Incentivizing Team Production by Indivisible Prizes

Electoral Competition under Proportional Representation*

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Abstract

This paper formulates proportional representation in a parliamental election as a multi-prize contest among political parties. In particular, we analyze the performance of commonly-used list rule, and investigate what the optimal list rule is when candidates differ in their abilities to contribute. We show that, in order to maximize the aggregated effort exerted by the party candidates, each party should assign the highest ability candidates to the middle of the list, while the top priority ranks and low priority ranks should be assigned to lower ability candidates under the optimal list rule. Turning to the optimal mechanism, we can show that the optimal list rule is indeed the optimal monotonic rule when individual effort cost function is not too convex and the complemantarities of individual efforts are not too strong. We also consider a situation in which some of the party members get extra benefits by the party's winning the majority of the parliament seats, and show that the party leader may place the highest ability group at the top of the list so that both the highest and the middle-level ability party members to exert the maximum efforts as the whole.

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1 Introduction

There are countries of which parliament seats are allocated proportionally by the number of votes each party collected. Each party announces a list of candidates with a priority order (a list rule), who compete as a team with other party candidates. Obvious questions that come up in our mind are: Is a list rule a desirable way to allocate prizes to the team members' efforts to collect votes for the party? If so, what is the best way for a party to order heterogeneous ability candidates in its list rule?

Crutzen and Sahuguet (2017) and Crutzen, Flamand, and Sahuguet (2017) are the first to analyze the incentive structure of the list rule in a parliamental election, or more generally in a contest between teams that compete for multiple indivisible prizes. They set up a multi-prize contest model with a CES team-effort aggregator function, and compare different electoral systems and different intra-team prize allocation rules. Their main analyses are on ex ante symmetric two party case with homogeneous candidates. In this paper, we extend this basic model, allowing for heterogeneous abilities of candidates in order to see the performance of the list rule and how a party leader should allocate heterogenous candidates on the priority list. More generally, this paper analyzes competition by the parties by employing a party-optimal rule that maximize the party's winning probability given other parties' effort levels.

Our basic model is to analyze each party's two stage decision problem: in stage one, a party decides which allocation rule to use, then in stage two, each party member (candidate) chooses his/her effort level to enhance the party's popularity, which improves the expected number of seats the party can win in the parliament. The equilibrium effort levels by the party members are characterized by the first-order conditions, and the second-order conditions are analyzed. Interestingly, with a CES effort aggregator function, the allocation characterized by the first-order best-response conditions is unique, but the second-order conditions for the allocation are not necessarily satisfied for an arbitrary seat assignment rule the party may adopt. However, we later show that the second-order conditions are likely to be satisfied under the *optimal* list rule.

We say a list rule the optimal list rule if and only if the expected seat maximizing rule among the family of list rules. We show that under the optimal list rule, the highest ability candidate is placed in the middle of the list whose probability of going to the parliament is more or less fifty-and-fifty. Then, the second and third highest ability candidates are placed before or after the highest ability candidate in a single-peaked manner. The lowest ability candidates are assigned to the top or the bottom of the optimal list rule which maximizes the aggregate effort exerted by all candidates. This is because the top of the list candidate does not have a strong incentive to make effort, since he/she will be able to go to the parliament without much effort, so it is not a good idea for the party leader to place the high ability candidate in such a position.

We also analyze the party-optimal rule without restricting the family of assignment rules to list rules. This analysis provides a good understanding about the properties of the optimal list rule. We show that the optimal rule is deterministic when individual effort cost function is not too convex and the complementarities of individual efforts are not too strong: i.e., when any k seats are won by a party, then the party assigns the seats to certain k candidates. We provide an algorithm to calculate the optimal mechanism in this case (weaker complementarity and not too convex cost function). The optimal rule assigns the highest ability candidate to about probability one half to get a parliament seat. This is to encourage highest ability person to exert more effort for the party. Note that the optimal (effort-maximizing) rule is not necessarily monotonic— even if a candidate could go to the parliament in the case that the party gets k seats, it does not mean that the same candidate can go to the parliament in the case of $k+1$ seats are won by the party. If the expected number of winning seats exceeds one, the lowest ability candidate may go to the parliament when only one seat is won in the realized state, exerting no effort. Then, we impose a natural monotonicity requirement on the mechanism (if a candidate goes to the parliament when k seats are won, then she will go to the parliament when more than k seats are won). This monotonicity requirement assures that all candidates exert effort to their parties. We show that the optimal monotonic rule is the optimal list rule when individual effort cost function is not too convex and the complementarities of individual efforts are not too strong. That is, in such a situation, the particular list rule has a very solid support from theoretical point of view. In contrast, when individual cost function is strongly convex or the complementarities of individual efforts are strong, the probabilities of candidates' going to the parliament are ranked by their abilities under the optimal assignment rule, and a deterministic rule is no longer optimal.

The readers may wonder that in many party list in proportional representation elections, well-known and powerful candidates are listed high on the lists. We may say that for those core party members, there may be additional incentives to make efforts other than just going to the parliament. For example, if a party has a good/reasonable chance to get power by winning the majority of the seats, then some top group party members would see a chance to become a member of the cabinet. Then, these member may work hard to win the majority of the parliament seats in addition to win a seat in parliament for him/her. Indeed, if there are such additional incentives for some candidates, it would make sense to bring up the highest ability candidates to the top of the list. The top group members make effort for the party to win the majority, and the middle group members do their best to secure their seats in the parliament. This can be the way to maximize the (expected) number of seats for the party.

For the above analysis, we need a mild regularity condition, requiring that an increase in a party's winning probability of each seat increases the party's aggregate effort. This is an intuitively reasonable condition, and it appears to be satisfied, unless the party leader use an assignment rule that is far from the optimal rule (say, the highest ability candidates are assigned to the highest ranks).

Finally, we consider J party case in which each party tries to choose the optimal assignment rule in the first stage simultaneously, foreseeing an effort contribution equilibrium by individual candidates across parties in the second stage. This extension is more complicated than the above analysis, since the above analysis assumes that a party assumes that other parties' effort levels are fixed. That is, the analysis is best response analysis. In contrast, if all parties choose their effort levels, then the problem becomes an equilibrium analysis, which is significantly more complicated. It turns out that we can replicate the same results in this general case, but under a stronger condition. We need to strengthen the regularity condition imposed above—the winning-probability elasticity of the aggregate effort needs to be more than just positive: we need it to be between zero and unity for all parties. This condition can be regarded as a stability condition, since if the elasticity exceeds unity then an increase in the party's winning probability enhances its aggregate effort even more, resulting in further increase in its winning probability. By strengthening the regularity condition, all the results previously obtained goes through.

To be completed.

1.1 Related Literature

(*)**To be completed.** Include references in the above two papers.

2 The Model

In the basic model, we analyze a party j's efforts to increase the expected number of seats in the parliament. There are n parliament seats (indivisible prizes), and each party j has n candidates who differ in their abilities (effectiveness) in contributing to her party by making effort. Candidate i in party j has ability a_{ij} and she decides how much effort e_{ij} to contribute to her party j. Party j's winning number of seats is a random variable through a Tullock-style contest among the parties based on the ratios of parties' efforts E_js. We assume that the sum of other parties' aggregate efforts is given as E_{-j},[1] and assume that party j's winning probability of each seat is statistically independent with binary distribution described by each seat's winning probability p_j:

$$p_j = \frac{E_j}{E_j + E_{-j}}.$$

Party j's aggregated effort is determined by a CES function

$$E_j = \left(\sum_{i=1}^{n} a_{ij} e_{ij}^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

where $0 < \sigma < 1$ is elasticity of substitution parameter, and $a_{ij} > 0$ represents member ij's ability in making effort e_{ij} for all $i = 1, ..., n$. Each candidate i's individual effort cost is specified as $\frac{1}{\beta} e_{ij}^{\beta}$.

The probability of party j's winning k seats is:

$$P_j^k = C(n, k) p_j^k (1 - p_j)^{n-k}.$$

[1] This E_{-j} can be regarded as $E_{-j} = \sum_{h \neq j} E_h$: that is, party j candidates make their effor decisions assuming that other party members' decisions as given.

It is not always the case that P_j^k increases with an increase in p_j. This can be seen by imagining P_j^0. It is obvious that the probability of winning no seat would decrease as p_j increases. Differentiating P_j^k with respect to p_j, we obtain

$$\begin{aligned}\frac{dP_j^k}{dp_j} &= C(n,k)p_j^k\,(1-p_j)^{n-k}\left(\frac{k}{p_j}-\frac{n-k}{1-p_j}\right)\\ &= C(n,k)p_j^{k-1}\,(1-p_j)^{n-k-1}\,(k-np_j)\end{aligned}$$

for $k = 1, ..., n-1$, and

$$\frac{dP_j^n}{dp_j} = np_j^{n-1}$$

That is, the probability of party j's winning k seats decreases with an increase in p_j for $k < k^* = \lfloor np_j \rfloor + 1$ (or $k < np_j$), and increases for $k \geq k^*$ (or $k > np_j$), if $k < n$, and is increasing in p_j if $n = k$. This aspect of proportional representation problem generates complications in incentive problem.

3 General Seat Allocation Rule

Party j decides how to allocate the number of seats won in a parliament election to the party candidates. One rule that is often used in a parliament election is a list rule: party announces the priority ordering of its candidates, and depending on the number of seats it wins, the highest priority candidates go to the parliament. In the basic model, we analyze this list rule, then later we investigate what the optimal rules are.

We can analyze each party's effort-maximizing rules by using a more general framework. A **general (stochastic) seat allocation rule** is a list of functions $(q^k)_{k=1}^n$ such that $q^k : \mathcal{S}(k) \to [0,1]$ such that $\mathcal{S}(k) \equiv \{S \subseteq N_j : |S| = k\}$ and $\sum_{S \in \mathcal{S}(k)} q(S) = 1$ for all $k = 1, ..., n$. A general seat allocation rule assigns probabilities to which subset of k candidates go to the parliament when k seats are won in the election.

When a general allocation rule is used, the member i of team j has the following benefit function

$$B_{ij} = V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q(S) P^k(p_j),$$

where $\mathcal{S}_i(k) = \{S \in \mathcal{S}_i(k) : i \in S\}$.

A **list rule** is a simple and commonly used rule in proportional representation parliament elections in many countries. Party j's candidates' names are listed with priority order, and if party j wins k seats then the top k candidates on the list go to the parliament. That is, the mth candidate on the list will go to the parliament with probability $\sum_{k=m}^{n} P_j^k(p_j)$. Clearly, a list rule is a special general seat allocation rule. Let $m : N_j \to \{1, ..., n\}$ be an onto mapping that describes the list of priority ordering of each candidate: that is, for each $i \in N_j$, $m(i) \in \{1, ..., n\}$ is the priority that candidate i is assigned. That is, for each k, let $S_k = \{i \in N_j : m(i) \leq k\} \in \mathcal{S}(k)$ and let $q^k(S_k) = 1$ and $q^k(S) = 0$ for all $S \in \mathcal{S}(k) \backslash \{S_k\}$, and all $k = 1, ..., n$.

There is an alternative way to describe general seat assignment rule by using a **matrix representation**. Let $r_i = (r_i^k)_{k=1}^n$ be a vector where r_i^k denotes the probability of candidate i's going to the parliament when the party wins k seats for each $k = 1, ..., n$. That is, an $n \times n$ assignment matrix $R = (r_i^k)_{i=1,...,n;\ k=1,...,n}$ with $r_i^k \in [0, 1]$ and $\sum_{i \in N_j} r_i^k = k$ for all $k = 1, ..., n$, describes allocation rule fully. It is easy to see that every general seat assignment rule can be represented by a matrix R by setting $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q(S)$ for each $i \in N_j$ and $k = 1, ..., n$. However, the converse is not obvious: for any assignment matrix R, is there a general seat assignment rule q that achieves R? The following lemma provides a positive answer.

Lemma 1. *Any $n \times n$ assignment matrix R such that (i) $r_i^k \in [0, 1]$ for all $i, k = 1, ..., n$, and (ii) $\sum_{i=1}^{n} r_i^k = k$ for all $k = 1, ..., n$, can be achieved by some allocation rule $q : S \to [0, 1]$ with $\sum_{S \in \mathcal{S}(k)} q^k(S) = k$ for all $k = 1, ..., n$.*

Remark. *In the matching literature, random assignments of indivisible goods often use the property known* *as the Birchoff=Newmann theorem (Birchoff 1946, and von Neumann 1953): any bistochastic matrix can be written as a convex combination of permutation matrices. Our lemma appears to be related to this theorem, but our problem has an aspect of public indivisible goods unlike their problem, and it is not clear if there is a formal relationship between the two.*

Obviously, a list rule can be represented by an assignment matrix as well. Let R be such that $r_i^k = 1$ if and only if $k \geq m(i)$ for all $i \in N_j$. This is the assignment matrix that represents list rule m. In the next section, we show how party j's winning probability p_j affects each candidate's payoff. Thus, without any loss of generality, we can work on assignment matrix to design a seat allocation rule.

3.1 Intra-Party Equilibrium Effort Allocations under Assignment Rule R

With this assignment matrix $R = (r_i^k)_{i \in N, k=1,\ldots,n}$, we can rewrite candidate i's benefit function as:

$$B_{ij} = V \sum_{k=1}^{n} r_i^k P^k(p_j).$$

First note that each player ij's benefit is affected by her exerting effort e_{ij} through an increase in p_j.

The impact of an increase in candidate i's effort on party j's aggregate effort is

$$\frac{\partial E_j}{\partial e_{hj}} = a_{hj} \left(\sum_{i=1}^{n} a_{ij} e_{hj}^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} e_{hj}^{-\sigma} = a_{hj} E_j^{\sigma} e_{hj}^{-\sigma},$$

thus, the impact of an increase in e_{ij} on p_j is written as:

$$\begin{aligned} \frac{\partial p_j}{\partial e_{ij}} &= \frac{E_{-j}}{(E_{-j} + E_j)^2} \frac{\partial E_j}{\partial e_{ij}} \\ &= p_j (1 - p_j) \frac{1}{E_j} \frac{\partial E_j}{\partial e_{ij}} \\ &= p_j (1 - p_j) \left(\frac{a_{ij} e_{ij}^{-\sigma}}{\sum_{h=1}^{n} a_{hj} e_{hj}^{1-\sigma}} \right) \end{aligned}$$

Differentiating $P^k(p_j)$, we obtain

$$\begin{aligned}
\frac{dP^k}{dp_j} &= C(n,k)\left\{kp_j^{k-1}(1-p_j)^{n-k} - (n-k)\,p_j^k(1-p_j)^{n-k-1}\right\} \\
&= C(n,k)p_j^{k-1}(1-p_j)^{n-k-1}(k-np_j)
\end{aligned}$$

Notice that the sign of the above is not necessarily positive. This can be seen by noting the special case of $k=0$. If p_j increases, it is obvious that $P^0(p_j)$ decreases. As the above formula says, $\frac{dP^k}{dp_j} \gtreqless 0$ if and only if $k \gtreqless np_j$.

Taking the derivative of $B_{ij} = V\sum_{k=1}^n r_i^k P^k(p_j)$ with respect to e_{ij}, we obtain,

$$\begin{aligned}
\frac{\partial B_{ij}}{\partial e_{ij}} &= V\sum_{k=1}^n r_i^k \frac{dP^k}{dp_j}\frac{\partial p_j}{\partial e_{ij}} \\
&= V\sum_{k=1}^n r_i^k C(n,k)\left\{kp_j^{k-1}(1-p_j)^{n-k} - (n-k)\,p_j^k(1-p_j)^{n-k-1}\right\}(1-p_j)\,p_j\left(\frac{a_{ij}e_{ij}^{-\sigma}}{\sum_{h=1}^n a_{hj}e_{hj}^{1-\sigma}}\right) \\
&= V\left(\frac{a_{ij}e_{ij}^{-\sigma}}{\sum_{h=1}^n a_{hj}e_{hj}^{1-\sigma}}\right)\sum_{k=1}^n r_i^k C(n,k)p_j^k(1-p_j)^{n-k}(k-np_j)
\end{aligned}$$

Thus, with effort cost function $c(e_{ij}) = e_{ij}$, the first order condition assuming an interior solution is

$$\frac{\partial B_{ij}}{\partial e_{ij}} - c'(e_{ij}) = V\left(\frac{a_{ij}e_{ij}^{-\sigma}}{\sum_{h=1}^n a_{hj}e_{hj}^{1-\sigma}}\right)\sum_{k=1}^n r_i^k C(n,k)p_j^k(1-p_j)^{n-k}(k-np_j) - e_{ij}^{\beta-1} = 0, \tag{1}$$

or

$$\begin{aligned}
e_{ij} &= \left[V\left(\frac{a_{ij}}{\sum_{h=1}^n a_{hj}e_{hj}^{1-\sigma}}\right)\sum_{k=1}^n r_i^k C(n,k)p_j^k(1-p_j)^{n-k}(k-np_j)\right]^{\frac{1}{\sigma+\beta-1}} \\
&= \left[V\left(\frac{a_{ij}}{\sum_{h=1}^n a_{hj}e_{hj}^{1-\sigma}}\right)\sum_{k=1}^n r_i^k \mu^k(p_j)\right]^{\frac{1}{\sigma+\beta-1}}
\end{aligned}$$

where

$$\begin{aligned}\mu^{k}(p_j) &\equiv C(n,k)p_j^k\,(1-p_j)^{n-k}\,(k-np_j)\\ &= \frac{dP^k(p_j)}{dp_j}p_j\,(1-p_j)\\ &= P^k(p_j)\,(k-np_j)\end{aligned}$$

denotes the impact of an increase in p_j on P^k. Note that the above solution for e_j through the first order condition makes sense only when $\sum_{k=1}^{n} r_i^k \mu^k(p_j) > 0$, since this expression means how candidate i's probability to go to the parliament is affected by an increase in p_j. If this is negative in its sign, then candidate i's making effort worsens her payoff and $e_{ij} = 0$ must hold. Thus, formally, we can write

$$e_{ij} = \left[V\left(\frac{a_{ij}}{\sum_{h=1}^{n} a_{hj}e_{hj}^{1-\sigma}}\right)\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1}{\sigma+\beta-1}}$$

Let candidate ij's (proxy of) **effective contribution share** to party j be

$$\theta_{ij} \equiv \frac{a_{ij}e_{ij}(p_j)^{1-\sigma}}{\sum_{h=1}^{n} a_{hj}e_{hj}(p_j)^{1-\sigma}}.$$

The following lemma summarizes the above results.

Proposition 1 (solution for the first-order conditions). *For each $p_j \in (0,1)$, and each assignment matrix $R = (r_i^k)_{i\in N_j, k=1,\ldots,n}$, there is a unique effort vector $e_j^*(p_j) \equiv (e_{ij}^*(p_j))_{i\in N_j}$ that is consistent with the candidates' first order conditions: for all $i \in N_j$,*

$$e_{ij}^*(p_j) = V^{\frac{1}{\beta}} \frac{a_{ij}^{\frac{1}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k\mu^k(p_j), 0\right\}\right]^{\frac{1}{\sigma+\beta-1}}}{\left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k\mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{1}{\beta}}}$$

the resulting party j's aggregated effort is

$$E_j(p_j) = V^{\frac{1}{\beta}} \left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{\beta(1-\sigma)}}$$

and candidate i's effective effort share is

$$\theta_{ij} \equiv \frac{a_{ij} e_{ij}(p_j)^{1-\sigma}}{\sum_{h=1}^{n} a_{hj} e_{hj}(p_j)^{1-\sigma}} = \frac{a_{ij} e_{ij}(p_j)^{1-\sigma}}{E_j(p_j)^{1-\sigma}} = \frac{a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}}{\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}}$$

Before moving on, we need to check whether or not the above solution generated from the first order conditions for candidates' payoff maximization satisfy the second order conditions as well. Since an increase in e_{ij} corresponds to p_j This involves checking the above solution $e_{ij}(p_j)$ is indeed the best response to $\mathbf{e}_{-ij}(p_j)$ and E_{-j} for each ij. Recall (1)

$$\begin{aligned} \frac{\partial B_{ij}}{\partial e_{ij}} - c'(e_{ij}) &= V \left(\frac{a_{ij} e_{ij}^{-\sigma}}{\sum_{h=1}^{n} a_{hj} e_{hj}^{1-\sigma}} \right) \sum_{k=1}^{n} r_i^k \mu^k(p_j) - e_{ij}^{\beta-1} \\ &= \frac{a_{ij} V}{a_{ij} e_{ij} + e_{ij}^{\sigma} \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma}} \times \left[\sum_{k=1}^{n} r_i^k \mu^k(p_j) - \left(e_{ij}^{\beta} + \frac{1}{a_{ij}} e_{ij}^{\sigma+\beta-1} \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma} \right) \right] \end{aligned}$$

To analyze the second order conditions, let us introduce the following functions:

$$f_{ij}(p_j) = V \sum_{k=1}^{n} r_i^k \mu^k(p_j)$$

and

$$g_{ij}(p_j) = (e_{ij}(p_j))^{\beta} + \frac{1}{a_{ij}} (e_{ij}(p_j))^{\sigma+\beta-1} \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma},$$

where $e_{ij}(p_j)$ is an inverse function of

$$p_j(e_{ij};\mathbf{e}_{-ij},E_{-j}) = \frac{\left(\sum_{h\neq i} a_{hj}e_{hj}^{1-\sigma} + a_{ij}e_{ij}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(\sum_{h\neq i} a_{hj}e_{hj}^{1-\sigma} + a_{ij}e_{ij}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + E_{-j}}$$

Since $f_{ij}(p_j) = g_{ij}(p_j)$ holds with the first order condition for candidate ij, the second order condition is satisfied when $f'_{ij}(p_j) < g'_{ij}(p_j)$ holds, regarding p_j is implicitly determined by e_{ij}. We write this as a lemma.

Lemma 2 (the second-order conditions). *The unique solution e^* of the system of the first order conditions of candidates' effort optimization satisfies the second order conditions if $f'_{ij}(p_j) < g'_{ij}(p_j)$ holds for all $i \in N_j$. Moreover, we have*

$$g'_{ij}(p_j) = \frac{g_{ij}(p_j)}{p_j\,(1-p_j)}\left[(\sigma+\beta-1)\,\frac{1}{\theta_{ij}} + (1-\sigma)\right]$$

The calculations for the last derivative can be found in the appendix. Unfortunately, unlike the movement of $g'_{ij}(p_j)$, the movement of $f'_{ij}(p_j)$ is not easy to analyze, since each $\mu^k(p_j)$ function changes its sign of the slope twice, and weighted sum of μ^ks can be a very complicated function in general. However, as is seen below, if the allocation rule is a list rule, it turns out to be rather well-behaved. In the next section, we discuss list rules and the optimal list rule.

4 Optimal List Rules

In this section, we focus on list rules. In order to analyze the family of list rules, the following weight is useful:

$$\omega^m(p_j) \equiv \sum_{k=m}^{n} \mu^k(p_j).$$

We can rewrite this mth ranked candidate's weight as follows:

Lemma 3 (weights on list rule). *Under a list rule, the mth-rank candidate's (effort) weight is denoted by*

$$\omega^m(p_j) = mC(n,m)p_j^m (1-p_j)^{n-m+1} = mP^m(p_j) \times (1-p_j) > 0.$$

and $\omega^m(p_j) \gtreqless \omega^{m+1}(p_j)$ *if and only if* $np_j \gtreqless m$. *That is, weight distribution over the ranking is single-peaked at* $m = np_j$.

Besides tractable formula in the lemma, there is an additional important implication: $\sum_{k=1}^n r_i^k \mu^k(p_j) = \sum_{k=m(i)}^n \mu^k(p_j) > 0$. Proposition 1 shows $e_{ij}(p_j) > 0$ as long as $\sum_{k=1}^n r_i^k \mu^k(p_j) > 0$. Using Lemma 3, we can show that $f_{ij}(p_j)$s can be written simply:

$$f_{ij}(p_j) = V\omega^{m(i)}(p_j).$$

Note that $f_{ij}(p_j)$ is not affected by the identity of candidate i or party j. It is determined solely by the ranking of candidate i, $m(i)$. Thus, under a list rule, f'_{ij} can be written as

$$f'_{ij}(p_j) = \frac{f_{ij}(p_j)}{p_j(1-p_j)}(m(i) - (n+1)p_j).$$

Using Lemma 2, we have the following.

Proposition 2 (the second-order conditions for a list rule). *Suppose that a list rule* $m : N_j \to \{1, ..., n\}$ *is adopted: that is candidate* i *is ranked* $m(i)$ *among* n *candidates. Then, the system of the first order conditions has a unique solution* $(e_{ij}(p_j))_{i \in N_j}$, *and the second order conditions are satisfied if*

$$(m(i) - (n+1)p_j) < (\sigma + \beta - 1)\frac{1}{\theta_{ij}(p_j)} + (1-\sigma)$$

holds for all $i \in N_j$.

Remark. *The above condition shows that if candidates ranked low in the list (large* $m(i)$*) have small effective contribution shares* θ_{ij}*s. If* β *and* σ *are large, the condition is easier to be satisfied. The condition says that* *high ranked candidates ($m(i) < (n+1)p_j$) has no problem in satisfying the second order condition, but for low-ranked candidates to satisfy the condition, her effective effort share $\theta_{ij}(p_j)$ should not be too large.*

We will assume the following natural regularity condition.

Regularity Condition. $\frac{dE_j(p_j)}{dp_j} > 0$

Remark. *Since $P^m(p_j) = mC(n,m)p_j^m (1-p_j)^{n-m+1}$, $\frac{dP^m(p_j)}{dp_j} = P^m \times \frac{1}{p_j(1-p_j)}(m-(n+1)p_j)$ holds. Even if a list rule is used, the rank m candidate's effort is increased (or decreased) by an increase in p_j iff $\frac{dP^m}{dp_j} > 0$ or $p_j > \frac{m}{n+1}$ ($\frac{dP^m}{dp_j} < 0$ or $p_j < \frac{m}{n+1}$). If the highest ability candidates are assigned to top ranks and if their abilities are much higher than others, then the regularity condition may be violated. (In such cases, high ability candidates get negative marginal incentives with high weights. Thus, the high ability candidates reduce efforts and $E_j(p_j) > E_j(p'_j)$ may happen for $p_j < p'_j$.)*

Under this assumption, Proposition 1 implies implications on the optimal list rule.

Proposition 3 (the optimal list rule). *Under the regularity condition, candidates should be assigned to the list according to their weights in a descending order. That is, if candidates are ordered by their abilities in a descending order, the optimal list rule satisfies $\omega^{m(1)}(p_j) \geq \omega^{m(2)}(p_j) \geq ... \geq \omega^{m(n)}(p_j)$.*

Remark. *By Lemma 3, this proposition shows that candidates should be assigned by their abilities in a single-peaked way peaked at $m = np_j$ (with appropriate integer treatments).*

Note that $m = np_j$ means that the mth ranked candidate's chance to go to parliament is more or less fifty-fifty. This implies that he/she needs to work really hard to open the door to the parliament. In contrast, the top-ranked candidate does not have a strong incentive to work hard since he/she is almost guaranteed to go to the parliament (though he/she still makes a positive effort by Lemma 3). Moreover, if the candidates are ranked according to the optimal list rule, then Proposition 2 and the regularity conditions are likely to be satisfied (see the remarks).

5 Party-Optimal Assignment Rules

In this section, we explore the optimal assignment rule without being restricted by list rules. Although such optimal assignment rules are unrealistic, this analysis sheds light on the properties of list rules. The purpose of the section is to get insights of the optimal rule, we ignore the second-order conditions. Party j maximizes its aggregate effort E_j by setting an assignment matrix R:

$$E_j = \left\{ V \left(\sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}}$$

where r_i^k is the probability of candidate i goes to the parliament when party j wins k seats, and $\sum_{i \in N_j} r_i^k = k$ for all $k = 1, ..., n$.

Party j's maximization problem is:

$$\max_{(r_i^k)} \sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \quad \text{s.t. } \sum_{i=1}^{n} r_i^k = k \text{ for all } k = 1, ..., n$$

where $\mu^k(p_j) = \frac{dP^k}{dp_j}(1-p_j)\,p_j$ with $\mu^k(p_j) \gtreqless 0$ if and only if $k \gtreqless np_j$ since $\mu^k(p_j) \equiv C(n,k) p_j^k (1-p_j)^{n-k} (k - np_j)$.

Notice that the bracket in the above formula has power $\frac{1-\sigma}{\sigma+\beta-1}$. It turns out to be essential whether this power is more than unity ($\beta < 2(1-\sigma)$: a convex function), or less than unity ($\beta > 2(1-\sigma)$: a concave function). We will analyze these two cases separately.

5.1 Convex case: $\beta < 2(1-\sigma)$

In this case, we have the following results.

Lemma 4. *Suppose that R is an optimal rule, and that $\beta < 2(1-\sigma)$ holds. Then, for any i and i' with $a_{ij} > a_{i'j}$ with $\sum_{k=1}^{n} r_i^k \mu^k(p_j) > 0$, $\sum_{k=1}^{n} r_i^k \mu^k(p_j) \geq \sum_{k=1}^{n} r_{i'}^k \mu^k(p_j)$ holds.*

Lemma 5. *Suppose that $\beta < 2(1-\sigma)$ holds. There is no optimal rule $R = (r_i^k)_{i,k}$ in which $r_i^{k'} \in (0,1)$ for all i with $\sum_{k=1}^{n} r_i^k \mu^k(p_j) > 0$.*

We will consider deterministic assignment rule R with $r_i^k \in \{0,1\}$ for all $i,k=1,...,n$, since we assume $\beta \leq 2(1-\sigma)$ (by Lemma 2). Rename candidates by their abilities in a descending order: $a_{1j} \geq a_{2j} \geq ... \geq a_{nj}$. By Lemma 3, we need to assign the highest of the following sum of weights to $i=1$, and the second highest to $i=2$, and so on:

$$\sum_{k=1}^{n} r_k^i \mu^k(p_j)$$

in order to maximize E_j (thus, to maximize p_j given E_{-j}).

We can describe an assignment rule by the following $n \times n$ matrix R such that (i) each row represents candidate $i=1,2,...,n$, and each column k represents the number of seats won in the election, (ii) each (i,k) argument r_{ik} is either 1 or 0, representing whether or not candidate i goes to the parliament when k seats are won, and (iii) for each column $k=1,...,n$, the elements in the kth column sum up to k. In order to describe the optimal (deterministic) mechanism, we will introduce some notations. Consider $k=1,...,n$ be the number of seats won by party j. For each $k=1,...,n$, party needs to send k candidates to the parliament. Let $\kappa(i)=(\kappa_1(i),...,\kappa_k(i),...,\kappa_n(i))$ be the number of seats available for each case k, and let $\nu(i)$ be the number of candidates left to be assigned when candidate i is going to be assigned: i.e., $\nu(i)=n-i+1$. We will assign seats to candidates in order starting from the highest ability candidate $i=1$ in a descending order. Let $\mathcal{M}(i) \equiv \{k \in \{1,...,n\} : \kappa_k(i) > 0\}$ be the set of cases k in which candidate i can be sent to the parliament, and let $\mathcal{L}(i) \equiv \{k \in \{1,...,n\} : \kappa_k(i) = \nu(i)\}$ be the set of cases k in which candidate i must be sent to the parliament (for feasibility: if not, k candidates cannot be sent to the parliament when k seats are won). Denote the effort-maximizing set of cases in which candidate i is sent to the parliament by $\zeta(i) \subseteq \{1,...,k,...,n\}$. Let $\kappa(i+1)=(\kappa_1(i+1),...,\kappa_k(i+1),...,\kappa_n(i+1))$ be such that $\kappa_k(i+1)=\kappa_k(i)-1$ if $k \in \zeta(i)$, and $\kappa_k(i+1)=\kappa_k(i)$ otherwise. Initially, $\kappa(1)=(\kappa_1(1),...,\kappa_k(1),...,\kappa_n(1))=(1,...,k,...,n)$, $\nu(1)=n$, $\mathcal{M}(1) \equiv \{1,...,n\}$, and $\mathcal{L}(1) \equiv \{n\}$ hold. The optimal set of cases for candidate i to go to the parliament is defined by

$$\zeta(i) = \arg \max_{\mathcal{L}(i) \subseteq K \subseteq \mathcal{M}(i)} \sum_{k \in K} \mu_j(k)$$

for $i=1,...,n$. This $\zeta(i)$ gives candidate i the largest aggregate weights $\sum_{k \in \zeta(i)} \mu_j(k)$ available for her. The matrix is completed by setting $r_{ik} = 1$ if and only if $k \in \zeta(i)$ for all $i = 1, ..., n$ and all $k = 1, ..., n$.

From Single-Crossingness on Winning Probabilities, it is clear that $\zeta(1) = \{k^*, k^*+1, ..., n\}$, since this set collects all positive $\mu^k(p_j)$s without having no negative $\mu^k(p_j)$s. How about $\zeta(2)$? It is still $\zeta(2) = \{k^*, k^*+1, ..., n\}$ as long as $k^* \geq 2$ ($\kappa_{k^*}(2) \geq 1$), since $\mathcal{M}(2) \equiv \{1, ..., n\}$. We consider two cases: (Case 1) $k^* \leq \frac{n+1}{2}$, and (Case 2) $k^* > \frac{n+1}{2}$.

(Case 1: $k^* \leq \frac{n+1}{2}$) In this case, we can assign the top k^* candidates to $\{k^*, k^*+1, ..., n\} = \zeta(1) = ... = \zeta(k^*)$. After that, as long as $i < n - k^* + 2$, we assign $\zeta(i) = \{i, i+1, ..., n\}$. When $i = n - k^* + 2$ comes, we assign $\zeta(i) = \{k^* - 1\} \cup \{i, i+1, ..., n\}$, and for $i = n - k^* + 3$, $\zeta(i) = \{k^* - 2, k^* - 1\} \cup \{i, i+1, ..., n\}$, and so on. When $i = n$, $\zeta(n) = \{1, ..., k^* - 1\} \cup \{n\}$.

(Case 2: $k^* > \frac{n+1}{2}$) In this case, we can only assign the top $n - k^*$ candidates to $\{k^*, k^*+1, ..., n\} = \zeta(1) = \zeta(n - k^*)$. Since $\kappa_{k^*-1}(n - k^* + 1) = \kappa_n(n - k^* + 1) = n - (n - k^* + 1) + 1 = \nu(n - k^* + 1)$, $\zeta(n - k^* + 1) = \{k^* - 1, k^*, ..., n\}$. Similarly, up to $i = n - k^* + 1$, $\zeta(i) = \{n - i + 1, ..., n\}$ is assigned. After that $\zeta(i) = \{n - i + 1, ..., k^* - 1\} \cup \{i, ..., n\}$.

Note that if $\sum_{k \in \zeta(i)} \mu_j(k) \leq 0$, then $e_{ij} = 0$ holds. The outcome of this algorithm is an **effort-maximizing rule**. This implies that the highest ability candidate 1 goes to the parliament if and only if party j wins $k^* = \lfloor np_j \rfloor + 1$ seats or more. That is, the highest ability candidate gets the same assignment between the optimal assignment rule and the optimal list rule.

Proposition 4 (the optimal assignment rule). *Suppose $\beta \leq 2(1 - \sigma)$. Then, the optimal assignment rule is described by matrix R with $r_{ik} = 1$ if and only if $k \in \zeta(i)$ for all $i = 1, ..., n$ and all $k = 1, ..., n$.*

In order to illustrate the optimal assignment rule, we provide an example below.

Example 1. Suppose $n = 7$. We consider three cases: $k^* = 3$, $k^* = 5$, and $k^* = 1$. The optimal assignment matrix is described by R_3^*, R_7^*, and R_1^* in the following:

$$R_3^* = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_7^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

When $k^* = 3$ (p_j is not high), the lowest ability candidate 7 goes to the parliament only when party j wins $k = 1, 2, 7$ seats, and candidate 6 goes to the parliament when party j wins $k = 2, 6, 7$ seats. When $k^* = 7$ (party j is a dominating party: high p_j), then the highest ability candidates 1, 2, and 3 can go to the parliament only when party j wins 5 or more seats. This is because the party wants the highest ability candidates work very hard to be elected.

5.2 Monotonic Allocation Rules

One desirable property we may impose on the assignment matrix is monotonicity. A rule described by an assignment matrix R is **monotonic** if and only if $r_i^{k+1} \geq r_i^k$ for all $i = 1, ..., n$ and $k = 1, ..., n$. As is seen in Example 1, the optimal assignment matrix R does not necessarily satisfy monotonicity when $k^* > 1$. However, monotonicity is a very reasonable requirement. A particularly appealing property of a monotonic rule is that everybody exert a positive effort. This can be seen easily by rewriting $\sum_{k=1}^n r_i^k \mu^k(p_j)$:

$$\sum_{k=1}^n r_i^k \mu^k(p_j) = r_i^1 \sum_{k=1}^n \mu^k(p_j) + \left(r_i^2 - r_i^1\right) \sum_{k=2}^n \mu^k(p_j) + ... + \left(r_i^n - r_i^{n-1}\right) \mu^n(p_j)$$

By the first order stochastic dominance, $\sum_{k=m}^{n} \mu^k(p_j) > 0$ for all $m = 1, ..., n$. By monotonicity, $r_i^k - r_i^{k-1} \geq 0$ for all $k = 1, ..., n$ ($r_0^k = 0$). Thus, monotonicity implies:

$$\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} = \sum_{k=1}^{n} r_i^k \mu^k(p_j) > 0$$

Proposition 5. *Under any monotonic rule, every candidate exerts effort.*

Under deterministic rules, monotonicity requires that if candidate i is sent to the parliament when m seats are won, she will be sent to the parliament if more than m seats are won. Then, the mth candidate's effort incentive is

$$M_m(p_j) = \sum_{k=m}^{n} \mu^k(p_j)$$

Order $M_m(p_j)$s by their values, and define a one-to-one mapping $m^* : N_j \to \{1, ..., n\}$ such that

$$M_{m^*(1)}(p_j) \geq .... \geq M_{m^*(i)} \geq ... \geq M_{m^*(n)}$$

The following result is straightforward.

Proposition 6. *Suppose $\beta \leq 2(1-\sigma)$. Then, the list rule $m^* : N_j \to \{1, ..., n\}$ is the optimal monotonic assignment rule.*

Note that $m^*(1) = k^*$. Thus, the top candidate's assignments are exactly the same in the optimal deterministic rule and the optimal list rule. Under the single-crossingness, it is easy to see that $m^*(2)$ is either $k^* + 1$ or $k^* - 1$, and m^* orders candidates in such a way that it forms a single-peaked way at peak $m^*(1) = k^*$. An interesting special case is $k^* = 1$. Since $\mu^k(p_j) > 0$ for all $k = 1, ..., n$ by single-crossingness, we have $m^*(i) = i$ for all $i = 1, ...n$. We have the following result.

Corollary 1. *Suppose $\beta \leq 2(1-\sigma)$. When $k^* = 1$, the optimal assignment rule is the list rule according to candidates' abilities.*

If we confine our attention to the class of deterministic rules (i.e., $r_i^k \in \{0,1\}$ for all i,k), we do not need convexity condition to show that the list rule $m^* : N_j \to \{1,...,n\}$ is the optimal monotonic rule.

Proposition 7. *If we confine or attention to the class of deterministic rules, then the list rule $m^* : N_j \to \{1,...,n\}$ is the optimal monotonic assignment rule.*

Note that the above result is specific to monotonic and deterministic rule. If we are trying to find the optimal deterministic rule, then the relative sizes of β and σ matter, and concave case will have different result. With monotonicity under deterministic rule, there is no freedom in playing with. This is why we have the above result.

5.3 Concave Case $\beta > 2(1-\sigma)$

With strong complementarity of team members' efforts, the reward should not be concentrated to small set of members.

Thus, party j's maximization problem becomes

$$\arg\max_{(r_i^k)} \left\{ V\left(\sum_{i=1}^n a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^n r_i^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}} \quad \text{s.t. } \sum_{i=1}^n r_i^k = k \text{ for all } k = 1,...,n$$

or

$$\max_{(r_i^k)} \sum_{i=1}^n a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\sum_{k=1}^n r_i^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \quad \text{s.t. } \sum_{i=1}^n r_i^k = k \text{ for all } k = 1,...,n$$

The optimal mechanism is the solution of the above (rather complicated) problem when $\beta > 2(1-\sigma)$ holds. As k increases the set r_i^k will face more strict constraints (when $k = n$, $r_i^n = 1$ must hold: every candidate needs to be sent to the parliament). We know, however, that $\mu^k(p_j) = \frac{dP^k}{dp_j}(1-p_j)p_j < 0$ for all $k < k^*$

and $\frac{dP^k}{dp_j}(1-p_j)\,p_j > 0$ for all $k > k^*$, and that what matters is just the weighted sum of the shares in the bracket in achieving the optimal allocation. Intuitively, there will be a plenty of freedom using r^ks for low ks to achieve unequal allocations.

Supposing that the sum of reward $\bar{R} = \sum_{k=1}^{n} k\frac{dP^k}{dp_j}(1-p_j)\,p_j$ can be allocated freely to the candidates according to their abilities, the optimal allocation is described by solving the following problem.

$$\arg\max_{(R_i)_{i=1}^n} \sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} R_i^{\frac{1-\sigma}{\sigma+\beta-1}} \qquad s.t. \qquad \sum_{i=1}^{n} R_i = \bar{R} = \sum_{k=1}^{n} k\frac{dP^k}{dp_j}(1-p_j)\,p_j$$

The first order conditions generate the optimality conditions:

$$\frac{1-\sigma}{\sigma+\beta-1} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} R_i^{\frac{1-\sigma}{\sigma+\beta-1}-1} = \frac{1-\sigma}{\sigma+\beta-1} a_{hj}^{\frac{\beta}{\sigma+\beta-1}} R_h^{\frac{1-\sigma}{\sigma+\beta-1}-1}$$

or

$$\frac{R_i}{R_h} = \left(\frac{a_{ij}}{a_{hj}}\right)^{\frac{\beta}{\beta-2(1-\sigma)}}$$

for all $i, h = 1, ..., n$.

Proposition 8. *Suppose $\beta > 2(1-\sigma)$. Then, whenever feasible, the optimal assignment rule tries to allocate the chances of candidates to get a seat in the parliament proportionally to candidates' abilities (with power $\frac{\beta}{\beta-2(1-\sigma)}$).*

Our Propositions 6 and 8 generate a generalized version of the result in Crutzen, Flamand, and Sahuguet (2017) as a corollary. When candidates are homogenous, $R_i = R_h$ holds for all $i, h = 1, ..., n$ when $\beta > 2(1-\sigma)$. Thus $q_{ik} = \frac{k}{n}$ for all $i, k = 1, ..., n$, which generates the egalitarian rule.

Corollary 2. (Crutzen, Flamand, and Sahuguet, 2017) *Suppose that all candidates have the same ability. Then, if $\beta \leq 2(1-\sigma)$ then the optimal monotonic rule is (any) list rule, while if $\beta > 2(1-\sigma)$ then the optimal rule is the egalitarian rule.*

6 Why Do Parties Assign Top Seats on the List to High Ability Candidates?

It was our prediction that the candidates whose names are on top of the list are not highest ability ones. The readers may say that the highest ranks in the list are usually occupied by highly qualified candidates in the real world. We have two answers for this comment. First, the abilities we are focusing here are the ones to attract voters to vote for the party she belongs to. Thus, these candidates are the ones who perform very well in town hall meetings and the ones who make convincing speeches. They are not necessarily good at proposing policies, working with bureaucrats, or negotiating with other party members. Second, even if candidates' abilities/qualifications are more or less unidimensional (various ability indicators are highly correlated with each other), we can explain the concentration of high ability candidates at top ranks of the list if top ranked candidates get some extra benefits from listed there.

In the following, we will consider the situation that the several top ranks on the list are special in the sense that listed on the top of the list means to get a chance to join the cabinet if their party wins the majority of the parliament's seats. Let $k^M \equiv \left\lfloor \frac{n}{2} \right\rfloor + 1$ be the number of seats needed for the majority of the parliament, and let the top k^C elected candidates would be invited to the cabinet if the party gets the majority of seats. Note that *we will be confining our attention to the class of list rules in this section.* Candidate m whose position on the list is also m has the following benefit function:

$$B_{mj} = \begin{cases} V \sum_{k=m}^{n} P_j^k & \text{if } m > k^C \\ V \sum_{k=m}^{n} P_j^k + W \sum_{k=k^{Maj}}^{n} P_j^k & \text{if } m \leq k^C \end{cases}$$

where $W > 0$ is the benefit from selected to be one of the cabinet members.[2] When $m > k^C$, the incentive is the same as before. We analyze efforts provided by candidates whose ranking m is less than k^C. The first

[2] We can introduce even more differentiated benefits depending on her ranking in the case of her party wins the majority and is going to play different role in the cabinet.

order condition for payoff maximization is:

$$\frac{\partial B_{mj}}{\partial e_{mj}} - \frac{dC_m}{de_{mj}} = \frac{a_{mj}}{e_{mj}} \left(\frac{e_{ij}}{E_j}\right)^{1-\sigma} \left[V \sum_{k=m}^{n} \mu_j^k + W \sum_{k=k^M}^{n} \mu_j^k\right] - e_{mj}^{\beta-1} = 0$$

This implies that for $m \leq k^C$ we have

$$e_{mj} = \left[a_{mj}\left(VM_j^m + WM_j^{k^M}\right)\left(\frac{1}{E_j}\right)^{1-\sigma}\right]^{\frac{1}{\sigma+\beta-1}}$$

where

$$M_j^m = \sum_{k=m}^{n} \mu_j^k$$

By substituting this into team production function, we have

$$\begin{aligned} E_j &= \left[\sum_{m=1}^{k^C} a_{mj}\left\{a_{mj}\left(VM_j^m + WM_j^{k^M}\right)\left(\frac{1}{E_j}\right)^{1-\sigma}\right\}^{\frac{1-\sigma}{\sigma+\beta-1}} + \sum_{m=k^C+1}^{n} a_{mj}\left\{a_{mj}VM_j^m\left(\frac{1}{E_j}\right)^{1-\sigma}\right\}^{\frac{1-\sigma}{\sigma+\beta-1}}\right]^{\frac{1}{1-\sigma}} \\ &= \left(\frac{1}{E_j}\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\left[\sum_{m=1}^{k^C} a_{mj}\left\{a_{mj}\left(VM_j^m + WM_j^{k^M}\right)\right\}^{\frac{1-\sigma}{\sigma+\beta-1}} + \sum_{m=k^C+1}^{n} a_{mj}\left(a_{mj}VM_j^m\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\right]^{\frac{1}{1-\sigma}} \end{aligned}$$

Solving with respect to E_j, we have

$$E_j = \left[\left\{\sum_{m=1}^{k^C} \alpha_{mj}\left(VM_j^m + WM_j^{k^M}\right)^{\frac{1-\sigma}{\sigma+\beta-1}} + \sum_{m=k^C+1}^{n} \alpha_{mj}\left(VM_j^m\right)^{\frac{1-\sigma}{\sigma+\beta-1}}\right\}^{\frac{\sigma+\beta-1}{1-\sigma}}\right]^{\frac{1}{\beta}}$$

Since we know that $M_j^m > 0$ and $M_j^{k^M} > 0$, and $M_j^{k^M} > M_j^m$ if $m \leq k^C < k^M$. Naturally, we can assume that $k^C < k^*$. Then, if W is much more than V and M^{k^M} is not much smaller than M^{k^*}, it is possible for us to have

$$VM_j^m + WM_j^{k^M} > VM^{k^*}$$

If this were the case, the k^C highest candidates will be assigned to the top k^C positions. Thus, the the highest ability candidates are listed at the top, and they also work very hard to attract voters to the party.

7 Some Equilibrium Analysis — J Party Case

Here, we show that our analysis can be generalized to J party case in which each party tries to maximize its winning probability. There are J parties (teams) that are competing for n parliament seats (indivisible prizes). Each party j has n candidates who differ in their abilities (effectiveness) in contributing to her party by making effort. Candidate i in party j has ability a_{ij} and she decides how much effort e_{ij} to contribute to her party j. Party j's winning number of seats is a random variable through a Tullock-style contest among J parties based on the ratios of parties' efforts E_js. In our basic model, we assume that seat allocation is determined through "winning probability" of each party j:

$$p_j = \frac{E_j}{E_1 + ... + E_J},$$

and we assume that p_j solely explain the number of seats party j wins as a random variable. The easiest example is to assume that each of n seats is allocated to parties with probabilities $(p_1, ..., p_J)$, and winning probabilities of each seat are i.i.d. random variables.

Formally, such a game can formulated as a two stage game. In the first stage, All party $j = 1, ..., n$ select party seat allocation rules R_j as assignment matrices simultaneously, then in the second stage all candidates in all parties simultaneously decide their effort levels. Let parties' winning probability vector be $p = (p_1, ..., p_J)$, and consider the probability of party $j = 1$'s winning k_1 seats. Let $P_j = (P_j^k)_{k=0}^n \in \Delta^{n+1}$ be probability distribution of party j's number of winning seats: i.e., P_j^k is the probability of party j's winning

k seats with $\sum_{k=0}^{n} P_j^k = 1$. Since we assume that seat allocation is determined by i.i.d., we have

$$
\begin{aligned}
P_1^k &= VC(n,k_1)p_1^{k_1} \\
&\times \left[\sum_{k_2=0}^{n-k_1} C(n-k_1,k_2)p_2^{k_2} \sum_{k_3=0}^{n-k_1-k_2} C(n-k_1-k_2,k_3)p_3^{k_3} \times ... \right. \\
&\quad \left. \times \sum_{k_{J-1}=0}^{n-k_1-...-k_{r-2}} C(n-k_1-...-k_{J-2},k_{J-1})p_{J-1}^{k_{J-1}} p_J^{n-k_1-...-k_{J-2}-k_{J-1}} \right]
\end{aligned}
$$

First note that $(p_i+p_j)^k = \sum_{\ell=0}^{k} C(k,\ell)p_i^\ell p_j^{k-\ell}$ for any k. Setting $k = n-k_1-...-k_{J-2}$, we have

$$
\sum_{k_{J-1}=0}^{n-k_1-\cdots-k_{J-2}} C(n-k_1-...-k_{J-2},k_{J-1})p_{J-1}^{k_{J-1}} p_J^{n-k_1-...-k_{J-2}-k_{J-1}} = (p_{J-1}+p_J)^{n-k_1-...-k_{J-2}}
$$

Repeatedly applying this, we have

$$
\begin{aligned}
P_1^{k_1} &= C(n,k_1)p_1^{k_1}(p_2+...+p_J)^{n-k_1} \\
&= C(n,k_1)p_1^{k_1}(1-p_1)^{n-k_1}
\end{aligned}
$$

Thus, even in J party case, party j's probability of winning k seats can be written as for each $k = 1,...,n$,

$$
P_j^k = C(n,k)p_j^k(1-p_j)^{n-k}
$$

as long as probabilities of winning are statistically independent of each other.

Here, we will investigate how competition by the parties work. In particular, what each party tries to maximize when parties are competing for the number of seats in the parliament. Since we assume that $(P^k(p_j))_{k=1}^n$ is first-order stochastically dominated by $(P^k(p_j'))_{k=1}^n$ for $p_j' > p_j$, each party j should try to maximize p_j. Since $p_j = \frac{E_j}{E_1+...+E_J}$, it seems to make sense for party j to choose rule $q = (q^k)_{k=1}^n$ in order to maximize E_j given E_{-j}. However, $p = (p_1,...,p_j,...,p_J)$ is actually determined in the interactions with other parties in equilibrium, and it is important to check our intuitive approach makes sense.[3]

We start with existence of equilibrium. It is easy to observe that E_j depends only on p_j — nothing else ($E_j = E_j(p_j)$). Thus, we can use the following fixed-point mapping to prove the existence of equilibrium. Let $p = (p_1, ..., p_J)$ and

$$f_j(p) = \frac{E_j(p_j)}{\sum_{k=1}^{J} E_k(p_k)}$$

for all $j = 1, ..., J$. Then $f(p) = (f_1(p), ..., f_J(p))$ is a fixed point mapping from simplex $\Delta^J \equiv \left\{ p \in \mathbb{R}_+^J : \sum_{k=1}^{J} p_k = 1 \right\}$ to itself, which is a continuous function. By Brouwer's fixed point theorem, there exists a fixed point $p^* = f(p^*)$.

Proposition 9 (existence of an effort profile that satisfies the system of the first-order conditions). There exists an effort strategy profile that satisfies the first order conditions of each candidate's effort optimization problem for any profile of list rules $a = (a_j)_{j=1}^J$, where $a_j = (a_{j1}, ..., a_{jn})$.

Let $\alpha_{ij} = a_{ij}^{\frac{\beta}{\sigma+\beta-1}}$. The following result is an immediate consequence of equilibrium condition.

Proposition 10 (winning probability ranking). In every equilibrium, the winning probabilities of parties j and h satisfy the following:

$$p_j \gtreqless p_h \Longleftrightarrow \sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \gtreqless \sum_{i=1}^{n} a_{ih}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_h), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}}$$

We can justify that each party would try to maximize its aggregate effort by conducting comparative static analysis. However, we need more than the regularity condition we have assumed in the previous sections. If $\eta_j = \frac{p_j}{E_j} \frac{\partial E_j}{\partial p_j} \in (0,1)$ is satisfied for all $j = 1, ..., n$ (naturally interpreted as stability of party j's effort), then when a candidate i's ability increases slightly, then E_j increases in equilibrium (a sufficient

[3] In the general equilibrium framework (in trade theory), we see cases of transfer paradox and immiserizing growth occurring.

condition). Without bounded above by unity, the system becomes unstable, and the comparative static analysis generates unintuitive results.

Proposition 11 (comparative statics). Suppose that $\eta_j \in (0,1)$ for all $j = 1, ..., J$. Then, for any $i = 1, ..., n$, $\frac{dp_j}{d\alpha_{ij}} > 0$ holds if and only if $\sum_{k=1}^{n} r_i^k \mu^k(p_j) > 0$.

8 Conclusion

To be written.

Appendix (Proofs)

Proof of Lemma 1. We will prove the statement by induction. Let's start with $n = 3$. In this case, it is easy to see (i) if $k = 1$, then we can set $q^1(\{i\}) = r_i^1$ for all $i = 1, 2, 3$, (ii) if $k = 2$, we can set $q^2(N\backslash\{i\}) = 1 - r_i^2$ for all $i = 1, 2, 3$, and (iii) $q^3(\{1, 2, 3\}) = 1$ since $r_i^3 = 1$ must hold for all $i = 1, 2, 3$. Thus, for $n = 3$ we can find $(q^k)_{k=1}^3$ for any feasible R.

Now, suppose that for $n = m$ we can find $(q^k)_{k=1}^m$ for any $m \times m$ R matrix with $r_i^k \in [0, 1]$ and $\sum_{i=1}^m r_i^k = k$ for all $k = 1, ..., m$ and $i = 1, ..., m$. We will show that for $n = m + 1$ we can find $(q^k)_{k=1}^{m+1}$ for any $(m+1) \times (m+1)$ R matrix with $r_i^k \in [0, 1]$ and $\sum_{i=1}^{m+1} r_i^k = k$ for all $k = 1, ..., m+1$ and $i = 1, ..., m+1$.

Let $n = m + 1$. As in the case of $n = 3$, we can see that for $k = 1$, 2, and $m + 1$, we can find q^ks. We will show for all other $k = 2, ..., m$, we can find $q^k : \mathcal{S}(k, N_j \cup \{m + 1\}) \to [0, 1]$ with $N_j = \{1, ..., m\}$ for all $(r_1^k, ..., r_{m+1}^k)$ with $\sum_{i=1}^{m+1} r_i^k = k$. Let $i^* \in \arg\max_i r_i^k$, and let $r_{-i^*}^k = (r_1^k, ..., r_{i^*-1}^k, r_{i^*+1}^k, ..., r_{m+1}^k)$.

First, let $\bar{r}^k = r_{-i^*}^k \times \frac{k}{|r_{-i^*}^k|}$. Since $\bar{r}^k$ has m arguments, we can find $\bar{q}^k : \mathcal{S}(k, N_j) \to [0, 1]$ with $|N_j| = m$ which supports $\bar{r}^k$ by our induction hypothesis. Then, we can create $\hat{q}^k : \mathcal{S}(k, N_j \cup \{i^*\}) \to [0, 1]$ which supports $\hat{r}^k = (\underbrace{\bar{r}^k}_{-i^*}, \underbrace{0}_{i^*})$ by setting $\hat{q}^k(S) = \bar{q}^k(S)$ for all $S \in (k, N_j)$ with $\bar{q}^k(S) > 0$, and $\hat{q}^k(S) = 0$ for any other $S \in \mathcal{S}(k, N_j \cup \{i^*\})$.

Second, let $\bar{r}^{k-1} = r^k_{-i^*} \times \frac{k-1}{|r^k_{-i^*}|}$. Since $\bar{r}^{k-1}$ has m arguments and $k \geq 2$, we can find $\bar{q}^{k-1} : \mathcal{S}(k-1, N_j) \to [0,1]$ with $|N_j| = m$ which supports $\bar{r}^{k-1}$ by our induction hypothesis. Then, we can create $\check{q}^k : \mathcal{S}(k, N_j \cup \{i^*\}) \to [0,1]$ which supports $\check{r}^k = (\underbrace{\bar{r}^{k-1}}_{-i^*}, \underbrace{1}_{i^*})$ by setting $\check{q}^k(S \cup \{i^*\}) = \bar{q}^{k-1}(S)$ for all $S \in (k, N_j)$ with $\bar{q}^{k-1}(S) > 0$, and $\hat{q}^k(S) = 0$ for any other $S \in \mathcal{S}(k, N_j \cup \{i^*\})$.

Clearly, $r^k = (r^k_{-i^*}, r^k_{i^*})$ can be written as a convex combination of $\hat{r}^k$ and $\check{r}^k$. This implies that r^k can be supported by a convex combination of $\hat{q}^k$ and $\check{q}^k$. Thus, we proved the induction hypothesis for $n = m+1$.

We have completed the proof.■

Proof of Proposition 1. Thanks to the CES effort aggregator function, we can solve the system of the first order conditions. Substituting the above equation for each ij into E_j, we obtain

$$\begin{aligned}
E_j &= \left\{\sum_{h=1}^{n} a_{hj} e_{hj}^{1-\sigma}\right\}^{\frac{1}{1-\sigma}} \\
&= \left\{\sum_{i=1}^{n} a_{ij} \left[V\left(\frac{a_{ij}}{E_j^{1-\sigma}}\right) \max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right\}^{\frac{1}{1-\sigma}} \\
&= \left(\frac{V}{E_j^{1-\sigma}}\right)^{\frac{1}{\sigma+\beta-1}} \left\{\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right\}^{\frac{1}{1-\sigma}}
\end{aligned}$$

or

$$E_j^{\frac{\beta}{\sigma+\beta-1}} = V^{\frac{1}{\sigma+\beta-1}} \left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{1}{1-\sigma}}$$

or

$$E_j = V^{\frac{1}{\beta}} \left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}} \left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{\beta(1-\sigma)}}$$

Substituting the above back to e_{ij}, we have:

$$\begin{aligned}
e_{ij}(p_j) &= \left[V\left(\frac{a_{ij}}{\sum_{h=1}^{n} a_{hj} e_{hj}^{1-\sigma}}\right)\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1}{\sigma+\beta-1}} \\
&= \left[\frac{Va_{ij}}{E_j^{1-\sigma}}\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1}{\sigma+\beta-1}} \\
&= \left[\frac{Va_{ij}\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}}{V^{\frac{1-\sigma}{\beta}}\left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{\beta}}}\right]^{\frac{1}{\sigma+\beta-1}} \\
&= V^{\frac{1}{\beta}}\frac{a_{ij}^{\frac{1}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1}{\sigma+\beta-1}}}{\left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{1}{\beta}}}
\end{aligned}$$

where $\mu^k(p_j) = C(n,k)p_j^k(1-p_j)^{n-k}(k-np_j) = \frac{dP_j^k}{dp_j}p_j(1-p_j)$, which is a change in $P^k(p_j)$ when p_j is increased slightly. By using μ^ks and r_i^ks, θ_{ij} can be written as:

$$\begin{aligned}
\theta_{ij} &= \frac{a_{ij}e_{ij}(p_j)^{1-\sigma}}{E_j(p_j)^{1-\sigma}} \\
&= \frac{a_{ij}V^{\frac{1-\sigma}{\beta}}\frac{a_{ij}^{\frac{1-\sigma}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}}{\left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{1-\sigma}{\beta}}}}{V^{\frac{1-\sigma}{\beta}}\left(\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}\right)^{\frac{\sigma+\beta-1}{\beta}}} \\
&= \frac{a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}}{\sum_{i=1}^{n} a_{ij}^{\frac{\beta}{\sigma+\beta-1}}\left[\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right]^{\frac{1-\sigma}{\sigma+\beta-1}}}
\end{aligned}$$

Substituting this into the above formulas, we obtain the results.□

Proof of Lemma 2. Since $p_j = \frac{E_j}{E_j+E_{-j}}$, we can describe e_{ij} as a function of p_j explicitly given $\mathbf{e}_{-ij}$:

$$e_{ij}(p_j) = \left(\frac{1}{a_{ij}}\right)^{\frac{1}{1-\sigma}}\left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma}E_{-j}^{1-\sigma} - \sum_{h\neq i} a_{hj}e_{hj}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$$

Thus, we can rewrite $g_{ij}(p_j)$ as

$$g_{ij}(p_j) = \left(\frac{1}{a_{ij}}\right)^{\frac{\beta}{1-\sigma}} \left\{ \left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} - \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma} \right]^{\frac{\sigma+\beta-1}{1-\sigma}} \left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} \right] \right\}$$

and g_{ij} is defined over $p_j \geq \underline{p_j}$, where

$$\underline{p_j} = \frac{\left(\sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}{\left(\sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} + E_{-j}}$$

since candidate ij can only choose her effort $e_{ij} \geq 0$.

Differentiating $g_{ij}(p_j)$, we obtain

$$\begin{aligned}
g'_{ij}(p_j) &= \left(\frac{1}{a_{ij}}\right)^{\frac{\beta}{1-\sigma}} \left(\frac{\sigma+\beta-1}{1-\sigma}\right) \left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} - \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma} \right]^{\frac{\beta}{1-\sigma}} \left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} \right] \\
&\quad \times (1-\sigma) \left(\frac{p_j}{1-p_j}\right)^{-\sigma} E_{-j}^{1-\sigma} \frac{1}{(1-p_j)^2} \\
&\quad + \left(\frac{1}{a_{ij}}\right)^{\frac{\beta}{1-\sigma}} \left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} - \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma} \right]^{\frac{\sigma+\beta-1}{1-\sigma}} (1-\sigma) \left(\frac{p_j}{1-p_j}\right)^{-\sigma} E_{-j}^{1-\sigma} \frac{1}{(1-p_j)^2} \\
&= \left(\frac{1}{a_{ij}}\right)^{\frac{\beta}{1-\sigma}} \left\{ \left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} - \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma} \right]^{\frac{\sigma+\beta-1}{1-\sigma}} \left[\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} \right] \right\} \\
&\quad \times \left[(\sigma+\beta-1) \frac{\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma}}{\left(\frac{p_j}{1-p_j}\right)^{1-\sigma} E_{-j}^{1-\sigma} - \sum_{h \neq i} a_{hj} e_{hj}^{1-\sigma}} + (1-\sigma) \right] \left(\frac{p_j}{1-p_j}\right)^{-1} \frac{1}{(1-p_j)^2} \\
&= \frac{g_{ij}(p_j)}{p_j(1-p_j)} \left[(\sigma+\beta-1) \frac{1}{\theta_{ij}} + (1-\sigma) \right]
\end{aligned}$$

It is easy to recognize that the contents of the bracket is large when θ_{ij} and σ are small. Thus, the slope of $g_{ij}(p_j)$ is steep positive under the same condition. If the slope $f'(p_j)$ is flatter than $g'_{ij}(p_j)$ then the second order condition for candidate i is satisfied.■

Proof of Lemma 3. A direct calculation shows

$$
\begin{aligned}
\omega^m(p_j) &= \sum_{k=m}^{n} C(n,k)p_j^k (1-p_j)^{n-k} (k-np_j) \\
&= p_j \sum_{k=m}^{n} C(n,k) \left[kp_j^{k-1} (1-p_j)^{n-k} - (n-k)\, p_j^k (1-p_j)^{n-k-1} \right] \\
&= p_j \sum_{k=m}^{n} \left[\frac{n!}{(k-1)!\,(n-k)!} p_j^{k-1} (1-p_j)^{n-k} - \frac{n!}{k!\,(n-k-1)!} p_j^k (1-p_j)^{n-k-1} \right] \\
&= mC(n,m)p_j^m (1-p_j)^{n-m+1}
\end{aligned}
$$

Calculating $\omega^{m+1}(p_j)$, we obtain,

$$
\begin{aligned}
\omega^{m+1}(p_j) &= (m+1)\, C(n,m+1)p_j^{m+1} (1-p_j)^{n-m} \\
&= \frac{(n-m)\, p_j}{m\,(1-p_j)} \omega^m(p_j)
\end{aligned}
$$

It is easy to see $\omega^m(p_j) \gtreqless \omega^{m+1}(p_j)$ if and only if $np_j \gtreqless m$.■

Proof of Lemma 4. Suppose $\sum_{k=1}^n r_i^k \mu^k(p_j) < \sum_{k=1}^n r_{i'}^k \mu^k(p_j)$. Then by swapping $(r_i^k)_{k=1}^n$ and $(r_{i'}^k)_{k=1}^n$, the weights on the two candidates are swapped as well. Since $\frac{1-\sigma}{\sigma+\beta-1} > 1$ (convex function) and $a_{ij} > a_{i'j}$, the desired inequality holds. We completed the proof.■

Proof of Lemma 5. Suppose not. Then, there is i' and k' such that $r_{i'}^{k'} \in (0,1)$, and $\sum_{k=1}^n r_i^k \mu^k(p_j) > 0$. Since $\sum_{i=1}^n r_i^{k'} = k'$, there is another i'' with $r_{i''}^{k'} \in (0,1)$. Without loss of generality, let $i' < i''$ (i' has a weakly higher ability). There are two cases: (i) $\mu^{k'}(p_j) > 0$, and (ii) $\mu^{k'}(p_j) < 0$.

Consider case (i) $\mu^{k'}(p_j) > 0$. There are two subcases. First, suppose that $\alpha_{i'} > \alpha_{i''}$. Since R is an optimal rule, $\sum_{k=1}^n r_{i'}^k \mu^k(p_j) \geq \sum_{k=1}^n r_{i''}^k \mu^k(p_j)$ holds (Lemma 3). Then, since $\frac{1-\sigma}{\sigma+\beta-1} > 1$, increasing $r_{i'}^{k'}$ by reducing $r_{i''}^{k'}$ by the same amount will improve E_j. This is a contradiction. Second suppose $\alpha_{i'} = \alpha_{i''}$. In this case, without loss of generality assume that $\sum_{k=1}^n r_i^k \mu^k(p_j) \geq \sum_{k=1}^n r_{i'}^k \mu^k(p_j)$ holds. Then, since $\frac{1-\sigma}{\sigma+\beta-1} > 1$, increasing $r_{i'}^{k'}$ by reducing $r_{i''}^{k'}$ by the same amount will improve E_j. Again, this is a contradiction.

Next consider case (ii) $\mu^{k'}(p_j) < 0$. The argument is totally symmetric. When $\alpha_{i'} > \alpha_{i''}$, decreasing

$r_{i'}^{k'}$ by increasing $r_{i''}^{k'}$ by the same amount will improve E_j. Similarly, when $\alpha_{i'} = \alpha_{i''}$, decreasing $r_{i'}^{k'}$ by increasing $r_{i''}^{k'}$ by the same amount will improve E_j. We have completed the proof.■

Proof of Proposition 7. Under deterministic rules, monotonicity requires a rule to be a list rule. That is, the optimization involves only assigning candidates to a list. The problem boils down to find $m : N_j \to \{1, ..., n\}$ to maximize

$$\max_{R^{dm}} \sum_{i=1}^{n} \alpha_{ij} \left[\sum_{k=1}^{n} r_i^k \mu^k(p_j) \right]^{\frac{1-\sigma}{\sigma+\beta-1}} = \max_{m} \sum_{i=1}^{n} \alpha_{ij} \left[M_{m(i)} \right]^{\frac{1-\sigma}{\sigma+\beta-1}}$$

By letting $\mathcal{M}_{m(i)} = \left[M_{m(i)} \right]^{\frac{1-\sigma}{\sigma+\beta-1}}$, we have $\mathcal{M}_{m(i)} \geq \mathcal{M}_{m(i')} \iff M_{m(i)} \geq M_{m(i')}$. Thus, the solution to above is again $m^* : N_j \to \{1, ..., n\}$ since $\alpha_{1j} \geq ... \geq \alpha_{nj}$. We have completed the proof.□

In the rest of the appendix, we provide an equilibrium analysis for J party case. In order to see how a party's rule choice affects equilibrium probability of winning, we analyze how equilibrium probability distribution responds to an increase in a party member's ability. Then, party j's aggregated effort is written as

$$E_j = \left\{ V \left(\sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}},$$

An equilibrium is described by the following system of equations:[4]

$$\begin{pmatrix} p_1 \\ \vdots \\ p_j \\ \vdots \\ p_J \end{pmatrix} = \begin{pmatrix} \frac{E_1(p_1)}{E_1(p_1)+E_{-1}(p_{-1})} \\ \vdots \\ \frac{E_j(p_j)}{E_j(p_j)+E_{-j}(p_{-j})} \\ \vdots \\ \frac{E_J(p_J)}{E_J(p_J)+E_{-J}(p_{-J})} \end{pmatrix}$$

[4]This analysis is valid for any prize allocation rule and for any functional form of the effort aggregator function.

We will consider a comparative static exercise of increasing α_{ij}. We drop the last equation from the system since $\sum_{j=1}^{J} p_j = 1$. Totally differentiating the system, we obtain

$$\begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} = \begin{pmatrix} \frac{\frac{\partial E_1}{\partial p_1}}{E} - \frac{E_1 \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & -\frac{E_1 \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & -\frac{E_1 \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \\ \vdots & \ddots & \vdots & & \vdots \\ -\frac{E_j \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & \frac{\frac{\partial E_j}{\partial p_j}}{E} - \frac{E_j \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & -\frac{E_j \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \\ \vdots & & \vdots & \ddots & \vdots \\ -\frac{E_{J-1} \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & -\frac{E_{J-1} \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & \frac{\frac{\partial E_{J-1}}{\partial p_{J-1}}}{E} - \frac{E_{J-1} \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \end{pmatrix} \begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial E_j}{\partial \alpha_{ij}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\alpha_{ij}$$

Since $\frac{\frac{\partial E_j}{\partial p_j}}{E} - \frac{E_j \frac{\partial E_j}{\partial p_j}}{E^2} = \frac{1}{E}\frac{\partial E_j}{\partial p_j} - \frac{p_j}{E}\frac{\partial E_j}{\partial p_j}$, we have

$$\begin{pmatrix} 1 - \frac{1}{E}\frac{\partial E_1}{\partial p_1} + \frac{p_1}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{p_1}{E}\frac{\partial E_j}{\partial p_j} & \cdots & \frac{p_1}{E}\frac{\partial E_{J-1}}{\partial p_{r-1}} \\ \vdots & \ddots & \vdots & & \vdots \\ \frac{p_j}{E}\frac{\partial E_1}{\partial p_1} & \cdots & 1 - \frac{1}{E}\frac{\partial E_j}{\partial p_j} + \frac{p_j}{E}\frac{\partial E_j}{\partial p_j} & \cdots & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\ \vdots & & \vdots & \ddots & \vdots \\ \frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & \cdots & 1 - \frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} + \frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \end{pmatrix} \begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial E_j}{\partial \alpha_{mj}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\alpha_{mj}$$

Let $\eta_i = \frac{1}{E}\frac{\partial E_j}{\partial p_j} = \frac{p_j}{E_j}\frac{\partial E_j}{\partial p_j}$ be party i's aggregated effort elasticity of its probability of winning. We can prove the following lemma.

Lemma A1. Suppose that candidate i's ability is increased slightly. Then, we have

$$\frac{dp_j}{d\alpha_{ij}} = \left[\left(1 - \eta_j\right) + \frac{\left(1 - p_j\right)\eta_j}{\sum_{i=1,\ i \neq j}^{J-1} \left(\frac{(1-p_i)\eta_i}{1-\eta_i} \right)} \right]^{-1} \frac{\partial E_j}{\partial \alpha_{mj}}$$

where $\frac{\partial E_j}{\partial \alpha_{mj}} = A \left(\max \left\{ \sum_{k=1}^{n} r_i^k \mu^k(p_j), 0 \right\} \right)^{\frac{1-\sigma}{\sigma+\beta-1}}$ for $A > 0$.

This technical lemma provides two important implications when the contents of the bracket is positive. First, an increase in α_{ij} increases party j's winning probability as long as she makes positive effort in equilibrium, which is dictated by the sign of candidate i's incentive term, $\sum_{k=1}^{n} r_i^k \mu^k(p_j)$. Second, the party can increase E_j by adjusting $q = (q^k)_{k=1}^{n}$ to shift weights $\sum_{k=1}^{n} r_i^k \mu^k(p_j)$ from low ability candidates to high ability candidates.

Proof of Proposition 11. Let the matrix in the left-hand side be D. Then, the determinant of D is

$$
|D| = \begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1}+\frac{p_1}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{p_1}{E}\frac{\partial E_j}{\partial p_j} & \cdots & \frac{p_1}{E}\frac{\partial E_{J-1}}{\partial p_{r-1}} \\
\vdots & \ddots & \vdots & & \vdots \\
\frac{p_j}{E}\frac{\partial E_1}{\partial p_1} & \cdots & 1-\frac{1}{E}\frac{\partial E_j}{\partial p_j}+\frac{p_j}{E}\frac{\partial E_j}{\partial p_j} & \cdots & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\
\vdots & & \vdots & \ddots & \vdots \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & \cdots & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}+\frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & & 0 & & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & & 0 & & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & \ddots & & & \vdots & \vdots \\
0 & 0 & & 1-\frac{1}{E}\frac{\partial E_j}{\partial p_j} & & 0 & -\frac{p_j}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}+\frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & & 0 & & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & & 0 & & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & \ddots & & & \vdots & \vdots \\
0 & 0 & & 1-\frac{1}{E}\frac{\partial E_j}{\partial p_j} & & 0 & -\frac{p_j}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix}
$$

$$
+ \begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & & 0 & & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & & 0 & & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & \ddots & & & \vdots & \vdots \\
0 & 0 & & 1-\frac{1}{E}\frac{\partial E_j}{\partial p_j} & & 0 & -\frac{p_j}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & & \frac{p_{J-1}}{E}\frac{\partial E_j}{\partial p_j} & & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & \frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix}
$$

$$
\begin{aligned}
&= \prod_{i=1}^{J-1}\left(1-\frac{1}{E}\frac{\partial E_i}{\partial p_i}\right)+\prod_{i=1}^{J-1}\left(1-\frac{1}{E}\frac{\partial E_i}{\partial p_i}\right)\sum_{i=1}^{J-1}\frac{\frac{p_i}{E}\frac{\partial E_i}{\partial p_i}}{1-\frac{1}{E}\frac{\partial E_i}{\partial p_i}} \\
&= \prod_{i=1}^{J-1}(1-\eta_i)\sum_{j=1}^{J-1}\left(1+\frac{p_j\eta_j}{1-\eta_j}\right),
\end{aligned}
$$

where $\eta_i = \frac{1}{E}\frac{\partial E_j}{\partial p_j} = \frac{p_j}{E_j}\frac{\partial E_j}{\partial p_j}$ is ith party's aggregated effort elasticity of its probability of winning. If we impose stability on equilibrium, then it is natural to assume $\eta_i \in (0,1)$ for all $i = 1, ..., J$. Thus, under stability, $|D| > 0$ is assured. Now, we can conduct a comparative static analysis:

$$
\begin{aligned}
\frac{dp_j}{d\alpha_{mj}} &= \frac{1}{|D|}\begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1}+\frac{p_1}{E}\frac{\partial E_1}{\partial p_1} & \cdots & 0 & \cdots & \frac{p_1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\
\vdots & \ddots & \vdots & & \vdots \\
\frac{p_j}{E}\frac{\partial E_1}{\partial p_1} & \cdots & \frac{\partial E_j}{\partial \alpha_{mj}} & \cdots & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\
\vdots & & \vdots & \ddots & \vdots \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \cdots & 0 & \cdots & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}+\frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix} \\
&= \frac{1}{|D|}\begin{vmatrix}
1-\frac{1}{E}\frac{\partial E_1}{\partial p_1} & 0 & & 0 & & 0 & -\frac{p_1}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
0 & 1-\frac{1}{E}\frac{\partial E_2}{\partial p_2} & & 0 & & 0 & -\frac{p_2}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\vdots & & \ddots & \vdots & & \vdots & \vdots \\
\frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} & & \frac{\partial E_j}{\partial \alpha_{mj}} & & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} & \frac{p_j}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}} \\
\vdots & & & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1-\frac{1}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & -\frac{p_{J-2}}{p_{J-1}}\left(1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}\right) \\
\frac{p_{J-1}}{E}\frac{\partial E_1}{\partial p_1} & \frac{p_{J-1}}{E}\frac{\partial E_2}{\partial p_2} & & 0 & & \frac{p_{J-1}}{E}\frac{\partial E_{J-2}}{\partial p_{J-2}} & 1-\frac{1}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}+\frac{p_{J-1}}{E}\frac{\partial E_{J-1}}{\partial p_{J-1}}
\end{vmatrix} \\
&= \frac{\frac{\partial E_j}{\partial \alpha_{mj}}}{|D|}\prod_{i=1,i\neq j}^{J-1}(1-\eta_i)\sum_{i=1,i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right) > 0
\end{aligned}
$$

Thus, by Cramer's rule, we have

$$
\begin{aligned}
\frac{dp_j}{d\alpha_{mj}} &= \frac{\prod_{i=1,\ i\neq j}^{J-1}(1-\eta_i)\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)\frac{\partial E_j}{\partial \alpha_{mj}}}{\prod_{i=1}^{J-1}(1-\eta_i)\sum_{i=1}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)} \\
&= \frac{\sum_{i=1}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)-1-\frac{p_j\eta_j}{1-\eta_j}}{\left(1-\eta_j\right)\sum_{i=1}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)}\times\frac{\partial E_j}{\partial \alpha_{mj}} \\
&= \frac{\prod_{i=1,\ i\neq j}^{J-1}(1-\eta_i)\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)}{\prod_{i=1}^{J-1}(1-\eta_i)\left[\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)+1+\frac{p_j\eta_j}{1-\eta_j}\right]}\times\frac{\partial E_j}{\partial \alpha_{mj}} \\
&= \frac{1}{\left(1-\eta_j\right)\left[1+\frac{1+\frac{p_j\eta_j}{1-\eta_j}}{\sum_{i=1,\ i\neq j}^{J-1}\left(1+\frac{p_i\eta_i}{1-\eta_i}\right)}\right]}\times\frac{\partial E_j}{\partial \alpha_{mj}} \\
&= \left[\left(1-\eta_j\right)+\frac{\left(1-p_j\right)\eta_j}{\sum_{i=1,\ i\neq j}^{J-1}\left(\frac{(1-p_i)\eta_i}{1-\eta_i}\right)}\right]^{-1}\times\frac{\partial E_j}{\partial \alpha_{mj}}
\end{aligned}
$$

Note that for all $m=1,...,n$ and all $j=1,...,J$, we have

$$
\frac{\partial E_j}{\partial \alpha_{mj}}=\frac{\sigma+\beta-1}{\beta\left(1-\sigma\right)}E_j\times\frac{\omega_j(m)}{\sum_{m'=1}^{n}\alpha_{m'j}\omega_j(m')}>0.
$$

Thus, we obtained natural comparative static results.□

References

[1] Crutzen, B.S.Y., S. Flamand, and N. Sahuguet, 2017, Prize allocation and incentives in team contests, Working Paper.

[2] Crutzen, B.S.Y., and N. Sahuguet, 2017, Electoral incentives: the interaction between candidate selection and electoral rules, Working Paper