Solving Recursive Utility Models with Nonstationary Consumption

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ABSTRACT. We study existence, uniqueness and computability of solutions for a class of discrete time recursive utilities models. By combining two streams of the recent literature on recursive preferences—one that analyzes principal eigenvalues of valuation operators and another that exploits the theory of monotone concave operators—we obtain conditions that are both necessary and sufficient for existence and uniqueness of solutions. We also show that the natural iterative algorithm is convergent if and only if a solution exists. Consumption processes are allowed to be nonstationary.

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1. INTRODUCTION

Recursive preference models such as those discussed in Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1990) play an important role in macroeconomic and financial modeling. For example, the long-run risk models analyzed in Bansal and Yaron (2004), Hansen et al. (2008), Bansal et al. (2012) and Schorfheide et al. (2017) have employed such preferences in discrete time infinite horizon settings with a variety of consumption path specifications to help resolve long-standing empirical puzzles identified in the literature.

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In recursive utilities models, the lifetime value of a consumption stream from a given point in time is typically expressed as the solution to a nonlinear forward-looking equation. While this approach is convenient and intuitive, it can be problematic in the sense that solutions fail to exist for some parameter values. In general, identifying restrictions that imply existence of a solution for an empirically relevant class of consumption streams is nontrivial. Moreover, even when a solution is known to exist, this solution lacks predictive content unless some form of uniqueness result can also be obtained.

The ideal case is where researchers have at hand results implying existence and uniqueness of solutions, as well as robust, globally convergent methods for computing them. While results along these lines can be found in the literature, a gap exists between the theoretical results established to date and the parameterizations of recursive utility and specifications of consumption paths actually encountered in empirical work at the current research frontier. The aim of the present paper is to close this gap by obtaining existence and uniqueness results that are as tight as possible in a range of empirically plausible settings, while restricting attention to practical conditions that can be tested in applications.

More specifically, we provide necessary and sufficient conditions for existence and uniqueness of solutions to the class of preferences studied in Epstein and Zin (1989), as well as a globally convergent method of computation, while admitting consumption paths that follow a relatively general multiplicative functional specification (see, e.g., Hansen and Scheinkman, 2009, 2012). The preferences in question take the form

$$V_t = \left[\zeta C_t^{1-1/\psi} + \beta \left\{\mathcal{R}_t\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}, \tag{1}$$

where $\{C_t\}$ is a consumption path, V_t is the utility value of the path extending on from time t, and $\mathcal{R}_t$ is the Kreps–Porteus certainty equivalent operator

$$\mathcal{R}_t(V_{t+1}) := (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}. \tag{2}$$

The parameter β is a time discount factor, while γ governs risk aversion and ψ is the elasticity of intertemporal substitution (EIS).

We assume that consumption growth can be expressed as

$$\ln(C_{t+1}/C_t) = \kappa(X_{t+1}, Y_{t+1}, X_t), \tag{3}$$

where κ is a continuous real-valued function, $\{X_t\}$ is a time homogeneous Markov process and $\{Y_t\}$ is an IID innovation process. The persistent component $\{X_t\}$ is required to be compact-valued, while the innovation $\{Y_t\}$ is allowed to be unbounded. As a result, consumption growth in any given period can be arbitrarily large or small.

Our conditions feature two quantities. The first is the composite parameter

$$\theta := \frac{1-\gamma}{1-1/\psi}. \tag{4}$$

The other is $r(K)$, the spectral radius of a "valuation operator" K determined by the primitives in (1)–(3). We show that when $\theta < 0$, a Markov solution exists for normalized utility V_t/C_t if and only if $r(K) > 1$. The same condition $r(K) > 1$ is also equivalent to existence of a unique solution, and to global converge of the successive approximations obtained by iterating with the Koopmans operator. When $\theta > 0$, the required inequality reverses, and a unique solution exists if and only if $r(K) < 1$. If $r(K) < 1$ fails, then no solution exists. This condition $r(K) < 1$ is also equivalent to the convergence of the sequence of successive approximations.

Aside from providing a test of existence and uniqueness of solutions, one consequence of these results is that, for all of the consumption processes we consider, the recursive expression (1) that defines Epstein–Zin utility has at most one solution. In other words, uniqueness is never problematic. Another consequence of our results is that, if the sequence of successive approximations converges, then a unique solution exists and this solution is equal to limit of the successive approximations. In all other situations, successive approximation will diverge. Thus, if computing the solution to the model at a given set of parameters is the primary objective, then convergence of the iterative method itself justifies the claim that the limit is a solution, and no other solution exists in the candidate space that we consider.

Our work builds on a growing literature on solutions to recursive preference models. The groundwork for Epstein–Zin preferences over infinite-horizon consumption streams was provided by Epstein and Zin (1989), who in turn built on the finite-horizon framework of temporal lotteries found in Kreps and Porteus (1978). Epstein and Zin (1989) obtained sufficient conditions for existence across a broad set of parameters, while allowing geometric consumption growth and eschewing a Markov assumption. These findings were strengthened by Marinacci and Montrucchio (2010), who provided sufficient conditions for both existence and uniqueness of solutions, as well as convergence of successive approximations. Their results were obtained via an innovative fixed point approach that exploits concavity and monotonicity properties possessed by Epstein–Zin preferences with empirically plausible parameterizations.

One issue with the conditions of Epstein and Zin (1989) and Marinacci and Montrucchio (2010) is that they require, at least asymptotically, a finite bound on b consumption growth $\ln(C_{t+1}/C_t)$ that holds with probability one. This fails for many standard consumption processes, such as Bansal and Yaron (2004), where

$$\ln(C_{t+1}/C_t) = \mu + z_t + \sigma_t \eta_{t+1}. \tag{5}$$

Here $\{z_t\}$ and $\{\sigma_t\}$ are stationary processes and $\{\eta_{t+1}\}$ is IID and $N(0,1)$.

In fact the problem is not so much that the bounded growth condition fails, since the shocks in (5) can be truncated at suitably large values and b can then be chosen as the maximal value taken by the right-hand side of (5). Rather, the issue is that the resulting restrictions on preference parameters, which are used to ensure finiteness of the solution to the recursive utility model, are excessively conservative. The intuition behind this is that the probability one bound b considers only the worst case in terms of pushing utility to infinity. Much sharper results can be obtained by considering what happens "on average." After all, recursive utility specifications, while nonlinear, are still defined using expectations. This means that the whole distribution matters, not just the bound on the right tail.

As stated above, the conditions presented in this paper are based around the spectral radius of the operator K, which captures the distributions of the persistent components and innovations in consumption specifications such as (5). Since the spectral radius depends on the full distributions and involves averaging by integration, it leads to conditions that are much sharper—in fact as sharp as possible for the setting we consider, being both necessary and sufficient. By applying our results to two empirical set ups from the recent literature (Bansal and Yaron (2004), Schorfheide et al. (2017)), we demonstrate that the gap between the necessary and sufficient conditions developed here and the sufficient conditions arising from probability one bounds is both large and significant for modern quantitative applications.

Here we should add two qualifications to the preceding discussion. First, Marinacci and Montrucchio (2010) treat a much larger range of recursive utility specifications than we consider here, as well as admitting non-Markovian state processes. Thus, while our results are sharper for the problems we consider, their study is far more comprehensive. Second, our condition is inherently computational, since the spectral radius cannot be derived analytically apart from very special cases. (In particular, once the Markov process X_t has been discretized, K becomes a matrix for which the spectral radius must be computed.) The computational—as opposed to analytical—nature of our condition aids us in obtaining tight results.

In addition to ourselves, a number of other researchers have sought to extend the work of Marinacci and Montrucchio (2010). In the realm of recursive utility specifications with Thompson aggregators, the idea of exploiting monotonicity and concavity of the Koopmans operator has been adapted and extended by Balbus (2016), Becker and Rincón-Zapatero (2017) and Bloise and Vailakis (2017). While these contributions do not resolve the issues associated with using probability one bounds on consumption growth discussed above, they carefully elucidate the links between the monotonicity and concavity properties of certain aggregators and fixed point results in partially ordered vector spaces. Such results also lie at the heart of this paper, and we have benefited extensively from the their ideas.[1]

There is one more closely related paper upon which we draw extensively: the study of Epstein–Zin utility models with fully unbounded consumption growth specifications in Hansen and Scheinkman (2012). Their approach is to draw a connection between the solution to the Epstein–Zin utility recursion and the Perron–Frobenius eigenvalue problem associated with a linear operator, denoted in their paper by $\mathbb{T}$, that is proportional to the operator K discussed above. Consumption growth obeys (3) and, unlike this paper, $\{X_t\}$ is not required to be compact-valued. In this very general setting they show that a solution exists when a joint restriction holds on the spectral radius of $\mathbb{T}$ and the preference parameters, in addition to as certain auxiliary restrictions. They also obtain a uniqueness result for the case $\theta \geqslant 1$.[2]

The advantage of the conditions in Hansen and Scheinkman (2012) is the lack of a compactness restriction on X_t. The advantages of our approach are as follows:

[1]Prior to Marinacci and Montrucchio (2010), important contributions to the literature on existence and uniqueness of solutions to recursively defined utility specifications (as well as the closely related problem of optimality of dynamic programs with general aggregators), were made by Koopmans (1960), Lucas and Stokey (1984), Becker et al. (1989), Streufert (1990), Boyd (1990), Ozaki and Streufert (1996), Le Van and Vailakis (2005) and Rincón-Zapatero and Rodríguez-Palmero (2007). We also note the important contributions of earlier authors who used monotonicity and concavity to obtaine fixed points of forward looking recursive models. See, for example, Coleman (1991), Datta et al. (2002) and Mirman et al. (2008).

[2]See proposition 6 of Hansen and Scheinkman (2012). Note that the symbol α in their study corresponds to θ here.

First, we obtain uniqueness of the solution for all θ, not just for $\theta \geqslant 1$. (One reason this matters is that empirical studies typically find that $\theta < 1$.) Second, we obtain conditions that are necessary as well as sufficient, both for existence and for uniqueness. Third, we obtain a globally convergent method of computation, and show that it converges if and only if a solution exists. Fourth, the auxiliary conditions in Hansen and Scheinkman (2012), which generalize our compactness assumption, involve testing integrability restrictions on the eigenfunctions of the operator $\mathbb{T}$. In general these kinds of conditions are difficult to test—unless, of course, one truncates the state space, which is exactly what we do here.[3]

Although one might still argue that the compactness assumption on the state process is excessively restrictive, we do not find this to be true—at least for the (rather mainstream) applications we consider. One reason is that, as discussed above, we do not exclude unbounded consumption growth, which can enter through the innovation vector. The compactness assumption is only applied to the persistent components of consumption growth, such as time varying volatility and the stochastic component of trend consumption growth. Since these are bounded in probability, one can always choose a compactification of the state space such that the overall impact on the stochastic process for consumption is arbitrarily small. In addition, computational results in section 4.3 show that discretization errors associated with the spectral radius calculations are small and diminish rapidly as grid size increases.

The other connection between our paper and that of Hansen and Scheinkman (2012) is both sets of results rely on information contained in the dominant eigenvalue and corresponding eigenfunction of the valuation operator. Indeed, the principal eigenpairs of valuation operators associated with future cash and utility payoffs have increasingly been used to understand long run risks and long run values in macroeconomic and financial applications (see, e.g., Alvarez and Jermann (2005); Hansen and Scheinkman (2009); Qin and Linetsky (2017)). In this paper we connect the principal eigenpairs of the valuation operator associated with Epstein–Zin preferences to the theory of monotone concave operators. In this way we link two active strands of research on present values associated with cash and utility flows.[4]

[3]Another recent paper that works in a noncompact state setting is Christensen (2017). There the focus is on robust decision makers, who can also be viewed as utility maximizers with risk-sensitive preferences.

[4]We also connect analysis of principle eigenvalues to a version of Banach's contraction mapping theorem, which is needed for the case $\theta \in (0, 1)$.

The paper is structured as follows: Section 2 develops a set of fixed point results in an abstract setting. Section 3 applies these results to the recursive utility models discussed above. Section 4 provides two applications. Code for all computations in the paper can be found at `https://github.com/jstac/recursive_utility_code`.

2. Fixed Point Results

We begin with an abstract fixed point problem for an operator A of the form $Ag(x) = \phi(Kg(x))$, where K is a linear operator and ϕ is a scalar function. We obtain a range of fixed point results depending on the properties of ϕ and K. Our motivation is that the recursive preference equation studied in subsequent sections can be mapped to such a fixed point problem after a continuous transformation. In that setting, the operator K corresponds to the valuation operator discussed in the introduction to this paper.

2.1. Definitions.

Let $\mathbb{U}, \mathbb{V}$ be topological spaces, let $F \colon \mathbb{U} \to \mathbb{U}$ and let $G \colon \mathbb{V} \to \mathbb{V}$. We call F *globally asymptotically stable* if F has a unique fixed point $u^* \in \mathbb{U}$ and $F^n(u) \to u^*$ as $n \to \infty$ for all $u \in \mathbb{U}$. The maps F and G are called *topologically conjugate* if there exists a homeomorphism h from $\mathbb{U}$ to $\mathbb{V}$ such that $h \circ F = G \circ h$ on $\mathbb{U}$. If F and G are topologically conjugate, then F is globally asymptotically stable on $\mathbb{U}$ if and only if G is globally asymptotically stable on $\mathbb{V}$ (see, e.g., Holmgren (2012), theorem 9.3).

Given a compact metric space $\mathbb{X}$, let $\mathscr{C}$ be all continuous real-valued functions on $\mathbb{X}$, let $\leqslant$ is the usual pointwise partial order on $\mathscr{C}$ and let $\| \cdot \|$ be the supremum norm. The symbol $\ll$ denotes strict pointwise inequality, so that $f \ll g$ means $f(x) < g(x)$ for all $x \in \mathbb{X}$. The statement $f < g$ means that $f \leqslant g$ and $f(x) < g(x)$ for some $x \in \mathbb{X}$. Given $f \leqslant g$ in $\mathscr{C}$, let

$$[f, g] := \{h \in \mathscr{C} : f \leqslant h \leqslant g\}.$$

Note that $\mathscr{C}$ is a Banach lattice with the supremum norm and partial order $\leqslant$. In particular, the metric induced by $\| \cdot \|$ on $\mathscr{C}$ is complete, $\mathscr{C}$ is closed under the taking of pairwise suprema and $|f| \leqslant |g|$ implies $\|f\| \leqslant \|g\|$ for all f, g in $\mathscr{C}$.[5]

The *positive cone* of $\mathscr{C}$, denoted below by $\mathscr{C}_+$, is all $g \in \mathscr{C}$ such that $0 \leqslant g$. As an order cone in $\mathscr{C}$, the set $\mathscr{C}_+$ is solid, normal and reproducing.[6] The interior of $\mathscr{C}_+$ is

[5]See Zaanen (1997) and Aliprantis and Border (2006) for more details.

[6]A cone C in $\mathscr{C}$ is called *solid* if it has nonempty interior, *normal* if there exists a constant N with $\|f\| \leqslant N\|g\|$ whenever $f, g \in C$ with $f \leqslant g$, and *reproducing* if every element f of $\mathscr{C}$ can be written as a linear combination of elements of C. See, for example, Du (2006).

denoted $\mathring{\mathscr{C}}_+$ and contains all $g \in \mathscr{C}$ with $0 \ll g$. An operator A from $\mathscr{C}_0 \subset \mathscr{C}$ into $\mathscr{C}$ is called *increasing* if $Af \leqslant Ag$ whenever $f, g \in \mathscr{C}_0$ and $f \leqslant g$. It is called *concave* if, for all $\alpha \in [0,1]$ and $f, g \in \mathscr{C}_0$,

$$\alpha Af + (1-\alpha)Ag \leqslant A\{\alpha f + (1-\alpha)g\}. \tag{6}$$

A linear operator L from $\mathscr{C}$ to itself is called *strongly positive* if $0 \ll Lg$ for every nonzero $g \in \mathscr{C}_+$. Note that, for any linear increasing map L on $\mathscr{C}$ we have $|Lf| \leqslant L|f|$ for every $f \in \mathscr{C}$.

Given a linear operator L from $\mathscr{C}$ to itself, the *operator norm* and *spectral radius* of L are defined by

$$\|L\| := \sup\{\|Lg\| : g \in \mathscr{C},\ \|g\| \leqslant 1\} \quad \text{and} \quad r(L) := \sup\{|\lambda| : \lambda \in \sigma(L)\}$$

respectively. Here $\sigma(L)$ is the spectrum of L.[7] The operator L is called *compact* if the image of the unit ball in $\mathscr{C}$ under L is relatively compact.

We will exploit the following fixed point theorem for monotone concave operators:

Theorem 2.1 (Du–Zhang)**.** *Let A be increasing and concave on $\mathscr{C}_+$. If, in addition, there exist functions $f_1 \leqslant f_2$ in $\mathscr{C}_+$ with $Af_1 \gg f_1$ and $Af_2 \leqslant f_2$, then A is globally asymptotically stable on $[f_1, f_2]$.*

Proof. This follows from corollary 2.1.1 of Zhang (2013), given that $\mathscr{C}_+$ is both normal and solid as an order cone in $\mathscr{C}$. □

2.2. **Set Up.** Let $K\colon \mathscr{C} \to \mathscr{C}$ be an increasing linear operator and let $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. Let Φ be the operator on $\mathscr{C}_+$ defined by $\Phi g = \phi \circ g$. Let A be the operator on $\mathscr{C}_+$ defined by $Ag = \Phi K g$, or, equivalently,

$$(Ag)(x) = \phi(Kg(x)) \qquad (x \in \mathbb{X}). \tag{7}$$

Note that $A\mathscr{C}_+ \subset \mathscr{C}_+$. Indeed, given that K is linear and increasing, for each fixed $g \in \mathscr{C}_+$ we have $Kg \in \mathscr{C}_+$. Since ϕ is continuous and nonnegative, ΦKg is also in $\mathscr{C}_+$. While A is a nonlinear operator, the nonlinear component enters only as a scalar transformation. Below we consider fixed points of A in $\mathscr{C}_+$ under a range of auxiliary assumptions. Assumptions placed on K and ϕ in this section are always in force.

[7] These definitions are standard. See, for example, Kolmogorov and Fomin (1975).

2.3. **Long Run Contractions.** Suppose for the purposes of this section that ϕ is Lipschitz of order ℓ for some $\ell > 0$. That is,

$$|\phi(s) - \phi(t)| \leqslant \ell\,|s - t| \text{ for all } s, t \geqslant 0. \tag{8}$$

First we consider the case where the spectral radius of K is small relative to $1/\ell$.

Proposition 2.1. *If $r(K) < 1/\ell$, then there exists an $n \in \mathbb{N}$ such that A^n is a contraction map on $\mathscr{C}_+$. In particular, A is globally asymptotically stable on $\mathscr{C}_+$.*

Proof of proposition 2.1. Fix $f, g \in \mathscr{C}_+$. As a first step we claim that

$$|A^n f - A^n g| \leqslant (\ell K)^n |f - g| \tag{9}$$

holds pointwise on $\mathbb{X}$ for all integers $n \geqslant 0$. It holds holds for $n = 0$, since A^0 and K^0 are by definition identity maps. Now suppose that it holds for some $n \geqslant 0$. We claim it also holds at $n + 1$. Indeed

$$\left|A^{n+1} f - A^{n+1} g\right| = |\Phi K\, A^n f - \Phi K\, A^n g| \leqslant \ell\,|K A^n f - K A^n g|\,,$$

where the inequality is due to (8). Using this bound and the linearity and monotonicity of K leads us to

$$\left|A^{n+1} f - A^{n+1} g\right| \leqslant \ell\,|K(A^n f - A^n g)| \leqslant \ell K\,|A^n f - A^n g| \leqslant (\ell K)^{n+1} |f - g|,$$

where the last inequality uses the induction hypothesis combined with the monotonicity of K. We have now confirmed that (9) holds for all $n \geqslant 0$.

Taking the supremum over (9) yields $\|A^n f - A^n g\| \leqslant \|(\ell K)^n |f - g|\|$. The definition of the operator norm then gives

$$\|A^n f - A^n g\| \leqslant \|(\ell K)^n\| \cdot \|f - g\| = \ell^n\,\|K^n\| \cdot \|f - g\|.$$

Applying Gelfand's formula $r(K) = \lim_{n\to\infty} \|K^n\|^{1/n}$ gives

$$(\ell^n\,\|K^n\|)^{1/n} \to \ell r(K) < 1.$$

Hence, for sufficiently large n, the map A^n is a contraction with modulus $\ell^n \|K^n\|$. All claims in proposition 2.1 follow from this property, completeness of $\|\cdot\|$, the fact that $\mathscr{C}_+$ is closed in $\mathscr{C}$, and a well-known extension to the Banach contraction mapping theorem (see, e.g., p. 272 of Wagner (1982)). □

Remark 2.1. Since $r(K) \leqslant \|K\|$, to obtain the spectral radius condition in proposition 2.1, it suffices to show that $\|K\| < 1/\ell$. In fact, as is clear from the proof of proposition 2.1, A itself becomes a contraction mapping under this condition.

2.4. **Fixed Points under Monotonicity and Concavity.** If $\phi(0) = 0$, then $0 \in \mathscr{C}_+$ is clearly a fixed point of A. In applications this fixed point is often trivial, and interest centers on positive fixed points. In such settings it turns out that notions of concavity and monotonicity are more helpful than contractivity arguments for establishing fixed point results. We begin with a simple lemma.

Lemma 2.1. *For the operator $A = \Phi K$ defined above, the following statements hold.*

(a) If ϕ is increasing on $\mathbb{R}_+$, then A is increasing on $\mathscr{C}_+$.
(b) If ϕ is concave on $\mathbb{R}_+$, then A is concave on $\mathscr{C}_+$.

Proof. Claim (a) is obviously true, given that K is already assumed to be increasing. Regarding claim (b), since K is linear, for any fixed $f, g \in \mathscr{C}_+$, $x \in \mathbb{X}$, and $\alpha \in [0, 1]$, we have

$$\begin{aligned}\alpha\phi(Kf(x)) + (1-\alpha)\phi(Kg(x)) &\leqslant \phi(\alpha Kf(x) + (1-\alpha)Kg(x)) \\ &= \phi[K(\alpha f(x) + (1-\alpha)g(x))].\end{aligned}$$

In other words, $\alpha Af(x) + (1-\alpha)Ag(x) \leqslant A[\alpha f(x) + (1-\alpha)g(x)]$ for any $x \in \mathbb{X}$, as was to be shown. □

Lemma 2.2. *If ϕ is strictly concave, K is strongly positive, and, in addition,*

$$\phi(0) = 0, \quad \lim_{t\downarrow 0} \frac{\phi(t)}{t} \leqslant 1 \quad \textit{and} \quad r(K) \leqslant 1, \tag{10}$$

then the only fixed point of A in $\mathscr{C}_+$ is 0*.*

Proof. Since K is linear we have $K0 = 0$. Hence, with the additional assumption $\phi(0) = 0$, we have $A0 = 0$. Thus, the zero element is a fixed point. It remains to show that no other fixed point exists in $\mathscr{C}_+$. In doing so we use the fact that ϕ must be strictly positive everywhere on $(0, \infty)$, since the existence of a positive t with $\phi(t) = 0$ violates our assumption that ϕ is both strictly concave and nonnegative.

Seeking a contradiction, suppose that $g \in \mathscr{C}_+$ is a nonzero fixed point of A. Observe that, since $Ag = g$ and g is nonzero, the fact that K is a strongly positive operator and ϕ is positive on $(0, \infty)$ implies that $g \gg 0$. In particular, the constant $m := \min g$ is strictly positive. Let $\eta := \phi(m)/m$. Note that, by strict concavity of ϕ and the assumption that $\frac{\phi(t)}{t} \to 1$ as $t \downarrow 0$, we have

$$\eta < 1 \quad \text{and} \quad t \geqslant m \implies \phi(t) \leqslant \eta t. \tag{11}$$

Observe that $Kg \gg g$ must hold. To see why, suppose that $Kg(x) \leqslant g(x)$ for some x. Invoking strict concavity and the limit in (10) again, we have $\phi(t) < t$ for any positive t, and hence $Ag(x) = \phi(Kg(x)) < Kg(x) \leqslant g(x)$. This contradicts the assumption that g is a fixed point of A. Our claim that $Kg \gg g$ is confirmed.

Next we claim that $A^n g \leqslant (\eta K)^n g$ for all n. Evidently this holds at $n = 0$, and, assuming it holds at n, we have

$$A^{n+1}g = \Phi K A^n g \leqslant \eta K A^n g \leqslant \eta K (\eta K)^n g = (\eta K)^{n+1} g.$$

In the first inequality we used the fact that $Kg \gg g$ and g is a fixed point of A, so that $KA^n g = Kg \gg g \geqslant m$ when m is as given in (11). In the second inequality we used the induction hypothesis and the monotonicity of K.

We have now shown by induction that $A^n g \leqslant (\eta K)^n g$ for all $n \in \mathbb{N}$. Hence

$$\|A^n g\| \leqslant \eta^n \|K^n g\| \leqslant \eta^n \|K^n\| \, \|g\| \tag{12}$$

for all n. Since $r(K) \leqslant 1$ and $\eta < 1$, Gelfand's formula implies the existance of an $n \in \mathbb{N}$ such that $\|K^n\|^{1/n} < 1/\eta$, or $\|K^n\| < (1/\eta)^n$. Evaluating (12) at this n gives $\|A^n g\| < \|g\|$, contradicting our assumption that g is a fixed point of A. □

Below we will make use of the following version of the Krein–Rutman theorem, the value of which for studing recursive preference models was identified and illustrated in Hansen and Scheinkman (2009, 2012).

Lemma 2.3 (Krein–Rutman)**.** *If, in addition to being linear and increasing, K is also strongly positive and compact, then $r(K) > 0$ and $r(K)$ is an eigenvalue of K. In particular, there exists an $e \in \mathscr{C}_+$ such that $Ke = r(K)e$, and $e \gg 0$.*

Lemma 2.3 follows directly from theorem 1.2 of Du (2006), given that $\mathscr{C}_+$ is both solid and reproducing. The element e in lemma 2.3 is unique up to a scale factor. In what follows we normalize by requiring that $\|e\| = 1$, and call e the *Perron–Frobenius* eigenfunction of K.

Lemma 2.4. *If K is strongly positive and compact, and ϕ and K jointly satisfy*

$$\lim_{t \downarrow 0} \frac{\phi(t)}{t} r(K) > 1 \quad \text{and} \quad \lim_{t \uparrow \infty} \frac{\phi(t)}{t} r(K) < 1, \tag{13}$$

then there exist positive constants $c_1 < c_2$ with the following properties:

(a) If $0 < c \leqslant c_1$ and $f = ce$, then $f \ll Af$.
(b) If $c_2 \leqslant c < \infty$ and $f = ce$, then $Af \ll f$.

Proof. Let $\lambda := r(K)$. Since K is strongly positive and compact, the Perron–Frobenius eigenfunction e discussed above is well defined. Regarding claim (a), observe that, in view of (13), there exists an $\epsilon > 0$ such that

$$\frac{\phi(t)}{t}\lambda > 1 \quad \text{whenever} \quad 0 < t < \epsilon.$$

Choosing c_1 such that $0 < c_1 < \epsilon/\lambda$ and $c \leqslant c_1$, we have $c\lambda e(x) \leqslant c_1\lambda\|e\| = c_1\lambda < \epsilon$, and hence

$$Ace(x) = \phi(cKe(x)) = \phi(c\lambda e(x)) = \frac{\phi(c\lambda e(x))}{c\lambda e(x)} c\lambda e(x) > ce(x).$$

Since $x \in \mathbb{X}$ was arbitrary, the first claim in the lemma is verified.

Turning to claim (b) and using again the hypotheses in (13), we can choose a finite M such that

$$\frac{\phi(t)}{t}\lambda < 1 \quad \text{whenever} \quad M < t.$$

Let m be the minimum of e on $\mathbb{X}$. Since $\mathbb{X}$ is compact and $e \gg 0$, we have $m > 0$. Let c_2 be a constant strictly greater than $\max\{M/(\lambda m), c_1\}$ and let c lie in $[c_2, \infty)$. By the definition of m we have $c\lambda e(x) \geqslant c_2\lambda e(x) > M$ for all $x \in \mathbb{X}$, from which it follows that

$$Ace(x) = \phi(c\lambda e(x)) = \frac{\phi(c\lambda e(x))}{c\lambda e(x)} \lambda ce(x) < ce(x).$$

By construction, $0 < c_1 < c_2$, so all claims are now established. □

Proposition 2.2. *let K be strongly positive and compact, and let the conditions in (13) hold. If, in addition, ϕ is increasing and concave, then A is globally asymptotically stable on $\mathring{\mathscr{C}}_+$.*

Proof. Given that ϕ is increasing and concave on $\mathbb{R}_+$, lemma 2.1 implies that A is increasing and concave on $\mathscr{C}_+$. Since lemma 2.4 implies existence of a pair f_1, f_2 such that $Af_1 \gg f_1$ and $Af_2 \leqslant f_2$, and since the function f_1 can be chosen from $\mathring{\mathscr{C}}_+$, theorem 2.1 implies that A has a fixed point g^* in $\mathring{\mathscr{C}}_+$.

Next we claim that

(14) $\forall\, g \in \mathring{\mathscr{C}}_+, \;\; \exists\, f_1, f_2 \in \mathring{\mathscr{C}}_+$ such that $f_1 \leqslant g, g^* \leqslant f_2, \;\; Af_1 \gg f_1$ and $Af_2 \leqslant f_2$.

To see this, fix $g \in \mathring{\mathscr{C}}_+$. Since $g \gg 0$ and $\mathbb{X}$ is compact, g attains a finite maximum and strictly positive minimum on $\mathbb{X}$. The same is true of the existing fixed point g^* and the Perron–Frobenius eigenfunction e. Hence, we can choose constants a_1 and a_2 such that $0 \ll a_1 e \leqslant g^*, g \leqslant a_2 e$. With a_1 chosen sufficiently small and a_2

sufficiently large, lemma 2.4 implies that $a_1 e \ll A(a_1 e)$ and $A(a_2 e) \ll a_2 e$. Thus (14) holds.

Turning to uniqueness, let g^{**} be a second fixed point of A in $\mathring{\mathscr{C}}_+$. By (14) there exist $f_1, f_2 \in \mathring{\mathscr{C}}_+$ such that $f_1 \leqslant g^{**}, g^* \leqslant f_2$ with $f_1 \ll Af_1$ and $Af_2 \ll f_2$. By theorem 2.1, the interval $[f_1, f_2]$ contains only one fixed point. Thus, $g^* = g^{**}$.

Finally, regarding convergence, let g be an element of $\mathring{\mathscr{C}}_+$. Invoking (14) establishes the existence of $f_1, f_2 \gg 0$ such that $f_1 \leqslant g, g^* \leqslant f_2$ with $f_1 \ll Af_1$ and $Af_2 \ll f_2$. By theorem 2.1, every element of $[f_1, f_2]$ converges to g^* under iteration of A. In particular, $A^n g \to g^*$ as $n \to \infty$. □

3. Models with Epstein-Zin Preferences

Consider the recursive utility specification of Epstein and Zin (1989). We take consumption $\{C_t\}$ to be a stochastic process with the multiplicative functional representation given in (3). We assume that $\{X_t\}$ is a time homogeneous Markov process taking values in metric space $\mathbb{X}$ and $\{Y_t\}$ is an IID innovation process independent of $\{X_t\}$ and taking values in topological space $\mathbb{Y}$. The stochastic kernel for $\{X_t\}$ will be denoted by Q,[8] while the common distribution of each Y_t is a Borel probability measure denoted by ν. The value V_t of the path $\{C_i\}_{i \geqslant t}$ is defined recursively by (1)–(2). The constants β and ζ are strictly positive. So that (1)–(2) are well defined, both γ and ψ are required to be distinct from 1.

Assumption 3.1. The state space $\mathbb{X}$ is compact and the consumption growth function κ in (3) is continuous.

3.1. The Fixed Point Problem.

Our interest centers on existence, uniqueness and computability of lifetime value V_t. We convert this into a fixed point problem following the steps in Hansen and Scheinkman (2012). Minor manipulations to (1) yield

$$\left(\frac{V_t}{C_t}\right)^{1-1/\psi} = \zeta + \beta \left\{ \mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right) \right\}^{1-1/\psi}.$$

With $\theta := (1-\gamma)/(1-1/\psi)$ and $W_t := (V_t/C_t)^{1-1/\psi}$, we can rewrite this as

$$W_t = \zeta + \beta \left\{ \mathbb{E}_t W_{t+1}^{\theta} \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] \right\}^{1/\theta}. \tag{15}$$

[8] In particular, $\mathbb{E}\left[h(X_{t+1}) \mid X_t = x \right] = \int h(x') Q(x, \mathrm{d}x')$ for all $x \in \mathbb{X}$ and $h \in \mathscr{C}$.

We seek a Markov solution $W_t = w(X_t)$ for some $w\colon \mathbb{X} \to \mathbb{R}$, and hence solutions to the functional equation

$$w(x) = \zeta + \beta \left\{ \int w(x')^\theta \int \exp[(1-\gamma)\kappa(x', y', x)] \nu(\mathrm{d}y') Q(x, \mathrm{d}x') \right\}^{1/\theta}. \tag{16}$$

Equivalently, we seek fixed points of the operator T defined on $\mathscr{C}_+$ by

$$Tw(x) = \zeta + [Kw^\theta(x)]^{1/\theta}, \tag{17}$$

where

$$Kg(x) := \beta^\theta \int g(x') \int \exp[(1-\gamma)\kappa(x', y', x)] \nu(\mathrm{d}y') Q(x, \mathrm{d}x'). \tag{18}$$

A fixed point w^* of T in $\mathscr{C}$ solves (15) via $W_t = w^*(X_t)$. A solution to the original problem (1) can then be found by reversing the change of variables used in the definition of W_t. While $w \equiv +\infty$ can be considered as a fixed point of T, this trivial solution is ignored. We focus in what follows on solutions in $\mathscr{C}_+$.

Assumption 3.2. The linear operator K defined in (18) is a strongly positive and compact operator from $\mathscr{C}_+$ to itself.

Assumption 3.2 holds in all applications we consider.

Example 3.1. Suppose that $\mathbb{X} \subset \mathbb{R}^n$ and $Q(x, \mathrm{d}x') = q(x, x')\mathrm{d}x'$ for some continuous positive density kernel $q\colon \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. Then K is an integral operator $Kg(x) = \int g(x')k(x, x')\mathrm{d}x'$ with kernel

$$k(x, x') := \beta^\theta \int \exp[(1-\gamma)\kappa(x', y', x)] \nu(\mathrm{d}y') q(x, x').$$

Evidently $g \geqslant 0$ implies $Kg \gg 0$. The fact that k is also jointly continuous, combined with compactness of $\mathbb{X}$, implies compactness of K as a linear operator on $\mathscr{C}$.[9] Hence assumption 3.2 is valid.

Example 3.2. Consider the setting of example 3.1, except that $\mathbb{X}$ is finite, endowed with the discrete topology, and q is a density with respect to the counting measure instead of Lebesgue measure (i.e., q is a stochastic matrix). The conditions of assumption 3.2 are again satisfied.

Since $\zeta > 0$, it is clear that T maps nonnegative functions into strictly positive functions.[10] Hence any fixed point of T must lie in the interior $\mathring{\mathscr{C}}_+$.

[9] See, for example, Kolmogorov and Fomin (1975), §24.

[10] Some ambiguity in (17) arises when $\theta < 0$ and $w(x) = 0$ for some x. The general rule we apply for exponent $\alpha < 0$ is $0^\alpha = \infty$ and $\infty^\alpha = 0$. Thus, when w is not in $\mathring{\mathscr{C}}_+$, we admit $Kw^\theta(x) = \infty$ at some x, and at these points $Tw(x) = \zeta$. When $Kw^\theta(x) < \infty$ we have $Tw(x) > \zeta$.

3.2. **A Topologically Conjugate Problem.** Rather than studying T directly, we study a continuous transformation of T and apply the concept of topological conjugacy. In particular, consider the operator A on $\mathring{\mathscr{C}}_+$ defined by

$$Ag(x) = \left\{ \zeta + (Kg(x))^{1/\theta} \right\}^{\theta}. \tag{19}$$

Let Θ be the transformation defined by $(\Theta w)(x) = w^\theta(x) := w(x)^\theta$. Since $t \mapsto t^\theta$ is a homeomorphism from $(0, \infty)$ to itself, the operator Θ is a homeomorphism from $\mathring{\mathscr{C}}_+$ to itself.

Lemma 3.1. *T and A are topological conjugate on $\mathring{\mathscr{C}}_+$ under Θ.*

Proof. We need to show that $\Theta T w = A \Theta w$ for all $w \in \mathring{\mathscr{C}}_+$. To see this, fix $w \in \mathring{\mathscr{C}}_+$ and observe that, for any $x \in \mathbb{X}$,

$$(\Theta T w)(x) = \left\{ \zeta + (Kw^\theta(x))^{1/\theta} \right\}^\theta = (Aw^\theta)(x) = A\Theta w(x). \qquad \square$$

This means that global asymptotic stability of one of these operators on $\mathring{\mathscr{C}}_+$ is logically equivalent to global asymptotic stability of the other. The next lemma slightly strengthens this result in terms of implications for T.

Lemma 3.2. *If A is globally asymptotically stable on $\mathring{\mathscr{C}}_+$, then T is globally asymptotically stable on $\mathscr{C}_+$ and the fixed point lies in $\mathring{\mathscr{C}}_+$.*

Proof. Let A be globally asymptotically stable on $\mathring{\mathscr{C}}_+$. From lemma 3.1, it follows that T is globally asymptotically stable on $\mathring{\mathscr{C}}_+$. Moreover, since T maps $\mathscr{C}_+$ into $\mathring{\mathscr{C}}_+$, we know that T has no fixed point in $\mathscr{C}_+ \setminus \mathring{\mathscr{C}}_+$. The fact that T maps $\mathscr{C}_+$ into $\mathring{\mathscr{C}}_+$ also implies that $T^n w$ lies in $\mathring{\mathscr{C}}_+$ for all $n \geqslant 1$. Given that T is globally asymptotically stable on $\mathring{\mathscr{C}}_+$, this trajectory converges to the fixed point $w^* \in \mathring{\mathscr{C}}_+$. In particular, T is globally asymptotically stable on all of $\mathscr{C}_+$. $\square$

3.3. **Results for Recursive Utility.** We begin with the case $\theta < 0$, which corresponds to most empirically relevant parameterizations.[11]

Theorem 3.1. *If $\theta < 0$, then the following statements are equivalent:*

(a) $r(K) > 1$.

(b) *T has a fixed point in $\mathscr{C}_+$.*

[11]In interpreting the action of T when $\theta < 0$, we use the convention $(0^\theta)^{1/\theta} = 0$.

(c) *There exists a* $w \in \mathscr{C}_+$ *such that* $\{T^n w\}_{n \geqslant 1}$ *is convergent in* $\mathscr{C}_+$.

(d) T *has a unique fixed point in* $\mathscr{C}_+$.

(e) T *is globally asymptotically stable on* $\mathscr{C}_+$.

Proof. It suffices to show that that $(a) \implies (e) \implies (d) \implies (c) \implies (b) \implies (a)$. Of these, $(e) \implies (d)$ is true by definition, and $(d) \implies (c)$ is obvious (just take w to be the fixed point). We now prove the remainder.

($(a) \implies (e)$) Suppose that $r(K) > 1$ and let A be defined by (19). By lemma 3.2, to establish (e) it suffices to show that A is globally asymptotically stable on $\mathring{\mathscr{C}}_+$. To see that this is so, let ϕ be the map from $\mathbb{R}_+$ to itself defined by

$$\phi(t) = \left\{\zeta + t^{1/\theta}\right\}^{\theta} \tag{20}$$

with $\phi(0) = 0$. With this definition of ϕ we can express A as $Ag(x) = \phi(Kg(x))$ for $g \in \mathring{\mathscr{C}}_+$ and $x \in \mathbb{X}$. To complete the proof of (e), we need only show that the conditions of proposition 2.2 hold with ϕ as defined in (20) and K as given in (18).

The operator K is strongly positive by assumption. Under the assumption that $\theta < 0$, the map ϕ is strictly increasing and strictly concave on $\mathbb{R}_+$. Thus, we need only show that the two inequalities in (13) are valid. Regarding the first inequality, we have

$$\frac{\phi(t)}{t} = \left\{\frac{\zeta}{t^{1/\theta}} + 1\right\}^{\theta} \uparrow 1 \quad \text{as } t \downarrow 0.$$

Since $r(K) > 1$, the first inequality holds. Regarding the second inequality in (13), evidently $\phi(t)/t \to 0$ as $t \to \infty$, so this bound certainly holds. The proof of (e) is therefore complete.

($(c) \implies (b)$) Let $w \in \mathscr{C}_+$ be such that $\{T^n w\}$ is convergent, with the limit denoted by $\bar{w}$. Since T maps into the interior of $\mathscr{C}_+$, it must be that $\bar{w} \gg 0$. At any such $\bar{w}$ the operator is continuous, being the composition of continuous mappings.[12] Thus, $T\bar{w} = T(\lim_{n\to\infty} T^n w) = \lim_{n\to\infty} T^{n+1} w = \bar{w}$. In particular, $\bar{w}$ is a fixed point of T.

($(b) \implies (a)$) Suppose to the contrary that $r(K) \leqslant 1$. It suffices to show that A has no fixed point in $\mathring{\mathscr{C}}_+$. To this end, recall lemma 2.2. We have $r(K) \leqslant 1$ by assumption and the other conditions (10) have already been checked. Hence lemma 2.2 applies, and A has no fixed point in $\mathring{\mathscr{C}}_+$. □

[12]Here we are using the fact that K is a compact operator and hence continuous.

We turn next to the case $\theta > 0$. The condition on the spectral radius then reverses, with $r(K) < 1$ implying stability. Unlike theorem 3.1, the sufficiency proof has to be broken down into two cases: $0 < \theta < 1$ and $1 \leqslant \theta$. In the first case the conjugate operator A defined in (19) has the property that A^n is a contraction for some n, but A is not a concave operator. In the case $\theta \geqslant 1$, the operator A is not generally contracting, but it is a concave operator, and theorem 2.1 can be applied.

Theorem 3.2. *If $\theta > 0$, then the following statements are equivalent:*

(a) $r(K) < 1$.
(b) *T has a fixed point in $\mathscr{C}_+$.*
(c) *There exists a $w \in \mathscr{C}_+$ such that $\{T^n w\}_{n \geqslant 1}$ is convergent in $\mathscr{C}_+$.*
(d) *T has a unique fixed point in $\mathscr{C}_+$.*
(e) *T is globally asymptotically stable on $\mathscr{C}_+$.*

Proof. As was the case for theorem 3.1, it suffices to show that $(a) \implies (e)$, $(c) \implies (b)$ and $(b) \implies (a)$.

($(a) \implies (e)$) In view of lemma 3.2, it suffices to show that A given by (19) is globally asymptotically stable on $\mathring{\mathscr{C}}_+$ when $r(K) < 1$. To this end, suppose first that $\theta \in (0, 1)$. The conditions of proposition 2.1 are then satisfied, since $r(K) < 1$ by assumption and ϕ is Lipschitz of order 1. From this proposition we see that A is globally asymptotically stable on $\mathscr{C}_+$, with a unique fixed point g^*. This fixed point lies in $\mathring{\mathscr{C}}_+$, since 0 is not a fixed point of A (because $A0 = \phi(K0) = \phi(0) > 0$), and, moreover, if g^* is nonzero then Kg^* is strictly positive, and hence so is $Ag^* = \Phi K g^*$. Thus, A is also globally asymptotically stable on $\mathring{\mathscr{C}}_+$. Hence (e) is valid.

Now consider the remaining case $\theta \geqslant 1$, while continuing to assume that $r(K) < 1$. For such θ the function ϕ is increasing and concave, and, since $r(K) < 1$, the conditions in (13) are both satisfied. Hence proposition 2.2 applies, and A is globally asymptotically stable on $\mathring{\mathscr{C}}_+$. Hence (e) holds

($(c) \implies (b)$) The proof of this implication is identical to the proof of the same implication given in theorem 3.1.

($(b) \implies (a)$) First we make some observations about $\phi(t) = (\zeta + t^{1/\theta})^\theta$ on $\mathbb{R}_+$ when $\theta > 0$. Evidently ϕ is continuous and increasing with $\phi(t) > t$ for all $t \in \mathbb{R}_+$. It is not difficult to see that, in addition,

$$0 \leqslant s \leqslant t \implies \phi(s) \leqslant \frac{\phi(t)}{t} s \quad \text{and} \quad \lim_{n \to \infty} \phi^n(t) = \infty, \; \forall\, t \geqslant 0. \tag{21}$$

Now suppose that T has a fixed point $g \in \mathscr{C}_+$ and yet $r(K) \geqslant 1$. Observe that g must lie in $\mathring{\mathscr{C}}_+$, since T maps $\mathscr{C}_+$ into $\mathring{\mathscr{C}}_+$. Let e be the Perron–Frobenius eigenfunction of K and let c be a positive constant such that $e_c := ce \leqslant g$. Such a c exists because $\min_{x \in \mathbb{X}} g(x)$ is strictly positive and $\max_{x \in \mathbb{X}} e(x)$ is finite. Let t_0 be a positive constant such that $e_c \leqslant t_0$ on $\mathbb{X}$. We claim that

$$\forall n \in \mathbb{N}, \quad A^n e_c \geqslant \frac{\phi^n(t_0)}{t_0} e_c \text{ on } \mathbb{X}. \tag{22}$$

To see this, observe that (22) holds at $n = 0$. Now suppose that (22) holds at some $n \geqslant 0$. We then have

$$A^{n+1} e_c(x) = \phi(K A^n e_c(x)) \geqslant \phi\left(\frac{\phi^n(t_0)}{t_0} K e_c(x)\right) \geqslant \phi\left(\frac{\phi^n(t_0)}{t_0} e_c(x)\right).$$

Here the first inequality is by the induction hypothesis, the monotonicity of ϕ and K and the linearity of K. The second is from $r(K) \geqslant 1$, which gives $K e_c = cKe = cr(K)e \geqslant ce = e_c$. Using $e_c(x) \leqslant t_0$ and the first property in (21), we have

$$A^{n+1} e_c(x) \geqslant \phi\left(\frac{\phi^n(t_0)}{t_0} e_c(x)\right) \geqslant \frac{\phi(\phi^n(t_0))}{\phi^n(t_0)} \frac{\phi^n(t_0)}{t_0} e_c(x) = \frac{\phi^{n+1}(t_0)}{t_0} e_c(x).$$

Thus, the statement in (22) is valid.

From (22) and the second property in (21) we conclude that $A^n e_c$ diverges to $+\infty$. Moreover, the fixed point g satisfies $g \geqslant e_c$, so $A^n g \geqslant A^n e_c$. Hence $A^n g$ eventually exceeds g, contradicting our assumption that g is a fixed point of A. □

4. Applications

In this section we apply the results presented above to some preference and consumption process specifications used in recent empirical studies. We investigate whether or not the conditions of the spectral radius-based tests in theorems 3.1–3.2 are satisfied in these applications. For comparison, we also investigate whether or not the conditions of the tests from earlier literature based on probability one bounds are satisfied.

Checking the conditions of theorems 3.1–3.2 requires computing the spectral radius of the operator K in (18) under each parameterization. First, we discretize the state process $\{X_t\}$, replacing $\mathbb{X}$ with finite set $\{x_m\}_{m=1}^M$. The points x_m can be vectors or scalars and the method of discretization varies across applications. Since finite sets are always compact, discretization also serves to compactify the state process. Our theoretical results address the compactified versions of the models that we study, rather than the original ones.

In the discrete setting, transition probabilities are represented by $\mathbf{Q}_{\ell m} := \mathbb{P}\{X_{t+1} = x_m \mid X_t = x_\ell\}$. The operator K in (18) then reduces to the matrix

$$\mathbf{K}_{\ell m} = \beta^\theta \int \exp[(1-\gamma)\kappa(x_m, y', x_\ell)]\nu(\mathrm{d}y')\mathbf{Q}_{\ell m}. \tag{23}$$

The spectral radius of $\mathbf{K}$ can be computed using standard eigenvalue routines.[13]

4.1. **Long-Run Risk.** Suppose first that consumption growth obeys the Bansal and Yaron (2004) long-run risk specification

$$\begin{aligned} \ln(C_{t+1}/C_t) &= \mu + z_t + \sigma_t\, \eta_{c,t+1}, \\ z_{t+1} &= \rho z_t + \phi_z\, \sigma_t\, \eta_{z,t+1}, \\ \sigma^2_{t+1} &= v\, \sigma^2_t + d + \phi_\sigma\, \eta_{\sigma,t+1}. \end{aligned}$$

Here $\{\eta_{i,t}\}$ are IID and standard normal for $i \in \{c, z, \sigma\}$. Consider the probability one test for existence and uniqueness of solutions based on the conditions in Epstein and Zin (1989) and Marinacci and Montrucchio (2010) and outlined in the introduction. Theorem 3 of Marinacci and Montrucchio (2010) shows that a unique solution to the recursive utility problem exists whenever

$$\exp(b)\beta^{1/(1-1/\psi)} < 1, \tag{24}$$

where b is an almost sure (i.e., probability one) upper bound on $\ln(C_{t+1}/C_t)$. Since consumption growth is in fact unbounded in this model, we truncate it by (i) discretizing the state space for the persistent components and (ii) truncating the innovation $\eta_{c,t+1}$. The upper bound of b of log consumption growth is then computed and the expression in (24) is evaluated.

The other test we consider is the spectral radius based test in theorems 3.1–3.2. We use the same discretization of the state space as for the probability one test. In particular, we represent the state for consumption as $X_t := (z_t, \sigma_t)$ and discretize it using two iterations of the Rouwenhorst method to produce a finite state space $\{x_m\}_{m=1}^M$. For any $x = (z, \sigma)$ in this set, let

$$m(x) := \exp\left\{(1-\gamma)(\mu + z) + \frac{(1-\gamma)^2\sigma^2}{2}\right\}.$$

The matrix $\mathbf{K}$ in (23) can then be written as $\mathbf{K}_{\ell m} = \beta^\theta m(x_\ell)\mathbf{Q}_{\ell m}$ and the spectral radius is readily computed.

In Bansal and Yaron (2004), the preference parameters are estimated to be $\gamma = 10.0$, $\beta = 0.998$ and $\psi = 1.5$. Here and in all subsequent cases $\zeta = 1 - \beta$. The

[13]Typically, the integral in (23) can be calculated analytically.

parameters in the consumption process are $\mu = 0.0015$, $\rho = 0.979$, $\phi_z = 0.044$, $v = 0.987$, d =7.9092e-7 and ϕ_σ =2.3e-6. To implement the probability one test, we first discretize the consumption process in the manner described above. To give the test the best chance of success, we truncate $\eta_{c,t+1}$ only two standard deviations above the mean. Evaluating the left-hand side of (24) at these parameters yields 1.0254. Thus, the Bansal–Yaron model fails the probability one-based sufficient condition. Nonetheless, $\theta = -27$ and the spectral radius $r(\mathbf{K})$ evaluates to 1.055. Hence, by theorem 3.1, a unique solution exists. This tells us (i) that the solution to the Bansal–Yaron specification is well-defined, unique and can be computed from any initial condition in $\mathscr{C}_+$ by successive approximation, and (ii) that the probability one condition is too strict to effectively treat this model.

Figure 1a illustrates further by showing the value of the expression on the left-hand side of (24) not just at the exact Bansal–Yaron parameterization, but also at neighboring parameterizations found by varying ψ and μ. Almost all values exceed unity, apart from a small measure of parameterizations to the left of the 1.0 contour line. Nonetheless, most of these models are in fact stable, with a unique, globally attracting solution. This is true because $\theta < 0$ in all cases and, as shown in figure 1b, only in the north-east corner do we find parameterizations with $r(\mathbf{K}) \leqslant 1$.

4.2. **Schorfheide–Song–Yaron Consumption Dynamics.** Next consider the consumption specification adopted in Schorfheide et al. (2017), where

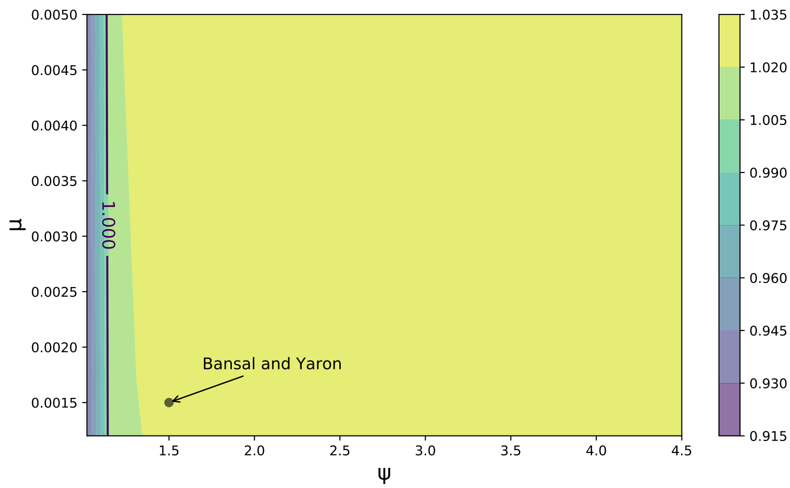
$$\begin{aligned} \ln(C_{t+1}/C_t) &= \mu + z_t + \sigma_{c,t}\,\eta_{c,t+1}, \\ z_{t+1} &= \rho\, z_t + \sqrt{1-\rho^2}\,\sigma_{z,t}\,\eta_{z,t+1}, \\ \sigma_{i,t} = \phi_i\,\bar\sigma \exp(h_{i,t}) \quad &\text{with} \quad h_{i,t+1} = \rho_{h_i} h_i + \sigma_{h_i}\eta_{h_i,t+1}, \quad i \in \{c,z\}. \end{aligned}$$

The innovations $\{\eta_{c,t}\}$ and $\{\eta_{h_i,t}\}$ are IID and standard normal for $i \in \{c,z\}$. The state can be represented as $X_t := (\sigma_{c,t}, \sigma_{z,t}, z_t)$. We discretize this state using the Rouwenhorst method in each of the three dimensions, obtaining the finite set $\{x_m\}_{m=1}^M$. Each x in this set is a pair (σ_c, σ_z, z). For such an x, let

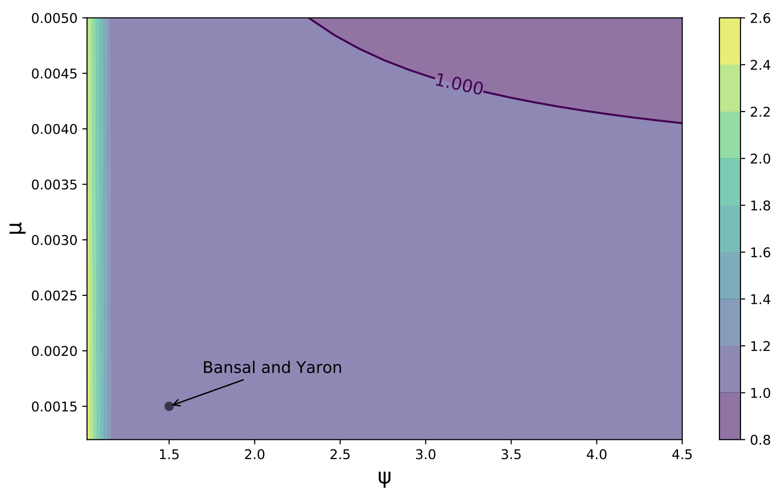
$$m(x) := \exp\left\{(1-\gamma)(\mu+z) + \frac{(1-\gamma)^2\sigma_c^2}{2}\right\}.$$

Then $\mathbf{K}$ can again be written as $\mathbf{K}_{\ell m} = \beta^\theta m(x_\ell)\mathbf{Q}_{\ell m}$.

In Schorfheide et al. (2017), the preference parameters are estimated to be $\gamma = 8.89$, $\beta = 0.999$ and $\psi = 1.97$. The parameters in the consumption process are $\mu = 0.0016$, $\rho = 0.987$, $\phi_z = 0.215$, $\bar\sigma = 0.0032$, $\phi_c = 1.0$, $\rho_{h_z} = 0.992$, $\sigma^2_{h_z} = 0.0039$,



(A) Probability one test

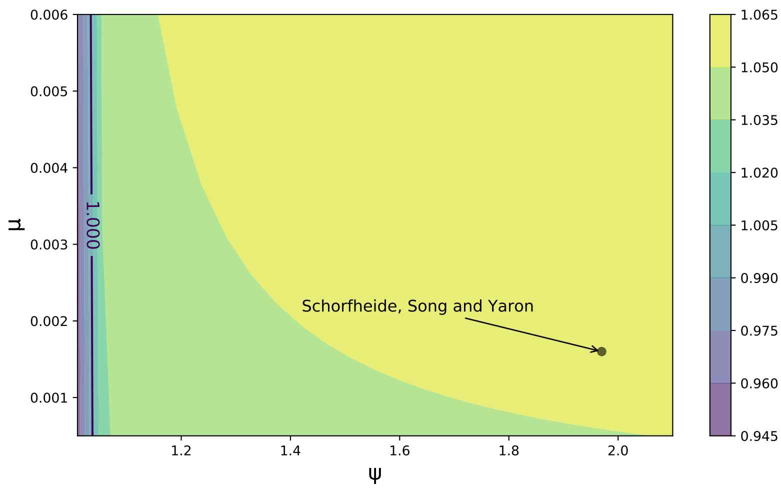


(B) Spectral radius test

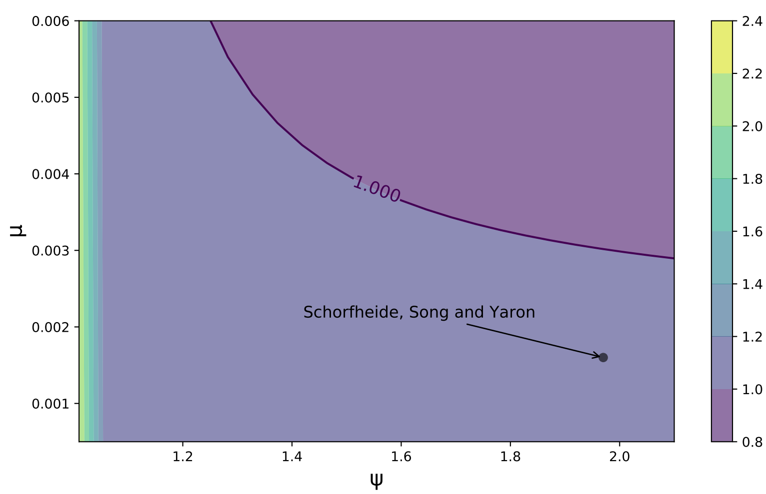
FIGURE 1. Stability tests for the Bansal–Yaron model

$\rho_{h_c} = 0.991$, and $\sigma^2_{h_c} = 0.0096$. The left-hand side of (24) evaluates to 1.051, so the sufficient condition for existence based on the probability one bound fails to hold. At the same time, $\theta = -16.02$ and $r(\mathbf{K}) = 1.011$, so a unique solution does exist, by theorem 3.1.

Figures 1a–1b illustrate further by repeating the exercise of examining the tests at neighboring parameterizations. The results are similar to those obtained for the



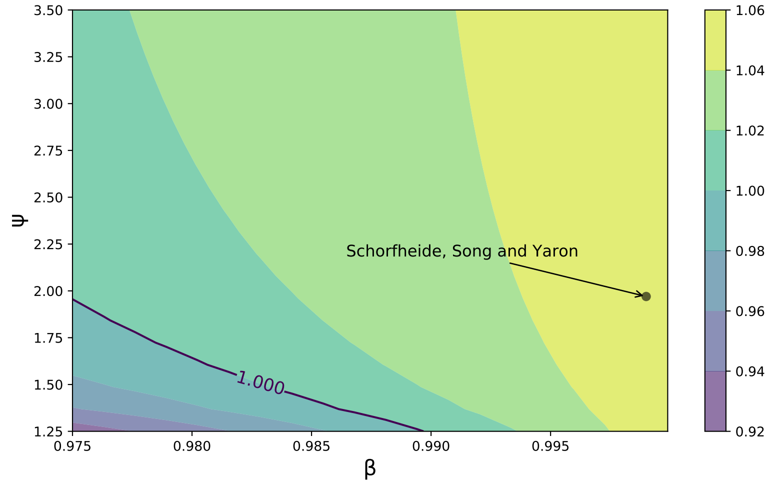
(A) Probability one test



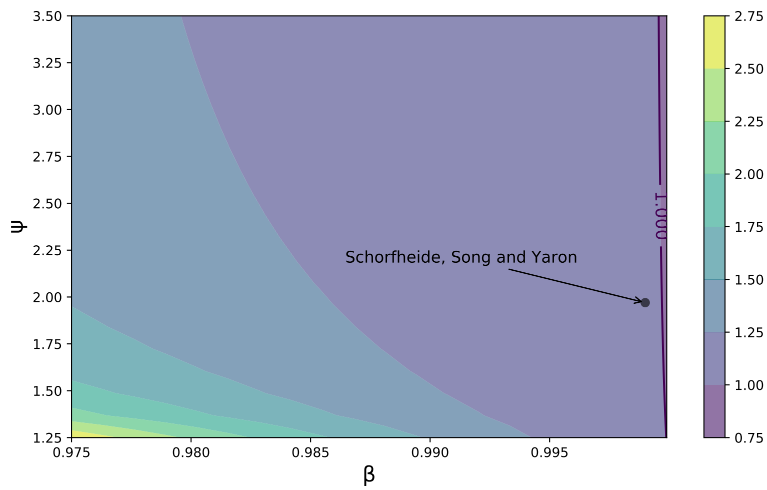
(B) Spectral radius test

FIGURE 2. Stability tests for the Schorfheide–Song–Yaron model

Bansal–Yaron parameterization: The probability one based test is too pessimistic, excluding many parameterizations that do in fact have unique, well-defined solutions. (Here $\theta < 0$ again occurs for all parameter values shown, so $r(\mathbf{K}) > 1$ corresponds to the stable case.) A similar outcome is shown in figures 3a–3b, when β is varied instead of μ.



(A) Probability one test



(B) Spectral radius test

FIGURE 3. Stability tests for the Schorfheide–Song–Yaron model

4.3. **Discretization Errors.** For the consumption specifications considered above, discretization errors appear to be small and rapidly diminishing. For example, figure 4 shows the spectral radius at the Bansal and Yaron (2004) specification of preferences across different levels of discretization. The value of I (which is 2 in the upper panel and 20 in the lower panel) is the number of states for σ, the time-varying volatility term. These states and the transition probabilities between them

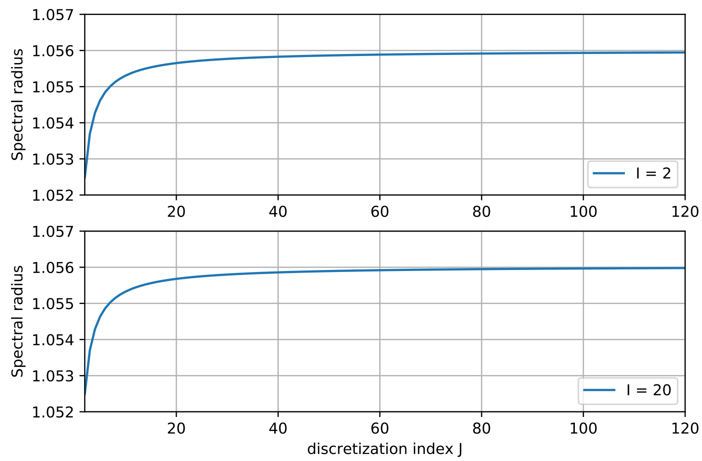


FIGURE 4. Spectral radius $r(\mathbf{K})$ as a function of level of discretization

are allocated according to the Rouwenhorst method. For each possible state for σ, we discretize the $\{z_t\}$ process to have J states. The values of J considered are shown on the horizontal axis, while the vertical axis shows the corresponding spectral radius $r(\mathbf{K})$.

Examining the values taken by the spectral radius, we see that, even for very small grid sizes, the spectral radius is accurate up to two decimal places relative to the limiting value. Once J is moderately large, additional grid points makes essentially no difference to value we obtain.

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