BARGAINING AND EXCLUSION WITH MULTIPLE BUYERS

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Abstract. A seller trades with q out of n buyers who have valuations $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ via sequential bilateral bargaining. When $q < n$, buyer payoffs vary across equilibria in the patient limit, but seller payoffs do not, and converge to

$$\max_{l \leq q+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{q+1} \right].$$

If l^* is the (generically unique) maximizer of this optimization problem, then each buyer $i < l^*$ trades with probability 1 at the fair price $a_i/2$, while buyers $i \geq l^*$ are excluded from trade with positive probability. Bargaining with buyers who face the threat of exclusion is driven by a *sequential outside option principle*: the seller can sequentially exercise the outside option of trading with the extra marginal buyer $q+1$, then with the new extra marginal buyer q, and so on, extracting full surplus from each buyer in this sequence and enhancing the outside option at every stage. A seller who can serve all buyers ($q = n$) may benefit from creating scarcity by committing to exclude some remaining buyers as negotiations proceed. An *optimal exclusion commitment*, within a general class, excludes a single buyer but maintains complete flexibility about which buyer is excluded.

1. Introduction

Consider a seller whose supply is valuable to multiple buyers. If the seller is a monopolist, this is a classical setting, which is well understood under various assumptions regarding information and price discrimination. Under complete information and perfect price discrimination, the monopolist extracts all surplus from every buyer. We investigate what happens in the complete information setting when the terms of trade are determined by *bargaining* between the seller and each individual buyer. What profits does the seller earn and which buyers does she trade with in a bargaining game with fixed supply? What payoffs do buyers get? If there is no scarcity and the seller serves all buyers, then the standard equal ("fair") division of surplus between the seller and each buyer should be expected. However, if there is scarcity and some buyers are necessarily "excluded," then the seller should be able to exploit the competition among buyers and obtain higher than fair prices. This suggests that the seller may benefit from limiting supply, and leads to a related question: if the seller may

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reduce supply or place more general restrictions on the sets of buyers she transacts with,[1] what restrictions will be most profitable and what outcomes will emerge?

We consider a market in which a seller contracts independently with q out of n individual buyers with respective values (net of seller cost) of $a_1 \geq a_2 \geq \ldots \geq a_n > 0$. For convenience, we use language suggesting that the seller is offering q units of the same good for sale, and each buyer has unit demand. However, the seller's transactions with each buyer may be idiosyncratic; the main restriction we impose is that there are no externalities between buyer valuations. We study the following bargaining game, which we refer to as the game with supply q. Negotiations occur over time, and players have a common discount factor $\delta \in (0, 1)$. In each round, the seller strategically picks a buyer to bargain with, and with equal probability each of the two players proposes a price to the other. If the proposal is accepted, the seller trades with the buyer at the proposed price, the buyer exits the game, and the seller continues to bargain with the remaining buyers in the next round. If the proposal is rejected, bargaining proceeds with the same set of buyers in the next round. The game ends when the seller has traded with q buyers.

We analyze Markov perfect equilibria (MPEs) of the game with supply q—subgame perfect equilibria in which each player's strategy in a round depends only on the set of buyers with whom the seller has not already traded, and actions taken within that round. Our main results concern limit MPE outcomes as δ goes to 1. We will frequently affix the qualifiers "limit" and "asymptotic" to describe limit outcomes in families of MPEs for $\delta \to 1$ (but drop qualifiers for brevity in some cases).

If $q = n$, so all buyers can be served, then the seller splits the surplus equally with each individual buyer, and her profits converge to $a_1/2+a_2/2+\ldots+a_n/2$ as $\delta \to 1$. This is closely related to the classic result on convergence of (symmetric) non-cooperative bargaining in the style of Rubinstein (1982) to the Nash (1950) bargaining solution (Binmore 1980; Binmore, Rubinstein and Wolinsky 1986).

Suppose next that supply is smaller than the number of buyers ($q < n$). For the remainder of the introduction (but not in the formal treatment), we assume for simplicity that buyer values are distinct. Consider first the case in which the seller has unit supply ($q = 1$). Proposition 1 in Manea (2018) characterizes MPEs in this simple case. If $a_2 \leq a_1/2$, then the seller bargains exclusively with buyer 1, and trade takes place at expected price $a_1/2$, reflecting the equal split that would be obtained in a standard bargaining game between the seller and buyer 1. In this case, the value of the outside option of trading with buyer 2 is too low to enhance the seller's bargaining power in negotiations with buyer 1. If $a_2 > a_1/2$, then for high δ, the seller randomizes between buyers 1 and 2 in equilibrium, and each buyer

[1]The importance of exclusion restrictions in the context of individually negotiated agreements with multiple agents has been examined in the applied literature. Gal-Or (1997) emphasizes the power of exclusion in an early paper. In the health economics literature, the phenomenon of insurance companies offering "narrow" hospital networks has been widely noted (e.g., Howard 2014; Liebman 2018; Ho and Lee 2019; Ghili 2022).

trades at prices converging to a_2, but the probability of bargaining (and trading) with buyer 2 converges to 0 as δ goes to 1. Now, the outside option of trading with buyer 2 is *binding*, and the seller *exercises* it with positive but vanishing probability as players become patient. An *outside option principle* emerges from this analysis of MPEs: the seller trades with buyer 1 with limit probability 1 at a limit price of $\max(a_1/2, a_2)$.[2] Therefore, when $q = 1$ trade is asymptotically efficient, and buyer 2 provides an *endogenous* outside option that has a limit equilibrium value of a_2.[3]

By analogy with the unit supply case, one might conjecture that when $q > 1$ the seller should attain asymptotic profits of

$$\sum_{i=1}^{q} \max\left(\frac{a_i}{2}, a_{q+1}\right). \tag{1}$$

However, this conjecture is incorrect. The formula above may be rationalized in terms of the following presumptions: (1) the seller trades efficiently (with limit probability 1) with buyers $1, \ldots, q$; (2) bargaining with each of the buyers $1, \ldots, q$ is driven by a fixed outside option provided by the *extra marginal* buyer $q+1$; (3) the (limit) value of the outside option provided by buyer $q+1$ in equilibrium is a_{q+1} (i.e., buyer $q+1$ has zero limit payoff). It turns out that the first two presumptions are incorrect, as they fail to take into account the dynamic nature of outside options under sequential bargaining. For instance, consider a setting with $n = 3, q = 2$ and suppose that $a_3 > a_1/2$, so that both buyers 2 and 3 constitute binding outside options in bargaining with buyer 1 in subgames where the seller has a single unit left. In this case, trading with buyer 2 in the first round at the highest possible price of a_2 is not (asymptotically) more profitable than trading with buyer 3 at a price of a_3. Indeed, in the next round, when bargaining with buyer 1, the seller obtains a price of a_2 if buyer 2 is available as an outside option, but a lower price of a_3 if buyer 3 is the outside option. In either case, the seller's profit would be $a_2 + a_3$. Hence, buyer 2 is valuable to the seller both directly as a trading partner, and indirectly as an outside option when bargaining with buyer 1 in the event that the seller trades with the lower value buyer 3 first. Therefore, buyer 2 might not necessarily manage to "outbid" buyer 3 in the first round. This suggests that trade need not be asymptotically efficient when $q > 1$, which we confirm in examples with $n = 3, q = 2$.

[2] The assumption of Markov equilibrium behavior is important for this conclusion. In Abreu and Manea (2022), we show that subgame perfect equilibria in the setting with $n = 2, q = 1$ are extremely permissive—the price may be above or below the outside option price, and the allocation may be asymptotically inefficient in either case. We proceed to propose refinements that are behaviorally plausible in the context of this bargaining environment and yield the intuitive predictions of the outside option principle. Although these refinements do not imply Markov behavior (they are weaker and not expressed in terms of stationarity), they provide support for Markovian predictions in the bargaining game considered here.

[3] In the original treatment (Binmore 1985; Binmore, Rubinstein and Wolinsky 1986; Sutton 1986; Binmore, Shaked and Sutton 1989), outside options were assumed to have exogenous values that can be obtained by traders without bargaining with third parties.

Despite the possibility that the extra marginal buyer $q+1$ trades with positive limit probability in a family of MPEs for $\delta \to 1$, we show that the seller extracts full surplus from buyer $q+1$ (hence, the third presumption above is correct). This property of MPEs allows us to replace the outside option principle when there is only one unit for sale with a *sequential outside option principle* in the context of our dynamic bargaining process with multiple units being traded bilaterally. The seller can sequentially *exercise* outside options by trading with the extra marginal buyer $q+1$ at limit price a_{q+1}, then trading with the new extra marginal buyer q at limit price a_q (buyer q becomes extra marginal in the subgame with supply $q-1$), and so on, thereby *enhancing* the outside option at every round. However, exercising outside options in this sequence implies that a more valuable buyer will be ultimately excluded, and it may be optimal for the seller not to pursue this strategy up to the exclusion of buyer 1, but instead interrupt it by excluding some buyer l. Absent the threat of replacing a buyer $i \leq l-1$ with some higher value buyer, the seller cannot extract full surplus from buyer i. However, we show that each buyer i must pay at least a fair limit price of $a_i/2$ in any family of MPEs for $\delta \to 1$. This leads to the following *lower bound* on the seller's asymptotic MPE profits:

$$M^{*q} := \max_{l \leq q+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{q+1} \right]. \tag{2}$$

Surprisingly, we find that the asymptotic lower bound M^{*q} also constitutes an *upper bound* on the seller's asymptotic MPE profits, and hence the seller's profits must converge to M^{*q} in any family of MPEs for $\delta \to 1$.

The static optimization problem displayed in (2) yields the seller's *payoffs* in the dynamic bargaining game with supply $q < n$. The optimization problem is also informative about the seller's *behavior*, in particular about which buyers get to trade with certainty and which buyers face the threat of exclusion in equilibrium. In the generic case in which the static optimization problem has a unique maximizer l^*, we show that in any MPE for high δ, buyers $i < l^*$ are guaranteed to be included—and trade at the fair price $a_i/2$—while buyers $i \geq l^*$ are excluded with positive probability. Furthermore, if $l^* \neq q+1$, then buyer l^* is included with limit probability 1 as $\delta \to 1$.

Our model does not yield unique MPEs. Even in the limit as $\delta \to 1$, the set of buyers who trade and buyers' payoffs can vary across convergent sequences of MPEs. Despite MPE multiplicity, we can determine buyer payoffs from coarse information about the structure of trading paths in a family of MPEs in question. In particular, given that the formula for seller profits applies in every subgame and that the seller is indifferent between the buyers she approaches with positive probability in any round, the price at which each buyer trades in equilibrium is reflected in the difference in seller profits before and after the trade. If the seller trades with positive probability with a buyer in a round, then the buyer's payoff in the subgame starting with that round can be inferred from the implied equilibrium price without knowing the precise probability of trade with the buyer in that round or granular details of possible paths of trade with the buyer in subsequent rounds.

We also consider a strategic situation in which the seller has unconstrained supply ($q = n$), but might find it profitable to sharpen competition between buyers by excluding some buyers in the course of negotiations. An *exclusion commitment* specifies the subset of buyers to be excluded from future negotiations depending on the set of buyers who have already traded. This general formulation of exclusion commitments allows for elaborate patterns of exclusion. We seek to identify optimal exclusion commitments in this framework. Given potential multiplicity of MPEs of the bargaining game induced by an exclusion commitment, the optimal commitment could vary depending on whether it is defined with respect to the supremum or the infimum of equilibrium seller payoffs (or some other selection rule). Nevertheless, we find that an optimal exclusion commitment can be defined unambiguously and takes a simple form: no buyer is excluded from bargaining until $n-1$ units are sold, and then the remaining buyer is always excluded. Under this commitment, the seller excludes a single buyer, but has complete flexibility through the sequential trading process regarding who is excluded. Therefore, maintaining a single unit of shortage at every stage allows the seller to extract all potential benefits of exclusion, and the seller does not benefit from exclusion commitments that treat buyers asymmetrically or create additional scarcity. The game with optimal exclusion commitment is identical to the game with exogenous supply $q = n-1$, and the results developed for the setting with exogenous supply characterize MPE outcomes under the optimal commitment.

Our permissive formulation of exclusion commitments implies that our conclusion is correspondingly strong, while the optimal commitment we identify is simple and does not exploit the permitted complexity. Thus, skeptics who feel that complex commitments are implausible may be reassured by the simplicity of the result, and others need not be concerned that allowing for additional complexity might lead to higher seller payoffs.

Finally, we briefly consider exclusion commitments in settings in which the seller has an exogenous supply constraint $q < n$. We argue that in this case the seller does not benefit from making commitments to exclude buyers at any stage before all available q units are sold. In particular, a reduction in supply is detrimental to the seller. This result echoes the intuition from the setting with unconstrained supply that any existing scarcity induces sufficient competition among buyers to deliver the gains of the sequential outside option principle.

We contrast our findings with those of Ho and Lee (2019), who were the first to analyze exclusion commitments in the context of a general model of bargaining in networks. In their formalization of exclusion commitments specialized to our setting, the seller "targets" a fixed network (subset of buyers).[4] After announcing the network, the seller simultaneously

[4]Also motivated by the questions of network endogeneity and optimal exclusion, Liebman (2018) considers a bargaining model between a health insurer and several hospitals in which the insurer commits to a network

dispatches independent "representatives" to bargain with a designated in-network buyer and *any* excluded buyer. The network is achieved in equilibrium with probability that converges to 1 as $\delta \to 1$. Their delegated-agent bargaining protocol delivers formula (1), with buyer $q+1$ providing a fixed outside option for each representative. Unlike in our setting, in the model of Ho and Lee the seller may benefit from reducing supply. Ho and Lee's representatives are compartmentalized and cannot effectively coordinate with one another, whereas in our model the seller internalizes the dynamic implications of *sequential* bilateral trades with individual buyers.[5] Our more conventional bargaining protocol enables the seller to extract higher profits via the sequential outside option principle embodied in formula (2).

The rest of the paper is organized as follows. Section 2 introduces the bargaining model, and Section 3 provides a preliminary lemma and an example. In Section 4 we develop some key bargaining theoretic principles that are used in Section 5 to establish uniqueness and deliver the formula for asymptotic seller profits. Section 6 characterizes included and excluded buyers, and Section 7 presents the result on buyer payoffs. Sections 8 and 9 formalize our notion of exclusion commitments, identify the optimal exclusion commitment, and contrast our results with those of Ho and Lee (2019). Section 10 concludes. Proofs omitted in the main body of the paper can be found in the Appendix, and some computations are relegated to an online Appendix.

2. Model

Consider a market where an agent, player 0, signs bilateral contracts with q out of n players from the set $N = \{1, 2, \ldots, n\}$. To fix terminology, we refer to player 0 as the *seller*, to the players in N as *buyers*, and to the bilateral contracts as *goods*. In this language, the seller has $q \leq n$ units of a good, and each of the n buyers has unit demand.[6] Assume that buyer i's *value* for the good (net of seller cost) is a_i, where $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, and these values are common knowledge. There are no externalities: buyer values are independent of who else gets a unit of the good.

The seller trades with individual buyers sequentially. In every round $t = 0, 1, \ldots$, the seller (strategically) selects a buyer i to bargain with (among those who have not yet traded). Bargaining between the seller and buyer i in round t proceeds via the random-proposer

size and then bargains with randomly selected hospitals. His analysis restricts attention to equilibria with immediate agreement, but equilibria with this property do not exist when bargaining frictions are small and hospitals ("buyers" in our setting) are heterogeneous. As this is the case we are primarily interested in, a direct comparison with his results is not possible. Taking a cooperative approach, Ghili (2022) studies network formation in the pairwise stability framework of Jackson and Wolinsky (1996) assuming that payoffs are determined by Nash bargaining. Manea (2021) discusses an example in which if buyers make offers more frequently than the seller, the seller is better off dealing with a single buyer instead of all buyers.

[5]Stole and Zwiebel (1996) and Arie, Grieco and Rachmilevitch (2018) also analyze bargaining models in which a player signs bilateral contracts with several others in sequence, but in their models the order in which negotiations proceed is exogenous, and exclusion does not occur in equilibrium.

[6]The seller may customize the "good" for each buyer upon purchase; the setting with multiple units of a homogenous good is a special case.

protocol: with probability 1/2 each of the two players proposes a price, and the other decides whether to accept or reject the proposal. If the proposal is accepted, the seller trades with buyer i at the proposed price, buyer i exits the game, and the seller continues to bargain with the remaining buyers in round $t+1$. Otherwise, bargaining proceeds with the same set of buyers in round $t+1$. The game ends when the seller trades all q units.[7] Players have a common discount factor $\delta \in (0,1)$: payoffs obtained in round t are discounted by δ^t. The game has perfect information.

We call this the *bargaining game with exogenous supply* q, or the *game with supply* q for short. We will also be interested in situations in which there is no inherent scarcity, i.e., $q = n$, but the seller may strategically commit to exclude some buyers in order to enhance competition. The model with exclusion commitments is described and analyzed in Section 8.

We analyze Markov perfect equilibria (MPEs) of the game with supply q, which are subgame perfect equilibria in which each player's strategy in every round depends only on the *state* S—the set of buyers with whom the seller has not already traded—and the actions taken within the round (including nature's random selection of proposer). By definition, in an MPE, behavior in any subgame that starts at the beginning of a bargaining round (before the seller's selection of a bargaining partner) in state S does not depend on the history of play prior to that round. We refer to any such subgame as *subgame* S. Our main results concern limit MPE outcomes as $\delta \to 1$.

3. A Preliminary Lemma and an Example

Our first lemma provides basic scaffolding for the arguments that follow. It establishes that in any MPE and from every state, the seller reaches an agreement with every buyer she chooses to bargain with in equilibrium. Hence, the game with supply q ends in q rounds. The result also provides recursive equations relating expected payoffs to the probabilities with which the seller bargains with buyers in different states, and shows that the difference in prices at which a buyer trades in a given state depending on which party wins the coin toss to propose vanishes as $\delta \to 1$.

Lemma 1. *Consider an MPE of the game with supply q and a state S with $|S| > n - q$. If the seller chooses to bargain with buyer i with positive probability in state S, then the seller and buyer i reach agreement with conditional probability 1 in that round. The expected payoff $u_j(S)$ of player $j \in S \cup \{0\}$ and the probability $\pi_i(S)$ with which the seller chooses to bargain*

[7] Proposition 4.ii in Rubinstein and Wolinsky (1990) introduced this "voluntary matching" bargaining protocol (their wording emphasizes the seller's strategic selection of bargaining partner, in contrast to random matching) in a setting with unit supply. We employed similar bargaining protocols in Abreu and Manea (2012, 2022) and Manea (2018). This bargaining protocol is distinct from the "random proposer" protocol of Elliott and Nava (2019) and Talamas (2019, 2020) whereby a "proposer" is randomly recognized in every round, and the proposer strategically selects a bargaining partner but also makes the offer.

with buyer i in state S jointly satisfy the following conditions:

$$u_0(S) \geq \frac{1}{2}\left(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)\right) + \frac{1}{2}\delta u_0(S), \text{ with equality if } \pi_i(S) > 0;$$

$$u_i(S) = \pi_i(S)\left(\frac{1}{2}\left(a_i + \delta u_0(S \setminus \{i\}) - \delta u_0(S)\right) + \frac{1}{2}\delta u_i(S)\right) + \sum_{k \in S \setminus \{i\}} \pi_k(S)\delta u_i(S \setminus \{k\}),$$

where $u_0(S \setminus \{i\}) = u_i(S \setminus \{k\}) = 0$ if $|S| = n - q + 1$. If the variables $u_j(S)$ and $\pi_j(S)$ derived from a family of MPEs for a sequence of discount factors going to 1 converge to limits denoted $\bar{u}_j(S)$ and $\bar{\pi}_j(S)$, and if $\pi_i(S) > 0$ along the sequence for some buyer $i \in S$, then any transaction between the seller and buyer i in state S takes place at the common limit price $a_i - \bar{u}_i(S)$ regardless of which player is the proposer.

The proof of Lemma 1 and other proofs omitted in the main body of the paper appear in the Appendix. To understand the inequality for the seller's expected payoff in state S of an MPE, note that the seller may select buyer i for bargaining in state S, and if chosen to propose, can offer a price arbitrarily close to $a_i - \delta u_i(S)$ that i will accept; following an agreement with buyer i, the seller obtains a continuation equilibrium payoff of $\delta u_0(S \setminus \{i\})$. When buyer i is chosen to propose, the seller may at worst reject i's offer and enjoy a continuation payoff of $\delta u_0(S)$; in equilibrium, buyer i will make an offer that makes the seller indifferent between accepting and rejecting. If the seller bargains with buyer i with positive probability in state S, then her realized payoff from trading with i should be equal to her equilibrium payoff $u_0(S)$. The buyer payoff equations have a similar interpretation.

For the rest of the paper, we use the notation $u_i(S)$ and $\pi_i(S)$ from Lemma 1 for the payoffs and mixing probabilities associated with state S in any MPE under consideration, with $\bar{u}_i(S)$ and $\bar{\pi}_i(S)$ denoting the corresponding limits of these variables (when they exist) in a family of MPEs for a sequence of discount factors going to 1. We simplify notation by writing $u_i, \pi_i, \bar{u}_i, \bar{\pi}_i$ for the variables $u_i(N), \pi_i(N), \bar{u}_i(N), \bar{\pi}_i(N)$ associated with the initial state N, respectively.

An example. With the preliminary analysis in place, we are able to solve simple examples. This exercise is helpful in developing appropriate conjectures and steering us away from plausible conjectures that turn out to be false. We are interested in the following questions, which concern limit equilibrium outcomes as $\delta \to 1$: Is the MPE unique? If not, does each buyer trade with the same probability in all MPEs? Are buyer payoffs constant across MPEs? Are seller payoffs constant across MPEs?

We consider an example in which a seller with supply $q = 2$ bargains with three buyers with values $a_1 = 4, a_2 = 3, a_3 = 1$. This example demonstrates that the answer to each of the first three questions is negative. The negative answer to the second question implies that MPEs are not always asymptotically efficient. Interestingly, the example is consistent with the answer to the fourth question being positive.

In this example, there are three classes of MPEs for δ close to 1: a first one in which $\pi_1 = 0$ and $\pi_2, \pi_3 > 0$; a second one in which $\pi_2 = 0$ and $\pi_1, \pi_3 > 0$; and a third one in which $\pi_1, \pi_2, \pi_3 > 0$. The limit values for $\delta \to 1$ of every player i's payoff $\bar{u}_i$, the seller's mixing probabilities $(\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$ in the initial state, and each buyer i's total trading probability $\bar{\Pi}_i$ in the three classes of MPEs are displayed in the table below. In every class of MPEs, the seller's payoff converges to 4, buyer 3's payoff converges to 0, and buyer 1's probability of trade converges to 1 for $\delta \to 1$. Moreover, after the first trade, the seller trades with the highest valuation remaining buyer with limit probability 1. It follows that the set of buyers the seller trades with is $\{1, 3\}$ with the corresponding limit probability $\bar{\pi}_3$, and $\{1, 2\}$ with complementary probability. That is, trade is inefficient with limit probability $\bar{\pi}_3$.

$a_1 = 4, a_2 = 3, a_3 = 1$	$\bar{u}_0$	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$(\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$	$\bar{\Pi}_1$	$\bar{\Pi}_2$	$\bar{\Pi}_3$
$\pi_1 = 0; \pi_2, \pi_3 > 0$	4	2	1	0	$(0, 1, 0)$	1	1	0
$\pi_2 = 0; \pi_1, \pi_3 > 0$	4	1.5	1.5	0	$(1, 0, 0)$	1	1	0
$\pi_1, \pi_2, \pi_3 > 0$	4	1.5	1	0	$(0.5, 0.25, 0.25)$	1	0.75	0.25

We walk the reader through the solution of this example in the online Appendix. The precise computation of MPEs for a fixed δ is challenging. Nevertheless, we can determine the asymptotic values of equilibrium variables in the different classes of MPEs in this example by taking the limit $\delta \to 1$ in the equilibrium conditions from Lemma 1. Here we establish uniqueness only of limit variables corresponding to each class of MPEs, but it can be shown that in this example there exist exactly three MPEs for sufficiently high δ.

We first describe the MPEs in subgames following trade with one buyer, in which the seller has a single unit remaining. Proposition 1 of Manea (2018) characterizes the unique MPE outcomes for such subgames. In states $\{i, 3\}$ $(i = 1, 2)$, the outside option of trading with buyer 3 is not sufficiently valuable to improve the seller's bargaining position with buyer i, and the seller sells the remaining unit with probability 1 to buyer i at expected price $a_i/2$: $\pi_i(\{i, 3\}) = 1, u_0(\{i, 3\}) = u_i(\{i, 3\}) = a_i/2, u_3(\{i, 3\}) = 0$. In state $\{1, 2\}$, the outside option of trading with buyer 2 is binding, and the seller randomizes between buyers 1 and 2 in equilibrium, but the probability of bargaining (and trading) with buyer 2 converges to 0 as δ goes to 1, and buyer 1 trades with limit probability 1 at limit price a_2: $\bar{\pi}_1(\{1, 2\}) = 1, \bar{u}_0(\{1, 2\}) = 3, \bar{u}_1(\{1, 2\}) = 1, \bar{u}_2(\{1, 2\}) = 0$.

Lemma 1 applied to the initial state $\{1, 2, 3\}$ implies that if MPE payoffs and bargaining probabilities in every state converge for a sequence of discount factors $\delta \to 1$, then the limit variables satisfy the following conditions:

$$\pi_i > 0 \text{ for all } \delta \implies \bar{u}_0 = a_i + \bar{u}_0(N \setminus \{i\}) - \bar{u}_i \tag{3}$$

$$\bar{\pi}_i < 1 \implies \bar{u}_i = \sum_{k \in N \setminus \{i\}} \frac{\bar{\pi}_k}{1 - \bar{\pi}_i} \bar{u}_i(N \setminus \{k\}). \tag{4}$$

Note that we already know the limit continuation values $\bar{u}_0(N \setminus \{i\})$ and $\bar{u}_i(N \setminus \{k\})$ for subgames following the first trade.

Consider the class of MPEs, in which $\pi_1 = 0$ and $\pi_2, \pi_3 > 0$. In this case, (3) implies that $\bar{u}_0 = a_2 + \bar{u}_0(\{1,3\}) - \bar{u}_2 = a_3 + \bar{u}_0(\{1,2\}) - \bar{u}_3$, which leads to $\bar{u}_2 - \bar{u}_3 = 1$. If $\bar{\pi}_2 < 1$, then (4) implies that $\bar{u}_2 = 0$, and hence $\bar{u}_3 = -1$, which is impossible. It follows that $\bar{\pi}_2 = 1$, which by (4) leads to $\bar{u}_1 = 2$ and $\bar{u}_3 = 0$, implying that $\bar{u}_2 = 1$ and $\bar{u}_0 = 4$.

The second class of MPEs is similar to the first, with the roles of buyers 1 and 2 interchanged. Analogous arguments imply that $\bar{\pi}_1 = 1$ and yield the limit payoffs for this class.

In the third family of MPEs, we have that $\pi_1, \pi_2, \pi_3 > 0$, and (3) implies that $\bar{u}_0 = a_1 + \bar{u}_0(\{2,3\}) - \bar{u}_1 = a_2 + \bar{u}_0(\{1,3\}) - \bar{u}_2 = a_3 + \bar{u}_0(\{1,2\}) - \bar{u}_3$, which leads to $\bar{u}_1 - \bar{u}_2 = 0.5$ and $\bar{u}_2 - \bar{u}_3 = 1$. If $\bar{\pi}_3 = 1$, then (4) implies that $\bar{u}_2 = 0$, and hence $\bar{u}_3 = -1$, which is impossible. Thus, $\bar{\pi}_3 < 1$, leading to $\bar{u}_3 = 0$ via (4). It follows that $\bar{u}_1 = 1.5$ and $\bar{u}_2 = 1$. Taking the limit $\delta \to 1$ in the payoff equations for buyers 1 and 2 from Lemma 1, we obtain:[8]

$$\begin{aligned} 1.5 &= \bar{\pi}_1 \times 1.5 + \bar{\pi}_2 \times 2 + \bar{\pi}_3 \times 1 \\ 1 &= \bar{\pi}_1 \times 1.5 + \bar{\pi}_2 \times 1 + \bar{\pi}_3 \times 0. \end{aligned}$$

Then, $\bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3 = 1$ leads to the unique solution $\bar{\pi}_1 = 0.5, \bar{\pi}_2 = \bar{\pi}_3 = 0.25$.

The first two MPEs are asymptotically efficient for $\delta \to 1$: in both cases the network of included buyers is $\{1,2\}$ with limit probability 1.[9] However, the third MPE is asymptotically inefficient because in this case the set of included buyers is $\{1,3\}$ with limit probability 0.25. Although limit buyer payoffs and probabilities of trade vary across the three classes of MPEs for this example, limit seller payoffs do not, and are equal to 4 in all MPEs. In Section 5, we prove that limit MPE seller payoffs are unique in general, and derive a formula for their value which in this example reduces to $\bar{u}_0 = a_2 + a_3$. Other common features of MPEs in this example, which will also be explained by our results, are that buyer 1 trades with limit probability 1 and that buyer 3 gets zero limit payoff.

4. Key Lemmas

We now develop some core results upon which our subsequent analysis builds. These results are intuitive, and indeed familiar in the case $q = 1$, but their complete proofs for the case $q > 1$ are not straightforward. We present proof sketches at the end of the section.

Lemma 2 shows that in any family of MPEs for the game with supply q for a sequence of discount factors going to 1, no buyer i can acquire the good for less than the "fair" price $a_i/2$ in the limit. This is intuitive because within each round in which the seller bargains

[8]The payoff equation for buyer 3 does not generate any restriction on limit mixing probabilities because buyer 3 gets limit payoff 0 in every state.

[9]It is not always the case that asymptotically efficient MPEs exist, as the example in the online Appendix illustrates.

with buyer i, the seller and buyer i make offers with equal probability, but the seller has the additional advantage of choosing her bargaining partner and possibly trading with other buyers if agreement is not reached in the current round.

Lemma 2 (Buyers pay at least fair prices)**.** *In any family of MPEs for the game with supply q for discount factors $\delta \in (0, 1)$,*

$$\limsup_{\delta \to 1} u_i \leq \frac{a_i}{2}.$$

Lemma 3 establishes that in the game with supply $q < n$, the payoffs of buyers $q+1, \ldots, n$ converge to 0 as $\delta \to 1$.[10] To get some perspective on this result, assume that buyer values are distinct. For $q = 1$, the result asserts that all buyers other than the buyer with the highest value have zero limit payoffs. This is an implication of Proposition 1 of Manea (2018). In this case, the highest valuation buyer trades with limit probability 1, and all other buyers with limit probability 0. The case $q > 1$ is more subtle: with sequential trade, a high value buyer is valuable to the seller both as a direct trading partner in the current round and as a better outside option when trading with other buyers in the future, and therefore might not necessarily manage to "outbid" a lower valuation buyer. In particular, when $q > 1$, buyer $q + 1$ may trade with positive limit probability in MPEs for $\delta \to 1$, as we have seen in the example from the previous section.

Lemma 3 (Buyers $q + 1, \ldots, n$ get zero payoffs under supply q)**.** *For any $q < n$ and any family of MPEs for the game with supply q and discount factors $\delta \in (0, 1)$, the payoffs of buyers $q + 1, \ldots, n$ converge to 0 as $\delta \to 1$.*

Lemma 4 below establishes that a buyer i who trades *with probability 1* in a sequence of MPEs for $\delta \to 1$—even when this occurs with some delay and perhaps stochastically in any given round—pays at most the fair price $a_i/2$ in the limit. This result may be viewed as a counterpoint to the outside option principle—a buyer who is never under the threat of exclusion in equilibrium cannot be exploited (relative to fair pricing) by the seller.

Lemma 4 (At most fair pricing with sure trade)**.** *Let $(\sigma^{\delta_z})_{z \geq 0}$ be a sequence of MPEs for the game with supply q in which the discount factors δ_z converge to 1 as $z \to \infty$. If the seller trades with buyer i with probability 1 under σ^{δ_z} for all $z \geq 0$, then*

$$\liminf_{z \to \infty} u_i \geq \frac{a_i}{2}.$$

We emphasize that "sure trade" in the naming of Lemma 4 refers to trade with *exact* probability 1 in a family of MPEs for a sequence of discount factors converging to 1. As discussed in the context of subgames in the example from the previous section, when $a_1 > a_2 > a_1/2$ in the setting with unit supply, trade with buyer 1 takes place with *limit probability*

[10]The result implies that every buyer $i \leq q$ with $a_i = a_{q+1}$ also gets a zero limit payoff (via an argument that exchanges the labels of buyers i and $q + 1$). Hence, buyers with values that do not exceed the extra marginal value get zero limit payoffs.

1 as $\delta \to 1$, but in this case the outside option of trading with buyer 2 is binding, and buyer 1 pays a limit price of a_2, which is above the fair price $a_1/2$.

Lemmata 2 and 4 have the following corollary.

Corollary 1 (Fair pricing with sure trade). *Let* $(\sigma^{\delta_z})_{z\geq 0}$ *be a sequence of MPEs for the game with supply* q *in which the discount factors* $\delta_z \to 1$ *as* $z \to \infty$*. If the seller trades with buyer* i *with probability 1 under* σ^{δ_z} *for all* $z \geq 0$*, then the expected payoff of buyer* i *converges to* $a_i/2$ *as* $z \to \infty$*.*

While this result echoes classic results on convergence to the Nash bargaining solution in Rubinstein-style alternating-offer bargaining (Binmore 1980; Binmore, Rubinstein and Wolinsky 1986), the argument here is more involved due to the presence of other buyers, the seller's strategic (and typically stochastic) selection of bargaining partner at every stage, and the resultant non-stationary interaction between the seller and each buyer. In general, the exact price a buyer pays in MPEs for a fixed δ depends on the state in which the buyer trades, but the result shows that if the buyer is certain to trade, then all these prices converge to the fair price as $\delta \to 1$.[11]

The example from the previous section demonstrates that although trading with exact probability 1 is a sufficient condition, it is not a necessary condition for fair pricing in the limit. Indeed, in the first class of MPEs in the example, buyer 1 trades with probability smaller than 1 but converging to 1 for $\delta \to 1$ and obtains a limit payoff of $a_1/2$ (buyer 2 is in an analogous situation in the second class of MPEs).

We briefly turn to the game with unconstrained supply, i.e., $q = n$. By Lemma 1 and Corollary 1, in every MPE of the game with supply $q = n$, the seller trades with each buyer i with probability 1 in one of the first n rounds at an expected discounted price converging to $a_i/2$ as $\delta \to 1$. This implies the following corollary.

Corollary 2. *In any family of MPEs of the game with supply* $q = n$ *for discount factors* $\delta \in (0, 1)$*, the seller's profit converges to* $\sum_{i \in N} a_i/2$ *as* $\delta \to 1$*.*

We conclude the section by sketching some key steps in the proofs of Lemmata 2-4. Readers satisfied with the intuitions provided above may proceed to the next section. Consider an MPE for the game with discount factor δ. An important implication of Lemma 1 that the

[11] We establish a result of a similar flavor for a network setting in earlier work (Abreu and Manea 2012). In that model, every link generates a unit surplus and each player needs to trade with a neighbor. We show that every player who is guaranteed to trade in equilibrium—even when trade occurs in an evolving network and potentially with different neighbors—obtains asymptotic payoffs of at least 1/2. Elliott and Nava (2019) also obtain a related result in a network setting with heterogeneous link values. In the efficient MPEs they analyze, every pair of players who trade with each other with probability 1 face a stationary environment of trading opportunities with other neighbors, but these outside options cannot be binding. Consequently, each such pair effectively trades in a stationary two-player bargaining game, and agreements reflect "Rubinstein payoffs" independent of the state of the network.

proofs rely on is that

$$u_i = \frac{2\pi_i(1-\delta)}{2-\delta-\delta\pi_i} \times \frac{a_i + \delta u_0(N \setminus \{i\})}{2} + \sum_{k \in N \setminus \{i\}} \frac{\pi_k(2-\delta)}{2-\delta-\delta\pi_i} \times \delta u_i(N \setminus \{k\}). \tag{5}$$

Moreover, we have that

$$\frac{2\pi_i(1-\delta)}{2-\delta-\delta\pi_i} + \sum_{k \in N \setminus \{i\}} \frac{\pi_k(2-\delta)}{2-\delta-\delta\pi_i} = 1. \tag{6}$$

Therefore, formula (5) expresses buyer i's payoff as a convex combination of a term reflecting his expected payoff $(a_i + \delta u_0(N \setminus \{i\}))/2$ in the event the seller bargains with him, and terms reflecting his payoff $\delta u_i(N \setminus \{k\})$ in the event the seller trades with another buyer k in the initial state. Indeed, a trade between the seller and buyer i creates a value of a_i for the buyer and profits $\delta u_0(N \setminus \{i\})$ in the continuation subgame for the seller. If $\pi_i = 1$, then the weight $2\pi_i(1-\delta)/(2-\delta-\delta\pi_i)$ on the first term equals 1, and the two players share the total gains from trade $a_i + \delta u_0(N \setminus \{i\})$ equally. Lemma 2 shows that the seller is able to avoid such hold-ups in equilibrium whenever the seller's continuation profits have a positive limit. More generally, it is possible that $\lim_{\delta \to 1} \pi_i = 1$ in a sequence of MPEs for $\delta \to 1$, and the weight $2\pi_i(1-\delta)/(2-\delta-\delta\pi_i)$ has a positive limit, which depends on π_i's rate of convergence to 1 as $\delta \to 1$. For instance, in the first class of MPEs for the example in the previous section, the weight corresponding to buyer 2 converges to 2/5 as $\delta \to 1$. By contrast, if $\lim_{\delta \to 1} \pi_i < 1$, then the weight converges to 0. In this case, buyer i's asymptotic payoffs are driven exclusively by his payoffs in subgames following trades with other buyers. Taking the limit $\delta \to 1$ in (5) for a sequence of MPEs in which all state variables converge, we derive formula (4) from the previous section, which we repeat here for convenience:

$$\bar{u}_i = \sum_{k \in N \setminus \{i\}} \frac{\bar{\pi}_k}{1-\bar{\pi}_i} \bar{u}_i(N \setminus \{k\}). \tag{7}$$

This formula facilitates inductive arguments in the proofs of Lemmata 2-4, with the case $\bar{\pi}_i = 1$ requiring separate treatment. While in the latter case, formula (5) is not informative about buyer i's limit payoff without knowledge of π_i's rate of convergence to 1 as δ goes to 1, it carries the information that $\bar{u}_k = \bar{u}_k(N \setminus \{i\})$ when applied for buyers $k \neq i$, which we leverage in the proofs.

The proof of Lemma 2 proceeds by induction on q (with base case $q = 0$). For the inductive step, it is sufficient to establish that $\bar{u}_i \leq a_i/2$ for all $i \in N$ for any sequence of MPEs in which the state variables converge as $\delta \to 1$. From the induction hypothesis, we know that $\bar{u}_i(N \setminus \{k\}) \leq a_i/2$ for all $k \neq i$. If $\bar{u}_0(N \setminus \{i\}) = 0$, then it is easy to reach the conclusion from (5) and (6): each term in the convex combination describing buyer i's payoff, including $(a_i + \delta u_0(N \setminus \{i\}))/2$, is asymptotically bounded above by $a_i/2$. If $\bar{\pi}_i < 1$, then the conclusion follows directly from (7). We are left with the case $\bar{\pi}_i = 1$ and $\bar{u}_0(N \setminus \{i\}) > 0$ (which, as noted earlier, arises for $i = 2$ in the first class of MPEs in the example from the previous section). The latter inequality implies that the seller trades with some buyer $k \in N \setminus \{i\}$ with positive limit probability in the second round of the game after an agreement with i, i.e., $\bar{\pi}_k(N \setminus \{i\}) > 0$. As $\bar{\pi}_i = 1$, the arguments above imply that $\bar{u}_k = \bar{u}_k(N \setminus \{i\})$. It follows $\bar{u}_0 = a_i - \bar{u}_i + a_k - \bar{u}_k + \bar{u}_0(N \setminus \{i, k\})$. The seller may deviate to first trading with buyer k at a price converging to $a_k - \bar{u}_k$, and then trading with buyer i at a price converging to $a_i - \bar{u}_i(N \setminus \{k\})$ to obtain a limit profit of $a_k - \bar{u}_k + a_i - \bar{u}_i(N \setminus \{k\}) + \bar{u}_0(N \setminus \{i, k\})$. For this deviation not to be profitable for the seller for high δ in the sequence of MPEs, it must be that $\bar{u}_i \leq \bar{u}_i(N \setminus \{k\})$, which proves the inductive step via the induction hypothesis.

The proof of Lemma 3 also proceeds by induction on q. For the inductive step, consider a buyer $i \geq q + 1$. We need to argue that $\bar{u}_i = 0$. As in the case of Lemma 2, it is sufficient to establish this for a sequence of MPEs in which state variables converge as $\delta \to 1$. A trade with any buyer $k \neq i$ leads to a game with supply $q - 1$ in which the induction hypothesis implies that $\bar{u}_i(N \setminus \{k\}) = 0$. If $\bar{\pi}_i < 1$, then (7) leads to $\bar{u}_i = 0$. To deal with the delicate case in which $\bar{\pi}_i = 1$, we consider a deviation whereby the seller switches the order of trades with buyer i and another buyer k if $q > 1$ like in the proof of Lemma 2 (or trades with another buyer j for which $a_j \geq a_i$ at limit price a_j if $q = 1$).

For Lemma 4, we argue inductively that $\bar{u}_i \geq a_i/2$ for every buyer i that trades with probability 1 in a sequence of MPEs with $\delta \to 1$. Consider such a buyer i. If $\pi_k > 0$ along a subsequence, then buyer i must trade with probability 1 in subgame $N \setminus \{k\}$, which by the induction hypothesis implies that $\bar{u}_i(N \setminus \{k\}) \geq a_i/2$. The inductive step follows from noting that the payoffs $(a_i + \delta u_0(N \setminus \{i\}))/2$ and $\delta u_i(N \setminus \{k\})$ in the convex combination (5) are asymptotically bounded below by $a_i/2$.

5. Seller Profits

The main result of this section establishes that the seller's MPE payoffs are essentially unique for δ close to 1, and provides a simple formula for the seller's limit profit. The uniqueness of asymptotic seller payoffs is unexpected in light of the example discussed in Section 3, which showcases multiple MPEs that are not asymptotically equivalent in terms of buyer payoffs or trading probabilities.

Theorem 1 (Seller profits)**.** *In any family of MPEs of the game with supply $q < n$ for discount factors $\delta \in (0, 1)$, the seller's expected profit converges as $\delta \to 1$ to*

$$M^{*q} := \max_{l \leq q+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{q+1} \right]. \tag{8}$$

To prove this theorem, we argue that M^{*q} constitutes both an upper and a lower bound on the seller's asymptotic profit in every sequence of MPEs for the game with supply q for $\delta \to 1$. The first result establishes the upper bound.

Lemma 5 (Upper bound on seller profits)**.** *In any family of MPEs for the game with supply $q < n$ for discount factors $\delta \in (0, 1)$,*

$$\limsup_{\delta \to 1} u_0 \leq M^{*q}.$$

We sketch the proof of Lemma 5 here. Consider an MPE of the game with supply $q < n$. Let l be the smallest index of a buyer who trades with probability smaller than 1 in the MPE. We have that $l \leq q + 1$. By Lemma 1, the MPE generates a probability distribution over sequences of q buyers that the seller trades with in the first q rounds of the game. By definition, there exists at least one such sequence $\mathbb{S}$ that excludes buyer l but includes buyers $1, 2, \ldots, l - 1$. Since choosing to bargain with buyers in the sequence $\mathbb{S}$ is optimal for the seller, it must be that the seller's MPE payoff is equal to her expected payoff from trading over $\mathbb{S}$. As $\mathbb{S}$ arises with positive probability in equilibrium, each buyer $j < l$ trades with probability 1 in the subgame following agreements with his predecessors in $\mathbb{S}$. Lemma 4 implies that the (limit) expected discounted price the seller collects from buyer j in the subgame is at most $a_j/2$. Hence, the seller's limit payoff from trading with buyers $1, \ldots, l-1$ over $\mathbb{S}$ does not exceed $a_1/2 + \ldots + a_{l-1}/2$. The seller receives no payment from buyer l along $\mathbb{S}$, and can at most extract all surplus from the remaining $q - l + 1$ buyers with the highest valuations. If follows that the seller's limit profit is bounded above by M^{*q}.

Surprisingly, it is also the case that the seemingly coarse upper bound M^{*q} constitutes a lower bound on the seller's asymptotic profits in MPEs for the game with supply q as $\delta \to 1$.

Lemma 6 (Lower bound on seller profits)**.** *In any family of MPEs of the game with supply $q < n$ for discount factors $\delta \in (0, 1)$,*

$$\liminf_{\delta \to 1} u_0 \geq M^{*q}.$$

To prove this result, let l^* be a maximizer in the optimization problem defining M^{*q}, and consider a family of MPEs of the game with supply $q < n$ for discount factors $\delta \in (0, 1)$. The seller may deviate from her equilibrium strategy to a strategy that generates trades with buyers in the sequence $q + 1, q, \ldots, l^* + 1, l^* - 1, \ldots, 1$ over a fixed but long enough time horizon with probability arbitrarily close to 1. Under this deviation, the seller bargains successively with each buyer in the sequence, rejecting all offers and waiting to become the proposer. Upon being selected to propose to buyer i, the seller makes an offer that buyer i accepts in equilibrium. By Lemma 3, for high enough δ, buyer $i = q + 1, q, \ldots, l^* + 1$ will accept price offers arbitrarily close to a_i when it is his turn to trade. Similarly, by Lemma 2, buyer $i = l^* - 1, \ldots, 1$ will accept price offers arbitrarily close to $a_i/2$. Over a long enough time horizon, the seller will win the coin toss against all buyers in the sequence with probability arbitrarily close to 1, and the deviation secures seller profits arbitrarily close to M^{*q} for high δ. We conclude that the seller's asymptotic profits in the family of MPEs are bounded below by M^{*q}.

Since the two bounds on the seller's asymptotic payoffs in the game with supply q delivered by Lemmata 5 and 6 coincide, they must be tight. Therefore, in any family of MPEs for the game with supply q, the seller's profits converge to M^{*q} as $\delta \to 1$, which proves Theorem 1.

We remark that while the strategy underlying the proof of Lemma 6 enables the seller to achieve her limit MPE payoff M^{*q} asymptotically in the game with supply q, it does not necessarily describe the seller's behavior in any MPE, and may even be played with limit probability 0 as $\delta \to 1$. Indeed, when the maximizer in (8) is unique and different from $q+1$, this is an implication of forthcoming Theorem 2.

Sequential outside option principle. Theorem 1 yields a *sequential outside option principle* for settings in which a seller trades sequentially with several, but not all, potential buyers. Recall that the standard outside option principle implies that if the seller has one unit for sale and there are multiple buyers, the second highest valuation is a lower bound on the price the seller can extract from the highest-value buyer. Similarly, if there are q units for sale and n buyers, if we think of the extra marginal buyer $q+1$ as a *static* outside option, $q \cdot a_{q+1}$ should be a lower bound on seller profits.[12] In our *dynamic* bargaining process, the seller can sequentially *exercise* the outside option by trading with the extra marginal buyer $q+1$ first, the new extra marginal buyer q next, and so on; the outside option provided by the extra marginal buyer improves every round. In particular, this argument implies that the seller can extract a profit of $a_2 + \ldots + a_{q+1}$ by trading in sequence with buyers $q+1, q, \ldots, 2$. This is the value of the maximand in (8) for $l = 1$. Our formula for seller profits (8) recognizes that it might be too costly to exclude buyers with high valuations, and combines Lemma 3 with Lemma 2. The latter implies that the seller can trade with buyers from a top interval of valuations at fair (or better) prices.

For another perspective on the sequential exercise of outside options, we revisit the example from the introduction in which $n = 3, q = 2$ and $a_3 > a_1/2$. As argued there, trading with buyer 2 in the first round even at the highest possible price of a_2 is not more valuable than trading with buyer 3 at a price of a_3 (which is feasible in the limit for $\delta \to 1$ by Lemma 3). This is because in the next round, when bargaining with buyer 1, the seller can demand a price of a_2 if buyer 2 is available as an outside option, but a lower price of a_3 if buyer 3 is the outside option. In either case, the seller's limit profit is $a_2 + a_3$. This example shows that buyers who are more valuable for inclusion may also be more valuable for exclusion when additional units remain to be sold to even more valuable buyers.

Extension to random matching. Our bargaining protocol allows the seller to strategically choose which buyer she bargains with in every round. An alternative protocol entails random matching between the seller and individual buyers according to exogenously given probabilities. The protocol with strategic choice of bargaining partner is easier to work with

[12] The model of Ho and Lee (2019) applied to our setting actually predicts limit seller payoffs of $q \cdot a_{q+1}$ when the outside option provided by buyer $q+1$ is binding for buyers $1, \ldots, q$. See Section 9 for further discussion.

and also seems more natural in our setting, in which the seller with multiple units may wish to trade only with a particular subset of buyers. An awkwardness of the random matching protocol is that the seller gets matched to bargain with buyers that she does not have an incentive to trade with, and such matches lead to delay in equilibrium. Nevertheless, our results extend: the seller can replicate strategic choice of bargaining partners simply by waiting to be matched with a desired buyer at an expected cost of delay that vanishes as $\delta \to 1$. At a high level, this is why Theorem 1 and the supporting lemmata extend with minor modifications. We provide details in the Appendix.

6. Included and Excluded Buyers

Theorem 1 reveals a close connection between the maximum M^{*q} in the simple static optimization problem displayed in (8) and the seller's profits in the complex dynamic bargaining game with supply $q < n$. As we have seen concretely in the example from Section 3, the seller can attain the total profits M^{*q} in a variety of ways and from different sets of buyers in equilibrium. Nevertheless, Theorem 2 below shows that the optimization problem is also informative—via its maximizers l—about which buyers are certain to trade and which buyers face the threat of "exclusion" in the game.

Generically, the static optimization problem has a unique maximizer l^*. For this generic case, we show that every buyer $i < l^*$ trades with probability 1 in any MPE for high enough δ. The converse is also true: every buyer $i \geq l^*$ trades with probability less than 1 in MPEs for high δ. Thus, buyers $i < l^*$ are guaranteed to be "included"—and hence by Corollary 1 trade at the fair price $a_i/2$ in the limit as $\delta \to 1$—while buyers $i \geq l^*$ are "excluded" with positive probability in equilibrium for high δ. We establish that if $l^* \neq q + 1$ and $a_{l^*} > a_{l^*+1}$, then buyer l^* trades with limit probability 1 in any family of MPEs with $\delta \to 1$. In this case, l^* is the buyer with the highest value that is excluded with positive probability in equilibrium, but the probability of excluding l^* vanishes as $\delta \to 1$. However, if $l^* = q + 1$, then in MPEs for high δ, the seller trades with the top q buyers with probability 1, and hence trades with buyer l^* with probability 0. We also prove that the seller trades only with buyers with the top $q + 1$ valuations, extending the logic of "two is enough for competition" to situations with multiple transactions: an extra buyer is enough for competition. In the Appendix, we state and prove a general version of the theorem that also deals with non-generic cases in which the static optimization problem (8) has multiple maximizers. Proofs for various parts of the result track the evolution of the formula for seller profits in subgames as trade takes place (and involve further use of the supporting lemmata).

Theorem 2 (Included and excluded buyers)**.** *Suppose that the optimization problem displayed in* (8) *has a unique maximizer* l^*. *Then, there exists* $\underline{\delta} < 1$ *such that the following statements hold for every MPE of the game with supply* q *and discount factor* $\delta > \underline{\delta}$.

- *The seller trades with buyer* i *with probability 1 if and only if* $i < l^*$.

- *If $l^* \neq q+1$ and $a_{l^*} > a_{l^*+1}$, then the seller trades with buyer l^* with limit probability 1 as $\delta \to 1$.*
- *If $l^* = q+1$, then the seller trades exclusively with buyers $1, \ldots, q$.*
- *The seller trades with probability 0 with any buyer i for which $a_i < a_{q+1}$.*

The result also highlights subtle differences between the static optimization problem defining M^{*q} and the equilibrium of the dynamic bargaining game: the missing term corresponding to the value of buyer l^* in the formula for M^{*q} does not translate into buyer l^* carrying all the burden of exclusion in the game. Indeed, buyer l^* is almost certain to be included in the limit $\delta \to 1$. In particular, this means that the strategy delivering the lower bound on limit seller profits in the proof of Theorem 1 is played with limit probability 0 in MPEs for $\delta \to 1$.

An example with $n = 3, a_1 = a_2 = 3, a_3 = 1$ shows that weakening the hypothesis $a_{l^*} > a_{l^*+1}$ to require that $a_{l^*} > a_n$ in Theorem 2 does not guarantee the conclusion that buyer l^* trades with limit probability 1. In this example, we have that $l^* = 1$ and $a_1 > a_3$, but there exists a family of MPEs with $\bar{\pi}_1 = \bar{\pi}_2 = 1/4$ and $\bar{\pi}_3 = 1/2$. In this family of MPEs, the seller trades with buyer $l^* = 1$ with limit probability $3/4 < 1$ as $\delta \to 1$.[13]

7. Buyer Payoffs

Suppose that $q < n$. A key conclusion of our analysis is that seller limit payoffs are unique across MPEs, and we have found a simple formula for this unique value expressed only in terms of buyer valuations. We have also seen by example that buyers' limit payoffs may vary across MPEs. What can then be said about a buyer's limit payoffs along a sequence of MPEs? Frequently quite a bit, even with relatively coarse information about the class of MPEs in question: we can often infer limit buyer payoffs from the *support* of seller's mixing probabilities in certain states without knowledge of these probabilities.

We will say that a *family of MPEs* of the game with supply q for a sequence of discount factors δ going to 1 is *convergent* if all associated variables $u_i(S)$ and $\pi_i(S)$ converge as $\delta \to 1$, and for every $i \in S \subseteq N$ either $\pi_i(S) = 0$ or $\pi_i(S) > 0$ uniformly for all δ in the sequence. Note that every family of MPEs contains a subfamily that is convergent according to this definition. As in earlier sections, we use bar notation for corresponding limit variables.

Lemma 1 implies that in any convergent family of MPEs,

$$\pi_i(S) > 0 \implies \bar{u}_i(S) = a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S). \tag{9}$$

Since Theorem 1 yields explicit formulae for $\bar{u}_0(S)$ and $\bar{u}_0(S \setminus \{i\})$, this allows us to compute buyer i's asymptotic payoff in state S without knowledge of the exact probability $\pi_i(S)$ (as long as this probability is positive) or granular details of the different paths of trade with

[13]Similarly to examples we discuss in Section 3, this example admits two other families of MPEs with $\pi_1 = 0$ and $\pi_2 = 0$, respectively.

buyer i starting from state S. We seek to express buyer i's limit payoff in the overall game as an expectation of $\bar{u}_i(S)$ over a minimal set of states S for which $\pi_i(S) > 0$.

Fix a convergent family of MPEs, and consider a (possibly empty) sequence of trades with buyers $i_1, \ldots, i_k$ distinct from i such that $\pi_i(N \setminus \{i_1, \ldots, i_k\}) > 0$ and $\pi_i(N \setminus \{i_1, \ldots, i_{k'}\}) = 0$ for $k' < k$. Let $\mathcal{I}_i$ denote the set of sequences $(i_1, \ldots, i_k)$ with this property. Note that every trade of buyer i occurs after one and only one sequence in $\mathcal{I}_i$, either immediately or following intermediate trades with other buyers. Hence, buyer i's limit payoff in the overall game can be expressed as an expected value of the payoffs $\bar{u}_i(N \setminus \{i_1, \ldots, i_k\})$—for which condition (9) delivers an explicit formula—over sequences $(i_1, \ldots, i_k)$ in $\mathcal{I}_i$.

To develop this analysis, let $\bar{\pi}_{i_1,\ldots,i_k} = \bar{\pi}_{i_1}(N)\bar{\pi}_{i_2}(N \setminus \{i_1\}) \ldots \bar{\pi}_{i_k}(N \setminus \{i_1, \ldots, i_{k-1}\})$ denote the probability that the seller trades in sequence with buyers $i_1, \ldots, i_k$ (with the value corresponding to the empty sequence understood to be 1), and define

$$\bar{\theta}_i(S) = \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_i:\{i_1,\ldots,i_k\}=N\setminus S} \bar{\pi}_{i_1,\ldots,i_k}.$$

We have that

$$\begin{aligned}\bar{u}_i = \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_i} \bar{\pi}_{i_1,\ldots,i_k}\bar{u}_i(N \setminus \{i_1, \ldots, i_k\}) &= \sum_{S\ni i} \bar{u}_i(S) \sum_{(i_1,\ldots,i_k)\in\mathcal{I}_i:\{i_1,\ldots,i_k\}=N\setminus S} \bar{\pi}_{i_1,\ldots,i_k} \\ &= \sum_{S\ni i} \bar{\theta}_i(S)\bar{u}_i(S) = \sum_{S\ni i} \bar{\theta}_i(S)(a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S)).\end{aligned}$$

We have established the following result.

Proposition 1 (Buyer payoffs)**.** *In any convergent family of MPEs for the game with supply $q < n$,*

$$\bar{u}_i = \sum_{S\ni i} \bar{\theta}_i(S)(a_i + \bar{u}_0(S \setminus \{i\}) - \bar{u}_0(S)).$$

Note that if $\pi_i(N) > 0$, then $\mathcal{I}_i$ consists only of the empty sequence, and vice versa. In this case, $\bar{\theta}_i(N) = 1$ and $\bar{\theta}_i(S) = 0$ for all other $S \subset N$ containing i, and Proposition 1 implies that $\bar{u}_i = a_i + \bar{u}_0(N \setminus \{i\}) - \bar{u}_0(N)$ (consistent with (9)). If the seller's randomization among buyers in the initial state has full support, then the result characterizes every buyer's limit payoff. Proposition 1 neatly summarizes what can be said more generally about buyer i's limit payoff using minimal information about the seller's mixing probabilities along paths of play that end with buyer i's *first chance to trade* (with positive probability in equilibrium). The computation of buyer i's limit payoff in a convergent family of MPEs requires knowledge of the seller's mixing probabilities for other buyers who get opportunities to trade before i has a chance, but not of the probabilities with which the seller bargains with buyer i in different states. In some cases, the seller's mixing probabilities for those other buyers may be inferred from their limit payoffs, which in turn can be determined from Proposition 1.

We revisit the example from Section 3 to illustrate how Proposition 1 (along with Theorem 1) can be used to quickly derive buyers' limit payoffs and trading probabilities. In that example, there are three classes of MPEs for high δ. In the third class of MPEs, the seller mixes with full support between the three buyers in the initial state, and Proposition 1 pins down limit payoffs for all buyers as discussed above. In the first class of MPEs, the support of the seller's mixing in the initial state is formed by buyers 2 and 3, and Proposition 1 immediately determines $\bar{u}_2$ and $\bar{u}_3$. This information can be plugged in the limit payoff equation of buyer 2 to infer that $\bar{\pi}_2 = 1$. Hence, $\bar{\theta}_1(\{1,3\}) = 1$, and Proposition 1 leads to $\bar{u}_1 = a_1 + \bar{u}_0(\{3\}) - \bar{u}_0(\{1,3\})$, where $\bar{u}_0(\{1,3\}) = a_1/2$ and, by definition, $\bar{u}_0(\{3\}) = 0$. Similarly, limit buyer payoffs and trading probabilities in the second class of MPEs can be directly derived via Proposition 1 and the limit buyer payoff equations. Nonetheless, limit buyer payoff equations do not always carry sufficient information about limit mixing probabilities, as an example we discuss in the online Appendix demonstrates.

8. Optimal Exclusion Commitments When $q = n$

We now turn to a strategic situation in which the seller has unconstrained supply $q = n$, but might find it profitable to increase competition between buyers via exclusion commitments. We model such commitments as follows. An *exclusion commitment* $\mathcal{E}$ is a function from the set of all subsets of N to itself such that $\mathcal{E}(S) \subseteq S$, $\mathcal{E}(\{i\}) = \{i\}$ for all $i \in N$, and $\mathcal{E}(S) \subseteq \mathcal{E}(S \setminus \{i\})$ for all $i \in S \setminus \mathcal{E}(S)$. In the *game with exclusion commitment* $\mathcal{E}$, bargaining proceeds like in the *game with supply* q, but trade is restricted by $\mathcal{E}$: after a history in which the seller has not yet traded with a subset of buyers S, she *excludes* the buyers in $\mathcal{E}(S)$, and may only bargain with buyers in $S \setminus \mathcal{E}(S)$; the game ends when $\mathcal{E}(S) = S$. The condition $\mathcal{E}(\{i\}) = \{i\}$ for $i \in N$ ensures that the seller ultimately excludes at least one buyer from trade. The condition $\mathcal{E}(S) \subseteq \mathcal{E}(S \setminus \{i\})$ for $i \in S \setminus \mathcal{E}(S)$ requires that exclusions be irreversible: if the seller is committed to exclude a buyer at a given stage, she eliminates that buyer from all future negotiations.[14] As in the case of the game with exogenous supply, the payoff relevant state for the definition of MPEs in the game with exclusion commitment $\mathcal{E}$ is given by S and the actions in the current round.

A salient class of exclusion commitments, which treats buyers symmetrically, is the $\tilde{q}$-*supply commitment* for $\tilde{q} < n$. This commitment, denoted by $\mathcal{E}^{\tilde{q}}$, is specified by $\mathcal{E}^{\tilde{q}}(S) = S$ if $|S| > n - \tilde{q}$, and $\mathcal{E}^{\tilde{q}}(S) = \emptyset$ otherwise. This means that the game ends exactly after $\tilde{q}$ trades. Hence, the game with $\tilde{q}$-supply commitment is identical to the game with supply $\tilde{q}$.

We seek to derive optimal exclusion commitments for the seller under the least and the most favorable selection of MPEs asymptotically as $\delta \to 1$. Let $\Sigma^\delta(\mathcal{E})$ denote the set of MPEs in the game with an exclusion commitment $\mathcal{E}$ in which players have a common discount factor

[14] If buyer j is excluded in state S but not in state $S \setminus \{i\}$ for some buyer i with whom trade is allowed in state S, then the potential competition offered by buyer j when bargaining with buyer i in state S is unnecessarily lost. For instance, in a situation where $\mathcal{E}(S) = S \setminus \{i\}$ and $j \in S \setminus \mathcal{E}(S \setminus \{i\})$, buyer i would be a "gateway" to accessing buyer j from state S and could "hold up" the seller for half of the profits she later collects from buyer j. Our formulation of exclusion commitments precludes such hold-ups (but allows for others; see footnote 15).

δ, and $u_0(\sigma, \delta)$ denote the seller's expected payoff under a strategy profile σ. We investigate the following bounds and their associated optimal exclusion commitments $\mathcal{E}$:

$$\begin{aligned}\underline{M} &= \max_{\mathcal{E}} \liminf_{\delta \to 1} \inf_{\sigma \in \Sigma^\delta(\mathcal{E})} u_0(\sigma, \delta)\\ \overline{M} &= \max_{\mathcal{E}} \limsup_{\delta \to 1} \sup_{\sigma \in \Sigma^\delta(\mathcal{E})} u_0(\sigma, \delta).\end{aligned}$$

Our main result about optimal exclusion commitments shows that the two bounds coincide, and are achieved by the same exclusion commitment: the $(n-1)$-supply commitment. As the game with the $(n-1)$-supply commitment is identical to the game with supply $n-1$, Theorem 1 implies that the common value of the bounds is $M^{*(n-1)}$.

Theorem 3 ($(n-1)$-supply commitment is optimal)**.** *The $(n-1)$-supply commitment solves the maximization problems associated with both $\underline{M}$ and $\overline{M}$, and furthermore $\underline{M} = \overline{M} = M^{*(n-1)}$.*

The proof leverages the body of results developed thus far. Since the $(n-1)$-supply commitment is one of the exclusion commitments $\mathcal{E}$ allowed in the optimization problem defining $\underline{M}$, and by Theorem 1, the seller's profit in any family of MPEs for the game with supply $n-1$ converges to $M^{*(n-1)}$ for $\delta \to 1$, it follows that $\underline{M} \geq M^{*(n-1)}$.

Lemmata 1 and 4 generalize to the game with any exclusion commitment without substantial changes in the proofs.[15] Then, a straightforward adaptation of the argument for Lemma 5 implies that in every family of MPEs for the game with any exclusion commitment $\mathcal{E}$ for discount factors $\delta \in (0,1)$, the limit superior of the seller's expected profit as $\delta \to 1$ does not exceed $M^{*(n-1)}$. Hence, $\overline{M} \leq M^{*(n-1)}$. As $\overline{M} \geq \underline{M}$, we conclude that $\underline{M} = \overline{M} = M^{*(n-1)}$, which means that the $(n-1)$-supply commitment is optimal for both optimization problems.

This *optimal exclusion commitment* entails that the seller commits to exclude a single buyer but allows the seller the flexibility to decide dynamically which buyer is excluded.[16]

[15] While Lemma 2 is not directly needed for the arguments here, we note parenthetically that it extends to the game with exclusion commitment $\mathcal{E}$ with straightforward proof modifications if $\mathcal{E}$ is *path independent*, that is, for every state S that can be reached in the game and all $i \neq j \in S$, we have that $j \in (S \setminus \{i\}) \setminus \mathcal{E}(S \setminus \{i\})$ if and only if $i \in (S \setminus \{j\}) \setminus \mathcal{E}(S \setminus \{j\})$ (a key step in the argument for Lemma 2 concerns a deviation by the seller to a strategy that changes the order of trade for a pair of buyers). An example of an exclusion commitment that violates path independence for which Lemma 2 does not hold is given by $\mathcal{E}(\{1,2,3\}) = \{3\}, \mathcal{E}(\{1,3\}) = \{1,3\}, \mathcal{E}(\{2,3\}) = \{3\}$ in a setting with $n = 3, q = 2$. Under this commitment, buyer 3 is always excluded, and the seller can trade with buyer 2 after buyer 1, but not the other way around. This game has a family of MPEs in which buyer 1 gets limit payoff $a_1/2 + a_2/4$.

[16] This is not always the only optimal commitment. For instance, if the optimization problem defining $M^{*(n-1)}$ has a maximizer $l^* > 1$, then modifying the $(n-1)$-supply commitment to rule out paths of trade that exclude buyer 1 generates another optimal exclusion commitment $\mathcal{E}$ ($\mathcal{E}$ differs from $\mathcal{E}^{n-1}$ only in that $\mathcal{E}(\{1,i\}) = \{i\}$ for $i \neq 1$). To achieve the asymptotic bound $M^{*(n-1)}$ in the game with exclusion commitment $\mathcal{E}$, the seller can first trade with buyer 1 at a limit price of at least $a_1/2$, which is feasible by the extension of Lemma 2 to path independent exclusion commitments (such as $\mathcal{E}$) mentioned in footnote 15, and then reach a subgame in which $\mathcal{E}$ reduces to a $(n-2)$-supply commitment, in which we know from Theorem 1 that the seller can obtain an asymptotic payoff of $a_2/2 + \ldots + a_{l^*-1}/2 + a_{l^*+1} + \ldots + a_n$.

Therefore, maintaining a single unit of shortage at every stage allows the seller to extract all potential benefits of exclusion, and the seller does not benefit from exclusion commitments that treat buyers asymmetrically or create additional scarcity.

Theorem 2 implies that the (generically unique) maximizer l in the optimization problem defining $M^{*(n-1)}$ represents a cutoff for the buyers who are included with certainty in MPEs under the optimal exclusion commitment for high δ. By Corollary 1, these buyers must trade at fair prices in the limit $\delta \to 1$. The other buyers face the risk of exclusion in MPEs and may have to pay higher than fair prices (as discussed in the context of Corollary 1, some of these buyers can also trade at fair prices).

By definition, an exclusion commitment requires that at least one buyer does not trade. It is possible that the seller attains higher profits without excluding any buyer: formally, this corresponds to the game with supply $q = n$, in which the seller obtains limit profits $\sum_{i \in N} a_i/2$ by Corollary 2. Theorem 3 implies that the seller is better off with an optimal exclusion commitment whenever $M^{*(n-1)} > \sum_{i \in N} a_i/2$. Note that this is often the case. The condition $M^{*(n-1)} \leq \sum_{i \in N} a_i/2$ is equivalent to $a_l \geq a_{l+1} + \ldots + a_n$ for all $l \leq n-1$, which in turn implies that $a_l \geq 2a_{l+2}$ for all $l \leq n-2$. This requires extreme differences in valuations be maintained consistently through the sequence of buyers: if there exist three consecutive buyers whose valuations do not drop by half, optimal commitments would strictly dominate having no commitments.

Similarly, the condition $l^* \neq n$ invoked in Theorem 2 for the game with supply $n-1$ is likely to be satisfied: $l^* = n$ implies that $M^{*(n-1)} = \sum_{i \in N \setminus \{n\}} a_i/2 < \sum_{i \in N} a_i/2$. When $l^* = n$, buyers $1, \ldots, n-1$ are served with certainty in the game with $(n-1)$-supply commitment. In this case the seller would be better off in the game without exclusion, in which she trades with all buyers with certainty.

When bargaining with an optimal commitment dominates bargaining without commitment, the threat of exclusion enables the seller to extract higher payoffs by flexibly serving $n-1$ of the group of n buyers than she would by serving any subset of $n-1$ buyers with certainty, and indeed by serving *all* n buyers with certainty. It follows directly that one or more buyers must trade with positive probability at higher than fair prices.

We conclude this section with a general MPE existence result.

Proposition 2 (Existence)**.** *An MPE exists for the game with any exogenous supply and for the game with any exclusion commitment.*[17]

9. Optimal Exclusion in the Game with Supply $q < n$

Does a seller with supply $q < n$ benefit from making exclusion commitments stricter than her exogenous supply constraint? An exclusion commitment $\mathcal{E}$ is *more restrictive* than the

[17] For $q < n$, the game with supply q is identical to the game with q-supply commitment, so the only game with exogenous supply outside the class of games with exclusion commitments is the game with supply $q = n$.

q-supply commitment $\mathcal{E}^q$ if $\mathcal{E}(S) = S$ whenever $|S| = n - q$ and, furthermore, $\mathcal{E}(S) = S$ for some S with $|S| > n - q$. Again, the argument for Lemma 5 can be easily adapted to show that M^{*q} is an upper bound on limit profits the seller can obtain using any exclusion commitment that is more restrictive than $\mathcal{E}^q$. On the other hand, Theorem 1 shows that the seller's limit profit in the game with supply q is M^{*q}. It follows that in the setting with supply $q < n$, the seller does not benefit from making commitments to exclude buyers at any stage before all available q units are sold.[18] In particular, for any $\tilde{q} < q$, the $\tilde{q}$-supply exclusion commitment is detrimental to a seller with supply q (this follows directly from noting that $M^{\tilde{q}} < M^q$). This conclusion reiterates the intuition from the previous section that any existing scarcity that persists through the trading process ($q < n$) is sufficient to create all the competition between buyers needed to capture the gains delivered by the sequential outside option principle, and further exclusion does not benefit the seller.

This result does not hold in Ho and Lee's (2019) delegated-agent model of bargaining with threat of replacement. In that model, the seller announces a set of buyers ("network") she will "target." The network consists of the most valuable $\tilde{q} \leq q$ buyers. The seller then assigns a representative to each buyer in the announced network, and instructs each representative to bargain only with her assigned buyer and any buyer outside the network. Ho and Lee show that the announced network forms in equilibrium with limit probability 1 as $\delta \to 1$, and the seller's limit profit is $\sum_{i=1}^{\tilde{q}} \max(a_i/2, a_{\tilde{q}+1})$. This expression may be rewritten as

$$\max_{l \leq \tilde{q}+1} \left[\frac{a_1 + a_2 + \ldots + a_{l-1}}{2} + (\tilde{q} - l + 1) a_{\tilde{q}+1} \right].$$

Observe that

$$\max_{l \leq \tilde{q}+1} \left[\frac{a_1 + \ldots + a_{l-1}}{2} + (\tilde{q} - l + 1) a_{\tilde{q}+1} \right] \leq \max_{l \leq \tilde{q}+1} \left[\frac{a_1 + \ldots + a_{l-1}}{2} + a_{l+1} + \ldots + a_{\tilde{q}+1} \right] = M^{*\tilde{q}}.$$

The difference $a_{l+1} + \ldots + a_{\tilde{q}+1} - (\tilde{q} - l + 1) a_{\tilde{q}+1} \geq 0$ in the expressions being maximized in the two optimization problems above is due to the fact that under Ho and Lee's bargaining protocol, every representative relies on the outside option provided by the extra marginal buyer $q + 1$ when bargaining with her assigned buyer. In particular, if a representative exercises the outside option of trading with buyer $q+1$, her assigned buyer does not become available to the other representatives as a more valuable outside option. In other words, the protocol followed by the seller's representatives rules out the strategy underlying our sequential outside option principle. For a fixed $\tilde{q}$, the total profits the seller achieves in the setting of Ho and Lee are lower than $M^{*\tilde{q}}$ in general due to both the difference in the maximand for every $l \leq \tilde{q} - 1$ and the possibility that the two optimization problems have different maximizers l.

[18] Note, however, that there are exclusion commitments $\mathcal{E}$ more restrictive than $\mathcal{E}^q$ that generate the same limit profits as $\mathcal{E}^q$. This is the case, for instance, if $\mathcal{E}(\{1, 3\}) = \{3\}$ and $\mathcal{E}(S) = \mathcal{E}^q(S)$ for all other states S in the example from Section 3. If $q \leq n - 2$, this is also the case if $\mathcal{E}(S) = \mathcal{E}^q(S) \cup \{q + 2, \ldots, n\}$ for all S.

A seller with supply $q < n$ may benefit from reducing supply to some $\tilde{q} < q$ in the setting of Ho and Lee. As noted above, the resulting total profits in this case are smaller than or equal to $M^{*\tilde{q}}$. In our setting, the seller cannot benefit from restricting supply because $M^{*\tilde{q}} < M^{*q}$. For a concrete example, suppose that $q = 4$ in a market with $n = 5, a_1 = a_2 = a_3 = a_4 = 3, a_5 = 2$. Under the protocol of Ho and Lee, a commitment to supply only three of the four units increases seller profits from 8 to 9. In our model, the optimal exclusion commitment does not require a supply reduction and generates profits of 11 in this example.

10. Conclusion

This paper analyses bilateral bargaining between a seller and multiple buyers. Our analysis applies equally to a buyer negotiating with multiple sellers. The results are most interesting when the seller is unable to serve all buyers either because supply is limited or because the seller commits to excluding some potential buyers. There are numerous examples of a buyer negotiating with multiple sellers (and vice versa). In such situations, our results reveal that commitments to operate with fewer than the available number of suppliers (respectively, buyers) could be a highly effective bargaining tool. We quantify the resultant benefits.

Our main results characterize seller profits, buyer payoffs and trading probabilities under exogenous supply constraints. We also investigate optimal exclusion commitments in the absence of supply constraints. In the process, we formalize exclusion commitments in a general way. Our analysis uncovers some key bargaining theoretic principles for the environments considered. On the one hand, buyers cannot hold up the seller in the sense of paying less than fair prices. On the other hand, buyers who are included with certainty must trade at exactly fair prices. Our theory yields a novel sequential outside option principle that captures the role of scarcity in inducing competition between buyers when several successive transactions are possible. With sequential trade, the outside option changes dynamically, and in particular may become increasingly more attractive, enabling a seller who contracts with multiple buyers to extract more surplus than if she were to threaten buyers with a static outside option, as assumed in preceding research on exclusion. We show that in equilibrium the seller optimally chooses a top segment of buyers to include with certainty at fair prices, and exploits the others via the sequential outside option principle.

In many applications, there are externalities between buyers. A buyer's marginal value may depend on the set of buyers that the seller ultimately contracts with. In future research, we seek to address this generalization. We also hope to explore extensions to settings with multiple sellers and multiple buyers.

Appendix

Proof of Lemma 1. Let $u_i(S)$ denote the expected payoff of player $i \in S \cup \{0\}$ in subgame S, and $\pi_i(S)$ the probability with which the seller chooses to bargain with buyer $i \in S$ in state S under an MPE.

Note that in subgame S the seller can trade only with buyers in S. It follows that the total surplus created in subgame S is bounded above by $\sum_{i \in S} a_i$. As $u_0(S) \geq 0$, we have that

$$\sum_{i \in S} u_i(S) \leq \sum_{i \in S \cup \{0\}} u_i(S) \leq \sum_{i \in S} a_i. \tag{10}$$

Hence, there exists $i \in S$ such that $u_i(S) \leq a_i$.[19] Since the seller has the option to bargain with buyer i in the first period of subgame S and make an acceptable offer that leaves buyer i with utility arbitrarily close to $\delta u_i(S)$, but otherwise demand positive prices and refuse all offers in the future, we have that

$$u_0(S) \geq \frac{1}{2}(a_i - \delta u_i(S)) > 0.$$

As every buyer $i \in S$ will reject offers that yield utility smaller than $\delta u_i(S)$ in state S of the MPE, the payoff the seller receives when making an offer is bounded above by $\max_{i \in S}(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S))$. Standard arguments demonstrate that the seller expects a payoff of $\delta u_0(S)$ in the event the buyer chosen for bargaining is selected to be the proposer (regardless of whether the offer is accepted or rejected). Then, $u_0(S) > 0$ implies that $\max_{i \in S}(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)) > \delta u_0(S)$. As the seller can obtain a payoff arbitrarily close to $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)$ by making an acceptable offer to buyer i, it must be that $\pi_i(S) > 0$ only if i maximizes the expression $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)$. For such i, we know that $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S) > \delta u_0(S)$.

Optimality of MPE strategies requires that if $\pi_i(S) > 0$, and the seller is selected to be the proposer, then she makes an offer that yields utility $\delta u_i(S)$ for buyer i and utility $a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S) > \delta u_0(S)$ for the seller, and buyer i must accept the offer with probability 1 in equilibrium. Similarly, if buyer i is the proposer, he makes an offer that yields utility $\delta u_0(S)$ for the seller, and the seller accepts it with probability 1. The payoff equations follow.

Finally, we prove the statement regarding limit prices. When the seller makes an offer to buyer i in subgame S under σ^{δ_z}, the price is given by $a_i - \delta_z u_i(S)$, which converges to $a_i - \bar{u}_i(S)$ as $z \to \infty$. If instead buyer i is the proposer, then the price is given by $\delta_z u_0(S) - \delta_z u_0(S \setminus \{i\})$, which converges to $\bar{u}_0(S) - \bar{u}_0(S \setminus \{i\})$ as $z \to \infty$. If $\bar{\pi}_i(S) > 0$, then we have that

$$u_0(S) = \frac{1}{2}(a_i + \delta_z u_0(S \setminus \{i\}) - \delta_z u_i(S)) + \frac{1}{2}\delta_z u_0(S),$$

which leads to $a_i - \bar{u}_i(S) = \bar{u}_0(S) - \bar{u}_0(S \setminus \{i\})$ by taking the limit $z \to \infty$. Hence, the transaction between the seller and buyer i in state S takes place at the common limit price $a_i - \bar{u}_i(S)$ regardless of which of the two players is selected to be the proposer. □

[19]The only change necessary to extend this proof to the game with an exclusion commitment $\mathcal{E}$ involves replacing the set S with the set of buyers $S \setminus \mathcal{E}(S)$ who are still permitted to trade in state S in this sequence of arguments. By definition, under any exclusion commitment $\mathcal{E}$, only buyers in $S \setminus \mathcal{E}(S)$ can trade in subgame S.

Proof of Lemma 2. We establish the result for all games with supply q by induction on q. The base case $q = 0$ is trivial as all buyers receive zero payoffs in a degenerate game in which no trade is possible.

For the inductive step, consider the game with supply q, and fix a corresponding family of MPEs $(\sigma^\delta)_{\delta\in(0,1)}$. We use the notation and conventions from the proof of Lemma 1. It is sufficient to show that if u_i converges over a sequence of δ's going to 1, then its limit is at most $a_i/2$ for every buyer i. We can assume by passing to a subsequence $(\delta_z)_{z\geq 0} \to 1$ that all equilibrium variables $u_j, u_j(S), \pi_j, \pi_j(S)$ converge as $z \to \infty$ to limits denoted by $\bar{u}_j, \bar{u}_j(S), \bar{\pi}_j, \bar{\pi}_j(S)$. We need to prove that $\bar{u}_i \leq a_i/2$ for all $i \in N$.

Following an agreement with buyer k, players reach subgame $N\setminus\{k\}$—a game with supply $q-1$, in which the induction hypothesis applies. Hence, $\bar{u}_i(N \setminus \{k\}) \leq a_i/2$ for all $k \neq i$.

Fix a discount factor δ belonging to the sequence (δ_z) and a buyer $i \in N$ such that $\pi_i > 0$ under σ^δ. By Lemma 1, we have that

$$
\begin{aligned}
(11) \qquad u_0 &= \frac{1}{2}\left(a_i + \delta u_0(N \setminus \{i\}) - \delta u_i\right) + \frac{1}{2}\delta u_0 \\
(12) \qquad u_i &= \pi_i \left(\frac{1}{2}\left(a_i + \delta u_0(N \setminus \{i\}) - \delta u_0\right) + \frac{1}{2}\delta u_i\right) + \sum_{k\in N\setminus\{i\}} \pi_k \delta u_i(N \setminus \{k\}).
\end{aligned}
$$

Solving the pair of equations (11) and (12) with unknowns u_0 and u_i and reorganizing terms, we obtain formula (5) from Section 4 (when $\pi_i = 0$, this formula follows directly from (12) even though (11) is not valid in this case). The identities (6) and (7) from Section 4 will also be useful.

If $\bar{\pi}_i < 1$, then (7) leads to

$$\bar{u}_i = \sum_{k\in N\setminus\{i\}} \frac{\bar{\pi}_k}{1-\bar{\pi}_i} \bar{u}_i(N \setminus \{k\}) \leq \frac{a_i}{2}.$$

If $\bar{u}_0(N\setminus\{i\}) = 0$, then for any $\varepsilon > 0$, there exists $\underline{z}$ such that if $z \geq \underline{z}$, then $\delta_z u_0(N\setminus\{i\}) \leq 2\varepsilon$ and $\delta_z u_i(N \setminus \{k\}) \leq a_i/2 + \varepsilon$ for all $k \in N \setminus \{i\}$. Equations (5) and (6) then lead to $u_i \leq a_i/2 + \varepsilon$ for all $z \geq \underline{z}$. Hence, $\bar{u}_i \leq a_i/2$.

For the rest of the proof, assume that $\bar{\pi}_i = 1$ and $\bar{u}_0(N \setminus \{i\}) > 0$. The latter inequality implies that the seller trades with some buyer $k \in N \setminus \{i\}$ with positive limit probability in the second round after reaching the agreement with i under σ^{δ_z}. Hence, $q \geq 2$ and $\bar{\pi}_k(N \setminus \{i\}) > 0$.

Since $\bar{\pi}_i = 1 > 0$, taking the limit $z \to \infty$ for in equation (11) for $\delta = \delta_z$ we obtain

$$(13) \qquad \bar{u}_0 = a_i + \bar{u}_0(N \setminus \{i\}) - \bar{u}_i.$$

Similarly, $\bar{\pi}_k(N \setminus \{i\}) > 0$ implies that

$$(14) \qquad \bar{u}_0(N \setminus \{i\}) = a_k + \bar{u}_0(N \setminus \{i, k\}) - \bar{u}_k(N \setminus \{i\}).$$

As $\bar{\pi}_i = 1$, it must be that

$$\bar{u}_k = \bar{u}_k(N \setminus \{i\}). \tag{15}$$

Putting equalities (13)-(15) together, we obtain

$$\bar{u}_0 = a_i + a_k + \bar{u}_0(N \setminus \{i,k\}) - \bar{u}_i - \bar{u}_k. \tag{16}$$

Since the seller may bargain with buyer k in state N and with buyer i in state $N \setminus \{k\}$, we have that

$$\begin{aligned} \bar{u}_0 &\geq a_k + \bar{u}_0(N \setminus \{k\}) - \bar{u}_k \\ \bar{u}_0(N \setminus \{k\}) &\geq a_i + \bar{u}_0(N \setminus \{i,k\}) - \bar{u}_i(N \setminus \{k\}), \end{aligned}$$

and hence

$$\bar{u}_0 \geq a_i + a_k + \bar{u}_0(N \setminus \{i,k\}) - \bar{u}_i(N \setminus \{k\}) - \bar{u}_k. \tag{17}$$

Then, (16) and (17) imply that $\bar{u}_i \leq \bar{u}_i(N \setminus \{k\})$. Since $\bar{u}_i(N \setminus \{k\}) \leq a_i/2$, we conclude that $\bar{u}_i \leq a_i/2$. □

Proof of Lemma 3. We prove the claim for all games with supply q and any number of buyers $n > q$ by induction on q, with the base case $q = 0$ being trivial like in the proof of Lemma 2 (applying the inductive hypothesis requires a reindexing of the buyers in decreasing order of valuations in subgames). For the inductive step, consider a family of MPEs $(\sigma^\delta)_{\delta \in (0,1)}$ of a game with supply q for $q \geq 1$, and a buyer $i \geq q+1$. If buyer i's payoff under σ^δ does not converge to 0 for $\delta \to 1$, then there exists a sequence of discount factors going to 1 for which i's payoff converges to a positive limit. By passing to a subsequence, we can assume that the other equilibrium variables also converge. We use bar notation for the limits of these variables over the subsequence. We will establish that $\bar{u}_i = 0$, contradicting the hypothesis above.

For any $k \in N \setminus \{i\}$, buyer i's value is among the highest q in subgame $N \setminus \{k\}$. Since subgame $N \setminus \{k\}$ is a game with supply $q-1$, the induction hypothesis implies that

$$\bar{u}_i(N \setminus \{k\}) = 0, \forall k \in N \setminus \{i\}. \tag{18}$$

If $\bar{\pi}_i < 1$, then (7) implies that

$$\bar{u}_i = \sum_{k \in N \setminus \{i\}} \frac{\bar{\pi}_k}{1 - \bar{\pi}_i} \bar{u}_i(N \setminus \{k\}).$$

Using (18), we conclude that $\bar{u}_i = 0$.

Consider now the case $\bar{\pi}_i = 1$. Applying (7) for buyers $j \neq i$, we obtain $\bar{u}_j = \bar{u}_j(N \setminus \{i\})$.

If $q = 1$,[20] then $\bar{u}_0 = a_i - \bar{u}_i \leq a_i \leq a_2$. As $n \geq 2$, there exists $j \in \{1,2\} \setminus \{i\}$ for which $\bar{u}_j = u_j(N \setminus \{i\}) = 0$. Since the seller may deviate to trading with such a buyer j at a limit price of a_j, it follows that $\bar{u}_0 \geq a_j$. We conclude that $a_2 \leq a_j \leq \bar{u}_0 = a_i - \bar{u}_i \leq a_i \leq a_2$,

[20] The case $q = 1$ follows from Manea's (2018) Proposition 1. Here we provide a self-contained treatment.

which is possible only if all weak inequalities hold with equality. In particular, $a_i - \bar{u}_i = a_i$ leads to $\bar{u}_i = 0$, as claimed.

Now suppose that $q \geq 2$. Then the game with supply q does not end after the seller trades with buyer i in the first round. In subgame $N \setminus \{i\}$, there exists a fixed $j \neq i$ such that $\bar{\pi}_j(N \setminus \{i\}) > 0$. The conditions $\bar{\pi}_i > 0$ and $\bar{\pi}_j(N \setminus \{i\}) > 0$ along with Lemma 1 lead to

$$\bar{u}_0 = a_i - \bar{u}_i + \bar{u}_0(N \setminus \{i\}) = a_i - \bar{u}_i + a_j - \bar{u}_j(N \setminus \{i\}) + \bar{u}_0(N \setminus \{i, j\}).$$

As $\bar{u}_j = \bar{u}_j(N \setminus \{i\})$, we obtain

$$\bar{u}_0 = a_i - \bar{u}_i + a_j - \bar{u}_j + \bar{u}_0(N \setminus \{i, j\}). \tag{19}$$

The seller has the option to deviate and trade with buyer j first at a limit price of $a_j - \bar{u}_j$, and with i second at a price of $a_i - \bar{u}_i(N \setminus \{j\}) = a_i$ (by (18), we have that $\bar{u}_i(N \setminus \{j\}) = 0$). Optimality of the seller's strategy in the sequence of MPEs requires that this deviation does not generate a higher limit profit for the seller:

$$\bar{u}_0 \geq a_j - \bar{u}_j + a_i + \bar{u}_0(N \setminus \{i, j\}). \tag{20}$$

Formula (19) and inequality (20) imply that $\bar{u}_i \leq 0$, and hence $\bar{u}_i = 0$. □

Proof of Lemma 4. We prove the result by induction on q, with the base case $q = 0$ being trivial as all buyers trade with probability 0, not 1, in a degenerate game. Following an agreement with buyer k, players reach subgame $N \setminus \{k\}$—a game with supply $q - 1$, in which the induction hypothesis applies.

For the inductive step, consider a game with supply q and a corresponding family of MPEs $(\sigma^{\delta_z})_{z \geq 0}$ with $\lim_{z \to \infty} \delta_z = 1$ such that the seller trades with buyer i with probability 1 under σ^{δ_z} for all $z \geq 0$. Again, we use the notation from the proof of Lemma 1 for variables associated with this family of MPEs.

It is sufficient to prove that if u_i converges along a subsequence of $(\delta_z)_{z \geq 0}$, then its limit is at least $a_i/2$. We can assume by passing to a subsequence that all equilibrium variables $u_k, u_k(S), \pi_k, \pi_k(S)$ converge as $z \to \infty$ to limits denoted by $\bar{u}_k, \bar{u}_k(S), \bar{\pi}_k, \bar{\pi}_k(S)$, and furthermore that the set $K = \{k \in N | \pi_k > 0 \text{ under } \sigma^{\delta_z}\}$ is constant for all $z \geq 0$.[21] We need to show that $\bar{u}_i \geq a_i/2$.

Fix $\varepsilon > 0$. For $k \in K$, we have that $\pi_k > 0$, and the assumption that the seller trades with buyer i with probability 1 under σ^{δ_z} for all z implies that the seller trades with buyer i with probability 1 in subgame $N \setminus \{k\}$ under σ^{δ_z} for all z. The induction hypothesis then shows that $\bar{u}_i(N \setminus \{k\}) \geq a_i/2$ for all $k \in K \setminus \{i\}$. Hence, there exists $\underline{z}$ such that if $z \geq \underline{z}$,

[21] The sequence $((u_k(S), \pi_k(S))_{k,S}, K)_{z \geq 0}$ derived from the family of MPEs $(\sigma^{\delta_z})_{z \geq 0}$ is contained in a compact subset of an Euclidean space, so by the Bolzano-Weierstrass theorem it admits a convergent subsequence. Since K can take only a finite set of values, convergence on component K of the subsequence is equivalent to K being constant starting at some point in the subsequence.

then $\delta_z u_i(N \setminus \{k\}) \geq a_i/2 - \varepsilon$ for all $k \in K$. Given the definition of K, note that the range $N \setminus \{i\}$ can be replaced by $K \setminus \{i\}$ in the summations from equations (5) and (6). Then,

$$\frac{a_i + \delta u_0(N \setminus \{i\})}{2} \geq \frac{a_i}{2} \text{ and } \delta_z u_i(N \setminus \{k\}) \geq a_i/2 - \varepsilon, \forall k \in K \setminus \{i\},$$

imply that $u_i \geq a_i/2 - \varepsilon$ for all $z \geq \underline{z}$. As $\varepsilon > 0$ was chosen arbitrarily, it follows that $\bar{u}_i \geq a_i/2$, as asserted. □

Proof of Lemma 5. Fix a family of MPEs $(\sigma^\delta)_{\delta\in(0,1)}$ for the game with supply $q < n$. For every $\delta \in (0, 1)$, there exists at least one buyer with whom the seller trades with probability smaller than 1 under σ^δ; let $l(\sigma^\delta) \in N$ be the smallest index among buyers with this property. Clearly, $l(\sigma^\delta) \leq q + 1$.

It is sufficient to prove that if u_0 converges along a sequence $(\delta_z)_{z\geq 0}$ going to 1, then its limit does not exceed M^{*q}. We can assume by passing to a subsequence that all equilibrium variables $u_k, u_k(S), \pi_k, \pi_k(S)$ converge as $z \to \infty$ to limits denoted by $\bar{u}_k, \bar{u}_k(S), \bar{\pi}_k, \bar{\pi}_k(S)$. Since N is finite, the subsequence can be selected to additionally satisfy $l(\sigma^{\delta_z}) = i$ for a fixed $i \leq q + 1$ and all $z \geq 0$. We need to establish that $\bar{u}_0 \leq M^{*q}$.

By Lemma 1, for every $z \geq 0$, the MPE σ^{δ_z} generates a probability distribution over sequences of q buyers the seller selects for bargaining and ensuing agreements. As $l(\sigma^{\delta_z}) = i$, there exists one such sequence $\mathbb{S}$ that arises with positive probability under σ^{δ_z} and excludes buyer i. By passing to a subsequence of $(\delta_z)_{z\geq 0}$ if necessary, we can assume that $\mathbb{S}$ is the same for all z. Since trading over $\mathbb{S}$ is a best response for the seller under the MPE σ^{δ_z}, the seller's equilibrium payoff is equal to her expected payoff from selecting bargaining partners in the sequence $\mathbb{S}$.[22]

As $\mathbb{S}$ arises with positive probability under σ^{δ_z} and $l(\sigma^{\delta_z}) = i$, each buyer $j < i$ is guaranteed to trade under σ^{δ_z} in the subgame following agreements with his predecessors in the sequence $\mathbb{S}$. Lemma 4 implies that the expected discounted price in the agreement with buyer j along $\mathbb{S}$ converges to a limit less than or equal to $a_j/2$ as $z \to \infty$.

Clearly, the seller cannot extract a price greater than a_j from any buyer $j > i$ in the sequence $\mathbb{S}$. Since the seller does not trade with buyer i over $\mathbb{S}$, and there are q buyers in $\mathbb{S}$, we have that

$$\bar{u}_0 \leq \frac{a_1 + a_2 + \ldots + a_{i-1}}{2} + a_{i+1} + \ldots + a_{q+1} \leq M^{*q}.$$

□

[22]To better understand this claim, note that every Markov behavior strategy of the seller can be decomposed into two dimensions: mixing probabilities between buyers in every state at the beginning of a round, and proposal and acceptance decisions at every state within a round. In an MPE, the seller's strategy must be optimal against buyer strategies (and moves by nature), and hence the seller's decisions on the first dimension should also be optimal when we fix her play on the second dimension and the others' strategies. This implies that the seller should be indifferent between all sequences of buyers that occur in equilibrium (given the expected payoffs derived from bargaining with each buyer over each such sequence).

Proof modifications for the game with random matching. Suppose that in every state S, each buyer $i \in S$ is randomly matched to bargain with the seller with probability $p_i(S) > 0$. Let $u_i(S)$ denote the expected payoff of player $i \in S \cup \{0\}$ in subgame S, and $\pi_i(S)$ be the probability that the seller trades with buyer i in state S (conditional on reaching state S, but not conditional on buyer i being randomly matched with the seller in state S; thus, $\pi_i(S) \le p_i(S)$). As in the benchmark model, it is sufficient to consider families of MPEs for discount factors $\delta \to 1$ in which the variables $u_i(S)$ and $\pi_i(S)$ converge. It is useful to focus on subfamilies of MPEs with the additional property that the support of $\pi(S)$ is constant for every state S, so that for any fixed pair $i \in S$, either $\pi_i(S) > 0$ or $\pi_i(S) = 0$ uniformly in the subfamily. With random matching, the seller may be matched with a buyer with whom agreement is not incentive compatible, and this will cause trading delay. The analogue of the immediate agreement property from Lemma 1 in the model with random matching is that in every state, there is a buyer with whom the seller trades with probability 1 conditional on being matched: for every S, there exists $i \in S$ such that $\pi_i(S) = p_i(S)$. The payoff equations under random matching can be written as follows:

$$
\begin{aligned}
u_0(S) &= \sum_{k \in S} \pi_k(S) \left(\frac{1}{2}(a_k + \delta u_0(S \setminus \{k\}) - \delta u_k(S)) + \frac{1}{2}\delta u_0(S) \right) + \left(1 - \sum_{k \in S} \pi_k(S) \right) \delta u_0(S) \\
u_i(S) &= \pi_i(S) \left(\frac{1}{2}\left(a_i + \delta u_0(S \setminus \{i\}) - \delta u_0(S)\right) + \frac{1}{2}\delta u_i(S) \right) \\
&\quad + \sum_{k \in S \setminus \{i\}} \pi_k(S) \delta u_i(S \setminus \{k\}) + \left(1 - \sum_{k \in S} \pi_k(S) \right) \delta u_i(S).
\end{aligned}
$$

While the seller is no longer indifferent between trading with every buyer $i \in S$ for which $\pi_i(S) > 0$, optimality of the seller's strategy implies that in every state S the seller should be indifferent between all buyers in the support of $\pi(S)$ in the patient limit. For a family of MPEs in which $\pi_i(S) > 0$ and state variables converge to limits denoted by a bar, this means that

$$\bar{u}_0(S) = a_i - \bar{u}_i(S) + \bar{u}_0(S \setminus \{i\}).$$

As in the case of the game with strategic choice of bargaining partner, in state S buyer i trades at an asymptotic price of $a_i - \bar{u}_i(S)$ regardless of whether he wins the coin toss to propose when getting matched. Taking the limit $\delta \to 1$ in buyer i's payoff equation for the initial state N, the asymptotic indifference property leads to the following counterpart to (7):

$$(21) \qquad \sum_{j \in N \setminus \{i\}} \bar{\pi}_j > 0 \implies \bar{u}_i = \sum_{k \in N \setminus \{i\}} \frac{\bar{\pi}_k}{\sum_{j \in N \setminus \{i\}} \bar{\pi}_j} \bar{u}_i(N \setminus \{k\}).$$

This condition plays a key role in extending the proofs of Lemmata 2-6 to the model with random matching.

Formulae (5) and (6) rely on the seller's exact indifference when mixing between buyers and do no have immediate analogues in the setting with random matching. The use of these formulae in the treatment of the case $\bar{u}_0(N \setminus \{i\}) = 0$ in the proof of Lemma 2 can be circumvented by noting that $\bar{u}_0(N \setminus \{i\}) = 0$ implies that $q = 1$. The game with random matching for $q = 1$ can be analyzed separately to argue that $\bar{u}_i \leq a_i/2$.

The proof of Lemma 4 relies more extensively on (5) and (6). We can deal with the case $\sum_{j\in N\setminus\{i\}} \bar{\pi}_j > 0$ via (21). Consider now the case $\sum_{j\in N\setminus\{i\}} \bar{\pi}_j = 0$. It must be that for high enough δ, we have that $\pi_i = p_i(N)$ and $\pi_j < p_j(N)$ for $j \neq i$. It follows that

$$a_i + \delta u_0(N \setminus \{i\}) - \delta u_i \geq \delta u_0 \geq a_j + \delta u_0(N \setminus \{j\}) - \delta u_j, \forall j \neq i.$$

Then, the seller's payoff equation leads to

$$u_0 \leq \sum_{k\in N} \pi_k \left(\frac{1}{2}(a_i + \delta u_0(N \setminus \{i\}) - \delta u_i) + \frac{1}{2}\delta u_0 \right) + \left(1 - \sum_{k\in N} \pi_k \right) \delta u_0.$$

This leads to an upper bound for u_0 that depends on u_i, which can be substituted in buyer i's payoff equation to obtain a lower bound on u_i similar to the right hand-side of (5):

$$u_i \geq \frac{2\pi_i(1-\delta)}{(1-\delta+\delta\sum_{j\in N}\pi_j)(2-2\delta+\delta\sum_{j\in N\setminus\{i\}}\pi_j)} \times \frac{a_i + \delta u_0(N \setminus \{i\})}{2} + \sum_{k\in N\setminus\{i\}} \frac{\pi_k(2-2\delta+\delta\sum_{j\in N}\pi_j)}{(1-\delta+\delta\sum_{j\in N}\pi_j)(2-2\delta+\delta\sum_{j\in N\setminus\{i\}}\pi_j)} \times \delta u_i(N \setminus \{k\}).$$

The sum of the coefficients in the equation above simplifies to

$$\frac{\sum_{j\in N}\pi_j}{1-\delta+\delta\sum_{j\in N}\pi_j},$$

which converges to 1 as $\delta \to 1$ (both the numerator and the denominator converge to $\bar{\pi}_i > 0$).[23] This makes it possible to proceed with the inductive proof of Lemma 4 as in the benchmark model.

For the game with random matching, the crucial step identifying the sequence of buyers $\mathbb{S}$ in Lemma 5 does not rely on exact indifference for the seller, but instead uses the seller's asymptotic indifference. We can construct a sequence over which trade occurs with positive probability (this can be defined based solely on the support of every $\pi(S)$, which is constant in the subfamily of MPEs under consideration)—and hence generates the seller's asymptotic MPE payoff—which excludes buyer l and includes buyers $1, \ldots, l-1$. This allows us to extend Theorem 1 to the model with random matching. □

[23]This expression can be interpreted as the present value of a prize of 1 received at a stochastic time in an environment where the probability of getting the prize at a given date conditional on not having received it earlier is $\sum_{j\in N}\pi_j$, which reflects the fact that the first trade takes place with probability $\sum_{j\in N}\pi_j$ in the game with random matching.

Theorem 2 (General version). *Let $\underline{l}^*$ and $\bar{l}^*$ be the smallest and the largest indices l that achieve the maximum in* (8)*, respectively, and let $l(\sigma)$ denote the lowest index of a buyer with whom the seller trades with probability less than 1 under strategy profile σ. There exists $\underline{\delta} < 1$ such that every MPE σ of the bargaining game with supply q for any discount factor $\delta > \underline{\delta}$ satisfies the following conditions and leads to the following outcomes:*

- *$l(\sigma)$ is a maximizer in the optimization problem* (8)*.*
- *If $i < \underline{l}^*$, then the seller trades with buyer i with exact probability 1.*
- *If $i \geq \bar{l}^*$, then the seller trades with buyer i with probability smaller than 1.*
- *If $a_i < a_{q+1}$, then the seller trades with buyer i with probability 0.*

Furthermore, every family of MPEs $(\sigma^\delta)_{\delta\in(0,1)}$ of the game with supply q for discount factors $\delta \in (0,1)$ has the following asymptotic properties for $\delta \to 1$:

- *If $i < \bar{l}^*$, then the probability that the seller trades with buyer i converges to 1, and the expected payoff of buyer i under σ^δ converges to $a_i/2$.*
- *If $\bar{l}^* \neq q+1$ and $a_{\bar{l}^*} > a_{\bar{l}^*+1}$, then the probability that the seller trades with buyer $\bar{l}^*$ also converges to 1.*
- *If $\bar{l}^* = q+1$, then the seller trades with buyers $1, \ldots, q$ with probability converging to 1.*

Proof of general version of Theorem 2. We prove the first part of the result by contradiction. If the claim is not true, then there exist a sequence of discount factors $\delta_z \to 1$ and associated equilibria σ^{δ_z} such that $l(\sigma^{\delta_z})$ is not a maximizer in the optimization problem (8). Moreover, the sequence may be selected such that $l(\sigma^{\delta_z}) = j$ for some fixed j and all $z \geq 0$. Then, the argument from Lemma 5 shows that

$$\bar{u}_0 \leq \frac{a_1 + a_2 + \ldots + a_{j-1}}{2} + a_{j+1} + \ldots + a_n \leq M^{*q}.$$

Since $\bar{u}_0 = M^{*q}$ by Theorem 3, it follows that j achieves the maximum M^{*q} in the optimization problem (8), contradicting the assumption $l(\sigma^{\delta_z}) = j$ is not a maximizer in (8).

The second part of the result follows from the first. Since $l(\sigma)$ is a maximizer in (8) for every MPE σ when $\delta > \underline{\delta}$, the definition of $\underline{l}^*$ implies that $\underline{l}^* \leq l(\sigma)$.

The proof of the third part proceeds by contradiction similarly to the first part. If the claim is not true, then there exists a buyer $i \geq \bar{l}^*$ and a sequence of discount factors $(\delta_z)_{z\geq 0}$ such that the seller trades with buyer i with probability 1 under σ^{δ_z} for all $z \geq 0$. As above, $(\delta_z)_{z\geq 0}$ can be selected so that $l(\sigma^{\delta_z}) = j$ for a fixed j and all z. Since i trades with probability 1 under σ^{δ_z}, we have that $j \neq i$, and hence $i > j$. Moreover, each buyer in the set $K = \{1, 2, \ldots, j-1, i\}$ trades with probability 1 under σ^{δ_z} for all z.

Following steps analogous to the proof of Lemma 5, the seller's payoff under σ^{δ_z} is equal to her expected payoff from selecting bargaining partners in a fixed sequence that excludes buyer j and includes each buyer $k \in K$ at a limit (discounted) price of at most $a_k/2$. This means that the seller obtains at most fair prices from buyers $1, 2, \ldots, j-1$ and can extract at most full surplus from a set of $q - j + 1$ buyers different from buyer j, with strictly less than full surplus extraction from buyer i. We conclude that

$$\bar{u}_0 < \frac{a_1 + \ldots + a_{j-1}}{2} + a_{j+1} + \ldots + a_{q+1} \leq M^{*q},$$

which contradicts Theorem 3.

For the fourth part, we argue by induction on q that for all $q \geq 0$, in every MPE of the game with supply q for high enough δ, any buyer i for which $a_i < a_{q+1}$ trades with probability 0 (applying the inductive hypothesis requires a reindexing of buyers in subgames as in Lemma 3). The base case $q = 0$ is trivial.

To prove the inductive step, assume that $q \geq 1$, and fix a buyer i for which $a_i < a_{q+1}$. Suppose that there exists a sequence of MPEs of the game with supply q for discount factors $(\delta_z)_{z \geq 0}$ converging to 1 along which the seller trades with buyer i with positive probability in the first period of the game. By passing to a subsequence of $(\delta_z)_{z \geq 0}$ along which all relevant MPE variables converge, we have that

$$\bar{u}_0 \leq a_i + \bar{u}_0(N \setminus \{i\}).$$

Since the seller has the option to first trade with buyer $q+1$ at a limit price of a_{q+1} by Lemma 3, the optimality of her equilibrium strategy implies that

$$\bar{u}_0 \geq a_{q+1} + \bar{u}_0(N \setminus \{q+1\}).$$

Note that both subgames $N \setminus \{i\}$ and $N \setminus \{q+1\}$ have supply $q-1$. When applied to each subgame, Theorem 1 implies that the seller's limit payoff depends only on the top q buyer values. Since $i > q+1$, buyers $k \leq q$ have the top q valuations in either subgame, and hence $\bar{u}_0(N \setminus \{i\}) = \bar{u}_0(N \setminus \{q+1\})$. However, $a_{q+1} > a_i$ generates a contradiction with the inequalities above. This argument establishes that for sufficiently high δ, the seller does not trade with buyer i in the first period of any MPE.

As buyer i does not have one of the top q values in subgame $N \setminus \{j\}$ for any $j \neq i$, the induction hypothesis implies that in all MPEs for high enough δ, the seller should trade with buyer i with probability 0 in every such subgame. Therefore, the seller trades with buyer i with probability 0 in any MPE for high enough δ.

For the second half of the result, fix a family of MPEs $(\sigma^\delta)_{\delta \in (0,1)}$ of the game with supply q for discount factors $\delta \in (0, 1)$.

We first prove the claim regarding payoffs in the first statement. For an argument by contradiction, assume that the expected payoff of buyer $i < \bar{l}^*$ does not converge to $a_i/2$ as $\delta \to 1$. Consider a sequence of discount factors $\delta_z \to 1$ such that buyer i's payoff under σ^{δ_z} converges to a different limit $\bar{u}_i$. By Lemma 2, $\bar{u}_i \leq a_i/2$, so it must be that $\bar{u}_i < a_i/2$. As $z \to \infty$, the seller can deviate from $\sigma_0^{\delta_z}$ to successively trade with buyer i at a limit price of $a_i - \bar{u}_i$, then with each buyer $j = q+1, q, \ldots, \bar{l}^* + 1$ at limit price a_j by Lemma 3, and then with each buyer $j = 1, \ldots, \bar{l}^* - 1$ different from i at a limit price of at least $a_j/2$ by Lemma

2. This deviation delivers the following lower bound on the seller's limit profit:

$$\bar{u}_0 \geq a_i - \bar{u}_i + a_{\bar{l}^*+1} + \ldots + a_{q+1} + \sum_{j=1, j\neq i}^{\bar{l}^*-1} \frac{a_j}{2} > \frac{a_1 + a_2 + \ldots + a_{\bar{l}^*-1}}{2} + a_{\bar{l}^*+1} + \ldots + a_{q+1} = M^{*q},$$

where the strict inequality is a consequence of $a_i - \bar{u}_i > a_i/2$, and the equality follows from the definition of $\bar{l}^*$. Thus, $\bar{u}_0 > M^{*q}$, contradicting Theorem 3.

We established that $\bar{u}_i = a_i/2$ for every buyer $i < \bar{l}^*$. Lemma 2 then implies that every such buyer trades with limit probability 1 under σ^δ as $\delta \to 1$.

We prove the next part also by contradiction. Assume that $\bar{l}^* \neq q+1$ and $a_{\bar{l}^*} > a_{\bar{l}^*+1}$, and suppose that the probability that the seller trades with buyer $\bar{l}^*$ under σ^δ does not converge to 1 for $\delta \to 1$. Then, there exists a sequence of discount factors $\delta_z \to 1$ such that the probability that the seller trades with buyer $\bar{l}^*$ under σ^{δ_z} converges to a limit less than 1, and MPE variables converge. It follows that there exists a path over which the seller trades under σ^{δ_z} with limit probability greater than 0 as $z \to \infty$ with a sequence of buyers $(i_1, \ldots, i_q)$ that does not include $\bar{l}^*$.

As argued above, each buyer $i < \bar{l}^*$ obtains a limit payoff of $a_i/2$ under σ^{δ_z} as $z \to \infty$. Since Lemma 2 implies that buyer i pays a limit price of at least $a_i/2$ in every state he trades with the seller, buyer i should pay a price that converges to $a_i/2$ in every subgame that arises with positive limit probability under σ^{δ_z} as $z \to \infty$.

Let k be the largest index such that $i_k > \bar{l}^*$. Note that k is well defined given the assumption that $\bar{l}^* \neq q+1$. Consider the subgame $S := N \setminus \{i_1, \ldots i_{k-1}\}$, which has supply $q-k+1$. It must be that $\bar{\pi}_{i_k}(S) > 0$.

Define $J = S \setminus \{\bar{l}^*, i_k\}$. For $j \in J$, we have that $j < \bar{l}^*$, so the seller obtains a limit price of exactly $a_j/2$ when trading with buyer j as argued above. The seller can extract a price of at most a_{i_k} from buyer i_k, so her limit profit in subgame S does not exceed

$$M(S) := \frac{a_{i_{k+1}} + \ldots + a_{i_q}}{2} + a_{i_k}.$$

Applying Theorem 1 to subgame S, we get that $\bar{u}_0(S) \geq M(S)$. Hence, $\bar{u}_0(S) = M(S)$.

Since $\bar{\pi}_{i_k}(S) > 0$, we have that $\bar{\pi}_{\bar{l}^*}(S) < 1$, and a version of formula (7) leads to

$$\bar{u}_{\bar{l}^*}(S) = \sum_{j \in S \setminus \{\bar{l}^*\}} \frac{\bar{\pi}_j(S)}{1 - \bar{\pi}_{\bar{l}^*}(S)} \bar{u}_{\bar{l}^*}(S \setminus \{j\}). \tag{22}$$

Subgame $S \setminus \{i_k\}$ has supply $q-k$, and contains $q-k$ buyers $i_{k+1}, \ldots, i_q > \bar{l}^*$. Lemma 3 implies that $\bar{u}_{\bar{l}^*}(S \setminus \{i_k\}) = 0$.

Consider now any $j \in J$ with $\bar{\pi}_j(S) > 0$, so that subgame $S \setminus \{j\}$ is reached with positive limit probability under σ^{δ_z} as $z \to \infty$. As argued above, the seller trades with buyer j at limit price $a_j/2$ in subgame S, and has to trade with every other buyer $j' \in J \setminus \{j\}$ with limit probability 1 at limit price $a_{j'}/2$ in subgame $S \setminus \{j\}$. Hence,

$$\bar{u}_0(S \setminus \{j\}) = M(S) - a_j/2 \quad \& \quad \bar{u}_{j'}(S \setminus \{j\}) = a_{j'}/2, \forall j' \in J \setminus \{j\}.$$

Since the maximum total surplus achievable in subgame $S \setminus \{j\}$ is $\sum_{j' \in J \setminus \{j\}} a_{j'} + a_{\bar{l}^*}$, it follows that

$$\bar{u}_{\bar{l}^*}(S \setminus \{j\}) \leq \sum_{j' \in J \setminus \{j\}} a_{j'} + a_{\bar{l}^*} - \bar{u}_0(S \setminus \{j\}) - \sum_{j' \in J \setminus \{j\}} \bar{u}_{j'}(S \setminus \{j\}) = a_{\bar{l}^*} - a_{i_k}.$$

As $\bar{\pi}_{i_k}(S) > 0$, $\bar{u}_{\bar{l}^*}(S \setminus \{i_k\}) = 0, \bar{u}_{\bar{l}^*}(S \setminus \{j\}) \leq a_{\bar{l}^*} - a_{i_k}$ for all $j \in J$ with $\bar{\pi}_j(S) > 0$, and $a_{\bar{l}^*} - a_{i_k} \geq a_{\bar{l}^*} - a_{\bar{l}^*+1} > 0$, equation (22) implies that

$$\bar{u}_{\bar{l}^*}(S) < a_{\bar{l}^*} - a_{i_k}.$$

However, in subgame S, the seller can deviate from $\sigma_0^{\delta_z}$ to bargain with $\bar{l}^*$ and trade at limit price $a_{\bar{l}^*} - \bar{u}_{\bar{l}^*}(S) > a_{i_k}$, and then bargain with each buyer $j \in J$ and trade at a limit price of at least $a_j/2$ by Lemma 2 for $z \to \infty$. This deviation generates a limit profit greater than $M(S)$ for the seller, contradicting $\bar{u}_0(S) = M(S)$.

The last bullet point follows from the fact that when $\bar{l}^* = q + 1$, the seller trades with each of the q buyers $i < \bar{l}^*$ with limit probability 1. If $\bar{l}^* = \underline{l}^* = q + 1$, then the seller must trade with buyers $1, \ldots, q$ with exact probability 1 for sufficiently high δ. □

Proof of Proposition 2. We establish the existence of an MPE for the bargaining game with exclusion commitment. The proof for the bargaining game with exogenous supply is analogous.

Consider the game with an exclusion commitment $\mathcal{E}$. It will be convenient to use the notation $\mathcal{I}(S) := S \setminus \mathcal{E}(S)$. We inductively construct MPE expected payoffs and bargaining probabilities for all players working backward from terminal states. Let m be the maximum number of trades possible under $\mathcal{E}$. In every subgame in which the seller has traded with exactly m buyers (terminal nodes), the payoffs of all players are 0. Assuming that we specified MPE strategies for subgames in which the seller has traded with at least $m' + 1$ buyers, we next construct MPE expected payoffs and bargaining probabilities for subgames in which the seller has traded with exactly m' buyers. Consider such a subgame S. We will argue that the constructed payoffs satisfy

$$u_i(S) \geq 0, \forall i \in \mathcal{I}(S) \cup \{0\} \tag{23}$$

$$\sum_{i \in \mathcal{I}(S) \cup \{0\}} u_i(S) \leq \sum_{i \in \mathcal{I}(S)} a_i. \tag{24}$$

Consider a candidate payoff profile $(u_i(S))_{i \in \mathcal{I}(S)}$ for the "active" buyers in state S contained in the simplex

$$\mathcal{U} = \{(u_i(S))_{i \in \mathcal{I}(S)} | u_i(S) \geq 0, \forall i \in \mathcal{I}(S); \sum_{i \in \mathcal{I}(S)} u_i(S) \leq \sum_{i \in \mathcal{I}(S)} a_i\}.$$

We construct a correspondence $F : \mathcal{U} \rightrightarrows \mathcal{U}$ as follows. For every $(u_i(S))_{i \in \mathcal{I}(S)} \in \mathcal{U}$, let $u_0'(S)$ be the payoff the seller can attain by making acceptable offers to optimally selected buyers and $\Pi(S) \subseteq \Delta(\mathcal{I}(S))$ the set of optimal bargaining probabilities for the seller in state S:

$$u_0'(S) = \frac{1}{2-\delta} \max_{i\in\mathcal{I}(S)} (a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)) \tag{25}$$

$$\Pi(S) = \Delta\big(\arg\max_{i\in\mathcal{I}(S)}(a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S))\big). \tag{26}$$

The correspondence F maps $(u_i(S))_{i\in\mathcal{I}(S)}$ to the set of profiles $(u_i'(S))_{i\in\mathcal{I}(S)}$ given by

$$u_i'(S) = \pi_i(S)\left(\frac{1}{2}(a_i + \delta u_0(S\setminus\{i\}) - \delta u_0'(S)) + \frac{1}{2}\delta u_i(S)\right) + \sum_{k\in\mathcal{I}(S)\setminus\{i\}} \pi_k(S)\delta u_i(S\setminus\{k\}) \tag{27}$$

for any selection of bargaining probabilities $\pi(S) \in \Pi(S)$.

F is convex-valued because $\Pi(S)$ is a convex set for every element of $\mathcal{U}$.

We next argue that the range of F is indeed included in $\mathcal{U}$. For any $(u_i(S))_{i\in\mathcal{I}(S)} \in \mathcal{U}$, there exists $i \in \mathcal{I}(S)$ such that $a_i > \delta u_i(S)$. Otherwise, $\sum_{i\in\mathcal{I}(S)} u_i(S) \geq (\sum_{i\in\mathcal{I}(S)} a_i)/\delta > \sum_{i\in\mathcal{I}(S)} a_i$. It follows that there exists $i \in \mathcal{I}(S)$ such that $a_i + \delta u_0(S\setminus\{i\}) - \delta u_i(S) > 0$, and hence $u_0'(S) > 0$.

Then, for any $\pi(S) \in \Pi(S)$, the condition $\pi_i(S) > 0$ implies that $u_0'(S) < a_i + \delta u_0(S \setminus \{i\}) - \delta u_i(S)$, which leads to $\delta u_0'(S) < a_i + \delta u_0(S\setminus\{i\}) - \delta u_i(S)$. Therefore, $a_i + \delta u_0(S \setminus \{i\}) - \delta u_0'(S) > \delta u_i(S) \geq 0$. Since all other terms appearing on the right-hand side of (27) are non-negative, we conclude that $u_i'(S) \geq 0$ for all $i \in \mathcal{I}(S)$.

We are left to show that $\sum_{i\in\mathcal{I}(S)} u_i'(S) \leq \sum_{i\in\mathcal{I}(S)} a_i$. Given conditions (25) and (26), $u_0'(S)$ solves the following equation for any $\pi(S) \in \Pi(S)$:

$$u_0'(S) = \sum_{i\in\mathcal{I}(S)} \pi_i(S)\left(\frac{1}{2}(a_i + \delta u_0(S\setminus\{i\}) - \delta u_i(S)) + \frac{1}{2}\delta u_0'(S)\right). \tag{28}$$

Summing up equations (27) over all $i \in \mathcal{I}(S)$ and equation (28), we obtain

$$\begin{aligned}\sum_{i\in\mathcal{I}(S)\cup\{0\}} u_i'(S) = \sum_{i\in\mathcal{I}(S)} \pi_i(S)(a_i + \delta \sum_{k\in(\mathcal{I}(S)\setminus\{i\})\cup\{0\}} u_k(S\setminus\{i\}))\\ \leq \max_{i\in\mathcal{I}(S)}\left(a_i + \delta\sum_{k\in\mathcal{I}(S\setminus\{i\})} a_k\right) \leq \sum_{i\in\mathcal{I}(S)} a_i.\end{aligned}$$

The first inequality follows from the fact that condition (24) holds for subgame $S \setminus \{i\}$ (formally, we set $u_k(S \setminus \{i\}) = 0$ for $k \in \mathcal{E}(S\setminus\{i\})$), and the second from the requirement that $\mathcal{E}$ satisfies $\mathcal{E}(S) \subseteq \mathcal{E}(S\setminus\{i\})$ for $i \in S\setminus\mathcal{E}(S) = \mathcal{I}(S)$, and hence $\mathcal{I}(S\setminus\{i\}) \subseteq \mathcal{I}(S)\setminus\{i\}$.

Since $u_0'(S)$ varies continuously with $(u_i(S))_{i\in\mathcal{I}(S)}$, and $\Pi(S)$ has closed graph as a correspondence defined on $\mathcal{U}$, it follows that F has closed graph. Kakutani's fixed-point theorem then implies that F has a fixed point $(u_i(S))_{i\in\mathcal{I}(S)}$. We then define $u_0(S)$ to be the corresponding $u_0'(S)$ and recover the probabilities $(\pi_i(S))_{i\in\mathcal{I}(S)}$ associated with the fixed point.

We can now construct an MPE. In state S, the seller chooses to bargain with buyer i with probability $\pi_i(S)$. When the seller bargains with buyer i, if the seller is selected to be the proposer, she offers an acceptable price that gives the buyer utility $\delta u_i(S)$, and similarly the buyer makes an acceptable offer that gives the seller utility $\delta u_0(S)$. A simple inductive argument combined with the payoff equations above proves that the constructed strategies generate the expected payoffs given by u. By the single-deviation principle, the specification of bargaining probabilities and offers in state S, in conjunction with the assumed behavior in subgames following a trade in state S, induces an MPE. □

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