

Recurrent Bubbles, Economic Fluctuations, and Growth*

Pablo A. Guerron-Quintana[†] Tomohiro Hirano[‡] Ryo Jinnai[§]

September 23, 2017

Abstract

We propose a model that generates permanent effects on economic growth following a recession (super hysteresis). Recurrent bubbles are introduced to an otherwise standard infinite-horizon business-cycle model with liquidity scarcity and endogenous productivity. In our setup, bubbles promote growth because they provide liquidity to constrained investors. Bubbles are sustained only when the financial system is under-developed. If the financial development is in an intermediate stage, recurrent bubbles can be harmful in the sense that they decrease the unconditional mean and increase the unconditional volatility of the growth rate relative to the fundamental equilibrium in the same economy. Through the lens of an estimated version of our model fitted to U.S. data, we argue that 1) there is evidence of recurrent bubbles; 2) the Great Moderation results from the collapse of the monetary bubble in the late 1970s; and 3) the burst of the housing bubble is partially responsible for the post-Great Recession dismal recovery of the U.S. economy.

1 Introduction

Recent crises have left a lasting effect on the level of output around the world. The Great Recession is an example of this scaring impact on economy activity. But recessions may also have an enduring impact on the growth rate of the economy, a phenomenon referred as super hysteresis (Ball (2014)). Indeed, Blanchard, Cerutti, and Summers (2014), using a sample of 23 advanced countries over 50 years, report that “about two thirds of recessions are followed by lower output relative to its pre-recession trend.” More important, “in about one half of those cases, the recession is followed not just by lower output, but by lower output growth relative to its pre-recession growth rate.”

*We are thankful to Dongho Song for extensive discussions and help with Markov switching estimation. We also benefited from useful comments by Susanto Basu, Bernard Dumas, Andrew Foerster, Nan Li, Vincenzo Quadrini, Rosen Valchev, and seminar participants at Bank of Canada, Boston College, Shanghai Jiao Tong University, the Norwegian Business School, and the NBER Summer Institute.

[†]Boston College and Espol, pguerron@gmail.com

[‡]University of Tokyo, tomohih@gmail.com

[§]Hitotsubashi University, rjinnai@ier.hit-u.ac.jp

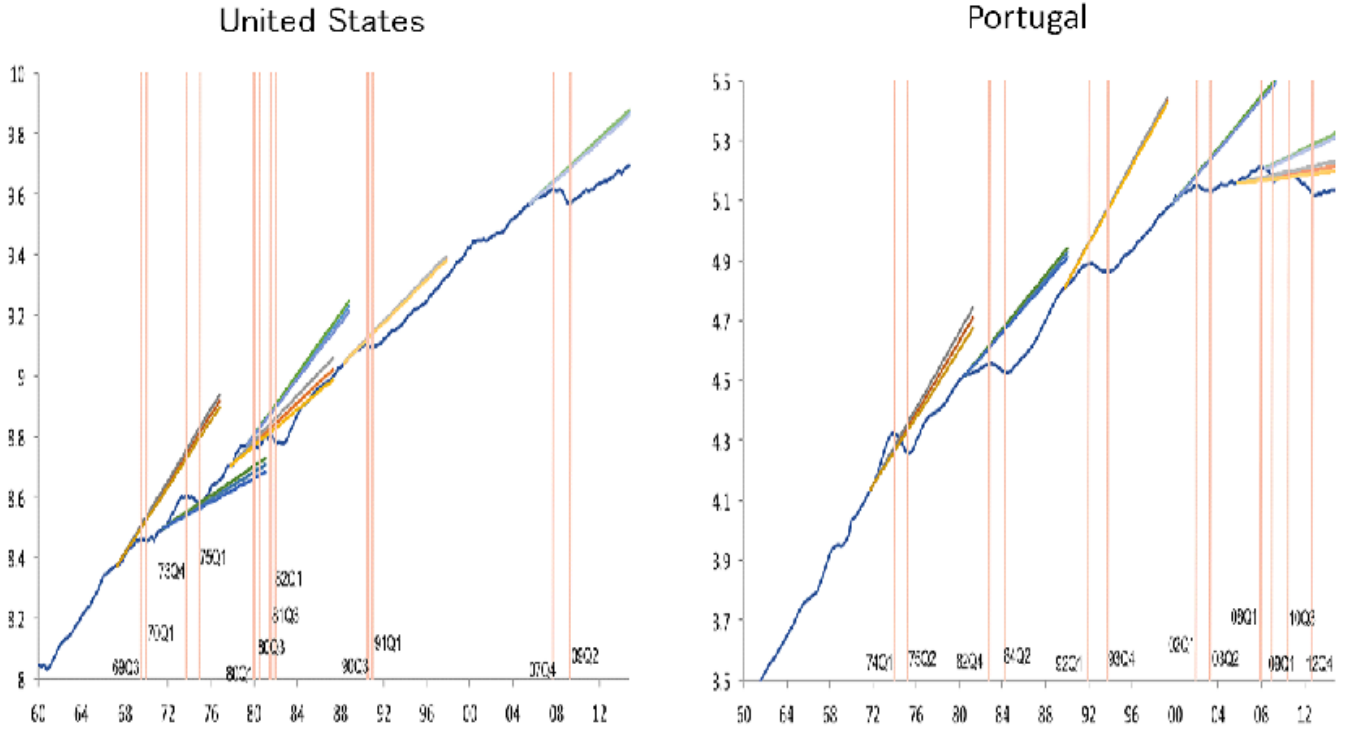


Figure 1: Super hysteresis in action. Real GDP taken from Blanchard, Cerutti, and Summers (2014). Straight lines correspond to pre-recession trends.

In the U.S., the 5-year average growth rate of GDP before the 2001 and 2008 crises were 4.3% and 2.9%, respectively. In the five years following the crisis, the average growth rates went down to 2.9% and 1.9% (right panel in Figure 1). Portugal offers a more sobering example of super hysteresis in action in recent years (left panel in Figure 1). Interestingly, Blanchard, Cerutti, and Summers argue that it is “difficult to think mechanisms that lead to super-hysteresis.” In this paper, we take on the task of understanding how and when hysteresis and super hysteresis arise in the economy.

We generate super hysteresis using a tractable model of recurrent bubbles, liquidity scarcity, and endogenous productivity. Investors are liquidity constrained in the sense of Kiyotaki and Moore (2012), resulting in depressed investment and low growth. In this environment, bubbles may mitigate the problem by providing extra liquidity, which in our endogenous growth model enhances economic growth. But if bubbles are helpful, their burst is harmful because it is followed by a sharp economic contraction and then prolonged low growth. The stagnation only ends when a new bubble emerges.

Bubbles are intrinsically useless assets in our model; they contribute neither to production nor households’ utility. However, bubbles are particularly liquid, nothing prevents their trades in spot markets when they exist, and liquidity service may convince people to hold them even though their returns are clearly dominated by other less liquid assets. Yet the existence of bubbles

requires special circumstances. They exist only if aggregate liquidity is in short supply, and only if everyone believes that bubbles are traded at a positive price. For tractability, we assume that there is a period of time in which bubbles cannot arise and be traded for exogenous reasons. Under this assumption, we analyze an interesting regime-switching equilibrium in which bubbles exist and are traded in one regime, and no such assets exist in the other.

In a calibrated version of our model, we find that the impact of bubbles on economic growth critically depends on the fundamentals of the economy. Particularly important is the degree of financial development, which is represented in the model by the tightness of the liquidity constraints. If the economy is financially underdeveloped, investors cannot get enough funds from selling equity on their capital. Because bubbles are liquid, they mitigate the problem of a weak financial system and hence they enhance both growth and employment. These benefits, however, come at the expense of volatility emanating from two sources. First, the economy switches between periods of bubbles with high growth and bubbleless periods with low growth. Second, the bubbly economy is more responsive to supply and demand shocks, implying that volatility is higher in the bubbly regime than in the fundamental.¹

In contrast, if the financial market is relatively developed from the beginning, bubbles lower the growth rate of the economy. This is because bubbles strengthen the household's incentive to raise the capacity utilization rate, which results from bubbles and capital being substitutes as sources of liquidity. As a result, investors depend less on capital to obtain funds. Excessive capital utilization leads to fast depreciation, lowering net investment, and hence the growth of the economy, even though gross investment increases. Interestingly, this channel operates not only when bubbles actually exist but also in the bubbleless period because the price of capital is affected by the possibility of bubbles arising in the future through the Euler equation.

We exploit these previous insights to map our model to the post World War II data in the U.S. Our estimation reveals the existence of a persistent bubble prior to 1980. As we move through the 1980s and forward, bubbles became less persistent with one coinciding with the housing boom and a second one at the end of our sample. Through the eyes of our model, lower volatility post-84 results in part from the absence of persistent bubbles. One possible interpretation of a pre-Moderation bubble is the central bank's attempt to exploit the Phillips curve by providing easy money. The Fed's realization that this was not possible led to the bubble burst, leading to lower volatility but also lower growth. When we estimate our model, we find that demand shocks are more volatile than technology shocks. In spite of the moderating effect of the bubble burst, there is also a significant decline in the volatility of shocks post-1984.

Our model also provides an intuitive explanation of the slowdown in growth over the past decades. As will become clear, bubbles enhances growth in our framework. To the extent that the 1970s, 1990s, and mid 2000s were periods associated with monetary, IT, and housing bubbles,

¹This is a natural consequence of the liquidity services provided by bubbles. In our model, bubbles increase liquidity so investors find it easier to invest more. During recessions, liquidity is tighter, leading to deeper downturns.

the collapse of these bubbles lead inevitably to slower growth. Furthermore, growth will remain depressed until a new bubble arises in the economy. Our model is rich enough that it can account for the post-Great Recession downward shift in the trend of economic activity in the U.S. Indeed, a temporary financial shock results in lower investment, which through an endogenous productivity channel leads to permanently lower trend in output even though the growth rate of the economy returns to its pre-crisis level.

Dealing with bubbles in DSGE models is intrinsically complicated. This is so because one must track the history of booms and bursts to characterize the current state of the economy. In our model, the states are capital, exogenous shocks, and an indicator of the regime: fundamental or bubble. Since the economy switches between the two regimes, capital is regime dependent. But because of endogenous productivity, capital is a sufficient statistic for the history of bubbles. So once we de-trend the model using capital, there is no longer regime dependence and the equilibrium conditions depend on only the exogenous states of the economy. This model can be easily solved by standard methods and is amenable to estimation.

The rest of the paper proceeds as follows. Next, we highlight the contributions of our model to the existing literature. We describe the baseline model in section 3. In section 4 and 5, we discuss issues such as existence of bubbles, their effect on growth and show dynamic responses implied by our model. The empirical results with a discussion of the Great Moderation and the Great Recession are in section 6.

2 Related Work in the Literature

Our paper is in line with the literature on rational bubbles in infinite horizon economies with imperfect financial markets. The seminal papers are Bewley (1980), Townsend (1980), Scheinkman and Weiss (1986), and Woodford (1990).² These papers study deterministic fiat money (or government bonds) in an endowment economy when borrowing and lending are not allowed. Although these studies prove the existence of deterministic bubbles in infinite horizon economies, they do not necessarily show the necessary conditions explicitly for the reason asset bubbles can occur. Kocherlakota (1992) derive the necessary conditions for deterministic bubbles in an endowment economy when borrowing is allowed. Kocherlakota (2009) extends Kocherlakota (1992) to include a production economy without growth, and examines the effects of land bubbles on production.

Based on these seminal papers, we develop an endogenous growth model with financial frictions, and examine recurrent asset bubbles and their impact on long run economic growth. In this regard, our paper is related to Hirano and Yanagawa (2017). There are, however, substantial differences. First, we consider recurrent bubbles, i.e., bubbles are expected to arise and to collapse recurrently

²Samuelson (1958) is the first paper showing rational bubbles in an overlapping generations model. Tirole (1982) extends the Samuelson model to include production. See Farhi and Tirole (2012), Miao (2014), and Allen, Barlevy, and Gale (2017) for the recent development on rational bubbles in overlapping generations models.

in the future, while Hirano and Yanagawa study the stochastic bubbles developed by Weil (1987). That is, a bubble is expected to collapse, but its reappearance is not expected at all. Second, the role of bubbles is different between Hirano and Yanagawa's paper and ours. Hirano and Yanagawa emphasize the role of bubbles as speculative vehicles. Agents buy and sell bubble assets mainly because they provide a high rate of return. In contrast, our paper emphasizes the role of bubbles as liquid assets, i.e., bubbles can be sold quickly compared with illiquid capital. Our formulation of bubbles is based on Kiyotaki and Moore (2012) where deterministic fiat money is described as a liquid asset. We show under what conditions recurrent bubbles with high liquidity can arise in equilibrium, and examine their impact on business cycles and the long-run economic growth rate.

Regarding recurrent bubbles, our paper is related to Gali (2014) and Miao and Wang (2017). In their papers, only a fraction of the existing bubbles collapses every period, and new bubbles are created right away so that aggregate supply of bubble assets is kept constant over time. This means that the economy is always in the bubbly regime. There is no entire collapse of bubbles. In our model, the emergence and entire collapse of bubbles is recurrent. As a consequence, the economy repeatedly switches between the bubbly regime and the bubbleless regime.

Moreover, Gali (2014) and Miao and Wang (2017) focus on a local analysis of the bubbly steady state. In our model, however, the entire breaking of bubbles implies that the economy no longer stays around the neighborhood of the bubbly steady-state. That is, the collapse of bubbles causes a sudden regime shift to the bubbleless economy, generating highly non-linear effects on macroeconomic activity. Importantly, these non-linear effects are anticipated by agents ex-ante and have major consequences on the model's dynamics. In this regard, our paper shares a similarity with the non-linear effects emphasized by Brunnermeier and Sannikov (2014), He and Krishnamurthy (2013), and Gertler and Kiyotaki (2015). In these papers, relatively large shocks to an economy cause the economy to jump far away from steady state, producing highly non-linear effects. They emphasize that this non-linearity is important to account for financial crisis phenomena.

The recurrent bubbles in Martin and Ventura (2012) are more similar to ours, in the sense that there is an entire collapse of bubbles. However, our papers differ in important dimensions. First, their model is based on an overlapping generations model, and agents live for only two periods. Hence anticipations about reappearance and recollapsing of bubbles in the future do not affect decisions of the current young agents at all. Their recurrent bubbles are essentially the same as the stochastic bubbles developed by Weil (1987), in which agents consider only the probability of the bubble bursts. Unlike theirs, in our model, infinitely-lived agents fully anticipate both the probability of reappearance and recollapsing in the future. Thus expectations about recurrent bubbles affect consumption, investment, and economic growth in the current period, which in turn affects bubble prices in the future. In this sense, there is a two-way feedback effect between macroeconomic activity and recurrent bubbles across time. This is a unique property in our model with infinitely-lived agents.

Furthermore, Martin and Ventura (2012) use a linear utility function, and agents consume only in old periods. Because of this assumption, agents do not care about the volatility arising from the collapse of bubbles. In our paper, however, agents are risk averse, and they fully anticipate the probability of recurrent bubbles. Hence, agents care about volatility arising from recurrent bubbles, which is crucial in our welfare analysis. Finally, Martin and Ventura’s model does not have mechanisms that are standard in the business cycle literature such as the intertemporal Euler equation, endogenous labor supply, and endogenous capacity utilization. In contrast, our model is a standard real business cycle model, in which both propagation through dynamic optimization and amplification through intra-temporal optimization of time allocation and capacity utilization are present. These features are important because we estimate our model using U.S. data.

Our paper is also related to the effects on long run economic growth of various types of financial crises. For example, Cerra and Saxena (2008) show that most financial crises are associated with a decline in growth that leaves output permanently below its pre-crisis trend. Our paper shows that the collapse of bubbles causes permanently lower output level than its pre-bubble burst trend, but also generates permanently lower long run economic growth, i.e., super-hysteresis. Furthermore we relate to studies on the role of financial development and growth as in Aghion, Howitt, and Mayer-Foulkes (2005). Unlike their paper, ours focuses on 1) the provision of liquidity as a way to overcome underdeveloped financial systems; and 2) the impact of bubbles in economic growth.

Our study of hysteresis is connected to previous work such as Gali (2016). This paper studies hysteresis in labor markets and the design of monetary policy. We view our papers as complementary since we highlight the role that bubbles may have in creating not only hysteresis but also super hysteresis in economic activity. Finally, we relate to the literature on the solution and estimation of Markov switching models as in Farmer, Waggoner, and Zha (2009), Bianchi (2013), and Hamilton (2016).

3 Model

Our description of the model consists of regimes, firms, households, and endogenous productivity.

3.1 Regimes

Let z_t denote a realization of the regime $z_t \in \{b, f\}$ where b and f denote a bubble and a fundamental regime, respectively. There are no bubble assets in the economy in a fundamental regime. If the regime switches to a bubble regime, M units of bubble assets are created and given to households in a lump-sum way. There is no bubble creation in other contingencies. Bubble assets last without depreciation as long as the economy stays in the bubble regime. They disappear suddenly and completely once regime switches back to a fundamental regime. We assume that z_t

follows a Markov process satisfying

$$\Pr(z_t = f | z_{t-1} = f) = 1 - \sigma_f \quad (1)$$

and

$$\Pr(z_t = b | z_{t-1} = b) = 1 - \sigma_b. \quad (2)$$

Bubble assets are intrinsically useless. They contribute to neither production nor households' utilities directly. Furthermore, they do not offer dividends to their owners either.

3.2 Firms

Output is produced using capital and labor services denoted by KS_t^D and L_t^D , respectively. The production function is

$$Y_t = A_t (KS_t^D)^\alpha (L_t^D)^{1-\alpha}$$

where A_t is the technology level which agents in the economy take as given. Competitive firms maximize profits defined as

$$Y_t - r_t KS_t^D - w_t L_t^D$$

by choosing KS_t^D and L_t^D , taking rental price of capital r_t and wage rate w_t as given. First order conditions are

$$r_t = \alpha \frac{Y_t}{KS_t^D}$$

and

$$w_t = (1 - \alpha) \frac{Y_t}{L_t^D}.$$

3.3 Households

The economy is populated by a continuum of households, with measure one. Each household has a unit measure of members who are identical at the beginning of a period. During the period, members are separated from each other, and each member receives a shock that determines the role of the member in the period. A member will be an investor with probability $\pi \in [0, 1]$ and will be a saver with probability $1 - \pi$. These shocks are i.i.d. among the members and across time.

A period is divided into four stages: household's decisions, production, investment, and consumption. In the household's decision stage, all members of a household are together and pool their assets: n_t units of equities and \tilde{m}_t units of bubble assets. An equity is the ownership of a unit of capital. Aggregate shocks to exogenous state variables are realized. The capacity utilization rate u_t is decided. Because all the members of the household are identical in this stage, the household head evenly divides the assets among the members. The household head also gives

contingency plans to each member as follows. If one becomes an investor, he or she spends i_t units of final goods to invest, and brings back home x_t^i units of final goods, n_{t+1}^i units of equity claims, and \tilde{m}_{t+1}^i units of bubble assets before the consumption stage. In contrast, if the member becomes a saver, he or she supplies l_t units of labor, and brings back home x_t^s units of final goods, n_{t+1}^s units of equity claims, and \tilde{m}_{t+1}^s units of bubble assets before the consumption stage. After receiving these instructions, members go to the market and remain separated from each other until the consumption stage.

At the beginning of the production stage, each member receives the shock determining his or her role in the period. Competitive firms produce final goods. Compensations to productive factors are paid to their owners. A fraction $\delta(u_t)$ of capital depreciates, where

$$\delta(u_t) = \delta(1) + \frac{\delta'(1)}{1+\zeta} \left(u_t^{1+\zeta} - 1 \right).$$

An advantage of this functional form is that the elasticity of $\delta(u_t)$ is constant at ζ ;

$$\frac{u_t \delta''(u_t)}{\delta'(u_t)} = \zeta$$

for all u_t .

Investors seek finance and undertake investment projects in the investment stage. The technology is linear; they transform any amount i_t units of final goods into i_t units of new capital. Asset markets close at the end of this stage.

Members of the household meet again in the consumption stage. An investor consumes c_t^i units of final goods and a saver consumes c_t^s units of final goods.

The instructions must meet a set of feasibility constraints. First, they have to satisfy the intra-temporal budget constraints, i.e.,

$$x_t^i + i_t + q_t \left(n_{t+1}^i - i_t - (1 - \delta(u_t)) n_t \right) + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t \left(\tilde{m}_{t+1}^i - \tilde{m}_t \right) = u_t r_t n_t \quad (3)$$

for an investor and

$$x_t^s + q_t \left(n_{t+1}^s - (1 - \delta(u_t)) n_t \right) + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t \left(\tilde{m}_{t+1}^s - \tilde{m}_t \right) = u_t r_t n_t + w_t l_t \quad (4)$$

for a saver, where q_t and \tilde{p}_t denote prices of equities and bubbles, respectively. The indicator function in front of \tilde{p}_t captures the idea that there is neither spot nor future market for bubbles in a fundamental regime. One interpretation of this restriction is that it is impossible to predict the nature of the next bubble while living in a fundamental state. Without markets, none can

purchase bubbles in a fundamental regime, which we formalize by the following constraints;

$$\mathbf{1}_{\{z_t=f\}}\tilde{m}_{t+1}^i = \mathbf{1}_{\{z_t=f\}}\tilde{m}_{t+1}^s = 0. \quad (5)$$

There is a feasibility constraint in the consumption stage given by

$$\pi x_t^i + (1 - \pi) x_t^s = \pi c_t^i + (1 - \pi) c_t^s. \quad (6)$$

An investor can issue new equity on at most a fraction ϕ of investment. In addition, she can sell at most a fraction ϕ of existing capital in the market.³ Effectively, these constraints introduce a lower bound to the capital holdings of an entrepreneur at the end of the period:

$$n_{t+1}^i \geq (1 - \phi) (i_t + (1 - \delta(u_t)) n_t). \quad (7)$$

Following Shi (2015), we call equation (7) a liquidity constraint. A similar constraint applies to savers, but we omit it because it does not bind in equilibrium (they are net buyers of equities). We also omit non-negativity constraints for u_t , c_t^i , i_t , n_{t+1}^i , x_t^s , c_t^s , l_t , n_{t+1}^s , and \tilde{m}_{t+1}^s for the same reason. Exceptions are both a short-sale constraint for investors

$$\tilde{m}_{t+1}^i \geq 0 \quad (8)$$

and a borrowing constraint for investors

$$x_t^i \geq 0. \quad (9)$$

The household problem is written as follows. They choose a sequence of u_t , x_t^i , c_t^i , i_t , n_{t+1}^i , \tilde{m}_{t+1}^i , x_t^s , c_t^s , l_t , n_{t+1}^s , and \tilde{m}_{t+1}^s to maximize

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{b_t} \left(\pi \frac{[c_t^i]^{1-\rho}}{1-\rho} + (1-\pi) \frac{[c_t^s (1-l_t)^\eta]^{1-\rho}}{1-\rho} \right) \right]$$

subject to (3), (4), (5), (6), (7), (8), (9), and the laws of motions of assets given by

$$n_{t+1} = \pi n_{t+1}^i + (1 - \pi) n_{t+1}^s \quad (10)$$

and

$$\tilde{m}_{t+1} = \pi \tilde{m}_{t+1}^i + (1 - \pi) \tilde{m}_{t+1}^s + \mathbf{1}_{\{z_t=f, z_{t+1}=b\}} M \quad (11)$$

for all $t \geq 0$. Initial portfolio is $\{n_0, \tilde{m}_0\} = \{K_0, \mathbf{1}_{\{z_t=b\}} M\}$ where K_t is capital stock in the economy in period t . b_t is a preference shock.

³These two constraints are different in nature. Kiyotaki and Moore (2012) carefully distinguish the two, calling the former the borrowing constraint and the latter the resalability constraint. We however let a single parameter ϕ govern them to simplify the model.

3.4 Learning-by-doing

We assume that the technology level A_t is endogenous;

$$A_t = \bar{A} (K_t)^{1-\alpha} e^{a_t}.$$

a_t is a productivity shock and \bar{A} is a scale parameter. Following Arrow (1962), Sheshinski (1967), and Romer (1986), we interpret the dependency of A_t on K_t as learning-by-doing; namely, knowledge is a by-product of investment and in addition, it is a public good that anyone can access at zero cost. With it, the long-run tendency for capital to experience diminishing returns is eliminated. We want to stress that the details behind the endogenous productivity mechanism are largely irrelevant for our purposes. Similar results would attain if we relied on expanding-variety or creative-destruction framework.

3.5 Market Clearing

Competitive equilibrium is defined in a standard way; all economic agents optimize given prices, and markets clear;

$$n_{t+1} = K_{t+1}, \tag{12}$$

$$L_t^D = (1 - \pi) l_t,$$

$$KS_t^D = u_t K_t,$$

and

$$\pi c_t^i + (1 - \pi) c_t^s + \pi i_t = Y_t$$

for all t , and

$$\pi \tilde{m}_{t+1}^i + (1 - \pi) \tilde{m}_{t+1}^s = M$$

in a bubble regime. Because $\pi \tilde{m}_{t+1}^i + (1 - \pi) \tilde{m}_{t+1}^s = 0$ holds in a fundamental regime, we have

$$\pi \tilde{m}_{t+1}^i + (1 - \pi) \tilde{m}_{t+1}^s = \mathbf{1}_{\{z_t=b\}} M \tag{13}$$

for all t . The law of motion for capital is

$$K_{t+1} = (1 - \delta(u_t)) K_t + \pi i_t,$$

which automatically holds by Walras' law.

4 Permanent Fundamental

We first consider a special case in which the economy is always in a fundamental regime; i.e., $z_0 = f$ and $\sigma_f = 0$, and hence $z_t = f$ for all $t \geq 0$. Guerron-Quintana and Jinnai (2015) use a variant of this fundamental model to study the implications of the 2008/2009 financial crisis on the level of output in the U.S. economy, showing that a temporary financial shock can trigger a secular stagnation in an estimated model.

4.1 Equilibrium with no binding liquidity constraint

Let us consider an equilibrium in which the price of capital is always equal to its marginal costs, i.e., $q_t = 1$. The household is indifferent between investing in capital in-house and purchasing capital in the market. Hence the liquidity constraint (7) does not bind in this equilibrium. The borrowing constraint (9) does not bind either; if it does, the household can make it loose without affecting other constraints or the amount of capital at the end of period t by increasing x_t^i by $\Delta > 0$, decreasing x_t^s by $(\pi/(1-\pi))\Delta$, decreasing n_{t+1}^i by Δ , and increasing n_{t+1}^s by $(\pi/(1-\pi))\Delta$.

With these observations, the household's problem can be simplified to

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{b_t} \left(\pi \frac{[c_t^i]^{1-\rho}}{1-\rho} + (1-\pi) \frac{[c_t^s (1-l_t)^\eta]^{1-\rho}}{1-\rho} \right) \right]$$

subject to

$$\pi c_t^i + (1-\pi) c_t^s + n_{t+1} - (1-\delta(u_t)) n_t = u_t r_t n_t + w_t (1-\pi) l_t.$$

This is a standard household problem, and so are the first conditions,

$$(c_t^i)^{-\rho} = (c_t^s)^{-\rho} (1-l_t)^{\eta(1-\rho)}, \quad (14)$$

$$\eta \frac{c_t^s}{1-l_t} = w_t, \quad (15)$$

$$r_t - \delta'(u_t) = 0,$$

and

$$1 = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (u_{t+1} r_{t+1} + 1 - \delta(u_{t+1})) \right].$$

The first equation states that the marginal utility from consumption has to be equalized across members. The second equation states that the marginal rate of substitution of leisure for consumption has to be equal to the real wage. The third equation states that the marginal benefit of raising the capacity utilization rate, i.e, the rental price of capital, has to be equal to its opportunity cost, i.e., the amount of capital depreciated at the margin. The last equation is the standard Euler equation for capital accumulation.

It will be reasonable to expect that this equilibrium realizes when the liquidity constraint is sufficiently loose (when ϕ is large). We confirm this intuition numerically in a subsequent section.

4.2 Equilibrium with binding liquidity constraint

Let us consider an equilibrium in which the price of capital always satisfies $1 < q_t < 1/\phi$, which guarantees that investing is profitable (return is bigger than marginal cost) but not enough to allow the entrepreneur to invest with no down-payments. Under this scenario, the inequality constraints (7) and (9) always bind in this equilibrium for the following reasons. If (7) is not binding, households can increase utility without violating any constraints or affecting their portfolio at the end of the period by increasing i_t by $\Delta > 0$, increasing n_{t+1}^i by $(q_t - 1)\Delta/q_t$, increasing both x_t^s and c_t^s by $(\pi/(1 - \pi))(q_t - 1)\Delta$, and decreasing n_{t+1}^s by $(\pi/(1 - \pi))((q_t - 1)/q_t)\Delta$, which is a contradiction to the household's optimization. If (9) is not binding, households can relax (7) without violating any constraints or affecting their portfolio at the end of the period by decreasing x_t^i by Δ , increasing x_t^s by $(\pi/(1 - \pi))\Delta$, increasing n_{t+1}^i by $(1/q_t)\Delta$, and decreasing n_{t+1}^s by $(\pi/(1 - \pi))(1/q_t)\Delta$. This is a contradiction to the household's optimization because they can increase utility if (7) is not binding.

Because (7) and (9) hold with equality, the optimal investment level is given by

$$i_t = \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] n_t}{1 - \phi q_t}. \quad (16)$$

Substituting (7) and (16) into (10), we find

$$n_{t+1} = \pi \frac{1}{q_t} (1 + \lambda_t) [u_t r_t + \phi q_t (1 - \delta(u_t))] n_t + \pi (1 - \phi) (1 - \delta(u_t)) n_t + (1 - \pi) n_{t+1}^s \quad (17)$$

where λ_t is defined as

$$\lambda_t = \frac{q_t - 1}{1 - \phi q_t}. \quad (18)$$

λ_t is the variable Shi (2015) calls the liquidity service. Substituting (6) and (17) into (4), we rewrite the household's problem as follows;

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{b_t} \left(\pi \frac{[c_t^i]^{1-\rho}}{1-\rho} + (1-\pi) \frac{[c_t^s (1-l_t)^\eta]^{1-\rho}}{1-\rho} \right) \right]$$

subject to

$$\pi c_t^i + (1 - \pi) c_t^s + q_t n_{t+1} = [u_t r_t + (1 - \delta(u_t)) q_t + \pi \lambda_t (u_t r_t + \phi q_t (1 - \delta(u_t)))] n_t + (1 - \pi) w_t l_t$$

for all t . The first order conditions with respect to both the intra-temporal consumption allocation (14) and the labor supply by savors (15) are the same as before. The optimality condition with

respect to the capacity utilization rate is

$$r_t - \delta'(u_t) q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0. \quad (19)$$

The capital Euler equation is

$$q_t = E_t \left[\beta e^{b_{t+1} - b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \right]. \quad (20)$$

Note how the price of capital q_t as well as the liquidity service λ_t affects (19) and (20). q_t appears in the second term in (19) because the opportunity costs of raising the capacity utilization rate is the market value of depreciated capital at the margin. λ_t appears in the third term in (19) because raising the capacity utilization rate provides additional liquidity to investors. In (20), λ_t appears in the right-hand side because capital plays dual roles in the current environment with binding liquidity constraints. Namely, capital is not only a production factor but also a means of providing liquidity to its owners. (20) states that the capital should be valued based on both of these contributions, i.e., dividends as well as liquidity services.

4.3 Calibration

The following parameters are directly set at standard values. We set $\pi = 0.05$, $\alpha = 0.4$, $\beta = 0.99$, and $\rho = 1$. We set the elasticity of $\delta'(u_t)$ at $\zeta = 0.33$, a standard value in the business-cycle literature (Comin and Gertler (2006)). We normalize the capacity utilization rate along the balanced growth path at $u = 1$, and set the capital depreciation rate along the balanced growth path at $\delta(1) = 0.025$.

The rest of parameters are calibrated using the model with no binding liquidity constraint as a benchmark. We calibrate η so that labor supply along the balanced growth path is $l = 0.25$. Finally, we set \bar{A} so that the rental rate of capital along the balanced growth path is $r = 0.05$. We view this calibration as a reasonable one that allows us to study the main properties of the alternative models. Furthermore, it is consistent with the settings in Del Negro, Eggertsson, Ferrero, and Kiyotaki (2016) and Guerron-Quintana and Jinnai (2015). Table 1 summarizes the parameter values.

4.4 Comparative Statics

Figure 2 plots the relation between the parameter affecting the liquidity constraint, ϕ , and the growth rate, g , along the balanced growth path, assuming that both productivity and preference shocks are constant at $a_t = b_t = 0$ for all t . The green flat line on the right part of the envelope shows that the growth rate is constant once ϕ reaches a certain threshold (indicated by a vertical black line). Beyond this threshold neither liquidity nor borrowing constraints bind because

Parameter	Value	Calibration Target
β	0.99	Exogenously Chosen
α	0.4	Capital Share=0.4
π	0.05	Exogenously Chosen
ρ	1	Exogenously Chosen
ζ	0.33	Exogenously Chosen
$\delta(1)$	0.025	Annual Depreciation=0.10
η	2.78	Labor Supply=0.25
\bar{A}	0.30	Rental Rate of Capital=0.05

Table 1: Parameters and Calibration Targets

investors can obtain enough liquidity by selling capital.

On the left part of the envelope (blue line), we see a nonlinear relation between liquidity and growth. That is, when liquidity is scarce in the economy, providing additional liquidity (a marginal increase in ϕ) enhances growth, but when it is relatively large, it is harmful to growth. We interpret ϕ as the degree of financial development in the economy because this parameter governs how much money investors can borrow from savers using capital as collateral. The result points to the existence of an optimal level of financial development for the purpose of promoting growth.

To understand this result, it is important to note that a marginal increase in ϕ has competing effects on growth. On one hand, it promotes investment as plotted in the top left panel in Figure 3. But on the other hand, it accelerates capital depreciation (the top right panel). Depreciation is high because capacity utilization rates are high (the bottom left panel). High capacity utilization is justified by the price of capital, which decreases with ϕ (the bottom right panel). The price of capital is low when ϕ is high because a large supply of liquidity in the aggregate reduces the value of each unit of capital as collateral. Because the cost of a marginal increase in capacity utilization rates is the market value of lost capital, the low price of capital means that households raise capacity utilization rates with less reluctance. The non-linearity arises because the growth enhancing effect through investment dominates if the liquidity shortage is severe, but otherwise, financial development is harmful to growth because it leads to too intensive use of capital and hence too fast capital depreciation.

The red dashed lines in Figures 2 and 3 confirm the aforementioned intuitions by plotting the same relationships in an otherwise identical economy with fixed capacity utilization rate. Note that financial development linearly enhances growth because more investment projects are funded. Fixed capacity utilization rate, however, is not only arguably unrealistic, but the model with this feature suffers from the well known comovement problem, i.e., a difficulty in generating aggregate comovement in response to shocks other than contemporaneous innovations to total factor productivity (Barro and King (1984)).

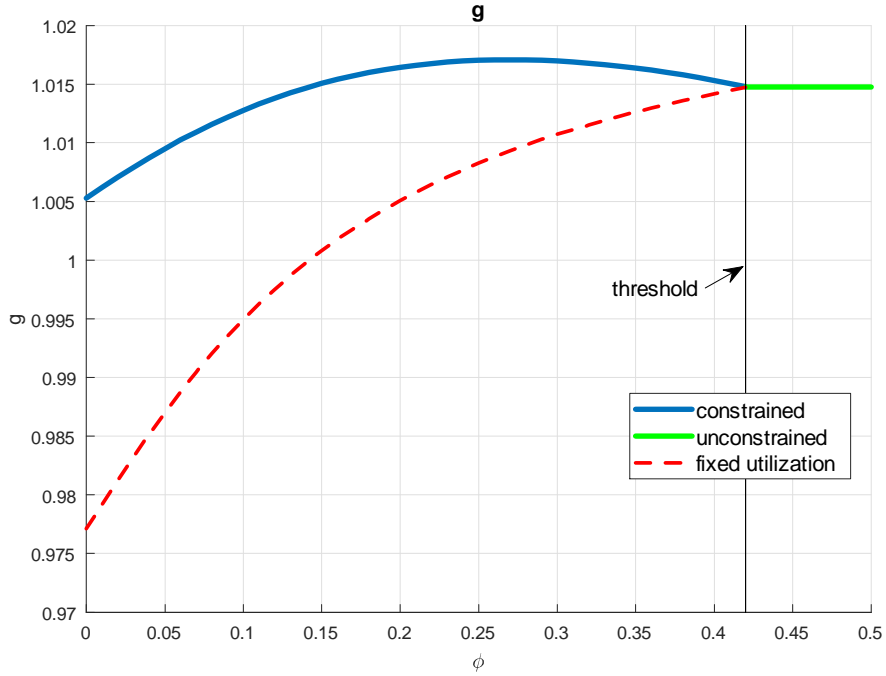


Figure 2: Liquidity and Growth in Permanent-Fundamental Model

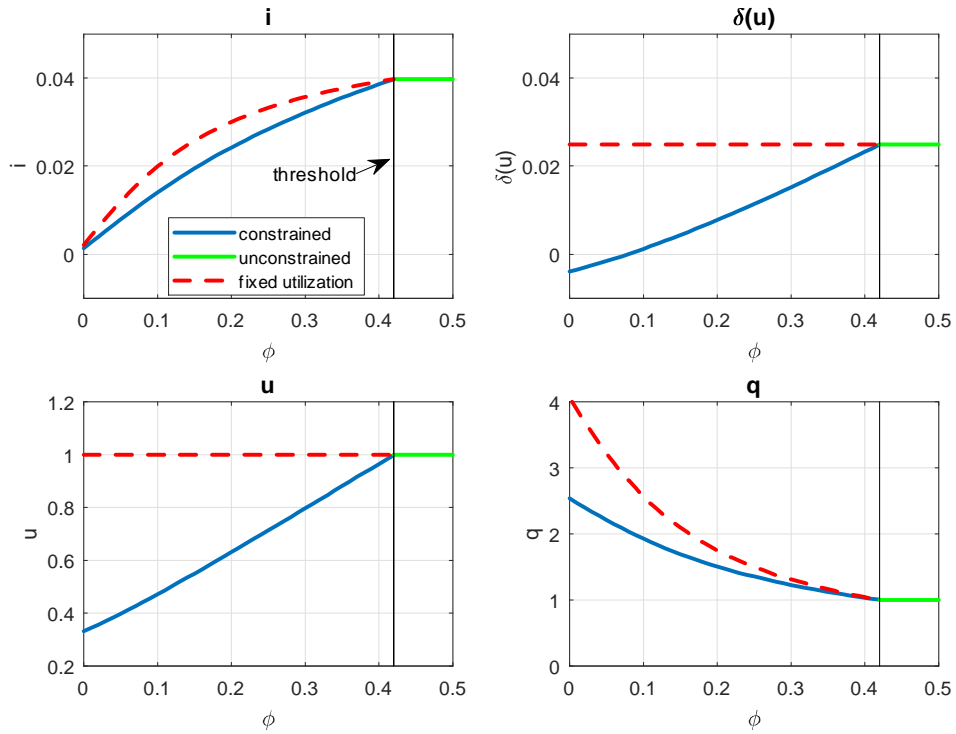


Figure 3: Effects of Liquidity in Permanent-Fundamental Model

5 Recurrent Bubble

We now flesh out the general case in which bubbles are recurrent. We first characterize equilibria of different properties.

5.1 Fundamental Equilibrium

If the price of bubble assets in the bubble regime is always $\tilde{p}_t = 0$, they are free but useless. This is an equilibrium price because households are indifferent to purchase such assets. We may call it the fundamental equilibrium regime, because the allocation in it is essentially the same as in the fundamental model we discussed in the previous section. But as Kiyotaki and Moore (2012) show, this equilibrium may not be the only one.

5.2 Bubble Equilibrium

We argue that if the price of capital is always equal to $q_t = 1$ (which is the case when liquidity is plentiful), the fundamental equilibrium is the unique competitive equilibrium. If $q_t = 1$, the household finds it indifferent between investing in capital in-house or purchasing capital in the market. As a result, the liquidity constraint (7) does not bind. Furthermore, the borrowing constraint (9) does not bind either. If it did, the household could make it loose without affecting other constraints or the amount of capital at the end of period t by increasing x_t^i by $\Delta > 0$, decreasing x_t^s by $(\pi/(1-\pi))\Delta$, decreasing n_{t+1}^i by Δ , and increasing n_{t+1}^s by $(\pi/(1-\pi))\Delta$. With both (7) and (9) unbound, the equilibrium price of bubble assets must be $\tilde{p}_t = 0$ because otherwise the demand for bubble assets is zero. This observation implies that bubbly assets may have a strictly positive value only if the price of capital is strictly greater than one.

From now on, we argue by construction that there is an equilibrium having the following properties if ϕ is sufficiently small: (i) $1 < q_t < 1/\phi$ always hold and (ii) $\tilde{p}_t > 0$ for all t with $z_t = b$. The inequality constraints (7), (8), and (9) always bind in such an equilibrium for the following reasons. If (7) is not binding, households can increase utility without violating any constraints or affecting their portfolio at the end of the period by increasing i_t by $\Delta > 0$, increasing n_{t+1}^i by $(q_t - 1)\Delta/q_t$, increasing both x_t^s and c_t^s by $(\pi/(1-\pi))(q_t - 1)\Delta$, and decreasing n_{t+1}^s by $(\pi/(1-\pi))((q_t - 1)/q_t)\Delta$, which is a contradiction to the household's optimization. (8) holds with equality in a fundamental regime due to (5). If (8) is not binding in a bubble regime, households can relax (7) without violating any constraints or affecting their portfolio at the end of the period by decreasing \tilde{m}_{t+1}^i by Δ , increasing \tilde{m}_{t+1}^s by $(\pi/(1-\pi))\Delta$, increasing n_{t+1}^i by $\tilde{p}_t\Delta/q_t$, and decreasing n_{t+1}^s by $(\pi/(1-\pi))(\tilde{p}_t/q_t)\Delta$. This is a contradiction to the household's optimization because they can increase utility if (7) is not binding. If (9) is not binding, households can relax (7) without violating any constraints or affecting their portfolio at the end of the period by decreasing x_t^i by Δ , increasing x_t^s by $(\pi/(1-\pi))\Delta$, increasing n_{t+1}^i by $(1/q_t)\Delta$, and decreasing

n_{t+1}^s by $(\pi/(1-\pi))(1/q_t)\Delta$. This is a contradiction to the household's optimization because they can increase utility if (7) is not binding.

Because (7), (8), and (9) hold with equality, optimal investment level is given by

$$i_t = \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] n_t + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t \tilde{m}_t}{1 - \phi q_t}. \quad (21)$$

Substituting (7) and (21) into (10), we find

$$n_{t+1} = \pi \frac{1}{q_t} (1 + \lambda_t) [(u_t r_t + \phi q_t (1 - \delta(u_t))) n_t + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t \tilde{m}_t] + \pi (1 - \phi) (1 - \delta(u_t)) n_t + (1 - \pi) n_{t+1}^s. \quad (22)$$

Substituting (6), (11), and (22) into (4), we rewrite the household's problem as follows;

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{bt} \left(\pi \frac{[c_t^i]^{1-\rho}}{1-\rho} + (1-\pi) \frac{[c_t^s (1-l_t)^\eta]^{1-\rho}}{1-\rho} \right) \right]$$

subject to

$$\begin{aligned} & \pi c_t^i + (1-\pi) c_t^s + q_t n_{t+1} + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t (1-\pi) \tilde{m}_{t+1}^s \\ &= [u_t r_t + (1-\delta(u_t)) q_t + \pi \lambda_t (u_t r_t + \phi q_t (1-\delta(u_t)))] n_t \\ & \quad + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t (1+\pi \lambda_t) [(1-\pi) \tilde{m}_t^s + \mathbf{1}_{\{z_{t-1}=f, z_t=b\}} M] + (1-\pi) w_t l_t \end{aligned} \quad (23)$$

and

$$\mathbf{1}_{\{z_t=f\}} \tilde{m}_{t+1}^s = 0 \quad (24)$$

for all t . The first order conditions with respect to the intra-temporal consumption allocation (14), the labor supply by savors (15), the capacity utilization rate (19), and the pricing equation for capital (20) are the same as before. The first order condition with respect to the demand for bubbly assets by savors \tilde{m}_{t+1}^s is

$$\mathbf{1}_{\{z_t=b\}} \tilde{p}_t = \mathbf{1}_{\{z_t=b\}} E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (1 + \pi \lambda_{t+1}) \tilde{p}_{t+1} \mathbf{1}_{\{z_{t+1}=b\}} \right].$$

This is a key equation in our model. If the economy is in the fundamental regime, this equation is trivial because both sides are zero. Furthermore, the demand for bubbly assets is zero in this case because the constraint (24) prohibits savors to purchase bubbly assets. If the economy is in the bubbly regime in period t , the equation is rewritten as

$$\tilde{p}_t = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (1 + \pi \lambda_{t+1}) \tilde{p}_{t+1} \mathbf{1}_{\{z_{t+1}=b\}} \right].$$

Two observations are worth noting. First, \tilde{p}_t can be strictly positive only if \tilde{p}_{t+1} in expectation

takes a strictly positive value in the bubbly regime. In other words, it is future resalability that justifies a positive price in the current period. Second, the liquidity parameter ϕ is absent in the equation. This is an attractive feature of bubbly assets that enables the no-arbitrage condition for savers to hold. Namely, even though bubbly assets do not provide dividends to their owners, savers find it indifferent to purchase bubbly assets and capital because bubbly assets carry a larger liquidity premium.

Equation (21) can be rewritten as follows in the equilibrium:⁴

$$i_t = \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1 - \phi q_t}. \quad (25)$$

The second term in the numerator shows that, other things being equal, the emergence of bubbles increase investment. As we will show momentarily, equation (25) plays a key role in determining whether bubbles are sustainable or not.

Competitive equilibrium is defined as a sequence of prices, w_t , r_t , q_t , and \tilde{p}_t , and quantities, Y_t , C_t , I_t , K_{t+1} , c_t^i , c_t^s , l_t , and u_t , that satisfies the following conditions:

$$Y_t = \bar{A} e^{a_t} u_t^\alpha K_t ((1 - \pi) l_t)^{1-\alpha},$$

$$(c_t^i)^{-\rho} = (c_t^s)^{-\rho} (1 - l_t)^{\eta(1-\rho)},$$

$$\eta \frac{c_t^s}{1 - l_t} = w_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0,$$

$$q_t = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \right],$$

$$\mathbf{1}_{\{z_t=b\}} \tilde{p}_t = \mathbf{1}_{\{z_t=b\}} E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (1 + \pi \lambda_{t+1}) \tilde{p}_{t+1} \mathbf{1}_{\{z_{t+1}=b\}} \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1 - \alpha) \frac{Y_t}{(1 - \pi) l_t},$$

$$Y_t = \pi c_t^i + (1 - \pi) c_t^s + \pi \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1 - \phi q_t},$$

⁴This is because the following relation holds in the equilibrium,

$$\begin{aligned} \mathbf{1}_{\{z_t=b\}} \tilde{m}_t &= \mathbf{1}_{\{z_t=b\}} [\pi \tilde{m}_t^i + (1 - \pi) \tilde{m}_t^s + \mathbf{1}_{\{z_{t-1}=f, z_t=b\}} M] \\ &= \mathbf{1}_{\{z_t=b\}} [\mathbf{1}_{\{z_{t-1}=b\}} M + \mathbf{1}_{\{z_{t-1}=f, z_t=b\}} M] \\ &= \mathbf{1}_{\{z_t=b\}} M. \end{aligned}$$

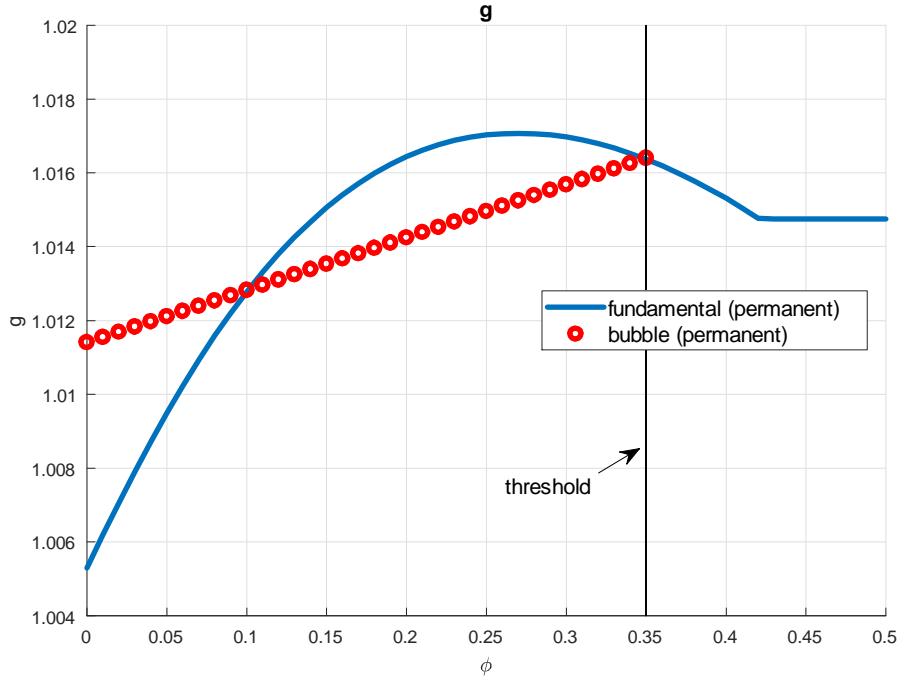


Figure 4: Liquidity and Growth with Permanent Bubble

$$K_{t+1} = (1 - \delta(u_t)) K_t + \pi \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1 - \phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1 - \phi q_t}$$

for all t . We examine this equilibrium numerically in the following section.

5.3 Comparative Statics

We first consider the special case in which the economy is always in the bubble regime ($z_0 = b$ and $\sigma_b = 0$ and hence $z_t = b$ for all $t \geq 0$). Figure 4 plots the relation between the parameter affecting the liquidity constraint ϕ and the growth rate g along the balanced growth path. The solid blue line shows the fundamental equilibrium in which the price of bubble assets is always zero. As we discussed, this equilibrium is essentially the same as the fundamental model in the previous section. The blue line in Figure 4 is therefore identical to the envelope in Figure 2.

The red circles in Figure 4 show the equilibrium in which the price of bubbly assets is always positive. Such an equilibrium exists only if ϕ is smaller than a threshold value shown in the figure by the vertical line. At low values of ϕ , growth is higher in the bubbly equilibrium, which is a consequence of the liquidity provision of bubbles. This feature in turn makes bubbles very valuable. The top left panel in Figure 5 plots the market value of bubbly assets relative to the

capital stock $m_t \equiv \tilde{p}_t M / K_t$ along the balanced growth path. It decreases with ϕ , implying that an economy with underdeveloped financial market (an economy with small ϕ) can sustain larger bubbles relative to its capital stock. As shown in the top right panel of Figure 5, investment is larger in the bubble equilibrium than the fundamental equilibrium at a given level of ϕ , confirming the intuition obtained from equation (25).

But larger investment does not necessarily mean higher growth. Going back to Figure 4, we see that the otherwise identical economy grows faster in the bubble equilibrium than in the fundamental equilibrium if the underlying financial system is very weak, but the opposite is true if the financial system is relatively developed. The price of capital and endogenous capital depreciation are key to understand this result. As shown in the middle left panel of Figure 5, the price of capital is lower in the bubbly equilibrium than in the fundamental equilibrium. This is because investors can obtain funds for investment either by selling capital or bubbly assets; in other words, the two types of assets are substitutes as a source of liquidity. Because of large supply of liquidity, the liquidity service of capital is lower in the bubbly equilibrium than in the fundamental equilibrium as shown in the middle right panel, and so is the price of capital. The low price of capital makes households less reluctant to raise capacity utilization rate (bottom left panel), leading to a faster capital depreciation (bottom right panel). Because it is net investment, not gross, that matters for the speed of capital accumulation, the growth rate of the economy can be slower in the bubble equilibrium than in the fundamental equilibrium.

Figure 6 plots the relation between the liquidity ϕ and the growth rate g_t in the general case, assuming that the probabilities of regime switches are $\sigma_f = \sigma_b = 0.01$.⁵ The red circles and crosses show regime-dependent growth rates observed in the recurrent-bubble equilibrium, in which the price of bubble assets in the bubble regime is always positive, $\tilde{p}_t > 0$. The red circles denote the growth rate in the bubble regime, whereas the red crosses denote the growth rate in the fundamental regime. This equilibrium exists only if ϕ is smaller than a threshold value shown in the figure by the vertical line.

One can see that the economy grows faster in the bubble regime (red circles) than in the fundamental regime (red crosses) in the recurrent-bubble equilibrium. This is in great contrast to the model without regime switches shown in Figure 4, in which we compare the growth rates in two different equilibria with and without bubbles, finding that the otherwise identical economy can grow more slowly with bubbles if its financial system is relatively developed, while in Figure 6, we compare growth rates in two different regimes in a single equilibrium, finding no growth reversal occurring across regimes. Intertemporal substitution is the driver of this result. Namely, households in the recurrent-bubble equilibrium have strong incentives to work and invest while bubbles exist because the return on investment is higher and household know that bubbles are only temporary. By the same token, households have weaker incentives to work and invest in the

⁵Martin and Ventura (2012) impose a similar symmetric restriction on the switching probabilities.

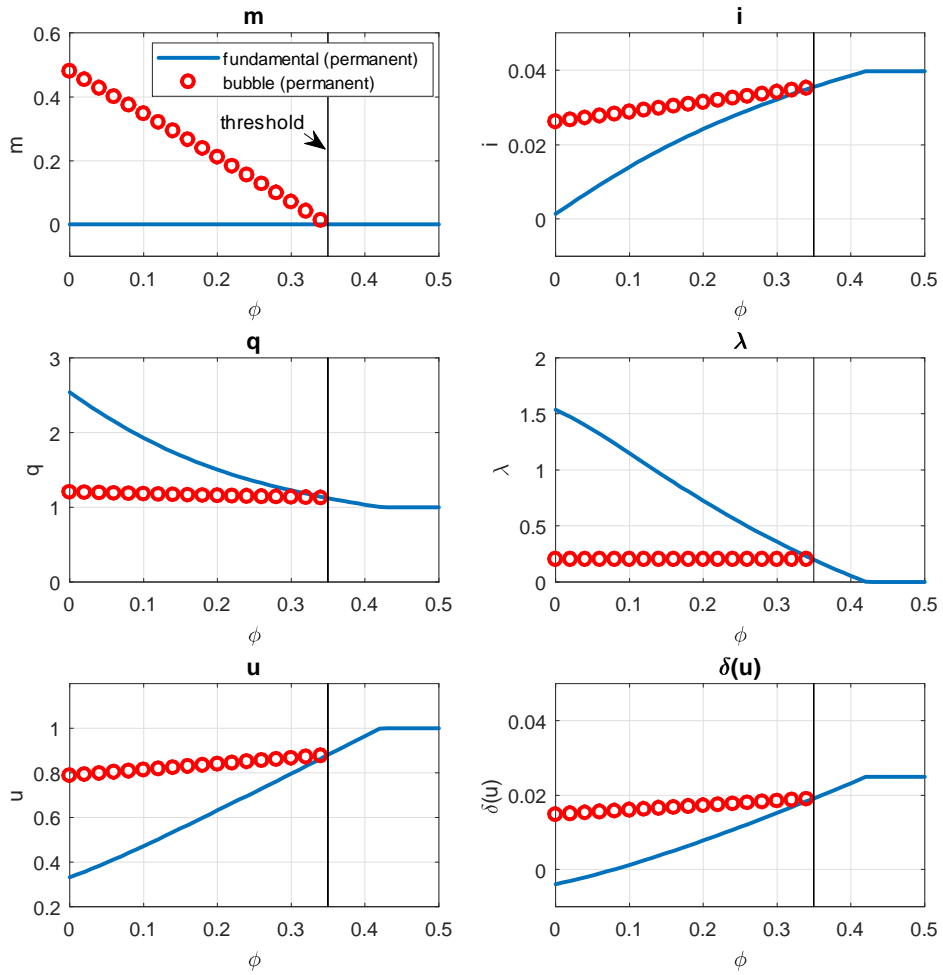


Figure 5: Effects of Liquidity in Permanent Bubble

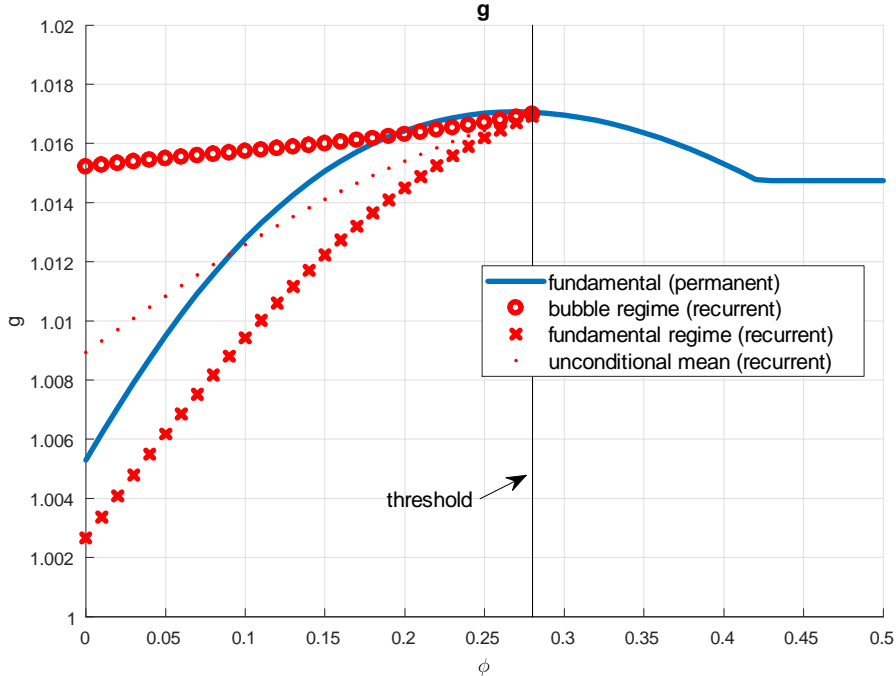


Figure 6: Liquidity and Growth in Recurrent-Bubble Model

fundamental regime because they know that bubbles will arise again. In other words, households substitute labor supply and consumption not only across time but also across regimes.

Figure 7 confirms the intuition discussed above. The top left panel shows that more investment projects are funded in the bubble regime than in the fundamental regime. This is because bubbles provide extra liquidity as shown in the top right panel. Utilization rate is higher in the bubble regime (the left panel in the second row) because the price of capital is lower (the right panel in the second row). The price of capital is lower because liquidity service is lower in the bubble regime (the left panel in the third row). Liquidity service is lower because capital and bubble assets are substitutes as sources of liquidity. Households work harder in the bubble regime than in the fundamental regime (the left panel in the fourth row) as a result of intertemporal substitution.

The red dots in Figure 6 show the unconditional mean of the growth rates in the recurrent-bubble equilibrium. Because we assume symmetric probabilities of the regime switch ($\sigma_f = \sigma_b$), the red dot is the middle point between the red circle and cross. On the left part of the figure, the red dots are located above the blue line. This means that conditional on having a weak financial system, the otherwise identical economy grows faster in the long run if it is in the recurrent-bubble equilibrium than in the fundamental equilibrium. This is because such an economy is greatly benefited by the liquidity provided by bubbles, achieving additional growth in the bubble regime. But this benefit comes at a cost. That is, because growth rates in the recurrent-bubble equilibrium are regime dependent, the economy has to endure occasional shifts in the growth rates

between a high level (red circle, bubble regime) and a low level (red cross, fundamental regime). In contrast, the growth rate in the fundamental equilibrium is not influenced by the regime at all. Therefore, if an economy with a weak financial system has bubbles, there is a trade-off between the long-run growth and the short-run volatility.

As we move toward a relatively more developed financial system (higher ϕ), the otherwise identical economy grows slower in the long run if it is in the recurrent-bubble equilibrium than in the fundamental equilibrium. In addition, the economy in the recurrent-bubble equilibrium has to endure changes in the growth rates associated with regime switches. Our model therefore sheds light on seemingly contradictory observations; namely, bubbles are often considered to be a potential subject of regulation in many developed economies, although bubbles raised growth compared to other periods in the same economies. Japan's high growth in the 80s with the real estate bubble is a prime example. Our model suggests that this is not surprising because recurrent bubbles may reduce the growth rate in the long run and increase the volatility in the short run. Importantly, our findings indicate that policy intervention is fraught with perils. This is so because policy makers need to know the existence of the bubble and the degree of financial development in the economy before they intervene. As Figure 6 shows, for highly illiquid economies, the benefits of allowing bubbles may overcome the excess volatility brought about by bubbles.

From Figure 6, we also learn that the objectively identical economy in the fundamental regime ($z_t = f$) grows faster if it is in the fundamental equilibrium (blue line) than in the recurrent-bubble equilibrium (red crosses). The expectation for future bubbles is the key to understand this result. First of all, it causes a wealth effect. Namely, anticipating that the economy will grow fast once bubbles arise, people in the fundamental regime invest less (the top left panel in Figure 7), consume more (the right panel in the third row), and work less (the left panel in the fourth row) in the recurrent-bubble equilibrium (red cross) than in the fundamental equilibrium (blue line).

The wealth effect is only a part of the story because our model features a price effect as well. Specifically, the price of capital in the fundamental regime is lower in the recurrent-bubble equilibrium than in the fundamental equilibrium (the right panel in the second row) to the extent that people anticipate that the emergence of bubbles will provide liquidity to the economy in the future. Low price of capital leads to higher capacity utilization rate (the left panel in the second row), larger output (the right panel in the fourth row), and higher capital depreciation rate (the bottom left panel). Both low investment due to the wealth effect and high capital depreciation rate due to the price effect lower the growth rate in the fundamental regime in the recurrent-bubble equilibrium.

Finally, Figure 8 shows the case when the economy starts with bubbles, but they will burst with a positive probability and no new bubbles will arise ever again ($z_0 = b$, $\sigma_b > 0$, and $\sigma_f = 0$). Comparing the pink and green circles, we see that the probability of bubble burst σ_b is positively associated with the growth rate while the economy is still in the bubble regime. The intertemporal substitution is crucial again because anticipating that bubbles will burst with a positive probability,

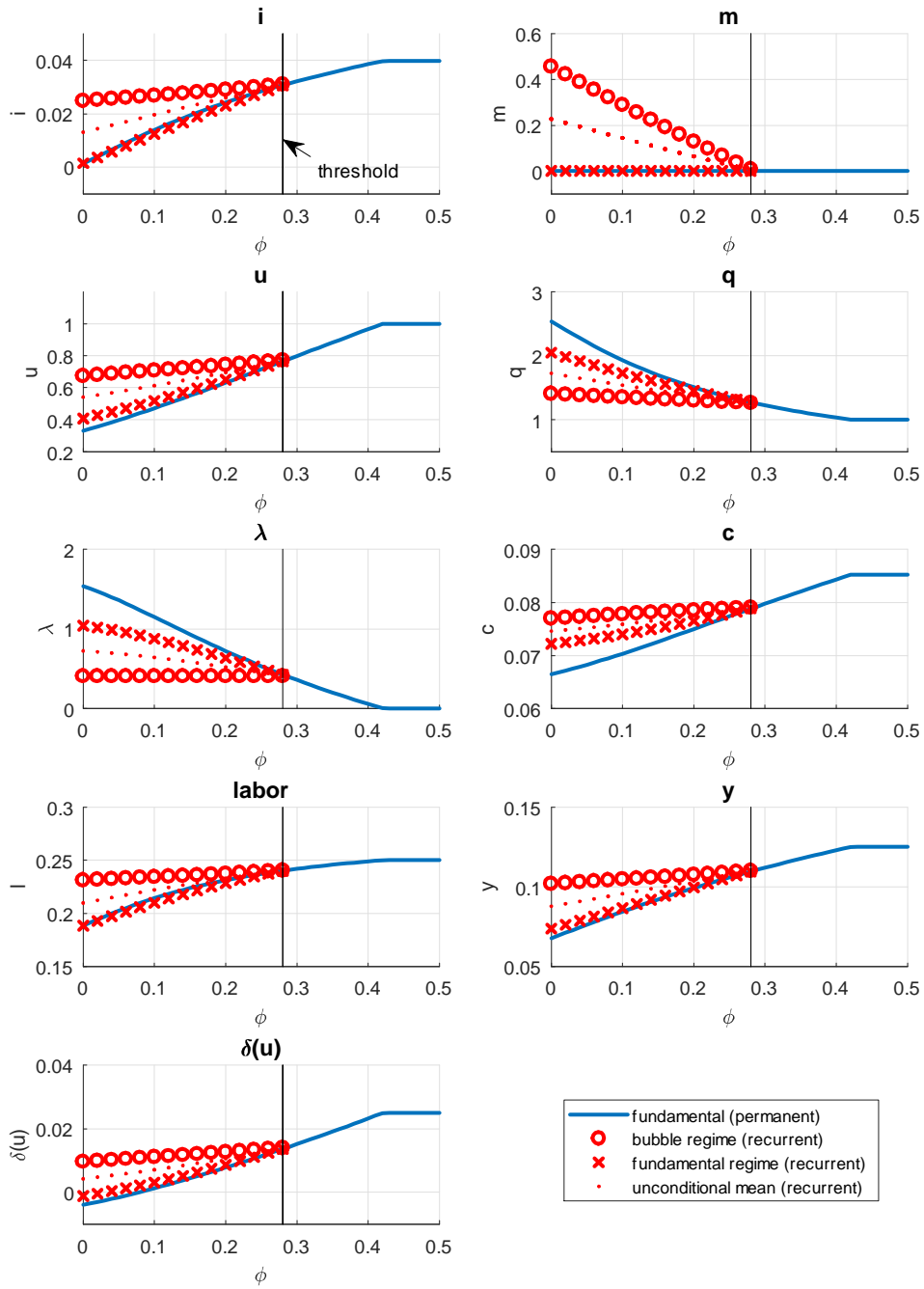


Figure 7: Effects of Liquidity in Recurrent-Bubble Model

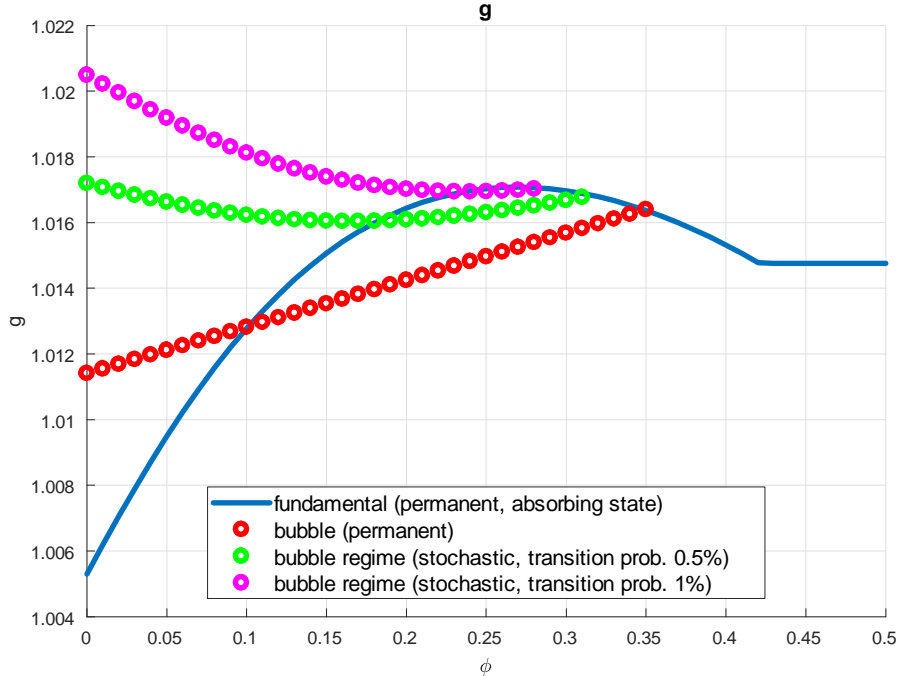


Figure 8: Stochastic Bubble with Absorbing Fundamental

households have strong incentives to work hard and invest much while bubbles are still present; and, this incentive is stronger as the bubbles become more short-lived. The model’s prediction with non-recurrent bubbles (stochastic bubbles with absorbing fundamental regime and initial condition $z_0 = f$) is similar to the permanent bubble.

5.4 Effects of a Regime Switch

Table 2 shows the regime-dependent steady state values in the recurrent-bubble equilibrium (for this section, we set the degree of liquidity constraint at $\phi = 0.15$). They are defined as the values of endogenous variables after detrending in an environment in which both productivity and preference shocks are at their steady state values. The first row corresponds to the steady state in the fundamental regime, whereas the second row corresponds to the steady state in the bubbly regime. The results confirm the intuition discussed above; namely, the growth rate of the economy, output, consumption, investment, labor hours, and capacity utilization rate are higher in the bubbly regime than in the fundamental one. In addition, real wages are higher and the rental price of capital is lower in the bubbly regime than in the fundamental regime.

Figure 9 shows the impact of regime switches to output, consumption, and investment in the recurrent-bubble equilibrium. We assume that the economy was originally in the bubbly regime, having experienced regime switches every 10 years thereafter by chance. A bubble burst causes a deep recession with output dropping as much as by 11 percentage points immediately after the

g_f	q_f	m_f	r_f	\hat{w}_f	\hat{Y}_f	\hat{c}_f	\hat{i}_f	l_f	λ_f	u_f	$\delta(u_f)$
1.012	1.58	0	0.064	0.27	0.093	0.075	0.018	0.22	0.76	0.58	0.006
g_b	q_b	m_b	r_b	\hat{w}_b	\hat{Y}_b	\hat{c}_b	\hat{i}_b	l_b	λ_b	u_b	$\delta(u_b)$
1.016	1.33	0.21	0.059	0.28	0.106	0.078	0.028	0.24	0.41	0.72	0.012

Table 2: Steady States in Recurrent-Bubble Equilibrium

bubble’s collapse, and investment even by 35 percentage points. To a lesser degree, consumption drops on impact too. Subsequently, the economy starts to grow again, but the recovery from the crisis is very weak, which shows that our model generates super-hysteresis following the bubble burst. The effects of bubble creation are symmetric in our model. That is, bubble creation brings a large boom to the economy followed by persistent robust growth.

If the capacity utilization rate is fixed, consumption increases on impact of a bubble burst. This result is not surprising because the difficulty of generating comovement to a shock other than a contemporaneous shock to total factor productivity is well known since Barro and King (1984) as well as recently in the literature on news shocks (Beaudry and Portier (2004) and Jaimovich and Rebelo (2009)). A regime switch in our model is similar to a news shock in the sense that it is neutral to the current technology. But unlike previous work in the news shock literature, our model is able to generate comovement to a regime switch without exotic utility functions but only with variable capacity utilization.

We conjecture that our model could be modified to accommodate additional regimes with bubbles providing different levels of liquidity but at the expense of losing tractability. Furthermore, we can consider bubbles that partially collapse but not completely. Specifically, a fraction of the bubble assets in the economy depreciates but new bubbles are injected every period too. The aggregate supply of bubble assets is always positive in this case. This extension corresponds to the bubble models studied, for example, by Miao and Wang (2017) and Gali (2014). These papers analyze the local dynamics around the steady state with bubbles, just as we did in the permanent-bubble version of our model. Our notion of recurrent bubbles, in contrast, admits both a complete collapse and a re-emergence of bubbles. The corresponding economic dynamics are highly non-linear because they are associated with transitions between steady states. Martin and Ventura (2012) study the non-linearity caused by the recurrent bubbles. But they study it in an overlapping generation model in which utility linearly depends on the old-age consumption alone. As a result, their framework is silent about the effects of the recurrent bubbles on the household’s inter-temporal substitution of consumption and labor.

5.5 Impulse Responses

Let us bring back supply and demand shocks into the analysis. We continue to assume that $\sigma_f = \sigma_b = 0.01$ and $\phi = 0.15$. There are multiple equilibria under these parameter values, i.e., the

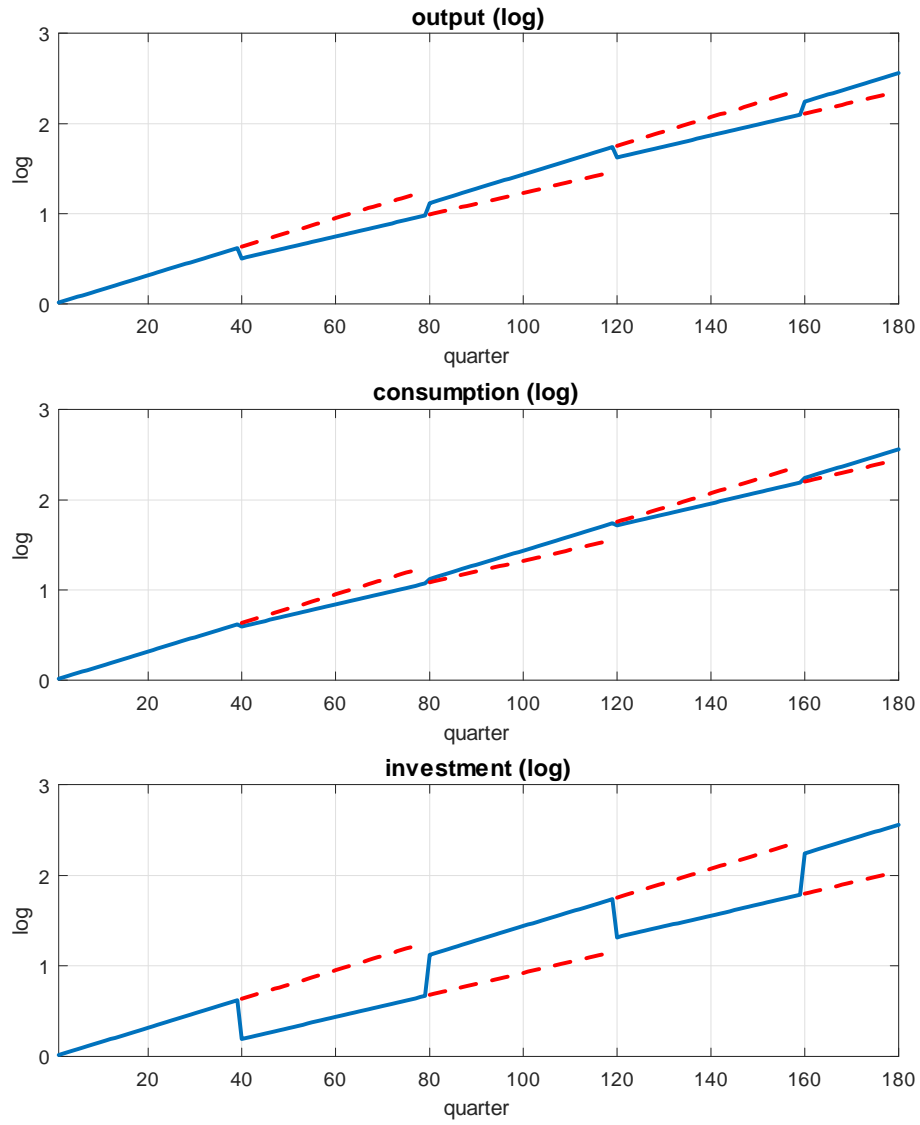


Figure 9: Effects of Regime Switch in Recurrent-Bubble Equilibrium

recurrent-bubble equilibrium in which the price of bubble assets is always positive in the bubble regime, and the fundamental equilibrium in which the price of bubble assets is always zero in the bubble regime. Computing the impulse response functions in the fundamental equilibrium is just standard. For the recurrent-bubble equilibrium, we compute impulse response functions by linearizing the system of equations summarizing the equilibrium around the regime-dependent steady states (please see the appendix for detail).

The top panel in Table 3 shows the impact of the productivity shock. We assume that the exogenous component of the productivity a_t increases by 1 percent in period t , slowly coming back to the steady state level thereafter with the autocorrelation coefficient being 0.95 quarterly. We report the contemporaneous responses in period t alone because they are enough to summarize the impulse responses. This is because there is no endogenous state variables in our model once endogenous variables are detrended by K_t , implying that both the regime $z_t \in \{f, b\}$ and the levels of the exogenous shocks $\{a_t, b_t\}$ are sufficient to pin down detrended-endogenous variables. Note, however, that the persistence of the shock does affect responses in period t because the model has forward looking variables and households are infinitely lived.

The first two columns show the IRFs in the recurrent-bubble equilibrium. A positive productivity shock generally raises macroeconomic variables in both regimes, but the magnitudes are different. Specifically, output, consumption, investment, hours worked, and capacity utilization all increase more in the bubble regime than in the fundamental regime. Asset prices play an important role. Namely, the size of the bubble increases when a positive productivity shock hits the economy in the bubble regime. This is because the demand for liquidity is strong when productivity is high. With more liquidity provided by bubbles, the price of capital does not rise as much as in the fundamental regime. With the price of capital cheaper, households are less reluctant to raise the capacity utilization rate, making the fluctuation larger in the bubble regime. The productivity shock, however, increases the growth rate of the capital $g_t = K_{t+1}/K_t$ because net investment increases.

The right column in Table 3 shows the impulse responses in the fundamental equilibrium. Broadly speaking, the responses are similar to those in the fundamental regime in the recurrent-bubble equilibrium. Looking closer, however, we see the wealth effects working in the recurrent-bubble equilibrium. Namely, households in the recurrent-bubble equilibrium enjoy more consumption and leisure because they understand that new bubble will arise in the future making them rich.

The bottom panel in Table 3 shows responses to the preference shock. b_t increases by 1 percent in period t , slowly coming back to the steady state level with the autocorrelation coefficient being 0.8 quarterly. Tilting the relative weights on the utility flow, this shock effectively makes the households impatient, consume more, work less, and invest less. But the magnitudes of the responses are again larger in the bubble regime than in the fundamental regime. Asset prices are important. That is, the size of the bubble shrinks in the bubble regime after the shock because the

Supply Shock ($\Delta a_t = 1\%$, $\text{Corr}(a_t, a_{t-1}) = 0.95$)			
	Recurrent-Bubble Equilibrium		Fundamental Equilibrium
	Bubble Regime	Fundamental Regime	Both Regimes
output	1.24%	1.09%	1.09%
consumption	1.08%	1.04%	1.03%
investment	1.69%	1.28%	1.32%
hours	0.12%	0.04%	0.05%
utilization	0.41%	0.16%	0.16%
capital price	0.74%	0.96%	0.98%
bubble size	2.29%	0%	0%
capital growth	0.033%	0.019%	0.022%

Demand Shock ($\Delta b_t = 1\%$, $\text{Corr}(b_t, b_{t-1}) = 0.8$)			
	Recurrent-Bubble Equilibrium		Fundamental Equilibrium
	Bubble Regime	Fundamental Regime	Both Regimes
output	0.03%	0.11%	0.09%
consumption	0.31%	0.30%	0.31%
investment	-0.78%	-0.71%	-0.73%
hours	-0.22%	-0.15%	-0.17%
utilization	0.39%	0.49%	0.47%
capital price	-0.53%	-0.60%	-0.60%
bubble size	-0.87%	0%	0%
capital growth	-0.034%	-0.024%	-0.025%

Table 3: Effects of Supply and Demand Shocks

households become impatient. With the amount of liquidity provided by the bubble decreases, the price of capital does not drop as much as it does in the fundamental regime. Because the price of capital is relatively high, the households are reluctant to raise the capacity utilization rate, making drops in investment and hours worked larger in the bubble regime than in the fundamental regime.

5.6 Existence Condition

From the discussion above, it should be apparent that depending on the degree of financial tightness bubbles may or may not be valuable. In this section, we highlight other elements that may affect bubbles' valuation.

5.6.1 Permanent Bubble

The steady state investment condition (25) is useful to understand when bubbles arise (are valued positively). To this end, let's re-write it as follows:

$$m = \hat{i}(1 - \phi q) - ur - \phi q(1 - \delta(u)). \quad (26)$$

Here, m and \hat{i} are the size of the bubbles and investment relative to the capital stock, i.e., $m_t = \tilde{p}_t M / K_t$ and $\hat{i}_t = i_t / K_t$, in the steady state respectively. The first term in the right-hand side of equation (26) is the down payment each investor pays for investment. The second term is the rental rate of capital, and the third term is the proceeds from selling capital up to the limit. Therefore, this equation says that bubbles have positive valuation (the left-hand side is positive) if and only if the amount of liquidity an investor can withdraw from capital is less than the amount of liquidity investors need to undertake investment project.

To convey more intuition, let's assume that utilization is 1 and there is full depreciation. Under these assumptions, equation (26) is rewritten as

$$m = g(1 - \phi q) - r \tag{27}$$

because $\hat{i} = g$ where g is the growth rate of the economy in the steady state. Bubbles are valued when the rental rate of capital is sufficiently low. This implication is in line with the previous work on bubbles; if we further assume that ϕ is equal to $\phi = 0$, the first term in the right-hand side collapses to g , and $g > r$ is the familiar dynamic inefficiency condition for the existence of bubbles in OLG models.

If ϕ is strictly positive, investors can borrow money from savers using capital as collateral. By making the first term in the right-hand side smaller, a larger value of ϕ makes it more difficult to support bubbles. This implication is also in line with previous work; i.e., Tirole (1982) shows that bubbles cannot arise in infinite horizon economies in which agents can borrow and lend freely. In other words, a tight enough friction in the financial market is necessary for the economy to have bubble equilibrium.

5.6.2 Recurrent Bubble

Let us briefly discuss the existence condition when bubbles come and go. Assuming full depreciation and fixing the utilization at one, we arrive to the following expression,

$$m_b = \hat{i}_b(1 - \phi q_b) - r_b.$$

Other things being equal, bubbles are sustained (m_b is positive) when the liquidity constraint is tight, the rental price of capital is high, and/or the investment (and hence the growth rate) in the bubble regime is high. These implications are similar to the permanent bubble model.

But people take the possibility of the bubble burst into account when they are in the bubble regime, evaluating assets accordingly. The opposite is true in the fundamental regime. Therefore, both prices and behaviors are affected not only by the actual occurrence of the regime switch but also by the sheer possibility of the regime switch. For instance, under full depreciation the steady state price of equity in the bubbly regime is

$$q_b = (1 - \sigma_b) \beta g_b^{-\rho} (r_b + \pi \lambda_b r_b) + \sigma_b \beta \left(\frac{\hat{c}_b^i}{\hat{c}_f^i} \frac{1}{g_b} \right)^\rho (r_f + \pi \lambda_f r_f).$$

Clearly, the dynamic link between the two regimes makes the existence condition complicated, but it sheds a new light on the study of bubbles.

6 Taking the model to the data

We use our model to revisit the post-world war II U.S.'s economic performance. Specifically, we use U.S. data on the growth rate of output and the consumption-to-investment ratio for the period 1947.Q2 - 2016.Q4 to estimate the paths of supply and demand shocks in our model. We choose these observables because in our model these variables are sensitive to both the regime switch and shock processes. Specifically, we estimate the persistence and volatility of productivity and preference shocks. For this section, except for the liquidity parameter, ϕ , all other parameter values are those in table 1. Recall that the liquidity parameter was a free parameter in the previous sections since our objective was to analyze its impact on different versions of our model. For our quantitative section, we choose $\phi = 0.19$, which is in line with Del Negro, Eggertsson, Ferrero, and Kiyotaki (2016).

Our model follows within the class of MS-DSGE models discussed in Farmer, Waggoner, and Zha (2009). We find a fundamental minimum-state-variable equilibrium. The absence of endogenous state variables greatly simplifies the solution method as otherwise we would have to rely on the methods in Farmer, Waggoner, and Zha (2011).

6.1 A Regime Switching World

As an initial step, we estimate the model using maximum likelihood and Kim's filter,⁶ assuming that the economy is in the recursive-bubble equilibrium. Our identification of the regimes relies on the implications we showed in both Table 3 and Figure 6; i.e., the bubbly regime is characterized by both higher volatility and higher economic growth. Although these two elements were present in the pre-1980s sample, there is a clear tension in the last decades. For instance, the housing boom epoch displayed higher growth but lower volatility. Hence, the importance of taking the model to the data to discipline the switches in the model.

The left upper panel in Figure 10 presents the filtered and smoothed probabilities of the economy being in a bubble regime. They suggest that the economy had been in a bubble regime prior to the 1980s, had moved to the fundamental regime and stayed until the late 1990s, and have returned to the bubble regime again. These patterns are reminiscent of the long-run trend in output growth reported by Comin and Gertler (2006). That is, as a secular trend, the U.S. output

⁶We thank Dongho Song for helping setting up the Markov Switching estimation routine.

growth was generally robust until the 1970s, was generally weak until the mid-1990s, and reversed course again until the Great Recession. Identifying the cause of these medium-term cycles is a challenging task. The pioneering work of Comin and Gertler (2006) attribute them to exogenous changes to wage markups, which in their model are amplified by prominent mechanisms in the growth literature such as product innovations and costly implementations of new ideas. Our model offers a novel explanation; the medium-term cycles might be caused by non-fundamental factors, i.e., regime switches between bubble and fundamental. It does not have to assume changes to the underlying technologies or the model parameters.

The timings of the regime switches in our model look different from estimates from alternative regime switching models. For example, Sims and Zha (2006) fit U.S. data to a regime switching VAR with drifting coefficients and variances. They report the existence of four distinct regimes: the Greenspan state prevailing during the 1990s and early 2000s; the second most common regime emerges in the early 1960s and parts of the 1970s; the last two regimes corresponds to sporadic events such as 9/11. Our regimes are unlike those estimated to account for the Great Moderation with a high volatility regime prior to 1984s and a calmer one post 1984 (Stock and Watson (2002)). Finally, our bubble regime bears little resemblance to recession regimes (See Hamilton (2016) for an extensive review of regime switching in macroeconomics).

Going into the details, the estimated probability path suggests that prior to the 1980s the likelihood of being in a bubble regime was consistently high. But as we move through the 1980s and forward, the fundamental regime became more prevalent. Indeed, the bubble regime is less likely during the 1990s with a short-lived increase during the mid 1990s. Importantly, the estimated model captures the rise of the housing bubble during the early 2000s and its subsequent collapse in 2008-2009. Additionally, our estimation points to the rise of a post Great Recession bubble, which some economic commentators have attributed to the extremely loose monetary policy of the last years. However, our model struggles to pinpoint the rise and collapse of the IT bubble. This is most likely a result of the short duration of this bubble and that the housing bubble arose fairly close. That is, the estimation gives more weight to the housing bubble over the IT bubble. The curious reader may have noticed the bubble's temporary and abrupt collapse in the early 1960s. This seems to capture Kennedy's Slide of 1962 (the stock market flash crash from December 1961 to June 1962).

The right upper panel in Figure 10 shows the filtered path of the bubble's price (in red the HP-filtered trend). It plots the expected value calculated by the probability of the economy being in the bubble regime in a period times the price of bubble assets realized if the economy is in the bubbly regime in the same period. Measured this way, a unit of bubble asset was priced highly during most of the pre-1980s sample. But with the arrival of the Great Moderation epoch (the mid-1980s), the bubble's price became more volatile. In addition, remember that trading bubble assets occurs only in the bubble regime. Because the estimated probabilities suggest that there were regime switches to the fundamental in the mid-1980s and to the bubble in the late-1990s,

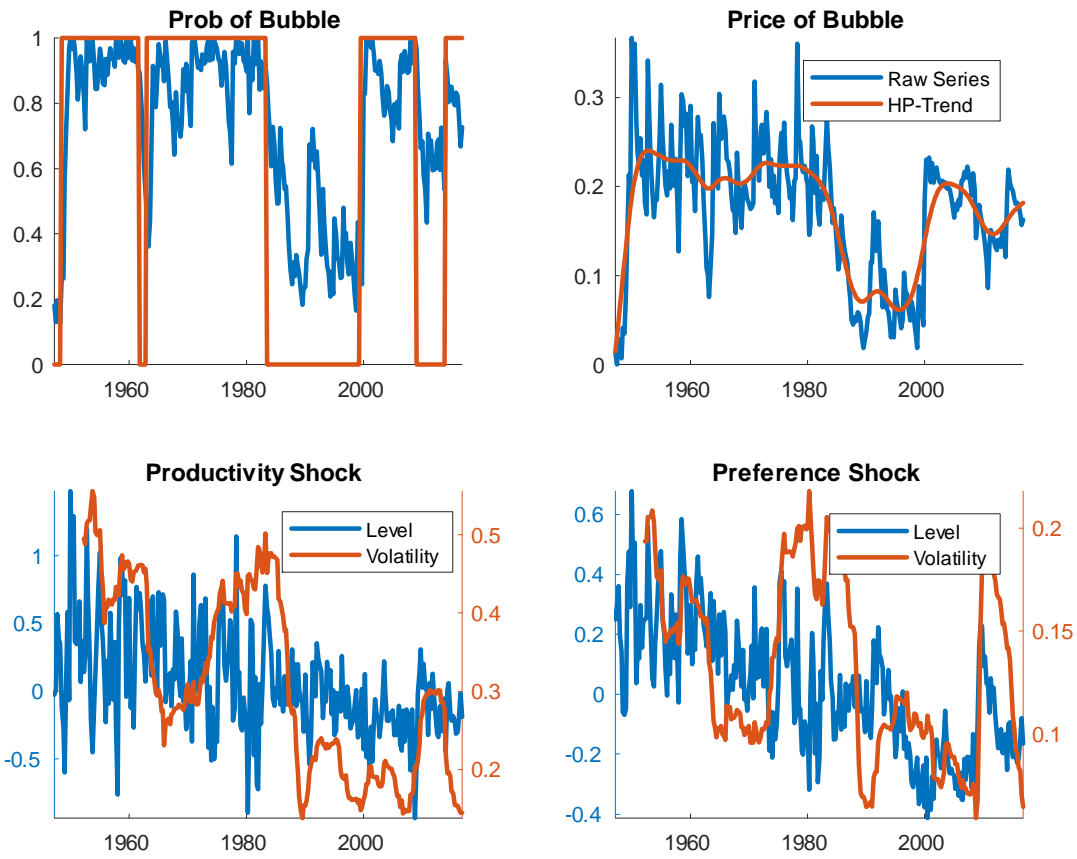


Figure 10: Variables from Recurrent Bubble Model

the actual trade volume of bubble assets is likely to be even more volatile in the latter half of the sample. Interestingly, both the estimated probabilities and the bubble price correctly capture the housing boom-bust episode. As a trend over the entire sample, we observe that the bubble price has been declining since the 1960s. This is precisely at the core of our model. Namely, because periods of high valuation are associated with periods of faster growth in our model, the growth slowdown of the recent decades could be attributed in part to smaller size the bubbles.

A natural question at this point is what these bubbles are capturing in reality. Although there is very little arguing about the housing and IT bubbles, it is less clear where the bubbles arose prior to 1980. For the 1970s, the obvious candidate is loose monetary policy (the Burns-Miller's dove regime estimated in Fernandez-Villaverde, Guerron-Quintana, and Rubio-Ramirez (2015)). Interestingly, Contessi and Kerdnunvong (2015) report stock and housing markets exuberance (based on cyclically adjusted price earning and cyclically adjusted price rent ratios) during the period 1965:Q3-1968Q4. Shiller (2015) also pointed out an instance of a high price-earnings ratio occurring in January 1966, calling it the "Kennedy-Johnson Peak."

The bottom panels in Figure 10 display the paths of productivity and preference shocks (the red lines correspond to 5-year rolling window volatilities). In spite of the moderating effect of the fundamental regime, there is still a role for less volatile shocks to account for the Great Moderation. By the same token, the high volatility episode during the 2008-2009 recession calls for larger disturbances, particularly so in the demand side of the economy.

6.2 A Permanent-Bubble World

We read the same observations through a different lens. Specifically, we assume that the economy is best described by the permanent bubble model. In this variant, the volatility of both supply and demand shocks declined by a factor of 2 during the Great Moderation. The bubble is about 6 times more volatile than output. The post-1984 moderation results from less volatile structural shocks. This view of the Great Moderation is consistent with Stock and Watson (2002).

Figure 11 displays the bubble's real valuation over the entire sample. In general, the bubble is more valuable during expansions like the 1970s, 1990s, and the first part of 2000s. Crucial for our purposes, the bubble's value declines in the early 1980s just as the Great Moderation started. It recovered during the technology bubble of the 1990s, which in our model implies higher growth. Except for this episode, the bubble's path is broadly consistent with the one estimated in our benchmark formulation (Figure 10).

The housing bubble in the early 2000s is captured by our model both during the pre-crisis years and the bust. At the end of the sample, we see some recovery but it is far from previous other recoveries. As we argued above, the less valuable the bubble is, the lower the liquidity services it provides, which results in weaker growth. By the end of our sample, we observe that the bubble's price is recovering. This finding is consistent with some economic observers' view

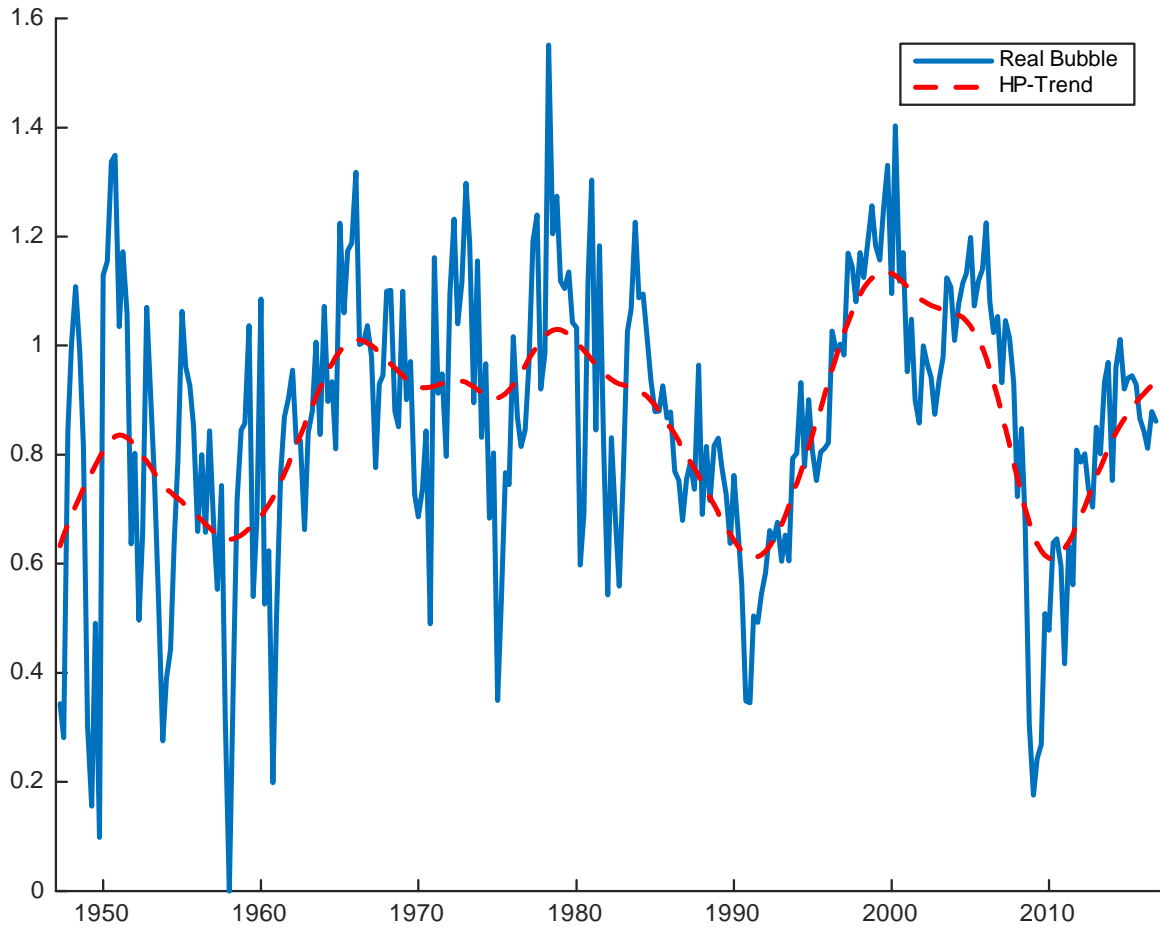


Figure 11: Implicit Real Value Bubble: $m_{b,t}$

that the quantitative easing measures implemented by the Federal Reserve Board are fueling a new bubble.⁷

6.3 A Bubbleless World

Now, let's imagine there was never a bubble in the economy. Supply and demand shocks became less volatile post-1984, with the volatility of the productivity declining by a factor of 1.7 while the volatility of the second disturbances shrinking by half. In all, this version of our model reflects the good-luck hypothesis behind the Great Moderation as in the permanent bubble model.

⁷See for example the PBS column <http://www.pbs.org/newshour/making-sense/column-the-monetary-bubble-to-end-all-bubbles-is-coming/>

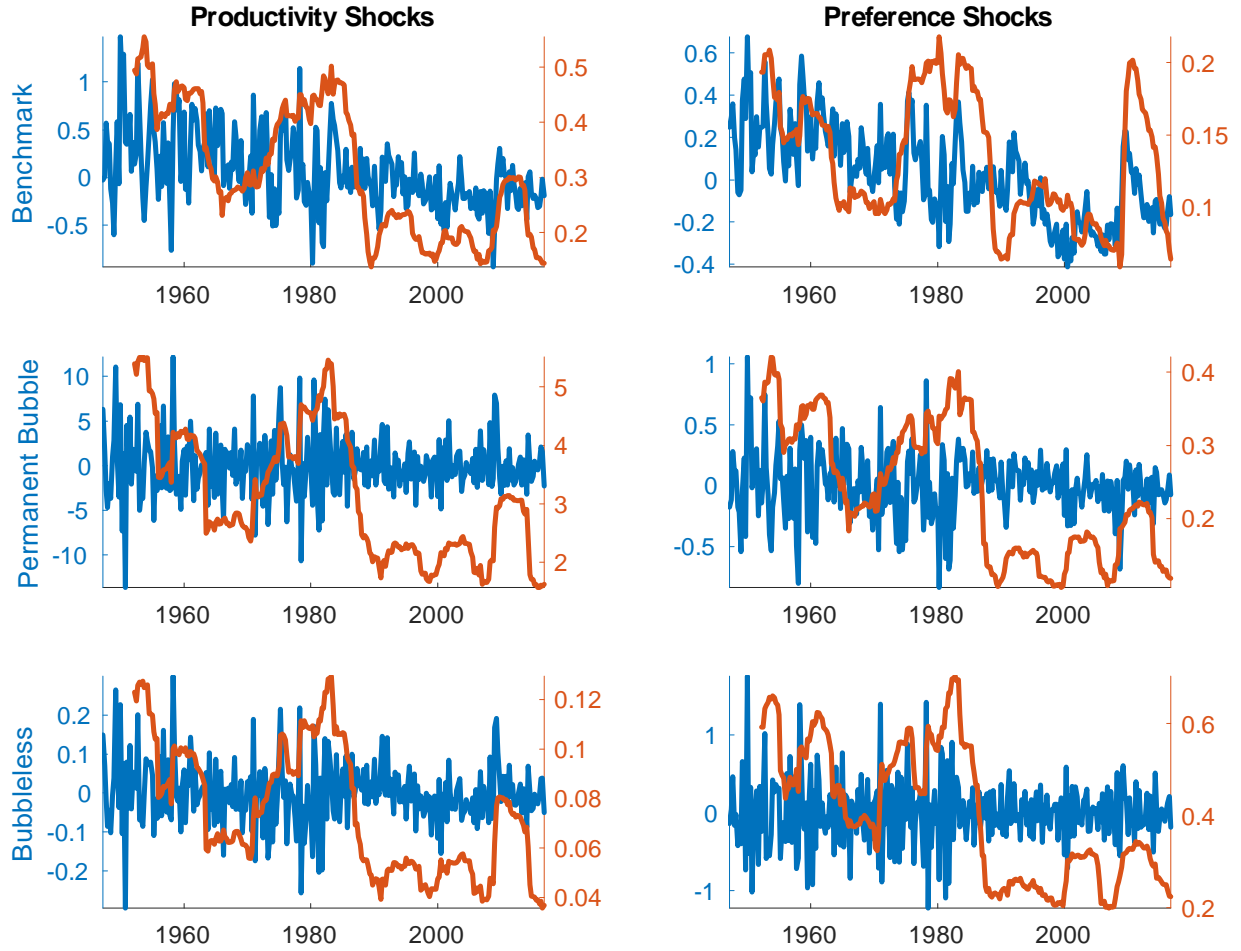


Figure 12: Smoothed Shocks

6.4 Three Models Side-By-Side

The smoothed shocks for the three versions of our model are in Figure 12. The dynamic paths for preference shocks are very similar across the different variants although the shocks from the bubbleless model are slightly more volatile. Productivity shocks in the permanent bubble model display significantly more variability than demand shocks. Moreover, supply shocks in this model are more volatile than the same shocks but in the other two variants. Finally, the post-1984 moderation is apparent in the two shocks.

7 Conclusions

We advance a model of recurrent bubbles, liquidity, and endogenous productivity. Unlike previous work in the literature (Martin and Ventura (2012)), we introduce recurrent bubbles in an infinite horizon business cycle model. We find that recurrent bubbles in this environment have non-trivial impact on the model’s dynamics because prominent mechanisms emphasized in the business cycle literature, such as the intertemporal substitution of consumption and leisure, the endogenous time allocation, and the endogenous capacity utilization rate, are greatly influenced by bubbles. We find that bubbles enhances long-run growth when the degree of financial development is limited. However, if the financial sector is developed enough from the beginning, bubbles may be detrimental to growth due to its general equilibrium effect through the price of capital and endogenous capacity utilization rate. Our model of recurrent bubbles and endogenous productivity attributes the slowdown post-1984 to the collapse of an unproductive bubble.

References

- AGHION, P., P. HOWITT, AND D. MAYER-FOULKES (2005): “The Effect of Financial Development on Convergence: Theory and Evidence,” *The Quarterly Journal of Economics*, 120(1), 173.
- ALLEN, F., G. BARLEVY, AND D. GALE (2017): “On Interest Rate Policy and Asset Bubbles,” Unpublished Manuscript.
- ARROW, K. J. (1962): “The Economic Implications of Learning by Doing,” *The Review of Economic Studies*, 29(3), pp. 155–173.
- BALL, L. M. (2014): “Long-Term Damage from the Great Recession in OECD Countries,” Working Paper 20185, National Bureau of Economic Research.
- BARRO, R. J., AND R. G. KING (1984): “Time-Separable Preferences and Intertemporal-Substitution Models of Business Cycles,” *The Quarterly Journal of Economics*, 99(4), 817–839.
- BEAUDRY, P., AND F. PORTIER (2004): “An Exploration into Pigou’s Theory of Cycles,” *Journal of Monetary Economics*, 51, 1183–1216.
- BEWLEY, T. (1980): “The optimum quantity of money,” in *Models of Monetary Economies*, ed. by J. Kareken, and N. Wallace, pp. 169–210. Federal Reserve Bank of Minneapolis.
- BIANCHI, F. (2013): “Regime Switches, Agents’ Beliefs, and Post-World War II U.S. Macroeconomic Dynamics,” *The Review of Economic Studies*, 80(2), 463–490.

- BLANCHARD, O., E. CERUTTI, AND L. SUMMERS (2014): “Inflation and Activity; Two Explorations, and Their Monetary Policy Implications,” Discussion paper, ECB Forum on Central Banking.
- BRUNNERMEIER, M. K., AND Y. SANNIKOV (2014): “A Macroeconomic Model with a Financial Sector,” *American Economic Review*, 104(2), 379–421.
- CERRA, V., AND S. C. SAXENA (2008): “Growth Dynamics: The Myth of Economic Recovery,” *American Economic Review*, 98(1), 439–57.
- COMIN, D., AND M. GERTLER (2006): “Medium-Term Business Cycles,” *The American Economic Review*, 96, 523–551.
- CONTESSI, S., AND U. KERDNUNVONG (2015): “Asset Bubbles: Detecting and Measuring Them Are Not Easy Tasks,” *The Regional Economist*, (July).
- DEL NEGRO, M., G. EGGERTSSON, A. FERRERO, AND N. KIYOTAKI (2016): “The Great Escape? A Quantitative Evaluation of the Fed’s Liquidity Facilities,” Working Paper 22259, National Bureau of Economic Research.
- FARHI, E., AND J. TIROLE (2012): “Bubbly Liquidity,” *The Review of Economic Studies*, 79(2), 678.
- FARMER, R. E., D. F. WAGGONER, AND T. ZHA (2009): “Understanding Markov-switching rational expectations models,” *Journal of Economic Theory*, 144(5), 1849 – 1867.
- (2011): “Minimal state variable solutions to Markov-switching rational expectations models,” *Journal of Economic Dynamics and Control*, 35(12), 2150 – 2166, *Frontiers in Structural Macroeconomic Modeling*.
- FERNANDEZ-VILLAVARDE, J., P. GUERRON-QUINTANA, AND J. F. RUBIO-RAMIREZ (2015): “Estimating dynamic equilibrium models with stochastic volatility,” *Journal of Econometrics*, 185(1), 216 – 229.
- GALI, J. (2014): “Monetary Policy and Rational Asset Price Bubbles,” *American Economic Review*, 104(3), 721–52.
- (2016): “Insider-outsider labor markets, hysteresis and monetary policy,” *Economics Working Papers 1506*, Department of Economics and Business, Universitat Pompeu Fabra.
- GERTLER, M., AND N. KIYOTAKI (2015): “Banking, Liquidity, and Bank Runs in an Infinite Horizon Economy,” *American Economic Review*, 105(7), 2011–43.

- GUERRON-QUINTANA, P. A., AND R. JINNAI (2015): “Financial Frictions, Trends, and the Great Recession,” Discussion paper series HIAS-E-14, Hitotsubashi Institute for Advanced Study, Hitotsubashi University.
- HAMILTON, J. (2016): “Macroeconomic Regimes and Regime Shifts,” vol. 2 of *Handbook of Macroeconomics*, chap. 3, pp. 163 – 201. Elsevier.
- HE, Z., AND A. KRISHNAMURTHY (2013): “Intermediary Asset Pricing,” *American Economic Review*, 103(2), 732–70.
- HIRANO, T., AND N. YANAGAWA (2017): “Asset Bubbles, Endogenous Growth, and Financial Frictions,” *The Review of Economic Studies*, 84(1), 406.
- JAIMOVICH, N., AND S. REBELO (2009): “Can News about the Future Drive the Business Cycle?,” *The American Economic Review*, 99, 1097–1118.
- KIYOTAKI, N., AND J. MOORE (2012): “Liquidity, Business Cycles, and Monetary Policy,” Working Paper 17934, National Bureau of Economic Research.
- KOCHERLAKOTA, N. (2009): “Bursting Bubbles: Consequences and Cures,” Unpublished Manuscript, University of Minnesota.
- KOCHERLAKOTA, N. R. (1992): “Bubbles and constraints on debt accumulation,” *Journal of Economic Theory*, 57(1), 245 – 256.
- MARTIN, A., AND J. VENTURA (2012): “Economic Growth with Bubbles,” *American Economic Review*, 102(6), 3033–58.
- MIAO, J. (2014): “Introduction to economic theory of bubbles,” *Journal of Mathematical Economics*, 53, 130 – 136, Special Section: Economic Theory of Bubbles (I).
- MIAO, J., AND P. WANG (2017): “Bubbles and Credit Constraints,” Unpublished Manuscript, Boston University.
- ROMER, P. M. (1986): “Increasing Returns and Long-Run Growth,” *Journal of Political Economy*, 94(5), pp. 1002–1037.
- SAMUELSON, P. A. (1958): “An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money,” *Journal of Political Economy*, 66(6), 467–482.
- SCHEINKMAN, J. A., AND L. WEISS (1986): “Borrowing Constraints and Aggregate Economic Activity,” *Econometrica*, 54(1), 23–45.
- SHESHINSKI, E. (1967): “Optimal accumulation with learning by doing,” in *Essays on the theory of optimal economic growth*, ed. by K. Shell, pp. 31–52. MIT Press, Cambridge, MA.

- SHI, S. (2015): “Liquidity, assets and business cycles,” *Journal of Monetary Economics*, 70(0), 116 – 132.
- SHILLER, R. J. (2015): *Irrational Exuberance*. Princeton University Press, Princeton, NJ, third edition edn.
- SIMS, C. A., AND T. ZHA (2006): “Were There Regime Switches in U.S. Monetary Policy?,” *American Economic Review*, 96(1), 54–81.
- STOCK, J. H., AND M. W. WATSON (2002): “Has the Business Cycle Changed and Why?,” *NBER Macroeconomics Annual*, 17, 159–218.
- TIROLE, J. (1982): “On the Possibility of Speculation under Rational Expectations,” *Econometrica*, 50(5), 1163–1181.
- TOWNSEND, R. M. (1980): “Models of Money with Spatially Separated Agents,” in *Models of Monetary Economies*, ed. by J. Kareken, and N. Wallace, pp. 265–303. Federal Reserve Bank of Minneapolis.
- WEIL, P. (1987): “Confidence and the Real Value of Money in an Overlapping Generations Economy,” *The Quarterly Journal of Economics*, 102(1), 1.
- WOODFORD, M. (1990): “Public Debt as Private Liquidity,” *The American Economic Review*, 80(2), 382–388.

8 Appendix

8.1 Model with no binding constraints

The household’s problem is

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{bt} \left(\pi \frac{[c_t^i]^{1-\rho}}{1-\rho} + (1-\pi) \frac{[c_t^s (1-l_t)^\eta]^{1-\rho}}{1-\rho} \right) \right]$$

subject to

$$\pi c_t^i + (1-\pi) c_t^s + n_{t+1} - (1-\delta(u_t)) n_t = u_t r_t n_t + w_t (1-\pi) l_t.$$

The equilibrium conditions are summarized as follows;

$$Y_t = \bar{A} e^{at} u_t^\alpha K_t ((1-\pi) l_t)^{1-\alpha},$$

$$(c_t^i)^{-\rho} = (c_t^s)^{-\rho} (1-l_t)^{\eta(1-\rho)},$$

$$\begin{aligned}
\eta \frac{c_t^s}{1-l_t} &= w_t, \\
\delta'(u_t) &= r_t, \\
1 &= E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{\hat{c}_{t+1}^i} \right)^\rho (u_{t+1} r_{t+1} + 1 - \delta(u_{t+1})) \right], \\
r_t &= \alpha \frac{Y_t}{u_t K_t}, \\
w_t &= (1-\alpha) \frac{Y_t}{(1-\pi) l_t},
\end{aligned}$$

and

$$\pi c_t^i + (1-\pi) c_t^s + K_{t+1} - (1-\delta(u_t)) K_t = Y_t$$

for all t .

Detrend variables by K_t ;

$$\begin{aligned}
\hat{Y}_t &= \bar{A} e^{a_t} u_t^\alpha ((1-\pi) l_t)^{1-\alpha}, \\
(\hat{c}_t^i)^{-\rho} &= (\hat{c}_t^s)^{-\rho} (1-l_t)^{\eta(1-\rho)}, \\
\eta \frac{\hat{c}_t^s}{1-l_t} &= \hat{w}_t, \\
\delta'(u_t) &= r_t, \\
1 &= E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \frac{1}{g_t} \right)^\rho (u_{t+1} r_{t+1} + 1 - \delta(u_{t+1})) \right], \\
r_t &= \alpha \frac{\hat{Y}_t}{u_t}, \\
\hat{w}_t &= (1-\alpha) \frac{\hat{Y}_t}{(1-\pi) l_t},
\end{aligned}$$

and

$$\pi \hat{c}_t^i + (1-\pi) \hat{c}_t^s + g_t - (1-\delta(u_t)) = \hat{Y}_t.$$

8.2 Fundamental Model

The household's problem is

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{b_t} \left(\pi \frac{[c_t^i]^{1-\rho}}{1-\rho} + (1-\pi) \frac{[c_t^s (1-l_t)^\eta]^{1-\rho}}{1-\rho} \right) \right]$$

subject to

$$\pi c_t^i + (1-\pi) c_t^s + q_t n_{t+1} = [u_t r_t + (1-\delta(u_t)) q_t + \pi \lambda_t (u_t r_t + \phi q_t (1-\delta(u_t)))] n_t + (1-\pi) w_t l_t$$

A competitive equilibrium is defined as a sequence of prices, w_t , r_t , and q_t , and quantities, Y_t , i_t , K_{t+1} , c_t^i , c_t^s , l_t , and u_t , that satisfy the following conditions:

$$Y_t = \bar{A} e^{a_t} u_t^\alpha K_t ((1 - \pi) l_t)^{1-\alpha},$$

$$(\hat{c}_t^i)^{-\rho} = (\hat{c}_t^s)^{-\rho} (1 - l_t)^{\eta(1-\rho)},$$

$$\eta \frac{\hat{c}_t^s}{1 - l_t} = \hat{w}_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0,$$

$$q_t = E_t \left[\beta e^{b_{t+1} - b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \right)^\rho (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$\hat{w}_t = (1 - \alpha) \frac{Y_t}{(1 - \pi) l_t},$$

$$Y_t = \pi \hat{c}_t^i + (1 - \pi) \hat{c}_t^s + \pi \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] K_t}{1 - \phi q_t},$$

and

$$K_{t+1} = (1 - \delta(u_t)) K_t + \pi \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] K_t}{1 - \phi q_t}$$

for all t .

Since the model displays endogenous productivity, it is necessary to detrend it before we solve it numerically. Dividing quantities by K_t , we obtain the following equations.

$$\hat{Y}_t = \bar{A} e^{a_t} u_t^\alpha ((1 - \pi) l_t)^{1-\alpha},$$

$$(\hat{c}_t^i)^{-\rho} = (\hat{c}_t^s)^{-\rho} (1 - l_t)^{\eta(1-\rho)},$$

$$\eta \frac{\hat{c}_t^s}{1 - l_t} = \hat{w}_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0,$$

$$q_t = E_t \left[\beta e^{b_{t+1} - b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \frac{1}{g_t} \right)^\rho (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \right],$$

$$r_t = \alpha \frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = (1 - \alpha) \frac{\hat{Y}_t}{(1 - \pi) l_t},$$

$$\hat{Y}_t = \pi \hat{c}_t^i + (1 - \pi) \hat{c}_t^s + \pi \frac{u_t r_t + \phi q_t (1 - \delta(u_t))}{1 - \phi q_t},$$

and

$$g_t = 1 - \delta(u_t) + \pi \frac{u_t r_t + \phi q_t (1 - \delta(u_t))}{1 - \phi q_t}$$

for all t , where hat variables denote the original variable divided by K_t , i.e., $\hat{Y}_t \equiv Y_t/K_t$ and so on, and $g_t \equiv K_{t+1}/K_t$.

8.3 Recurrent Bubble Model

Competitive equilibrium is summarized by the following equations;

$$Y_t = \bar{A} e^{a_t} u_t^\alpha K_t ((1 - \pi) l_t)^{1-\alpha},$$

$$(\hat{c}_t^i)^{-\rho} = (c_t^s)^{-\rho} (1 - l_t)^{\eta(1-\rho)},$$

$$\eta \frac{c_t^s}{1 - l_t} = w_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0,$$

$$q_t = E_t \left[\beta e^{b_{t+1} - b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \right],$$

$$\mathbf{1}_{\{z_t=b\}} \tilde{p}_t = \mathbf{1}_{\{z_t=b\}} E_t \left[\beta e^{b_{t+1} - b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (1 + \pi \lambda_{t+1}) \tilde{p}_{t+1} \mathbf{1}_{\{z_{t+1}=b\}} \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1 - \alpha) \frac{Y_t}{(1 - \pi) l_t},$$

$$Y_t = \pi \hat{c}_t^i + (1 - \pi) c_t^s + \pi \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1 - \phi q_t},$$

$$K_{t+1} = (1 - \delta(u_t)) K_t + \pi \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1 - \phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1 - \phi q_t}.$$

Dividing variables by K_t , we find

$$\hat{Y}_t = \bar{A} e^{a_t} u_t^\alpha ((1 - \pi) l_t)^{1-\alpha},$$

$$(\hat{c}_t^i)^{-\rho} = (\hat{c}_t^s)^{-\rho} (1 - l_t)^{\eta(1-\rho)},$$

$$\eta \frac{\hat{c}_t^s}{1 - l_t} = \hat{w}_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0,$$

$$q_t = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \frac{1}{g_t} \right)^\rho (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \right],$$

$$m_t = \mathbf{1}_{\{z_t=b\}} E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \frac{1}{g_t} \right)^\rho (1 + \pi \lambda_{t+1}) m_{t+1} g_t \right],$$

$$r_t = \alpha \frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = (1 - \alpha) \frac{\hat{Y}_t}{(1 - \pi) l_t},$$

$$\hat{Y}_t = \pi \hat{c}_t^i + (1 - \pi) \hat{c}_t^s + \pi \frac{u_t r_t + \phi q_t (1 - \delta(u_t)) + m_t}{1 - \phi q_t},$$

$$g_t = 1 - \delta(u_t) + \pi \frac{u_t r_t + \phi q_t (1 - \delta(u_t)) + m_t}{1 - \phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1 - \phi q_t}$$

where $m_t \equiv \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M/K_t$.

It is convenient to make the dependence on the regime explicit;

$$\hat{Y}_{f,t} = \bar{A} e^{at} (u_{f,t})^\alpha ((1 - \pi) l_{f,t})^{1-\alpha}, \quad (28)$$

$$\hat{Y}_{b,t} = \bar{A} e^{at} (u_{b,t})^\alpha ((1 - \pi) l_{b,t})^{1-\alpha}, \quad (29)$$

$$(\hat{c}_{f,t}^i)^{-\rho} = (\hat{c}_{f,t}^s)^{-\rho} (1 - l_{f,t})^{\eta(1-\rho)}, \quad (30)$$

$$(\hat{c}_{b,t}^i)^{-\rho} = (\hat{c}_{b,t}^s)^{-\rho} (1 - l_{b,t})^{\eta(1-\rho)}, \quad (31)$$

$$\eta \frac{\hat{c}_{f,t}^s}{1 - l_{f,t}} = \hat{w}_{f,t}, \quad (32)$$

$$\eta \frac{\hat{c}_{b,t}^s}{1 - l_{b,t}} = \hat{w}_{b,t}, \quad (33)$$

$$r_{f,t} - \delta'(u_{f,t}) q_{f,t} + \pi \lambda_{f,t} (r_{f,t} - \phi q_{f,t} \delta'(u_{f,t})) = 0, \quad (34)$$

$$r_{b,t} - \delta'(u_{b,t}) q_{b,t} + \pi \lambda_{b,t} (r_{b,t} - \phi q_{b,t} \delta'(u_{b,t})) = 0, \quad (35)$$

$$q_{f,t} = E_t \left[\begin{aligned} & (1 - \sigma_f) \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_{f,t}^i}{\hat{c}_{f,t+1}^i} \frac{1}{g_{f,t}} \right)^\rho \\ & (u_{f,t+1} r_{f,t+1} + (1 - \delta(u_{f,t+1})) q_{f,t+1} + \pi \lambda_{f,t+1} (u_{f,t+1} r_{f,t+1} + \phi q_{f,t+1} (1 - \delta(u_{f,t+1})))) \end{aligned} \right] \quad (36)$$

$$+ E_t \left[\begin{aligned} & \sigma_f \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_{f,t}^i}{\hat{c}_{b,t+1}^i} \frac{1}{g_{f,t}} \right)^\rho \\ & (u_{b,t+1} r_{b,t+1} + (1 - \delta(u_{b,t+1})) q_{b,t+1} + \pi \lambda_{b,t+1} (u_{b,t+1} r_{b,t+1} + \phi q_{b,t+1} (1 - \delta(u_{b,t+1})))) \end{aligned} \right],$$

$$q_{b,t} = E_t \left[\begin{array}{c} (1 - \sigma_b) \beta e^{b_{t+1} - b_t} \left(\frac{\hat{c}_{b,t}^i}{\hat{c}_{b,t+1}^i} \frac{1}{g_{b,t}} \right)^\rho \\ (u_{b,t+1} r_{b,t+1} + (1 - \delta(u_{b,t+1})) q_{b,t+1} + \pi \lambda_{b,t+1} (u_{b,t+1} r_{b,t+1} + \phi q_{b,t+1} (1 - \delta(u_{b,t+1})))) \end{array} \right] \quad (37)$$

$$+ E_t \left[\begin{array}{c} \sigma_b \beta e^{b_{t+1} - b_t} \left(\frac{\hat{c}_{b,t}^i}{\hat{c}_{f,t+1}^i} \frac{1}{g_{b,t}} \right)^\rho \\ (u_{f,t+1} r_{f,t+1} + (1 - \delta(u_{f,t+1})) q_{f,t+1} + \pi \lambda_{f,t+1} (u_{f,t+1} r_{f,t+1} + \phi q_{f,t+1} (1 - \delta(u_{f,t+1})))) \end{array} \right],$$

$$m_{f,t} = 0, \quad (38)$$

$$m_{b,t} = E_t \left[(1 - \sigma_b) \beta e^{b_{t+1} - b_t} \left(\frac{\hat{c}_{b,t}^i}{\hat{c}_{b,t+1}^i} \frac{1}{g_{b,t}} \right)^\rho (1 + \pi \lambda_{b,t+1}) m_{b,t+1} g_{b,t} \right] \quad (39)$$

$$+ E_t \left[\sigma_b \beta e^{b_{t+1} - b_t} \left(\frac{\hat{c}_{b,t}^i}{\hat{c}_{f,t+1}^i} \frac{1}{g_{b,t}} \right)^\rho (1 + \pi \lambda_{f,t+1}) m_{f,t+1} g_{b,t} \right],$$

$$r_{f,t} = \alpha \frac{\hat{Y}_{f,t}}{u_{f,t}}, \quad (40)$$

$$r_{b,t} = \alpha \frac{\hat{Y}_{b,t}}{u_{b,t}}, \quad (41)$$

$$\hat{w}_{f,t} = (1 - \alpha) \frac{\hat{Y}_{f,t}}{(1 - \pi) l_{f,t}}, \quad (42)$$

$$\hat{w}_{b,t} = (1 - \alpha) \frac{\hat{Y}_{b,t}}{(1 - \pi) l_{b,t}}, \quad (43)$$

$$\hat{Y}_{f,t} = \pi \hat{c}_{f,t}^i + (1 - \pi) \hat{c}_{f,t}^s + \pi \frac{u_{f,t} r_{f,t} + \phi q_{f,t} (1 - \delta(u_{f,t})) + m_{f,t}}{1 - \phi q_{f,t}}, \quad (44)$$

$$\hat{Y}_{b,t} = \pi \hat{c}_{b,t}^i + (1 - \pi) \hat{c}_{b,t}^s + \pi \frac{u_{b,t} r_{b,t} + \phi q_{b,t} (1 - \delta(u_{b,t})) + m_{b,t}}{1 - \phi q_{b,t}}, \quad (45)$$

$$g_{f,t} = 1 - \delta(u_{f,t}) + \pi \frac{u_{f,t} r_{f,t} + \phi q_{f,t} (1 - \delta(u_{f,t})) + m_{f,t}}{1 - \phi q_{f,t}}, \quad (46)$$

$$g_{b,t} = 1 - \delta(u_{b,t}) + \pi \frac{u_{b,t} r_{b,t} + \phi q_{b,t} (1 - \delta(u_{b,t})) + m_{b,t}}{1 - \phi q_{b,t}}, \quad (47)$$

$$\lambda_{f,t} = \frac{q_{f,t} - 1}{1 - \phi q_{f,t}}, \quad (48)$$

and

$$\lambda_{b,t} = \frac{q_{b,t} - 1}{1 - \phi q_{b,t}} \quad (49)$$

where subscripts f and b denote realizations of the variables in a fundamental and bubble regime, respectively; for instance, $\hat{Y}_{f,t}$ is the realization of \hat{Y}_t in a fundamental regime.

The impulse response functions are calculated by linearizing the equations (28) to (49) around

$$\hat{Y}_{f,t} = \hat{Y}_f, \hat{c}_{f,t}^i = \hat{c}_f^i, \hat{c}_{f,t}^s = \hat{c}_f^s, l_{f,t} = l_f, g_{f,t} = g_f, q_{f,t} = q_f, \lambda_{f,t} = \lambda_f, u_{f,t} = u_f, r_{f,t} = r_f, \hat{w}_{f,t} = \hat{w}_f, \\ \hat{Y}_{b,t} = \hat{Y}_b, \hat{c}_{b,t}^i = \hat{c}_b^i, \hat{c}_{b,t}^s = \hat{c}_b^s, l_{b,t} = l_b, g_{b,t} = g_b, q_{b,t} = q_b, \lambda_{b,t} = \lambda_b, u_{b,t} = u_b, r_{b,t} = r_b, \hat{w}_{b,t} = \hat{w}_b \text{ and} \\ m_{b,t} = m_b.$$