\}_{j \in N_U}$.

Let $j \in N_C$. By Lemma 1(ii), for each $x \in M_U$, $z_j \, P_j \, (x, p_x)$. By (1) and (3), for each $x \in M_U$, $z'_j \, I_j \, z_j \, P_j \, (x, p_x) = (x, p'_x)$. Thus, we have: (7) for each $j \in N_C$, $D_j(p') \cap M_U = \emptyset$.

For each $j \in N_U$, by (1) and (6), $p'_{x'_j} = p_{x'_j} > 0$. Thus, by (6), (7) and Fact C.6, we have $N_U \subseteq N'_U$.

Thus, by (5), $N_U = N'_U$, and so by (6), $M_U = M'_U$.

Case II-1-2: $V'_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$

By $i^* \in N_C$, $k+1 \in M_C$, Fact C.5, and Theorem 1, $\widehat{z}_{N_U \cup \{i^*\}} \in Z^{\min}(N_U \cup \{i^*\}, M_U \cup \{k+1\}, R_{N_U \cup \{i^*\}}, \widehat{r})$ where $\widehat{r}_x = C^1_+(R_{N_C \setminus \{i^*\}}, x; z^*)$ for each $x \in M_U \cup \{k+1\}$, and $\widetilde{z}_{N'_U} \in Z^{\min}(N'_U, M'_U, R'_{N_U}, \widetilde{r})$ where $\widetilde{r}_x = C^1_+(R_{N'_C}, x; z^*)$ for each $x \in M'_U$.

Thus if $N_U \cup \{i^*\} = N'_U$ and $M_U \cup \{k+1\} = M'_U$, then $N_C \setminus \{i^*\} = N'_C$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U \cup \{i^*\} = N'_U$, $R'_{N_U \cup \{i^*\}} = (R'_i, R_{N_U})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus we need to show $N_U \cup \{i^*\} = N'_U$ and $M_U \cup \{k+1\} = M'_U$.

By ($*$3), and $V'_i(k+1; z^*_i) > C^2_+(R, k+1; z^*)$,

$$p'_{k+1} = C^2_+(R', k+1; z^*) \geq \min\{V'_i(k+1; z^*_i), V_{i^*}(k+1; z^*_{i^*})\} > C^2_+(R, k+1; z^*) = p_{k+1}.$$

Thus we have: (1) for each $j \in N_C \setminus \{i^*\}$, $V_j(k+1; z^*_j) \leq p_{k+1} < p'_{k+1}$. Thus, by Fact C.2(ii), we have: (2) $p'_{k+1} > p_{k+1}$ and $p'_{M(k)} = p^{\min} = p_{M(k)}$.

By Lemma C.2(i), for each $j \in N_C \setminus \{i^*\}$, $x'_j \in M_C \cup \{0, k+1\}$. By (1) and Definition 8, for each $j \in N_C \setminus \{i^*\}$, $x'_j \neq k+1$. Thus we have: (3) $\{x'_j\}_{j \in N_C \setminus \{i^*\}} \subseteq (M_C \setminus \{k+1\}) \cup \{0\}$. By Fact C.2(i), we have: (4) for each $j \in N_C \setminus \{i^*\}$, $z'_j \, I_j \, z^*_j \, I_j \, z_j$. Thus, by (2) and (4), we have: (5) for each $j \in N_C \setminus \{i^*\}$, $\{x_j, x'_j\} \subseteq D_j(p') \subseteq D_j(p)$. Recall $R_{N_C \setminus \{i^*\}} = R'_{N_C \setminus \{i^*\}}$. Thus,

$$\begin{aligned} &\text{Lemma C.1(ii) (by ($*$3), condition (a) holds)} \\ \Rightarrow\;& (z_{N_C \setminus \{i^*\}}, p_{M_C \setminus \{k+1\}}) \in W^{\min}(N_C \setminus \{i^*\}, M_C \setminus \{k+1\}, R_{N_C \setminus \{i^*\}}) \\ \Rightarrow\;& (z_{N_C \setminus \{i^*\}}, p'_{M_C \setminus \{k+1\}}) \in W^{\min}(N_C \setminus \{i^*\}, M_C \setminus \{k+1\}, R'_{N_C \setminus \{i^*\}}) \text{ by (2)} \\ \Rightarrow\;& (z'_{N_C \setminus \{i^*\}}, p'_{M_C \setminus \{k+1\}}) \in W^{\min}(N_C \setminus \{i^*\}, M_C \setminus \{k+1\}, R'_{N_C \setminus \{i^*\}}) \text{ by (3), (5)} \end{aligned}$$

Thus, by $N_C \setminus \{i^*\} \subseteq N$, $M_C \setminus \{k+1\} \subseteq M$ and Fact C.3(i), we have: $N_C \setminus \{i^*\} \subseteq N'_C$, which implies (6) $N_U \cup \{i^*\} \supseteq N'_U$.

In the following, we show $N_U \cup \{i^*\} \subseteq N'_U$.

By $p'_{k+1} > p_{k+1}$, $k+1$ must be assigned. Thus by (3) and Lemma C.2(ii), for each $x \in M_U \cup \{k+1\}$, there is $j \in N_U \cup \{i^*\}$ such that $x'_j = x$, which implies $M_U \cup \{k+1\} \subseteq \{x'_j\}_{j \in N_U}$. Thus, by $|M_U| = |N_U|$ (Lemma 1(i)), $i^* \in N_C$, and $k+1 \in M_C$, $|M_U \cup \{k+1\}| = |N_U \cup \{i^*\}|$. Thus we have: (7) $M_U \cup \{k+1\} = \{x'_j\}_{j \in N_U \cup \{i^*\}}$.

Let $j \in N_C \backslash \{i^*\}$. Then by Lemma 1(ii), for each $x \in M_U$, $z_j \, P_j \, (x, p_x)$. Thus, by (2) and (3), for each $x \in M_U \subseteq M(k)$, $z'_j \, I_j \, z_j \, P_j \, (x, p_x) = (x, p'_x)$ and by (1), $z'_j \, P_j \, (k+1, p'_{k+1})$. Thus, we have: (8) for each $j \in N_C \backslash \{i^*\}$, $D_j(p') \cap (M_U \cup \{k+1\}) = \emptyset$.

For each $j \in N_U \cup \{i^*\}$, by (2) and (7), $p'_{x'_j} = p_{x'_j} > 0$. By (7), (8) and Fact C.6, we have $N_U \cup \{i^*\} \subseteq N'_U$.

Thus, by (6), $N_U \cup \{i^*\} = N'_U$, and so by (7), $M_U \cup \{k+1\} = M'_U$.

Case II-2: $i^* \in N_U$.

By $i^* \in N_U$ and $x_{i^*} = k+1$, there is $i' \in N_U \backslash \{i^*\}$ such that

$$V_{i'}(k+1; z^*_{i'}) = C^2_+(R, k+1; z^*) \leq V_{i^*}(k+1; z^*_{i^*}) \qquad (*4)$$

By $i^* \in N_U$ and $x_{i^*} = k+1$, we have: $(*5)$ $C^2_+(R, k+1; z^*) > C^1_+(R_{N_C}, k+1; z^*)$. To see $(*5)$, by contradiction, suppose not, i.e., $C^2_+(R, k+1; z^*) = C^1_+(R_{N_C}, k+1; z^*)$. In case of $C^2_+(R, k+1; z^*) = 0$, by Definition 3, $i^* \in N_C$, contradicting $i^* \in N_U$. In case of $C^2_+(R, k+1; z^*) > 0$, there is $j \in N_C$ such that $V_j(k+1; z^*_j) = C^1_+(R_{N_C}, k+1; z^*)$. Thus, $k+1$ is connected by agent j's demand and so by $j \in N_C$, $i^* \in N_C$, contradicting $i^* \in N_U$. By $(*4)$ and $(*5)$, we have: $(*6)$ $V_{i^*}(k+1; z^*_{i^*}) \geq V_{i'}(k+1; z^*_{i'}) > C^1_+(R_{N_C}, k+1; z^*)$.

The proof are divided into four cases, Cases II-2-1, II-2-2, II-2-3, and II-2-4. We group them in two parts. Part A treats Cases II-2-1, II-2-2, and II-2-3. Part B treats Case II-2-4.

Part A: By Fact C.5 and Theorem 1, $\widehat{z}_{N_U} \in Z^{\min}(N_U, M_U, R_{N_U}, \widehat{r})$ where $\widehat{r}_x = C^1_+(R_{N_C}, x; z^*)$ for each $x \in M_U$, and $\widetilde{z}_{N'_U} \in Z^{\min}(N'_U, M'_U, R'_{N_U}, \widetilde{r})$ where $\widetilde{r}_x = C^1_+(R_{N'_C}, x; z^*)$ for each $x \in M'_U$.

Thus if $N_U = N'_U$ and $M_U = M'_U$, then $N_C = N'_C$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U = N'_U$, $R'_{N_U} = (R'_i, R_{N_U \backslash \{i\}})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus we need to show $N_U = N'_U$ and $M_U = M'_U$.

By $i^* \in N_U$ and $x_{i^*} = k+1$, if $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$, then by Lemma C.3(iii), $N_U = N'_U$ and $M_U = M'_U$. Thus, in Cases II-2-1, II-2-2, and II-2-3, we show that $p'_{k+1} > C^1_+(R_{N_C}, k+1; z^*)$, respectively.

Case II-2-1: $i \neq i^*$ and $i \neq i'$

By $i' \in N_U \backslash \{i^*\}$, $i' \neq i^*$. Thus,

$$\begin{aligned} p'_{k+1} &= C^2_+(R', k+1; z^*) \underset{i' \neq i^*, i \neq i^*, \, i \neq i'}{\geq} \min\{V_{i'}(k+1; z^*_{i'}), V_{i^*}(k+1; z^*_{i^*})\} \\ &\underset{(*6)}{>} C^1_+(R_{N_C}, k+1; z^*). \end{aligned}$$

Case II-2-2: (a) $i = i^*$ or $i = i'$, and

(b) $\left|\{j \in N_U \backslash \{i', i^*\} : V_j(k+1; z_j^*) > C_+^1(R_{N_C}, k+1; z^*)\}\right| \geq 1$

By (b), there is $j \in N_U \backslash \{i', i^*\}$ such that (1) $V_j(k+1; z_{i^*}^*) > C_+^1(R_{N_C}, k+1; z^*)$. In case of $i = i^*$, by $i' \in N_U \backslash \{i^*\}$, we have $i \neq i'$, and

$$\begin{aligned} p'_{k+1} &= C_+^2(R', k+1; z^*) \underset{j \neq i', i' \neq i,\ j \neq i}{\geq} \min\{V_{i'}(k+1; z_{i'}^*), V_j(k+1; z_{i^*}^*)\} \\ &\underset{(1),(*6)}{>} C_+^1(R_{N_C}, k+1; z^*). \end{aligned}$$

We can treat the case of $i = i'$ by the same way, and so we omit it.

Case II-2-3: (a) and (c) $V_i'(k+1; z_i^*) > C_+^1(R_{N_C}, k+1; z^*)$.

In case of $i = i^*$, by $i' \in N_U \backslash \{i^*\}$, we have $i \neq i'$ and $i' \neq i^*$, and

$$\begin{aligned} p'_{k+1} &= C_+^2(R', k+1; z^*) \underset{i' \neq i^*, i \neq i^*, i \neq i'}{\geq} \min\{V_{i'}(k+1; z_{i'}^*), V_{i^*}(k+1; z_i^*)\} \\ &\underset{(c),(*6)}{>} C_+^1(R_{N_C}, k+1; z^*). \end{aligned}$$

We can treat the case of $i = i'$ by the same way, and so we omit it.

Part B: This part treats Case II-2-4.

Case II-2-4: (a), (d) $\left|\{j \in N_U \backslash \{i', i^*\} : V_j(k+1; z_j^*) > C_+^1(R_{N_C}, k+1; z^*)\}\right| = 0$, and (e) $V_i'(k+1; z_i^*) \leq C_+^1(R_{N_C}, k+1; z^*)$.

By (a), $i = i'$ or $i = i^*$. We consider only the case of $i = i'$ here. We can treat the case of $i = i^*$ by the same way, and so we omit it.

By Fact C.5 and Theorem 1, $\widehat{z}_{N_U} \in Z^{\min}(N_U, M_U, R_{N_U}, \widehat{r})$ where for each $x \in M_U$, $\widehat{r}_x = C_+^1(R_{N_C}, x; z^*)$, and if $i^* \in N_C'$ and $x_{i^*}' = k+1$, then $\widetilde{z}_{N_U' \cup \{i^*\}} \in Z^{\min}(N_U' \cup \{i^*\}, M_U' \cup \{k+1\}, R'_{N_U' \cup \{i^*\}}, \widetilde{r})$ where for each $x \in M_U' \cup \{k+1\}$, $\widetilde{r}_x = C_+^1(R_{N \backslash [N_U' \cup \{i^*\}]}, x; z^*)$.

Thus if $i^* \in N_C'$, $x_{i^*}' = k+1$, $N_U = N_U' \cup \{i^*\}$ and $M_U = M_U' \cup \{k+1\}$, then $N_C = N \backslash [N_U' \cup \{i^*\}]$ and $\widehat{r} = \widetilde{r}$, and so by $i \in N_U = N_U' \cup \{i^*\}$, $R'_{N_U' \cup \{i^*\}} = (R_i', R_{N' \backslash \{i\}})$, and Fact C.1, $\widehat{z}_i \, R_i \, \widetilde{z}_i$. Thus, we show that $i^* \in N_C'$, $x_{i^*}' = k+1$, $N_U = N_U' \cup \{i^*\}$ and $M_U = M_U' \cup \{k+1\}$.

By (d), we have: (1) $V_j(k+1; z_j^*) \leq C_+^1(R_{N_C}, k+1; z^*)$ for each $j \in N_U \backslash \{i', i^*\}$. By $i' \in N \backslash \{i^*\}$, $i = i' \neq i^*$. Thus,

$$V_i'(k+1; z_i^*) \underset{\text{(e)}}{\leq} C_+^1(R_{N_C}, k+1; z^*) \underset{(*6), i \neq i^*}{<} V_{i^*}(k+1; z_{i^*}^*),$$

and so by (1),

$$V_{i^*}(k+1; z_{i^*}^*) = C_+^1(R', k+1; z^*) > C_+^1(R_{N_C}, k+1; z^*) = C_+^2(R', k+1; z^*).$$

By $V_{i^*}(k+1; z_{i^*}^*) = C_+^1(R', k+1; z^*) > C_+^2(R', k+1; z^*)$, we have: (2) $x_{i^*} = x_{i^*}' = k+1$. By $C_+^2(R', k+1; z^*) = C_+^1(R_{N_C}, k+1; z^*)$, $p'_{k+1} = C_+^1(R_{N_C}, k+1; z^*)$. Thus by (2), we have: (3) $i^* \in N_C'$ and $k+1 \in M_C'$. By $p'_{k+1} = C_+^1(R_{N_C}, k+1; z^*)$,

$$p'_{k+1} = C_+^1(R_{N_C}, k+1; z^*) \underset{(*5)}{<} C_+^2(R, k+1; z^*) = p_{k+1}.$$

Thus, by Fact C.2(ii), we have: (4) $p'_{M(k)} = p^{\min} = p_{M(k)}$ and $p'_{k+1} < p_{k+1}$.

By (2), (3), and Lemma C.2(i), we have: (5) $\{x'_j\}_{j \in N_C} \subseteq M_C \cup \{0\}$. By $i^* \in N_U$ and Fact C.2(i), we have: (6) for each $j \in N_C$, $z'_j \, I_j \, z^*_j \, I_j \, z_j$. Thus, by (4), we have: (7) for each $j \in N_C$, $\{x_j, x'_j\} \subseteq D_j(p) \subseteq D_j(p')$. By $i \in N_U$, $R_{N_C} = R'_{N_C}$. By $i^* \in N_U$, $x_{i^*} = k+1 \in M_U$ and (4), $p_{M_C} = p'_{M_C}$. Thus,

$$
\begin{aligned}
& \text{Lemma C.1(i)} \\
\Rightarrow \quad & (z_{N_C}, p_{M_C}) \in W^{\min}(N_C, M_C, R_{N_C}) \\
\Rightarrow \quad & (z_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \quad p_{M_C} = p'_{M_C},\ R_{N_C} = R'_{N_C} \\
\Rightarrow \quad & (z'_{N_C}, p'_{M_C}) \in W^{\min}(N_C, M_C, R'_{N_C}) \quad (5),\ (7)
\end{aligned}
$$

Thus, by $N_C \subseteq N$, $M_C \subseteq M(k)$ and Fact C.3(i), we have: $N_C \subseteq N'_C$. Thus by (3), $N_C \cup \{i^*\} \subseteq N'_C$, which implies (8) $N'_U \cup \{i^*\} \subseteq N_U$.

In the following we show $N_U \backslash \{i^*\} \subseteq N'_U$, which implies $N'_U \cup \{i^*\} \supseteq N_U$.

By Lemma C.2(ii) and (2), we have $M_U \backslash \{k+1\} \subseteq \{x'_j\}_{j \in N_U \backslash \{i^*\}}$. Thus, by $|M_U| = |N_U|$ (Lemma 1(i)), $i^* \in N_U$, and $x_{i^*} = k+1 \in M_U$, we have: (9) $M_U \backslash \{k+1\} = \{x'_j\}_{j \in N_U \backslash \{i^*\}}$.

Note that for each $j \in N_C$ and each $x \in M_U \backslash \{k+1\}$,

$$z'_j \underset{(6)}{I_j} z_j \underset{\text{Lemma 1(ii)}}{P_j} (x, p_x) \underset{(4)}{=} (x, p'_x).$$

Also note that for each $x \in M_U \backslash \{k+1\}$,

$$z'_{i^*} \underset{(2)}{=} (k+1, p'_{k+1}) \underset{(4)}{P_{i^*}} (k+1, p_{k+1}) \underset{x_{i^*} = k+1}{=} z_{i^*} \, R_{i^*} \, (x, p_x) \underset{(4)}{=} (x, p'_x).$$

Thus we have: (10) for each $j \in N_C \cup \{i^*\}$, $D_j(p') \cap (M_U \backslash \{k+1\}) = \emptyset$.

By (2), (4) and (9), for each $j \in N_U \backslash \{i^*\}$, $p'_{x'_j} = p_{x'_j} > 0$. Thus, by (9), (10) and Fact C.6, $N_U \backslash \{i^*\} \subseteq N'_U$. Thus by (8), $N_U = N'_U \cup \{i^*\}$, and so by (9), $M_U = M'_U \cup \{k+1\}$. **Q.E.D.**

Online Appendix (Not for publication)

OA.1: The exact and approximate auction of Demange et al. (1986)

A set of objects M' is *minimally overdemanded* at p if M' is overdemanded at p, and no proper subset of M' is overdemanded at p.

Definition OA.1: **The exact ascending auction** is defined as follows. Let the increment $\varepsilon = 1$ be given.

Starting with reserve prices, agents report their demand sets at the current price. If there is a set of objects that are minimally overdemanded, then the auctioneer raises the prices of those objects by 1. Otherwise, the auctioneer stops the auction at the current price.

In the assignment market in Section 3.1, the reserve prices are $(0, 0)$. Then all three agents demand b so b is minimally overdemanded and the price of b increases by 1. At this moment, the market price is $(0, 1)$. Both agents 1 and 2 will only demand b till the price of b reaches 6. At the market price $(0, 6)$, b is minimally overdemanded and the price of b increases by 1. Thus at the price $(0, 7)$, agent 1 exits the auction and both agents 2 and 3 only demands a. Both agents 2 and 3 will only demand a till the price of a reaches 5. At the market price $(5, 7)$, a is minimally overdemanded and the price of a increases by 1. Therefore the market price is $(6, 7)$ at which all the agents exit the auction and demand object 0.

Definition OA.2: **The approximate ascending auction** is defined as follows.

Let some increment $\varepsilon > 0$ be given. Initially all the agents are uncommitted and stand in a given queue. Agents are called to bid one by one. When agent i is to make a bid, she encounters one of the three cases.

Case 1: If agent i bids for an unassigned object, she becomes committed to getting that object at its reserve price.

Case 2: If agent i bids for an object x that is tentatively assigned to agent j at price p_x, the price of object x is increased by ε and agent i becomes committed to getting object l by paying $p_x + \varepsilon$. Meanwhile, agent j becomes uncommitted and stands in the first position in the queue of remaining uncommitted agents.

Case 3: Agent i drops out by bidding the dummy.

The auction terminates when all uncommitted agents drop out.

In the assignment market in Section 3.1, the reserve prices are $(0, 0)$. The order of agents to bid is 1, 2, and 3. Agents 1 and 2 compete for object b. Agent 1 tentatively gets b at price 0, then agent 2 competes with agent 1 and gets b at price 1. Later on, agent 1 competes with agent 2 and gets b at 2. The competition continues till agent 1 gets b at 6. At this moment, the price of a remains 0 and that of b is 6.

For agent 2, if she gets b, agent 2 needs to pay 7, which brings a negative utility to her so agent 2 gets object a at 0. Then agent 3 enters the market. Notice that if agent 3 gets b, agent 3 needs to pay 7, which also brings a negative utility to her so agent 3 will compete with agent 2 for object a. Then agent 3 gets a at 1, and agent 2 competes and gets a at 2. The competition continues till agent 3 gets a at 5. If agent 2 gets competes and gets a at 6, she has a negative utility so agent 2 drops. The final price is $(5, 6)$.

OA.2: Demand-connectedness-path-finding (DCP-finding) process

Let $x \in M$ be a connected object that is assigned to agent i, i.e., $x_i = x$, at $(z, p) \in Z \times \mathbb{R}^m_+$.
Phase 1: *Round 1:* Set $N_1 \equiv \{i\}$.
If $p_x = 0$, stop the process.
If $p_x > 0$, set $N_2 \equiv \{j \in N \backslash N_1 : x \in D_j(p)\}$ and go to Round 2.
Round $s(\geq 2)$*:* Set $L_s \equiv \{y \in L : x_j = y \text{ for some } j \in N_s\}$.
If there is $y \in L_s$ s.t. $y = 0$ or $p_y = 0$, stop Phase 1 and go to Phase 2.
Otherwise, set $N_{s+1} \equiv \{j \in N \backslash \cup_{k=1}^{s} N_k : D_j(p) \cap L_s \neq \emptyset\}$ and go to Round $s+1$.[17]
Phase 2: Let S be the final round of the process. Then, construct a sequence $\{i_s\}_{s=1}^{S}$ of distinct agents as follows: (i) Choose $i_1 \in N_S$ such that $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, and (ii) for each $j \in \{2, \cdots, S\}$, choose $i_j \in N_{S+1-j}$ such that $x_{i_j} \in L_{S+1-j}$ and $x_{i_j} \in D_{i_{j-1}}(p)$.

The DCP-finding process works as follows: pick a connected object x that is assigned agent i. In Phase 1, if the price of x is zero, we are done and it contains a trivial DCP. If not, i.e., $p_x > 0$, we collect the demanders N_2 of x by excluding agent i. Since x is connected and $p_x > 0$, $N_2 \neq \emptyset$. If there is some agent in N_2, say, agent j, who obtains an object x_j with zero price, then agent j is connected to x by her demand, i.e., $\{x, x_j\} \in D_j(p)$ and we are done. Otherwise, we collect the set L_2 of objects assigned to agents in N_2, and repeat the process till some agent obtains an object with zero price. In Phase 2, we trace back from the agent who gets an object with zero price to object x via the DCP.

Proposition OA.1: Let $(z, p) \in Z \times \mathbb{R}^m_+$. Let $x \in M$ be an assigned connected object.
(i) Phase 1 of DCP-finding process stops in a finite number of rounds.
(ii) The sequence $\{i_s\}_{s=1}^{S}$ of distinct agents of Phase 2 is a DCP for object x.

By the finiteness of N, (i) holds. By construction, the sequence $\{i_s\}_{s=1}^{S}$ in Lemma (ii) satisfies (ii-1) to (ii-4) in Definition 2. The demand sets used to identify the DCPs can be derived from the agents' reported indifference prices at the given allocation.

OA.3 Connected-agent-identifying process

Let (z, p) be a Walrasian equilibrium.
Round 1: Let $N_1 \equiv \{i \in N : p_{x_i} = 0\}$. If $N_1 = \emptyset$, set $N^* = \emptyset$ and stop the process.
Otherwise, go to Round 2.
Round $s(\geq 2)$*:* Let

$$M^{s-1} \equiv \{y \in M \backslash \{x_i : i \in \cup_{k=1}^{s-1} N_k\} : p_y > 0,\, y \in D_i(p) \backslash \{x_i\} \text{ for some } i \in N_{s-1}\}.$$

[17] In such a case, since for each $y \in L_s$, y is connected and $p_y > 0$, $N_{s+1} \neq \emptyset$.

If $M^{s-1} = \emptyset$, set $N^* = \cup_{k=1}^{s-1} N_k$ and stop the process.
Otherwise, let $N_s \equiv \{i \in N : x_i = y \text{ for some } y \in M^{t-1}\}$, and go to Round $s+1$.

The above mechanism works as follows: at a given Walrasian equilibrium (z, p), we collect a set of the agents N_1 who get objects with zero prices. Then we collect a set of objects M^1 in the demand sets of agents in N_1 with positive prices, except for their assigned objects. Since objects in M^1 have positive prices, they must be assigned to some agents. Then we identify the set of agents N_2 who are assigned objects from M^1, and collect a set of objects M^2 in the demand sets of agents in N_2 with positive prices, except for their assigned objects. We repeat such a process. The collection of the identified agents in N_1, N_2,..., are connected agents. Formally, we get the result below.

Proposition OA.2: The connected-agent-identifying mechanism identifies all the connected agents N^* at the given Walrasian equilibrium (z, p) in a finite number of rounds via agents' reports of indifference prices in finitely many times.

Proof: Let T be the final round of the process. By the finiteness of agents and objects, $T < +\infty$.

In the following, we show $N^* = N_C$. If $N_C = \emptyset$, by Remark 1, there is no agent $i \in N$ such that $p_{x_i} = 0$. Thus, the mechanism stops at $N'_1 = \emptyset$, i.e., $N_C = \emptyset$. Let $N_C \neq \emptyset$.

First, we show that $\overset{T}{\underset{k=1}{\cup}} N'_k \subseteq N_C$. By Remark 1, there is some $i \in N$ such that $p_{x_i} = 0$. Thus, $N'_1 \neq \emptyset$ and $N'_1 \subseteq N_C$. If $T = 2$, i.e., $N'_T = \emptyset$, then $\overset{2}{\underset{k=1}{\cup}} N'_k \subseteq N_C$. Let $T > 2$. Thus $N'_2 \neq \emptyset$. By the definition of N'_2, for each $i \in N'_2$, $p_{x_i} > 0$ and there is $j \in N_1$ such that $x_i \in D_j(p)$. By Definition 3, $N'_2 \subseteq N_C$. By induction argument, for each $t = 1, \cdots, T-1$, $N'_t \neq \emptyset$ and $N'_t \subseteq N_C$. Recall that $N'_T = \emptyset$. Thus $\overset{T}{\underset{k=1}{\cup}} N'_k \subseteq N_C$.

Then, we show that $\overset{T}{\underset{k=1}{\cup}} N'_k = N_C$. We proceed by contradiction. Suppose that there is $i \in N_C \setminus \overset{T}{\underset{k=1}{\cup}} N'_k$. Then $i \notin N'_1$ and $p_{x_i} > 0$. By Definition 2, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of $\Lambda(\Lambda \geq 2)$ distinct agents such that (a) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, (b) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$, (c) $x_{i_\Lambda} = x_i$, and (d) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

By Definition 6, (a) implies $i_1 \in N'_1$. By (d), there is $i \in \{i_2, \cdots, i_\Lambda\}$ such that $x_i \in D_j(p)$ for some $j \in N'_1$, e.g., $i = i_2$. Thus, $i \in N'_2$ and $N'_2 \neq \emptyset$. Let $i_{l_1} \in \{i_2, \cdots, i_\Lambda\}$ be such that there is no $l' > l_1$ such that $i_{l'} \in N'_2$, i.e., agent i_{l_1} is the agent who belongs to N'_2 with the largerst index in $\{i_\lambda\}_{\lambda=2}^{\Lambda}$. By (d), there $i \in \{i_{l_1+1}, \cdots, i_\Lambda\}$ such that $x_i \in D_j(p)$ for some $j \in N'_2$, e.g., $i = i_{l_1+1}$. Thus $i \in N'_3$ and $N'_3 \neq \emptyset$. By same reasoning, we can select $i_{l_2} \in \{i_{l_1+1}, \cdots, i_\Lambda\}$ such that $i_{l_2} \in N'_3$ with the largest index in $\{i_\lambda\}_{\lambda=l_1+1}^{\Lambda}$. Repeating such argument, we can show $i = i_\Lambda \in \overset{T}{\underset{k=1}{\cup}} N'_k$, contradicting $i \in N_C \setminus \overset{T}{\underset{k=1}{\cup}} N'_k$. **Q.E.D.**