

Communication and Cooperation in Repeated Games*

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May 24, 2017

Abstract

We study the role of communication in repeated games with private monitoring. We first show that without communication, the set of Nash equilibrium payoffs in such games is a subset of the set of ε -coarse correlated equilibrium payoffs (ε -*CCE*) of the underlying one-shot game. The value of ε depends on the discount factor and the quality of monitoring. We then identify conditions under which there are equilibria with "cheap talk" that result in nearly efficient payoffs outside the set ε -*CCE*. Thus, in our model, communication is necessary for cooperation.

1 Introduction

The proposition that communication is necessary for cooperation seems quite natural, even self-evident. Indeed, in the old testament story of the Tower of Babel, God thwarted the mortals' attempt to build a tower reaching the heavens merely by dividing the languages. The inability to communicate with each other was enough to doom mankind's building project. At a more earthly level, antitrust laws in many countries prohibit or restrict communication among firms. Again, the premise is that limiting communication limits collusion.

Despite its self-evident nature, it is not clear how one may formally establish the connection between communication and cooperation. One option is to consider the effects of *pre-play* communication in one-shot games. This allows players to coordinate and even correlate their play in the game.¹ But in many games of interest,

*The research reported here was supported by a grant from the National Science Foundation (SES-1626783). We are grateful to Gabriel Carroll, Tim Roughgarden and Satoru Takahashi for very valuable comments.

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¹Indeed, as Barany (1998) has shown, with four or more players, the set of equilibria with pre-play communication coincides with set of correlated equilibria of the original game.

this does not enlarge the set of equilibria to allow for cooperation. For instance, pre-play communication has no effect in the prisoners' dilemma. It also has no effect in a differentiated-product price-setting oligopoly with linear demands.²

In this paper we study the role of *in-play* communication in repeated games—the basic framework for analyzing the prospects for cooperation among self-interested parties. The basic idea is that players are willing to forgo short-term gains in order to reap future rewards. But this relies on the ability of players to monitor each other well. If monitoring is poor, cooperative outcomes are hard to sustain because players can cheat with impunity and we study whether, in these circumstances, communication can help. Specifically, we show how *in-play* communication improves the prospects for cooperation in repeated games with imperfect *private* monitoring. In such settings, players receive only noisy private signals about the actions of their rivals.³

Our main result (Theorem 6.1) identifies monitoring structures—the stochastic mapping between actions and signals—with the property that communication is necessary for cooperation.

Theorem *For any high but fixed discount factor, there exists an open set of monitoring structures such that there is an equilibrium with communication whose welfare exceeds that from any equilibrium without communication.*

What kinds of monitoring structures lead to this conclusion? Two conditions are needed. First, private signals should be rather noisy so that in the absence of communication monitoring is poor. Second, private signals of players should be strongly correlated (actually, affiliated) when they cooperate and become less so following a deviation.⁴ The second condition is natural in many economic environments: price-setting oligopoly (Aoyagi, 2002), principal and multi-agent settings (Gromb and Martimort, 2007 or Fleckinger, 2012) and relational contracts (Deb et. al, 2016).

How exactly does communication facilitate cooperation? The basic idea is that players can monitor each other not only by what they "see"—the signals—but also by what they "hear"—the messages that are exchanged. But since the messages are just cheap talk—costless and unverifiable—one may wonder how these can be used for monitoring. We construct equilibria in which the messages are cross-checked to ensure truthful reporting of signals. Moreover, the cross-checking is sufficiently accurate so that deviating players find it difficult to lie effectively. These essential properties rely on the second condition on the monitoring structure mentioned above. We use these ideas to construct an equilibrium that is nearly efficient.

²This is a *potential game* in the sense of Monderer and Shapley (1996) and a result of Neyman's (1997) then implies that the set of correlated equilibria coincides with the unique Nash equilibrium.

³A classic example is Stigler's (1964) model of secret price cuts where firms choose prices that are not observed by other firms. The prices (actions) then stochastically determine firms' sales (signals). Each firm only observes its own sales and must infer its rivals' actions only via these.

⁴Example 2 below shows that communication does not help if the reverse is true—that is, if affiliation actually increases following a deviation.

Our main result requires two steps. The first task is to find an effective bound on equilibrium payoffs that can be achieved without communication. But the model we study is that of a repeated game with private monitoring and there is no known characterization of the set of equilibrium payoffs. This is because with private monitoring each player knows only his own history (of past actions and signals) and has only noisy information about the private histories of other players. Since players' histories are not commonly known, these cannot be used as state variables in a recursive formulation of the equilibrium payoff set. In Section 4, we borrow an equilibrium notion from algorithmic game theory—that of a coarse correlated equilibrium—and are able to relate (in Proposition 4.1) the Nash equilibrium payoffs of the *repeated* game to the *coarse correlated* equilibrium payoffs of the *one-shot* game.⁵ The set of coarse correlated equilibria is larger than the set of correlated equilibria and so has less predictive power in one-shot games. But because it is very easy to compute, in some cases it is nevertheless useful in bounding the set of Nash equilibria—for instance, in congestion games (Roughgarden, 2016). Here we show that it is useful in bounding the set of Nash equilibria of repeated games as well. Precisely,

Proposition *The set of Nash equilibrium payoffs of the repeated game without communication is a subset of the set of ε -coarse correlated equilibria of the one-shot game.*

The " ε " is determined by the discount factor and the monitoring structure of the repeated game and we provide an explicit formula for this. When the monitoring quality is poor—it is hard for other players to detect a deviation— ε is small and the set of ε -coarse correlated equilibrium payoffs provides an effective bound to the set of equilibrium payoffs in the repeated game.

The second task is to show that with communication, equilibrium payoffs above the bound can be achieved. To show this, we construct a cooperative equilibrium explicitly in which, in every period, players publicly report the signals they have received (see Proposition 5.1). Players' reports are aggregated into a single "score" and the future course of play is completely determined by this summary statistic. Deviations result in low scores and trigger punishments with higher probability. The construction of the score function relies on an auxiliary result (Proposition B.1) that is of some independent interest: Every generic affiliated distribution is a strict correlated equilibrium of a game with identical payoffs.

Note that the game is one with *identical* interests even if the distribution is asymmetric. This result in turn lets us construct an equilibrium with communication to be of a particularly simple "trigger-strategy" form. When signals are "very informative" about each other, the constructed equilibrium is nearly efficient.

⁵While a formal definition appears in Section 2, a coarse correlated equilibrium is a joint distribution over players' actions such that no player can gain by playing a pure action under the assumption that the other players will follow the marginal distribution over their actions. The notion originates in the work of Moulin and Vial (1979).

We emphasize that the analysis in this paper is of a different nature than that underlying the so-called "folk theorems" (see Sugaya, 2015). These show that for a fixed monitoring structure, as players become increasingly patient, near-perfect collusion can be achieved in equilibrium. In this paper, we keep the discount factor fixed and change the monitoring structure so that the set of equilibria with communication is substantially larger than the set without. A difficulty here is that monitoring structures—which are stochastic mappings from actions to signals—are high-dimensional objects. We show, however, that only *two* easily computable parameters—one measuring how noisy the signals are and the other how strongly correlated they are—suffice to identify monitoring structures for which communication is necessary for cooperation.

Related literature

There is a vast literature on repeated games under different kinds of monitoring. Under *perfect* monitoring, given any fixed discount factor, the set of perfect equilibrium payoffs with and without communication is the same. Under *public* monitoring, again given any fixed discount factor, the set of (public) perfect equilibrium payoffs with and without communication is also the same. Thus, in these settings communication does not affect the set of equilibria.

Compte (1998) and Kandori and Matsushima (1998) study repeated games with *private* monitoring when there is *communication* among the players. In this setting, they show that the folk theorem holds—any individually rational and feasible outcome can be approximated as the discount factor tends to one. This line of research has been pursued by others as well, in varying environments (see Fudenberg and Levine (2007) and Obara (2009) among others). Particularly related to the current paper is the work of Aoyagi (2002) and Zheng (2008) concerning correlated signals. They show that efficient outcomes can be approximated as the discount factor tends to one, again with communication. All of these papers thus show that communication is *sufficient* for cooperation when players are sufficiently patient. But as Kandori and Matsushima (1998) recognize, “One thing which we did *not* show is the *necessity* of communication for a folk theorem” (p. 648, their italics).

In a remarkable paper, Sugaya (2015) shows the surprising result that in very general environments, the folk theorem holds without any communication. Thus, in fact, communication is *not* necessary for a folk theorem. The analysis of repeated games with private monitoring is known to be difficult—and more so if communication is absent. Although Sugaya’s result was preceded by folk theorems for some limiting cases where the monitoring was almost perfect or almost public, the generality of its scope was unanticipated.

Unlike the folk theorems, in our work we do not consider the limit of the set of equilibrium payoffs as players become arbitrarily patient. We study the set of equilibrium payoffs for a fixed discount factor. Key to our result is a method of bounding the

set of payoffs without communication using the easily computable set of ε -coarse correlated equilibria. Pai, Roth and Ullman (2014) also develop a bound that depends on a measure of monitoring quality based on the computer-science notion of "differential privacy." But the bound so obtained applies to equilibrium payoffs with communication as well as those without, and so does not help in distinguishing between the two. Sugaya and Wolitzky (2017) find sufficient conditions under which the equilibrium payoffs with private monitoring are bounded by the equilibrium payoffs with perfect monitoring. This bound again applies whether or not there is communication and so is also unable to distinguish between the two.

Spector (2015) shows that communication can be beneficial in a model of price competition with private monitoring. Firms see their own current sales but, unlike in our model, can see other firms' sales with some delay. Communication is helpful in reducing this delay in monitoring. In our model, all communication is pure cheap talk—private signals remain so forever.

The current paper builds on our earlier work, Awaya and Krishna (2016), where we explored some of the same issues in the special context of Stigler's (1964) model of secret price cuts in a symmetric duopoly with (log-) normally distributed sales. Much of the analysis there relied on some important properties of the normal distribution as well as the symmetry. This paper considers *general* n -person finite games and *general* signal distributions. More important, the bound on payoffs without communication that is developed here is tighter than the bound constructed in the earlier paper.⁶ Finally, the construction of equilibria with communication is entirely different and does not rely on any symmetry among players.

The role of communication in fostering cooperation has also been the subject of numerous experiments in varied informational settings. Of particular interest is the work of Ostrom et al. (1994, Chapter 7) who find that in-play communication in repeated common-pool resource games leads to greater cooperation than does pre-play communication.

The remainder of the paper is organized as follows. The next section outlines the formal model of repeated games with private monitoring. To motivate the subsequent analysis, in Section 3 we present some of the main ideas, as well as some subtleties, by means of some simple examples. Section 4 analyzes the repeated game without communication whereas Section 5 does the same with communication. The main result is stated in Section 6. Appendix A contains omitted proofs from Section 4. Appendix B contains an auxiliary result regarding correlated equilibria that forms the basis of the equilibrium constructed in Section 5.

2 Preliminaries

As mentioned in the introduction, we study repeated games with private monitoring.

⁶An example showing this is available from the authors.

Stage game The underlying game is defined by $(I, (A_i, Y_i, w_i)_{i \in I}, q)$ where $I = \{1, 2, \dots, n\}$ is the set of players, A_i is a finite set of *actions* available to player i and Y_i is a finite set of *signals* that i may observe. The actions of all the players $a \equiv (a_1, a_2, \dots, a_n) \in A \equiv \times_i A_i$ together determine $q(\cdot | a) \in \Delta(Y)$, a probability distribution over the signals of all players.⁷ A vector of signals $y \in Y$ is drawn from this distribution and player i only observes y_i . Player i 's payoff is then given by the function $w_i : A_i \times Y_i \rightarrow \mathbb{R}$ so that i 's payoff depends on other players' actions only via the induced signal distribution $q(\cdot | a)$. We will refer to $w_i(a_i, y_i)$ as i 's *ex post* payoff.⁸

Prior to any signal realizations, the *expected* payoff of player i is then given by the function $u_i : A \rightarrow \mathbb{R}$, defined by

$$u_i(a) = \sum_{y_i \in Y_i} w_i(a_i, y_i) q_i(y_i | a)$$

where $q_i(\cdot | a) \in \Delta(Y_i)$ is the marginal distribution of $q(\cdot | a)$ on Y_i so that $q_i(y_i | a) = \sum_{y_{-i} \in Y_{-i}} q(y_i, y_{-i} | a)$. As usual, $\|u\|_\infty$ denotes the sup-norm of u . In what follows, we will merely specify the expected payoff functions u_i not the underlying ex post payoff functions w_i . The latter can be derived from the former for generic signal distributions—specifically, as long as $\{q_i(\cdot | a) : a \in A\}$ is a linearly independent set of vectors.

We refer to $G \equiv (A_i, u_i)_{i \in I}$ as the *stage game*. The set of *feasible* payoffs in G is $\mathcal{F} = \text{co } u(A)$, the convex hull of the range of u . A payoff vector $v^* \in \mathcal{F}$ is (strongly) *efficient* if there does not exist a feasible $v \neq v^*$ such that $v \geq v^*$.

The collection $\{q(\cdot | a)\}_{a \in A}$ is referred to as the *monitoring structure*. We suppose throughout that $q(\cdot | a)$ has *full support*, that is, for all $y \in Y$ and $a \in A$,

$$q(y | a) > 0 \tag{1}$$

Quality of monitoring Let $q_{-i}(\cdot | a) \in \Delta(Y_{-i})$ be the marginal distribution of $q(\cdot | a) \in \Delta(Y)$ over the joint signals of the players $j \neq i$. The *quality* of a monitoring structure q is defined as

$$\eta = \max_i \max_{a, a'_i} \|q_{-i}(\cdot | a) - q_{-i}(\cdot | a'_i, a_{-i})\|_{TV} \tag{2}$$

⁷We adopt the following notational conventions throughout: capital letters denote sets with typical elements denoted by lower case letters. Subscripts denote players and unsubscripted letters denote vectors or cartesian products. Thus, $x_i \in X_i$ and $x = (x_1, x_2, \dots, x_n) \in X = \times_i X_i$. Also, x_{-i} denotes the vector obtained after the i th component of x has been removed and (x'_i, x_{-i}) denotes the vector where the i th component of x has been replaced by x'_i . Finally, $\Delta(X)$ is the set of probability distributions over X .

⁸This ensures that knowledge of one's ex post payoff does not carry any information beyond that in the signal. For instance, in Stigler's (1964) model a firm's profits depend only on its own actions (prices) and its own signal (sales).

where $\|\mu - \nu\|_{TV}$ denotes the *total variation* distance between the probability measures μ and ν .⁹ It is intuitively clear that when the quality of monitoring is poor, it is hard for players other than i to detect a deviation by i .

Coarse correlated equilibrium The distribution $\alpha \in \Delta(A)$ is a *coarse correlated equilibrium (CCE)* of G if for all i and all $a_i \in A_i$,

$$u_i(\alpha) \geq u_i(a_i, \alpha_{-i})$$

where $\alpha_{-i} \in \Delta(A_{-i})$ denotes the marginal distribution of α over A_{-i} .¹⁰ The distribution $\alpha \in \Delta(A)$ is an ε -*coarse correlated equilibrium (ε -CCE)* of G if for all i and all $a_i \in A_i$,

$$u_i(\alpha) \geq u_i(a_i, \alpha_{-i}) - \varepsilon \quad (3)$$

Define ε -CCE(G) = $\{u(\alpha) \in \mathcal{F} : \alpha \text{ is an } \varepsilon$ -CCE of $G\}$ to be the set of ε -coarse correlated equilibrium payoffs of G .

Repeated game We will study an infinitely repeated version of G , denoted by G_δ , defined as follows. Time is discrete and indexed by $t = 1, 2, \dots$ and in each period t , the game G is played. Payoffs in the repeated game G_δ are discounted averages of per-period payoffs using the common discount factor $\delta \in (0, 1)$. Precisely, if the sequence of actions taken is (a^1, a^2, \dots) , player i 's ex ante expected payoff is $(1 - \delta) \sum_t \delta^t u_i(a^t)$. A (behavioral) *strategy* for player i in the game G_δ is a sequence of functions $\sigma_i = (\sigma_i^1, \sigma_i^2, \dots)$ where $\sigma_i^t : A_i^{t-1} \times Y_i^{t-1} \rightarrow \Delta(A_i)$. Hence, a strategy determines a player's current, possibly mixed, action as a function of his private history—his own past actions and past signals.

Repeated game with communication We will also study a version of G_δ , denoted by G_δ^{com} , in which players can communicate with each other after every period by sending *public* messages m_i from a finite set M_i . The communication phase in period t takes place after the signals in period t have been observed. Thus, a strategy of player i in the game G_δ^{com} consists of two sequences of functions $\sigma_i = (\sigma_i^1, \sigma_i^2, \dots)$ and $\rho_i = (\rho_i^1, \rho_i^2, \dots)$ where $\sigma_i^t : A_i^{t-1} \times Y_i^{t-1} \times M^{t-1} \rightarrow \Delta(A_i)$ determines a player's current action as a function of his own past actions, past signals and past messages from *all* the players. The function $\rho_i^t : A_i^t \times Y_i^t \times M^{t-1} \rightarrow \Delta(M_i)$ determines a player's current message as a function of his own past and *current* actions and signals as well as past messages from *all* the players. The messages m_i themselves have no direct payoff consequences.

⁹The total variation distance between two probability measures μ and ν on X is defined as $\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in X} |\mu(x) - \nu(x)|$. This is, of course, equivalent to the metric derived from the L^1 norm.

¹⁰A coarse correlated equilibrium differs from a correlated equilibrium in that the latter requires that no player have a deviation at the interim stage—after a recommendation to play a particular action. A coarse correlated equilibrium requires that no player have a deviation at the ex ante stage.

Equilibrium notion We will consider *sequential* equilibria of the two games. For G_δ , the repeated game without communication, the full support condition (1) ensures that the set of sequential equilibrium payoffs coincides with the set of Nash equilibrium payoffs (see Sekiguchi, 1997). In both situations, we suppose that players have access to public randomization devices.

3 Some examples

Before beginning a formal analysis of equilibrium payoffs in the repeated game G_δ and its counterpart with communication, G_δ^{com} , it will be instructive to consider a few examples. The first example illustrates, in the simplest terms, the main result of the paper. The other examples then point to some complexities. A word of warning is in order. All of the following examples have the property that the marginal distributions of players' signals are the same regardless of players' actions. This means that the expected payoff functions $u_i(a)$ cannot be derived from underlying ex post payoff functions $w_i(a_i, y_i)$. The examples have this property only to illustrate some features of the model in the simplest way possible. This is not essential—the examples can easily be amended so that underlying ex post payoff functions exist.

Example 1: Communication is necessary for cooperation. Consider the following prisoners' dilemma as the stage game:

	c	d
c	2, 2	−1, 3
d	3, −1	0, 0

Each player has two possible signals y' and y'' and suppose that the monitoring structure q is

$$q(\cdot | cc) = \begin{array}{c} y' \\ y'' \end{array} \begin{array}{|c|c|} \hline \frac{1}{2} - \varepsilon & \varepsilon \\ \hline \varepsilon & \frac{1}{2} - \varepsilon \\ \hline \end{array} \quad q(\cdot | \neg cc) = \begin{array}{c} y' \\ y'' \end{array} \begin{array}{|c|c|} \hline \frac{1}{4} & \frac{1}{4} \\ \hline \frac{1}{4} & \frac{1}{4} \\ \hline \end{array}$$

where $\neg cc$ denotes any action profile other than cc .

We argue below that without communication, it is impossible for players to "cooperate"—that is, to play cc —and that the unique equilibrium payoff is $(0, 0)$. With communication, however, it is possible for the players to cooperate (with high probability) and, in fact, attain average payoffs close to $(2, 2)$.

The monitoring structure here has two key features. First, the marginal distributions $q_i(\cdot | a)$ are *identical* no matter what action a is played, so that the quality of monitoring η (defined in (2)) is zero. Second, if cc is played, each player's signal is

very *informative* about the other player's signal. If something other than cc is played, a player's signal is completely uninformative about the other player's signal.¹¹

Claim 1 *Without communication, cooperation is not possible—for all δ , the unique equilibrium payoff of G_δ is $(0, 0)$.*

Fix any strategy of player 2. Since the marginal distribution on player 2's signals is not affected by what player 1 does, his *ex ante* belief on what player 2 will play in any future period is also independent of what he plays today. Thus, in any period, player 1 is better off playing d rather than c . Cooperation is impossible.¹²

Claim 2 *With communication, cooperation is possible—given any $\delta > \frac{1}{2}$, there exists an $\bar{\varepsilon}$ such that for all $\varepsilon < \bar{\varepsilon}$, there exists an equilibrium of G_δ^{com} whose payoffs are close to $(2, 2)$.*

Now suppose that players report their signals during the communication phase—that is, $M_i = Y_i$. Consider the following variant of a "trigger strategy": play c in period 1 and in the communication phase, report the signal that was received. In any period t , play c if in all past periods, the reported signals have agreed—that is, if both players reported y' or both reported y'' . If the reports disagreed in any past period, play d . In the communication phase, report your signal.

To see that these strategies constitute an equilibrium, note first that if all past reports have agreed, and a player has played c in the current period, then there is no incentive to misreport one's signal. Misreporting only increases the probability of triggering a punishment from 2ε to $\frac{1}{2}$ and so there is no gain from deviating during the communication phase.

Finally, if all past reports agreed, a player cannot gain by deviating by playing d . Such a deviation will trigger a punishment with probability $\frac{1}{2}$, no matter what he reports in the communication phase. It is routine to verify that when ε is small, this is not profitable. Each player's payoff in this equilibrium is

$$v = \frac{1 - \delta}{1 - \delta + \delta\varepsilon} \times 2$$

which converges to 2 as ε converges to zero.

¹¹The term "informative" is used in the sense of Blackwell (1951). From player i 's perspective, the signals of the other players y_{-i} constitute the "state of nature" and his own signal carries information about this.

¹²The equality of the marginals violates one of Sugaya's (2015) conditions and so his folk theorem does not apply.

Example 2: Cooperation is impossible even with communication The first example exhibited some circumstances in which cooperation was not possible without communication but with communication, it was. Does communication *always* facilitate cooperation? As the next example shows, this is not always the case—the signal structure q matters.

Consider the prisoners' dilemma of Example 1 again but with the following "flipped" signal structure:

$$q(\cdot | \neg dd) = \begin{array}{c} y' \\ y'' \end{array} \begin{array}{|c|c|} \hline y' & y'' \\ \hline \frac{1}{4} & \frac{1}{4} \\ \hline \frac{1}{4} & \frac{1}{4} \\ \hline \end{array} \quad q(\cdot | dd) = \begin{array}{c} y' \\ y'' \end{array} \begin{array}{|c|c|} \hline y' & y'' \\ \hline \frac{1}{2} - \varepsilon & \varepsilon \\ \hline \varepsilon & \frac{1}{2} - \varepsilon \\ \hline \end{array}$$

where again, $\neg dd$ denotes any action profile other than dd .

The marginal distribution of signals $q_i(\cdot | a)$ is, as before, unaffected by players' actions— η is zero again. But now the signal distribution when dd is played is *more* informative than when any other action is played—in fact, the former is completely informative.

Claim 3 *With or without communication, cooperation is not possible—for all δ , the unique equilibrium payoff in both G_δ and G_δ^{com} is $(0, 0)$.*

First, suppose that there is an equilibrium with communication in which after some history, player 1 is supposed to play c with probability one and report his signal truthfully. Suppose player 1 plays d instead of c and at the communication stage, regardless of his private signal, reports with probability one-half that his signal was y' and with probability one-half that his signal was y'' . Now regardless of whether player 2 plays c or d in that period, the joint distribution over player 1's reports and player 2's signals is the same as if player 1 had played c —that is, $q(\cdot | \neg dd)$. Thus, player 1 can deviate and "lie" in a way that his deviation cannot be statistically detected.

More generally, suppose that there an equilibrium with truthful communication in which after some history, player 1 is supposed to play c with probability $p > 0$. Again, suppose that player 1 plays d instead of c and at the communication stage reports as follows: if his private signal is y' , report y' with probability $1 - \frac{1}{2}p$ and y'' with probability $\frac{1}{2}p$; if his private signal is y'' , report y' with probability $\frac{1}{2}p$ and y'' with probability $1 - \frac{1}{2}p$. It is easy to verify that this "lying" strategy induces the same joint distribution over player 1's reports and player 2's signals regardless of player 2's action. Precisely, if player 2 plays c , the distribution is $q(\cdot | \neg dd)$ and when player 2 plays d , it is $q(\cdot | \neg dd) + (1 - p)q(\cdot | dd)$. As above, player 1 can thus deviate and then lie in a way that his deviation cannot be statistically detected.¹³

Thus, with the "flipped" monitoring structure no cooperation is possible even with communication. A fortiori, no cooperation is possible without communication either. In this example, therefore, communication is unable to facilitate cooperation.

¹³Here we have assumed that the equilibrium communication consists only of reporting one's current signal truthfully. A slight extension of this argument, available from the authors, shows that even with arbitrary messages, there is no equilibrium with cooperation.

Example 3: Communication is not necessary for cooperation Our final example illustrates the possibility that "full" cooperation is possible without communication even though monitoring is very poor—in fact, non-existent.¹⁴ Consider the following version of "rock-paper-scissors":

	r	p	s
r	10, 10	0, 11	11, 0
p	11, 0	10, 10	0, 11
s	0, 11	11, 0	10, 10

The stage game has a unique Nash equilibrium in which players randomize equally among the three actions and results in a payoff of (7, 7). Suppose the monitoring structure is:

$$q(\cdot \mid a_1 = a_2) = \begin{array}{c} y^r \\ y^p \\ y^s \end{array} \begin{array}{|c|c|c|} \hline \frac{1}{3} & 0 & 0 \\ \hline 0 & \frac{1}{3} & 0 \\ \hline 0 & 0 & \frac{1}{3} \\ \hline \end{array} \quad q(\cdot \mid a_1 \neq a_2) = \begin{array}{c} y^r \\ y^p \\ y^s \end{array} \begin{array}{|c|c|c|} \hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \end{array}$$

so that if players coordinate on the same action, then the signals are perfectly informative; otherwise, they are uninformative. Once again $\eta = 0$ since the marginal distributions of signals are not affected by players' actions.

Claim 4 *Cooperation is possible without communication—for any $\delta \geq \frac{3}{11}$, there exists an equilibrium of G_δ with a payoff of (10, 10).*

In what follows, we will say that the players are "coordinated" if they take the same action and so the resulting signal distribution is $q(\cdot \mid a_1 = a_2)$. Otherwise, they are said to be "miscoordinated."

Consider the following strategy: in period 1, play r . In period t , play action $a \in \{r, p, s\}$ if the signal received in the last period was y^a .

The average payoff from this strategy is clearly 10 since the players are always coordinated. Now suppose player 1 deviates once from the prescribed strategy and then reverts back to it.

Player 1's immediate payoff from the deviation is 11. But the deviation also causes the players to become miscoordinated. So no matter what signal player 1 receives, player 2 is equally likely to play each of his actions. As a result, once the players are miscoordinated, the continuation payoff is

$$w = \frac{1}{3}10 + \frac{1}{3}((1 - \delta)11 + \delta w) + \frac{1}{3}((1 - \delta)0 + \delta w)$$

This is because with probability $\frac{1}{3}$, the players will become coordinated again in the next period and then remain coordinated thereafter. With probability $\frac{1}{3}$, they will

¹⁴This trivially holds in games where there is an efficient one-shot Nash equilibrium, of course. In this example that is not the case.

remain miscoordinated, player 1 will get 11 and then the continuation payoff w ; with probability $\frac{1}{3}$, he will get 0 and then w again. Thus the continuation payoff after miscoordination is

$$w = \frac{21 - 11\delta}{3 - 2\delta}$$

The original deviation is not profitable as long as

$$(1 - \delta) 11 + \delta w \leq 10$$

and this holds as long as $\delta \geq \frac{3}{11}$. The one-deviation principle (Mailath and Samuelson, 2006), then ensures that the prescribed strategies constitute an equilibrium.

4 Equilibrium without communication

In this section, we develop a method to bound the set of equilibrium payoffs in games with private monitoring. We will show that the set of equilibrium payoffs of the *repeated* game without communication G_δ is contained within the set of ε -coarse correlated equilibrium payoffs of the *stage* game G . In our result, we give an explicit formula for ε involving (a) the discount factor; and (b) the quality of monitoring (as defined in (2)).

The main result of this section is:

Proposition 4.1

$$NE(G_\delta) \subseteq \varepsilon\text{-CCE}(G)$$

where $\varepsilon = 2 \frac{\delta^2}{1-\delta} \eta \times \|u\|_\infty$.

The proposition is of independent interest because repeated games with private monitoring do not have a natural recursive structure and so a characterization of the set of equilibrium payoffs seems intractable. So one is left with the task of finding effective bounds for this set. Proposition 4.1 provides such a bound and one that is easy to compute explicitly: the set $\varepsilon\text{-CCE}$ is defined by $\sum_i |A_i|$ linear inequalities. Moreover, the bound does not use any detailed information about the monitoring structure—it depends only on the monitoring quality parameter η .¹⁵

The " ε -coarse correlated equilibrium" in the statement cannot simply be replaced with " ε -correlated equilibrium." Precisely, if the set of ε -correlated equilibrium payoffs of G is denoted by $\varepsilon\text{-CE}(G)$, then the statement $NE(G_\delta) \subseteq \varepsilon\text{-CE}(G)$ is false for the same value of ε as above. For instance, in Example 3 the set of correlated equilibrium payoffs $CE(G) = \{(7, 7)\}$ while the set of coarse correlated equilibria is as depicted

¹⁵Sugaya and Wolitzky (2017) show that the set of equilibrium payoffs with perfect monitoring and a mediator is a bound for large enough δ . Our bound applies for all δ and is tighter when η is relatively small. Moreover, it does not require a fixed point computation.

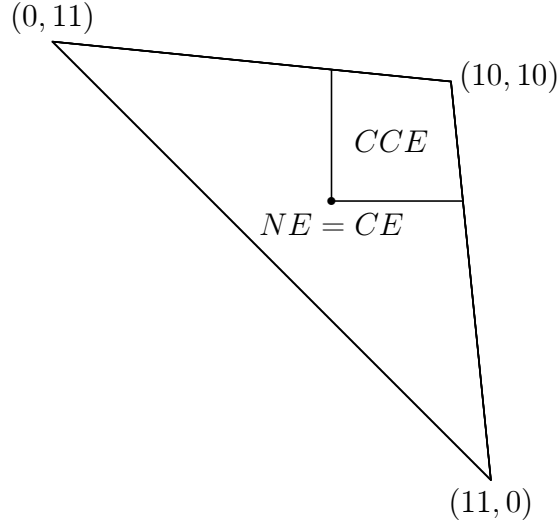


Figure 1: CCE of Rock-Scissors-Paper

in Figure 1. For the monitoring structure in Example 3, $\eta = 0$ and hence $\varepsilon = 0$ as well. But for $\delta \geq \frac{3}{11}$, repeated game has an equilibrium payoff of $(10, 10)$ which is in $CCE(G)$ but not in $CE(G)$.

Before proceeding with the formal proof of this result, we briefly sketch the main ideas. Suppose that σ is a strategy profile with a payoff $v(\sigma)$ that is not an ε - CCE payoff in the one-shot game. Suppose that some player i deviates to a strategy $\bar{\sigma}_i$ in which i chooses \bar{a}_i in every period regardless of history—that is, $\bar{\sigma}_i$ consists of a *permanent* deviation to \bar{a}_i . We decompose the (possible) gain from such a deviation into two bits. Consider a fictitious situation in which the players $j \neq i$ are replaced by a non-responsive *machine* that, in every period, and regardless of history, plays the ex ante distribution $\alpha_{-i}^t \in \Delta(A_{-i})$ that would have resulted from the candidate strategy σ . In the fictitious situation, player i 's deviation is unpunished in the sense that the machine continues to play as if no deviation had occurred. We can then write

$$\underbrace{v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\sigma)}_{\text{Gain from deviation}} = \underbrace{v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\bar{\sigma}_i, \alpha_{-i})}_{\text{Loss from punishment}} + \underbrace{v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\sigma)}_{\text{Gain when unpunished}} \quad (4)$$

The first component on the right-hand side represents the payoff difference from facing the real players $j \neq i$ versus facing the non-responsive machine. If σ_{-i} is an effective deterrent to the permanent deviation then this should be negative and in Lemma 4.1 we calculate a lower bound to this loss. The second component is the gain to player i when his permanent deviation goes unpunished. As we will show below in Lemma 4.2, this gain can be related to the coarse correlated equilibria of the one-shot game (see (3)).

We begin with a formal definition of the *non-responsive* strategy played by the fictitious "machine." Given a strategy profile σ , the induced ex ante distribution over

A in period t is

$$\alpha^t(\sigma) = E_\sigma [\Pi_j \sigma_j^t (h_j^{t-1})] \in \Delta(A)$$

and the corresponding marginal distribution over A_{-i} in period t is

$$\alpha_{-i}^t(\sigma) = E_\sigma [\Pi_{j \neq i} \sigma_j^t (h_j^{t-1})] \in \Delta(A_{-i})$$

where the expectation is defined by the probability distribution over $t - 1$ histories determined by σ . Note that α_{-i} depends on the whole strategy *profile* σ and not just on the strategies σ_{-i} of players other than i . Note also that because players' histories are correlated, it is typically the case that $\alpha_{-i}^t(\sigma) \notin \Pi_{j \neq i} \Delta(A_j)$. Given σ , let $\alpha_{-i}(\sigma)$ denote the (correlated) strategy of players $j \neq i$ in which they play $\alpha_{-i}^t(\sigma)$ in period t following *any* $t - 1$ period history. The strategy α_{-i} , which is merely a sequence $\{\alpha_{-i}^t\}$ of joint distributions in $\Delta(A_{-i})$, replicates the *ex ante* distribution of actions of players $j \neq i$ resulting from σ but is non-responsive to histories.

We now proceed to decompose the gain from a permanent deviation.

4.1 Loss from punishment

In this subsection we provide a bound on the absolute value of $v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\bar{\sigma}_i, \alpha_{-i})$, the difference in payoffs between being punished by strategy σ_{-i} of the real players $j \neq i$ versus not being punished by the fictitious machine. It is clear that the magnitude of this difference depends crucially on how responsive σ_{-i} is compared to the α_{-i} and this in turn depends on how well the players $j \neq i$ can detect i 's permanent deviation.

We show below that this loss can in fact be bounded by a quantity that is a positive linear function of η . Moreover, the bound is increasing in δ . The following result provides an exact formula for the trade-off between the quality of monitoring and the discount factor.

Lemma 4.1 *Suppose i plays \bar{a}_i always. The difference in i 's payoff when others play σ_{-i} versus when they play the non-responsive strategy α_{-i} derived from σ satisfies*

$$|v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\bar{\sigma}_i, \alpha_{-i})| \leq 2 \frac{\delta^2}{1 - \delta} \eta \times \|u\|_\infty$$

Proof. See Appendix A. ■

4.2 Gain when unpunished

We now relate the second component in (4) to the ε -coarse correlated equilibria of the one-shot game (see (3)). We begin by characterizing the set of ε -CCE payoffs.

For any $v \in \mathcal{F}$, define

$$\Theta(v) \equiv \min_{\beta \in \Delta(A)} \max_i \max_{\bar{a}_i} [u_i(\bar{a}_i, \beta_{-i}) - u_i(\beta)]$$

subject to

$$u(\beta) = v$$

where $\beta_{-i} \in \Delta(A_{-i})$ denotes the marginal distribution of β over A_{-i} .

In words, no matter how the payoff v is achieved via a correlated action, at least one player can gain at least $\Theta(v)$ by deviating. It is easy to see that $v \in \varepsilon\text{-CCE}(G)$ if and only if $\Theta(v) \leq \varepsilon$. This is because $\Theta(v) \leq \varepsilon$ is the same as: there exists a $\beta \in \Delta(A)$ satisfying $u(\beta) = v$ such that

$$\max_i \max_{\bar{a}_i \in A_i} [u_i(\bar{a}_i, \beta_{-i}) - u_i(\beta)] \leq \varepsilon$$

and this is equivalent to $v \in \varepsilon\text{-CCE}(G)$.

Note that $\Theta(v)$ is also the value of an artificial two-person zero-sum game Γ in which player I chooses a pair (i, \bar{a}_i) and player II chooses a joint distribution $\beta \in \Delta(A)$ such that $u(\beta) = v$. The payoff to player I is then $u_i(\bar{a}_i, \beta_{-i}) - u_i(\beta)$. The fact that $\Theta(v) \leq 0$ is the same as $v \in \text{CCE}(G)$ is analogous to a result of Hart and Schmeidler (1989) on correlated equilibria.

The following important result shows that the function Θ , which measures the *static* incentives to deviate, also measures the *dynamic* incentives to deviate from a non-responsive strategy. It shows that $\Theta(v)$ is also the value of a different two-person zero-sum game Γ_δ in which player I chooses a pair $(i, \bar{\sigma}_i)$ where $\bar{\sigma}_i$ denotes the constant sequence \bar{a}_i and player II chooses a sequence $\alpha \in \Delta(A)^\infty$ such that $v(\alpha) = v$. The payoff to player I in Γ_δ is $v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\alpha)$.

Lemma 4.2

$$\Theta(v) = \min_{\alpha \in \Delta(A)^\infty} \max_i \max_{\bar{\sigma}_i} [v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\alpha)]$$

subject to

$$v(\alpha) = v$$

where $v_i(\bar{\sigma}_i, \alpha_{-i})$ is i 's payoff when he plays \bar{a}_i always and others play the non-responsive strategy $\alpha_{-i} = (\alpha_{-i}^1, \alpha_{-i}^2, \dots) \in \Delta(A_{-i})^\infty$ derived from $\alpha = (\alpha^1, \alpha^2, \dots) \in \Delta(A)^\infty$.

Proof. It is clear that $\Theta(v)$ is at least as large as the right-hand side of the equality above. This is because the set of strategies available to player II in Γ_δ includes all stationary strategies and the latter are equivalent to all strategies in Γ . The set of strategies available to player I in Γ and Γ_δ are the same.

Given a sequence $\alpha = (\alpha^1, \alpha^2, \dots) \in \Delta(A)^\infty$, let $v^t = u(\alpha^t)$ be the ex ante payoffs in period t . Then, if $v(\alpha) = v$ we have $(1 - \delta) \sum_{t=1}^\infty \delta^t v^t = v$.

For any payoff vector $w \in \mathcal{F}$, define $\theta_i(w, \bar{a}_i) = \min_{\beta \in \Delta(A)} [u_i(\bar{a}_i, \beta_{-i}) - u_i(\beta)]$ subject to $u(\beta) = w$. Then, for for all i and all \bar{a}_i ,

$$\begin{aligned} v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\alpha) &= (1 - \delta) \sum_{t=1}^\infty \delta^t [u_i(\bar{a}_i, \alpha_{-i}^t) - u_i(\alpha^t)] \\ &\geq (1 - \delta) \sum_{t=1}^\infty \delta^t \theta_i(v^t, \bar{a}_i) \end{aligned} \tag{5}$$

where the second inequality follows from the definition of θ_i .

Now since $\theta_i(\cdot, \bar{a}_i)$ is convex¹⁶, it is the case that a solution to the problem:

$$\min_{v^t} (1 - \delta) \sum_{t=1}^{\infty} \delta^t \theta_i(v^t, \bar{a}_i)$$

subject to

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^t v^t = v$$

is to set $v^t = v$ for all t . Thus, we have that

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^t \theta_i(v^t, \bar{a}_i) \geq \theta_i(v, \bar{a}_i)$$

which when combined with (5) yields that for all i and \bar{a}_i ,

$$v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\alpha) \geq \theta_i(v, \bar{a}_i) = \min_{\beta: u(\beta)=v} [u_i(\bar{a}_i, \beta_{-i}) - u_i(\beta)]$$

This implies that

$$\min_{\alpha: v(\alpha)=v} [v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\alpha)] \geq \min_{\beta: u(\beta)=v} [u_i(\bar{a}_i, \beta_{-i}) - u_i(\beta)]$$

and thus

$$\max_{\pi \in \Delta(\cup A_j)} \min_{\alpha: v(\alpha)=v} E_{\pi} [v_i(a_i, \alpha_{-i}) - v_i(\alpha)] \geq \max_{\pi \in \Delta(\cup A_j)} \min_{\beta: u(\beta)=v} E_{\pi} [u_i(a_i, \beta_{-i}) - u_i(\beta)]$$

Applying the minmax theorem (Sion, 1958) in the game Γ_{δ} on the left-hand side and in the game Γ on the right-hand side, we obtain

$$\begin{aligned} \min_{\alpha: v(\alpha)=v} \max_i \max_{\bar{a}_i} v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\alpha) &\geq \min_{\beta: u(\beta)=v} \max_i \max_{\bar{a}_i} [u_i(\bar{a}_i, \beta_{-i}) - u_i(\beta)] \\ &= \Theta(v) \end{aligned}$$

■

4.3 Payoff bound

With Lemmas 4.1 and 4.2 in hand, we can now complete the proof of the result (Proposition 4.1) that the set of Nash equilibrium payoffs of the repeated game is contained in the set of ε -coarse correlated equilibrium payoffs of the one-shot game.

Suppose σ is a strategy profile in G_{δ} such that $v \equiv v(\sigma) \notin \varepsilon\text{-CCE}(G)$ for $\varepsilon = 2 \frac{\delta^2}{1-\delta} \eta \times \|u\|_{\infty}$. Then we know that $\Theta(v) > \varepsilon$. Lemma 4.2 implies that

$$\min_{\alpha \in \Delta(A)^{\infty}} \max_i \max_{\bar{\sigma}_i} [v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\alpha)] > \varepsilon$$

¹⁶The convexity of $\phi_i(\cdot, \bar{a}_i)$ is a consequence of the fact that u is linear in α .

and so

$$\max_i \max_{\bar{\sigma}_i} [v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\sigma)] > \varepsilon$$

where α is the non-responsive strategy derived from σ as above. Thus, there exists a player i and a permanent deviation for that player such that $v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\sigma) > \varepsilon$. Applying Lemmas 4.1 and 4.2 we have

$$\begin{aligned} v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\sigma) &= v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\bar{\sigma}_i, \alpha_{-i}) + v_i(\bar{\sigma}_i, \alpha_{-i}) - v_i(\sigma) \\ &> -2 \frac{\delta^2}{1-\delta} \eta \|u\|_\infty + \varepsilon \\ &= 0 \end{aligned}$$

Thus, σ is not a Nash equilibrium of G_δ . This completes the proof of Proposition 4.1.

4.4 Effective bound

Proposition 4.1 provides a bound to the set of equilibrium payoffs of the repeated game without communication. But in games which have an efficient coarse correlated equilibrium, the bound is ineffective. We will impose the condition

Condition 1 *G does not have an efficient coarse correlated equilibrium.*

Most games of interest satisfy the condition: the prisoners' dilemma, "chicken," Cournot oligopoly (with discrete quantities), Bertrand with or without differentiated products, etc. The rock-paper-scissors game in Example 3, on the other hand, does not satisfy the condition. In that example, $\alpha(rr) = \alpha(pp) = \alpha(ss) = \frac{1}{3}$ constitutes a coarse correlated equilibrium (but not a correlated equilibrium) that is efficient.

5 Equilibrium with communication

In what follows, we will consider efficient actions a^* that Pareto dominate some Nash equilibrium of the stage game, that is, $u(a^*) \gg u(\alpha^N)$ where $\alpha^N \in \times_i \Delta(A_i)$ is a (possibly mixed) Nash equilibrium of the stage game.¹⁷ We will display a particular strategy profile for the game with communication and identify conditions on the signal structure q and the discount factor δ that guarantee that the profile constitutes an equilibrium which is "nearly" efficient. We emphasize that our result is not a "folk theorem." In the latter, the signal structure is held fixed and the discount factor is raised sufficiently so that any feasible outcome can arise in equilibrium. In our result, the discount factor is held fixed (perhaps at some high level) and the monitoring structure is varied so that efficient outcomes can be sustained in equilibrium.

¹⁷This condition can be easily weakened to require only that $u(a^*)$ Pareto dominate some convex combination of one-shot Nash equilibrium payoffs.

5.1 Strategies

The strategy profile is similar to the well-known grim trigger strategy with the additional feature that at the end of period t , players report the signals, y_i , they received in that period. The play in any period is governed by a state variable that takes on two values—"normal" and "punishment." The players' strategies depend only on the state and are very simple: if the state in period t is "normal," play a_i^* ; if the state is "punishment," play the one-shot Nash action α^N . The state transitions from period t to $t + 1$ are determined solely by the players' reports of their signals in period t .

In period 1, the state is "normal." If the state is "normal" in period t , and the players report signals y at the end of the period, then the probability that the state remains "normal" in period $t + 1$ is $p^*(y)$ and how the reports y are aggregated to yield $p^*(y)$ is specified in detail below. The "punishment" state is absorbing—if the state is "punishment" in any period, it remains so in every subsequent period.

Specifically, for all i , consider the following strategy (σ_i^*, ρ_i^*) in the repeated game with communication. The set of messages is the same as the set of signals, that is, $M_i = Y_i$.

The actions chosen according to σ_i^* are as follows:

- In period 1, choose a_i^* .
- In any period $t > 1$, if the state is "normal," choose $a_i^t = a_i^*$; otherwise, choose α_i^N .

The messages sent according to ρ_i^* are as follows:

- In any period $t \geq 1$, if the action chosen $a_i^t = a_i^*$, then report $m_i^t = y_i^t$.
- In any period $t \geq 1$, if the action chosen $a_i^t = a_i \neq a_i^*$ and the signal received is y_i then report $m_i^t \in \arg \max_{z_i \in Y_i} E[p^*(z_i, \tilde{y}_{-i}) \mid y_i]$ where the expectation is taken with respect to the distribution $q(\cdot \mid (a_i, a_{-i}^*))$.

We will show that there exists a function $p^*(y)$ with the following key properties:

1. along the equilibrium path, p^* induces truthful reporting of signals;
2. if a player deviates from a_i^* in the normal state, then the expected value of p^* falls (the probability of going to the punishment state increases) regardless of his or her report;
3. if the signals are close to being perfectly informative along the equilibrium path, then the probability of punishment is small.

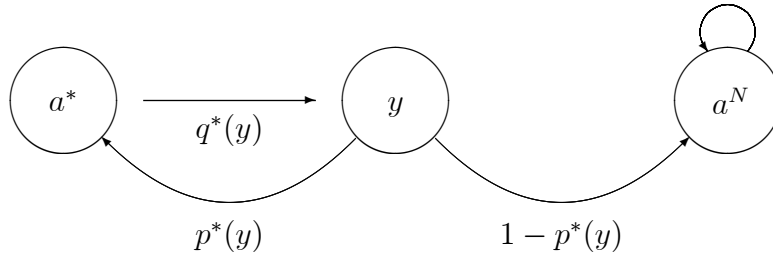


Figure 2: Equilibrium strategy

The first two properties are required for the strategies outlined above to constitute an equilibrium. The third property guarantees that the equilibrium is "nearly" efficient. Figure 2 is a schematic depiction of the play resulting from the given strategies.

Until now we have made no assumptions about the signals and their distribution. In particular, the bound on payoffs without communication, obtained in the previous section, applies without any specific assumptions about the signal structure. In this section, however, we will use some specific features of the signals—that they are *ordered* and (pairwise) *affiliated*.

For ease of notation, suppose the set of signals for each player is a *finite, ordered* set. Moreover, we will suppose that these sets of signals are of the same cardinality, say K . Let $q^* = q(\cdot | a^*)$ be the joint distribution of signals resulting from the action profile $a^* \in A$ and denote by E^* all expectations with respect to q^* . We will suppose that the signals are pairwise *affiliated*.¹⁸

Condition 2 (Affiliation) q^* is (pairwise) affiliated.

We can then suppose, without loss of generality, that for all i , the signals are consecutive integers¹⁹

$$Y_i = \{1, 2, \dots, K\}$$

We also suppose that no two signals of player i result in the same beliefs about the signals of player j .

Condition 3 (Non-equivalent signals) For all i , $y_i \neq y'_i$ and $j \neq i$, $q_j^*(\cdot | y_i) \neq q_j^*(\cdot | y'_i)$.

Affiliation implies that if $y_i < y'_i$, then the distribution $q_j^*(\cdot | y'_i)$ first-order stochastically dominates $q_j^*(\cdot | y_i)$ which in turn implies that $E^*[\tilde{y}_j | y_i] \leq E^*[\tilde{y}_j | y'_i]$.

¹⁸The joint distribution q over $\times_i Y_i$ is pairwise affiliated if for all i, j and all $y_i < y'_i$ and $y_j < y'_j$, $q_{ij}(y'_i, y'_j)q_{ij}(y_i, y_j) \geq q_{ij}(y'_i, y_j)q_{ij}(y_i, y'_j)$, where q_{ij} is the marginal distribution of q over $Y_i \times Y_j$.

¹⁹This is because monotone transformations affiliated random variables are also affiliated.

The condition of non-equivalent signals now implies that the inequality must be strict, that is, if $y_i < y'_i$, then $E^*[\tilde{y}_j | y_i] < E^*[\tilde{y}_j | y'_i]$.

5.2 Incentive compatibility

We now turn to the construction of a p^* which will be used to support the equilibrium with communication. This requires that players have the incentive to report their signals truthfully.

Given $q^* \in \Delta(Y)$, consider the following auxiliary one-shot common-interest game $(Y_i, P^*)_{i=1}^{i=n}$. In this game, the set of player i 's *actions* is the set of *signals* Y_i . When players choose $y \in Y$, the common payoff is

$$P^*(y) = \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} y_i y_j - \sum_{i=1}^n \sum_{x_i < y_i} E^* \left[\sum_{j \neq i} \tilde{y}_j | x_i \right] - \frac{1}{2} \sum_{i=1}^n E^* \left[\sum_{j \neq i} \tilde{y}_j | y_i \right]$$

where E^* is the expectation operator with respect to the distribution q^* .

Appendix B proves that *any* affiliated and non-equivalent $q^* \in \text{int } \Delta(Y)$ is a *strict* correlated equilibrium of the common interest game defined above. The normalized version of P^* , called the *score function*, is

$$p^*(y) = \frac{P^*(y) - \min P^*}{\max P^* - \min P^*} \quad (6)$$

where $\min P^*$ (resp. $\max P^*$) denotes the minimum (resp. maximum) value of P^* over Y . Since Y is finite, the minimum and maximum exist and since q^* is a strict correlated equilibrium, $\max P^* > \min P^*$. Thus, for all y , $p^*(y) \in [0, 1]$. Since p^* is just an affine transform of P^* it is the case that if q^* is a correlated equilibrium of the game $(Y_i, P^*)_{i=1}^{i=n}$, it is also a correlated equilibrium of the game $(Y_i, p^*)_{i=1}^{i=n}$.

This has the following implication for the proposed strategy in the repeated game with communication. Since $p^*(y)$ is the probability of continuing to cooperate if the publicly reported signals are $y \in Y$, every player's continuation payoff is proportional to p^* and so it is optimal to report signals truthfully if other players are doing so. Note that even though the proposed strategies rely on the function p^* , they depend only face-to-face communication among the players and not on any private messages via a mediator. Moreover, even the randomization implicit in p^* can be achieved via communication (by making use of jointy controlled lotteries, as in Aumann and Maschler, 1995).

5.2.1 A limiting case

Definition 5.1 *A distribution $q \in \Delta(Y)$ is perfectly informative if $q(y) > 0$ if and only if $y_1 = y_2 = \dots = y_n$.*

When the distribution is perfectly informative, then every player's individual signal provides perfect information about the signals of the other players. Thus, the terminology is consistent with that of Blackwell (1951). Note also that the definition requires that *all* "diagonal" signal profiles occur with positive probability.

Now consider the function,

$$P^0(y) = -\frac{1}{4} \sum_{i=1}^n \sum_{j \neq i} (y_i - y_j)^2 \quad (7)$$

We claim that *any* perfectly informative distribution q^0 is a correlated equilibrium of the game $(Y_i, P^0)_{i=1}^n$. To see this, note that if i receives a recommendation y_i , then he knows that all the other players received exactly the same recommendation. If all other players follow the recommendation, then choosing y_i is strictly better for i than any other choice.

Clearly $\max P^0 = 0$ and $\min P^0 < 0$. When normalized, the resulting score function is

$$\begin{aligned} p^0(y) &= \frac{P^0(y) - \min P^0}{\max P^0 - \min P^0} \\ &= 1 - \frac{P^0(y)}{\min P^0} \\ &= 1 + \frac{\frac{1}{4} \sum_{i=1}^n \sum_{j \neq i} (y_i - y_j)^2}{\min P^0} \end{aligned} \quad (8)$$

and because of perfect informativeness, the expected score

$$E^0 [p^0(\tilde{y})] = 1$$

where E^0 is the expectation with respect to q^0 . This implies that the strategies have the property that if signals are perfectly informative then the probability of a punishment is zero.

Lemma 5.1 *Let $\{q^l\} \in \text{int } \Delta(Y)$ be a sequence such that $q^l \rightarrow q^0$ and $q^0 \in \Delta(Y)$ is a perfectly informative distribution. Then the corresponding score functions $p^l \rightarrow p^0$.*

Proof. From the definition of P^l (see (15)), we have that

$$P^l(y) = \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} y_i y_j - \sum_{i=1}^n \sum_{x_i < y_i} E^l \left[\sum_{j \neq i} \tilde{y}_j \mid x_i \right] - \frac{1}{2} \sum_{i=1}^n E^l \left[\sum_{j \neq i} \tilde{y}_j \mid y_i \right] \quad (9)$$

where E^l denotes expectations under q^l . On the other hand, notice that

$$P^0(y) = \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} y_i y_j - \sum_{i=1}^n \sum_{x < y_i} (n-1)x - \frac{1}{2} \sum_{i=1}^n (n-1)y_i \quad (10)$$

because the right-hand side equals

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} y_i y_j - (n-1) \sum_{i=1}^n \frac{(y_i - 1) y_i}{2} - \frac{1}{2} (n-1) \sum_{i=1}^n y_i \\
&= \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} y_i y_j - \frac{1}{2} (n-1) \sum_{i=1}^n y_i^2 \\
&= -\frac{1}{4} \sum_{i=1}^n \sum_{j \neq i} (y_i - y_j)^2
\end{aligned}$$

Comparing terms in (9) and (10) shows that as $q^l \rightarrow q^0$, $E^l[\tilde{y}_j \mid \tilde{y}_i = x] \rightarrow x$ (recall that each x occurs with positive probability).

Finally, note that since $\max P^0 - \min P^0 > 0$, $\lim (\max P^l - \min P^l) > 0$, and $\lim p^l$ is well defined. Thus, p^l (the normalized version of P^l) converges to p^0 (the normalized version of P^0) as well. ■

5.3 Equilibrium

We show below that when q^* is "very informative"—that is, it is close to being perfectly informative, there is an equilibrium with communication that is nearly efficient. We begin by considering the limit case when $q^* = q(\cdot \mid a^*)$ is itself perfectly informative, say $q^* = q^0$.

5.3.1 Perfectly informative signals

Consider the strategy profile given above. With perfectly informative signals, the probability of continuing in the "normal" state, when the reported signals are y , is given by the score function $p^0(y)$. This means that the payoff from the given strategy profile is

$$(1 - \delta) u_i(a^*) + \delta \left[E^0[p^0(\tilde{y})] u_i(a^*) + (1 - E^0[p^0(\tilde{y})]) u_i(a^N) \right] = u_i(a^*)$$

since the expected score $E^0[p^0(\tilde{y})] = 1$.

Now suppose player i deviates to \bar{a}_i in period t and thereby induces the signal distribution $\bar{q} = q(\cdot \mid \bar{a}_i, a_{-i}^*)$. Denote by \bar{E} all expectations with respect to the new signal distribution \bar{q} with full support over Y . Further, suppose that after the deviation, player i follows the strategy of reporting that his signal is $z_i(y_i)$ when it is actually y_i . It is easy to verify that no matter what reporting strategy z_i the deviating player follows, the expected score

$$\begin{aligned}
\bar{E}[p^0(z_i(\tilde{y}_i), \tilde{y}_{-i})] &= 1 + \frac{1}{2 \min P_0} \left(\sum_{j \neq i} \bar{E}[(z_i(\tilde{y}_i) - \tilde{y}_j)^2] + \sum_{k \neq i} \sum_{j \neq k, i} \bar{E}[(\tilde{y}_k - \tilde{y}_j)^2] \right) \\
&< 1
\end{aligned}$$

because we have assumed that \bar{q} has full support and so $\bar{E}[(z_i(\tilde{y}_i) - \tilde{y}_j)^2] > 0$ and $\min P^0 < 0$.

Player i 's expected payoff from such a deviation is therefore

$$(1 - \delta) u_i(\bar{a}_i, a_{-i}^*) + \delta \left[\bar{E} [p^0(z_i(\tilde{y}_i), \tilde{y}_{-i})] u_i(a^*) + (1 - \bar{E} [p^0(z_i(\tilde{y}_i), \tilde{y}_{-i})]) u_i(\alpha^N) \right] \quad (11)$$

and this is strictly smaller than $u_i(a^*)$ once δ is large enough.

Choose $\underline{\delta}_i$ such that all possible deviations $\bar{a}_i \neq a_i^*$ and all misreporting strategies z_i for i are unprofitable. Since both the actions and the signals are finite in number, such a $\underline{\delta}_i$ exists. The same is true for all players j . Let $\underline{\delta} = \max_j \underline{\delta}_j$. Then we know that for all $\delta > \underline{\delta}$, the proposed strategy profile constitutes an equilibrium. Note that $\underline{\delta}$ depends on $q(\cdot | a)$ for $a \neq a^*$ and not on $q(\cdot | a^*)$.

5.3.2 Nearly informative signals

Recall that after renormalization, we can suppose that each $Y_i = \{1, 2, \dots, K\}$. A signal profile $y \in Y$ is then said to be *diagonal* if $y_1 = y_2 = \dots = y_n$, that is, if all players' signals are identical. Denote by Y^D the set of all diagonal profiles. Of course, a perfectly informative distribution $q^0 \in \Delta(Y)$ has the property that $q^0(y) > 0$ if and only if $y \in Y^D$. Let $Q^0 \subset \Delta(Y)$ be the set of all perfectly informative distributions.

Fix $q(\cdot | a^*) \equiv q^* \in \text{int } \Delta(Y)$. First, find a perfectly informative distribution $q^0 \in Q^0$ such that:

$$q^0 \in \arg \min_{q \in Q^0} \|q^* - q\|_{TV}$$

and let

$$\gamma = \|q^* - q^0\|_{TV} \quad (12)$$

It is routine to verify that $\gamma = 1 - \sum_{y \in Y^D} q^*(y)$, the total probability mass that q^* assigns to non-diagonal signal profiles. Moreover, $\gamma = \|q^* - q\|_{TV}$ for any $q \in Q^0$ such that for all $y \in Y^D$, $q(y) \geq q^*(y)$. Thus, in the total variation metric, q^* is equidistant from any perfectly informative distribution which places greater probability mass on all diagonal profiles. This means that given $q^* \in \text{int } \Delta(Y)$ we can always choose a q^0 such that $q^0(y) > 0$ for all $y \in Y^D$.

Now as $\gamma \rightarrow 0$, Lemma 5.1 guarantees that for all $y \in Y$, the score functions

$$p^*(y) \rightarrow p^0(y)$$

and so

$$E^*[p^*(\tilde{y})] \rightarrow 1$$

as well. Moreover,

$$\max_{z_i} \bar{E} [p^*(z_i(\tilde{y}_i), \tilde{y}_{-i})] \rightarrow \max_{z_i} \bar{E} [p^0(z_i(\tilde{y}_i), \tilde{y}_{-i})]$$

Let v_i^* denote player i 's payoff in the equilibrium described above. Then we have that,

$$v_i^* = (1 - \delta) u_i(a^*) + \delta \left[E^* [p^*(\tilde{y})] v_i^* + (1 - E^* [p^*(\tilde{y})]) u_i(\alpha^N) \right]$$

or

$$v_i^* = \frac{(1 - \delta) u_i(a^*) + \delta (1 - E^* [p^*(\tilde{y})]) u_i(\alpha^N)}{1 - \delta E^* [p^*(\tilde{y})]} \quad (13)$$

Thus, there exists a γ^0 such that for all $\gamma < \gamma^0$, for all i ,

$$\begin{aligned} & (1 - \delta) u_i(\bar{a}_i, a_{-i}^*) \\ & + \delta \left[\max_{z_i} \bar{E} [p^*(z_i(\tilde{y}_i), \tilde{y}_{-i})] v_i^* + \left(1 - \max_{z_i} \bar{E} [p^*(z_i(\tilde{y}_i), \tilde{y}_{-i})] \right) u_i(\alpha^N) \right] \\ & < (1 - \delta) u_i(a^*) + \delta \left[E^* [p^*(\tilde{y})] v_i^* + (1 - E^* [p^*(\tilde{y})]) u_i(\alpha^N) \right] \end{aligned} \quad (14)$$

We have thus established

Proposition 5.1 *Fix $q(\cdot | a)$ for all $a \neq a^*$ and let $\underline{\delta}$ be defined by (11). For all $\delta > \underline{\delta}$, there exists a $\bar{\gamma}$ such that for all $\gamma < \bar{\gamma}$, there is an equilibrium of the game with communication whose payoffs are nearly efficient.*

6 Main result

Suppose that a^* is an efficient action and $u(a^*) \gg u(\alpha^N)$ where α^N is a Nash equilibrium of the one-shot game. Once Condition 1 is applied, the bound developed in Proposition 4.1 becomes effective—that is, if the quality of monitoring is zero ($\eta = 0$), equilibrium welfare without communication are bounded away from $u(a^*)$. On the other hand, when $q(\cdot | a^*)$ is perfectly informative ($\gamma = 0$), there is an equilibrium with communication that results in payoffs equal to $u(a^*)$. Our main result then follows:

Theorem 6.1 *Fix $q(\cdot | a)$ for all $a \neq a^*$ and let $\underline{\delta}$ be defined by (11). For any $\delta > \underline{\delta}$, there exist $\bar{\eta}$ and $\bar{\gamma}$ such that if the quality of monitoring (see (2)) $\eta < \bar{\eta}$ and the informativeness of $q(\cdot | a^*)$ (see (12)) $\gamma < \bar{\gamma}$, then there is an equilibrium with communication whose welfare exceeds that from any equilibrium without communication.*

We end with an example where the mechanics of Theorem 1 can be seen at work.

Example 4 Again consider the prisoners' dilemma

	c	d
c	2, 2	-1, 3
d	3, -1	0, 0

but now with a monitoring structure where for both players, the set of signals is $Y_i = \{1, 2, 3\}$ and the signal distributions are as follows:

$$q^* \equiv q(\cdot | cc) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{|c|c|c|} \hline 1 & \frac{1}{3}(1-\gamma) & \frac{1}{6}\gamma & \frac{1}{6}\gamma \\ \hline 2 & \frac{1}{6}\gamma & \frac{1}{3}(1-\gamma) & \frac{1}{6}\gamma \\ \hline 3 & \frac{1}{6}\gamma & \frac{1}{6}\gamma & \frac{1}{3}(1-\gamma) \\ \hline \end{array}$$

and

$$\bar{q} \equiv q(\cdot | \neg cc) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{|c|c|c|} \hline 1 & (\frac{1}{3}-\eta)^2 & \frac{1}{9}-\eta^2 & \frac{1}{9}-\frac{1}{3}\eta \\ \hline 2 & \frac{1}{9}-\eta^2 & (\frac{1}{3}+\eta)^2 & \frac{1}{9}+\frac{1}{3}\eta \\ \hline 3 & \frac{1}{9}-\frac{1}{3}\eta & \frac{1}{9}+\frac{1}{3}\eta & \frac{1}{9} \\ \hline \end{array}$$

The parameter γ is such that $\|q^* - q^0\|_{TV} = \gamma$ where q^0 is a perfectly informative distribution that places equal weight only on the diagonal elements. The parameter $\eta < \frac{1}{3}$ is such that for both i , $\|q_i^* - \bar{q}_i\|_{TV} = \eta$. Affiliation requires that $\gamma < \frac{2}{3}$.

No communication bound For the prisoners' dilemma, the set of ε -coarse correlated equilibrium payoffs is depicted in Figure 3. From Proposition 4.1 it follows that in any equilibrium of G_δ , the repeated game without communication, symmetric equilibrium payoffs $v_1 = v_2$ satisfy

$$\begin{aligned} v_i &\leq 2 \times 2 \frac{\delta^2}{1-\delta} \eta \times \|u\|_\infty \\ &= 12 \frac{\delta^2}{1-\delta} \eta \end{aligned}$$

since in the prisoners' dilemma, $\|u\|_\infty = 3$.

Equilibrium with communication Routine calculations show that for the monitoring structure given above, the score function $p^* : Y_1 \times Y_2 \rightarrow [0, 1]$ corresponding to q^* is

$$p^* = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{array}{|c|c|c|} \hline 1 & 1 & \frac{3}{4} - \frac{3}{8}\gamma & 0 \\ \hline 2 & \frac{3}{4} - \frac{3}{8}\gamma & 1 - \frac{3}{4}\gamma & \frac{3}{4} - \frac{3}{8}\gamma \\ \hline 3 & 0 & \frac{3}{4} - \frac{3}{8}\gamma & 1 \\ \hline \end{array}$$

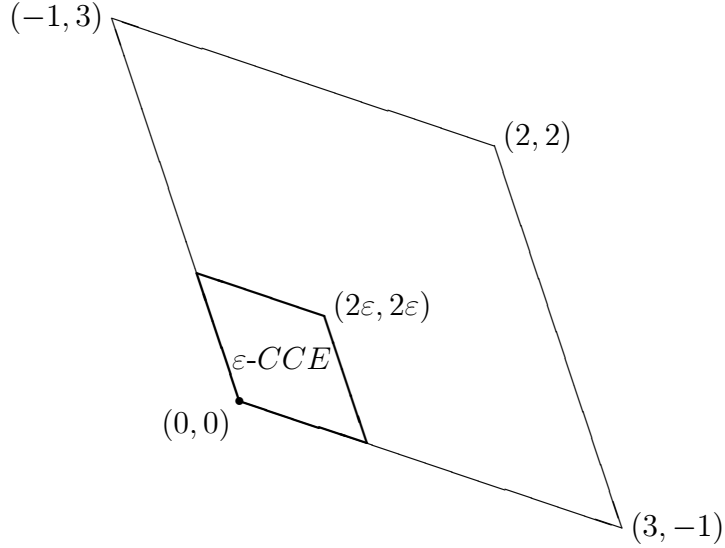


Figure 3: ε -CCE of Prisoners' Dilemma

and so that the expected score if no one deviates is $E^*[p^*(\tilde{y})] = 1 - \frac{3}{4}\gamma$ and using (13), we obtain that the payoff in the equilibrium constructed in Section 5 is

$$v_i^* = \frac{2(1-\delta)}{1-\delta(1-\frac{3}{4}\gamma)}$$

If player 1, say, deviates from cc in a "normal" state, then regardless of his true signal, it is best for him to report $y_1 = 2$ in the communication phase (assuming that both γ and η are small, specifically, $4\eta + 3\gamma < 2$). This implies that, after a deviation, the probability of the state being "normal" in the next period is

$$\begin{aligned} \max_{z_1} \bar{E}[p^*(z_1(\tilde{y}_1), \tilde{y}_2)] &= \bar{E}[p^*(2, \tilde{y}_2) \mid \tilde{y}_1 = y_1] \\ &= \frac{5}{6} + \frac{1}{2}\eta(1 - \frac{3}{2}\gamma) - \frac{11}{12}\gamma \end{aligned}$$

When used in (14) this implies that the suggested strategies form an equilibrium if and only if

$$\begin{aligned} B &\equiv (1-\delta)(u_1(cc) - u_1(dc)) + \delta \left[E^*[p^*(\tilde{y})] - \max_{z_i} \bar{E}[p^*(z_i(\tilde{y}_i), \tilde{y}_{-i})] \right] v_i^* \\ &\geq 0 \end{aligned}$$

Using the expressions derived above, it is easy to verify that a necessary condition for this to hold is that $\delta \geq \frac{3}{4}$.

Payoff comparison For there to be gains from communication—equilibrium payoffs from communication exceed those from no communication—it is sufficient that

$$C \equiv v_i^* - 12 \frac{\delta^2}{1-\delta} \eta \geq 0$$

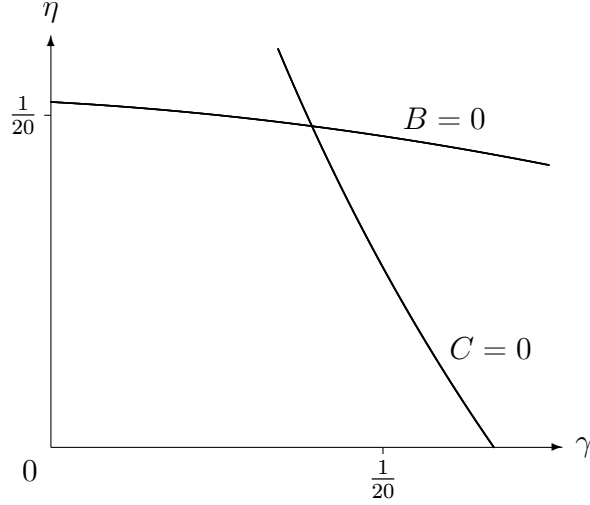


Figure 4: Monitoring structures for which the conclusion of Theorem 1 holds in Example 4 when $\delta = 0.8$.

For $\delta = 0.8$, Figure 4 depicts the parameters (γ, η) for which both $B \geq 0$ and $C \geq 0$. In the region below the two curves, there is an equilibrium with communication whose payoffs exceed the no-communication bound. For instance, if $\gamma = 0.02 = \eta$, equilibrium payoffs without communication are at most 0.768 whereas with communication, there is an equilibrium with payoffs of 1.887.

A Appendix

This appendix contains omitted and auxiliary results from Section 4.

For a fixed strategy profile σ , let $\mu^t(h^t)$ be the probability that a history h^t is realized. Then

$$\mu^t(h^t) = \mu^{t-1}(h^{t-1}) \sigma(a^t | h^{t-1}) q(y^t | a^t)$$

where $h^t = (h^{t-1}, a^t, y^t)$. Let

$$\begin{aligned} \mu_{-i}^t(h_{-i}^t) &= \sum_{h_i^t} \mu^t(h^t) \\ &= \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h^{t-1}) \sigma(a^t | h^{t-1}) q_{-i}(y_{-i}^t | a^t) \end{aligned}$$

be the marginal distribution of μ^t on the private histories $h_{-i}^t = (h_{-i}^{t-1}, a_{-i}^t, y_{-i}^t)$ of players $j \neq i$.

Similarly, let $\bar{\mu}^t$ be the probability of history h^t that results when i permanently deviates to \bar{a}_i , that is, from the strategy profile $(\bar{\sigma}_i, \sigma_{-i})$. Then,

$$\bar{\mu}^t(h^t) = \begin{cases} \bar{\mu}^{t-1}(h^{t-1}) \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q(y^t | a^t) & \text{if } a_i^t = \bar{a}_i \\ 0 & \text{otherwise} \end{cases}$$

and analogously let $\bar{\mu}_{-i}^t$ be the marginal distribution of $\bar{\mu}^t$ on player $-i$'s private histories h_{-i}^t so that

$$\begin{aligned} \bar{\mu}_{-i}^t(h_{-i}^t) &= \sum_{h_i^t} \bar{\mu}^t(h^t) \\ &= \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1}) \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t) \end{aligned}$$

Lemma A.1 For all t , $\|\mu_{-i}^t - \bar{\mu}_{-i}^t\|_{TV} \leq t\eta$.

Proof. The proof is by induction on t . For $t = 1$, we have

$$\begin{aligned} \mu_{-i}^1(h_{-i}^1) - \bar{\mu}_{-i}^1(h_{-i}^1) &= \sum_{a_i^1} \sigma(a^1) q_{-i}(y_{-i}^1 | a^1) - \sigma_{-i}(a_{-i}^1) q_{-i}(y_{-i}^1 | \bar{a}_i, a_{-i}^1) \\ &= \sum_{a_i^1} \sigma(a^1) q_{-i}(y_{-i}^1 | a^1) - \sum_{a_i^1} \sigma(a^1) q_{-i}(y_{-i}^1 | \bar{a}_i, a_{-i}^1) \\ &= \sum_{a_i^1} \sigma(a^1) (q_{-i}(y_{-i}^1 | a^1) - q_{-i}(y_{-i}^1 | \bar{a}_i, a_{-i}^1)) \end{aligned}$$

since $\sigma_{-i}(a_{-i}^1) = \sum_{a_i^1} \sigma(a^1)$ and so

$$\begin{aligned} \|\mu_{-i}^1 - \bar{\mu}_{-i}^1\|_{TV} &= \frac{1}{2} \sum_{h_{-i}^1} |\mu_{-i}^1(h_{-i}^1) - \bar{\mu}_{-i}^1(h_{-i}^1)| \\ &\leq \frac{1}{2} \sum_{h_{-i}^1} \sum_{a_i^1} \sigma(a^1) |q_{-i}(y_{-i}^1 | a^1) - q_{-i}(y_{-i}^1 | \bar{a}_i, a_{-i}^1)| \\ &= \frac{1}{2} \sum_{a_{-i}^1} \sum_{a_i^1} \sigma(a^1) \sum_{y_{-i}^1} |q_{-i}(y_{-i}^1 | a^1) - q_{-i}(y_{-i}^1 | \bar{a}_i, a_{-i}^1)| \\ &= \sum_{a^1} \sigma(a^1) \|q_{-i}(\cdot | a^1) - q_{-i}(\cdot | \bar{a}_i, a_{-i}^1)\|_{TV} \\ &\leq \eta \end{aligned}$$

the last inequality follows from our assumption that the quality of the monitoring does not exceed η .

Now suppose that the statement of the lemma holds for $t - 1$. We can write

$$\begin{aligned}
\Delta &= \mu_{-i}^t(h_{-i}^t) - \bar{\mu}_{-i}^t(h_{-i}^t) \\
&= \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h^{t-1}) q_{-i}(y_{-i}^t | a^t) \\
&\quad - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1}) \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t) \\
&= \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h^{t-1}) q_{-i}(y_{-i}^t | a^t) \\
&\quad - \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t) \\
&\quad + \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t) \\
&\quad - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1}) \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t)
\end{aligned}$$

Combining terms, we have

$$\begin{aligned}
\Delta &= \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h^{t-1}) [q_{-i}(y_{-i}^t | a^t) - q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t)] \\
&\quad + \left[\sum_{h_i^{t-1}} \mu^{t-1}(h_{-i}^{t-1}) \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1}) \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) \right] q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t) \\
&= \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h^{t-1}) [q_{-i}(y_{-i}^t | a^t) - q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t)] \\
&\quad + [\mu_{-i}^{t-1}(h_{-i}^{t-1}) - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1})] \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t)
\end{aligned}$$

where we have used the fact that $\sum_{a_i^t} \sigma(a^t | h^{t-1}) = \sum_{a_i^t} \sigma_i(a_i^t | h_i^{t-1}) \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) = \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1})$.

Thus,

$$\begin{aligned}
&\|\mu_{-i}^t - \bar{\mu}_{-i}^t\|_{TV} \\
&= \frac{1}{2} \sum_{h_{-i}^t} |\mu_{-i}^t(h_{-i}^t) - \bar{\mu}_{-i}^t(h_{-i}^t)| \\
&\leq \frac{1}{2} \sum_{h_{-i}^t} \sum_{h_i^{t-1}} \sum_{a_i^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h^{t-1}) |q_{-i}(y_{-i}^t | a^t) - q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t)| \\
&\quad + \frac{1}{2} \sum_{h_{-i}^t} |\mu_{-i}^{t-1}(h_{-i}^{t-1}) - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1})| \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t)
\end{aligned}$$

Since $h_{-i}^t = (h_{-i}^{t-1}, a_{-i}^t, y_{-i}^t)$, the first term equals

$$\begin{aligned} & \sum_{h_{-i}^{t-1}} \sum_{a^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h_{-i}^{t-1}) \|q_{-i}(\cdot | a^t) - q_{-i}(\cdot | \bar{a}_i, a_{-i}^t)\|_{TV} \\ & \leq \sum_{h_{-i}^{t-1}} \sum_{a^t} \mu^{t-1}(h_{-i}^{t-1}) \sigma(a^t | h_{-i}^{t-1}) \eta \\ & = \eta \end{aligned}$$

The second term equals

$$\begin{aligned} & \frac{1}{2} \sum_{h_{-i}^{t-1}} \sum_{a_{-i}^t} \sum_{y_{-i}^t} |\mu_{-i}^{t-1}(h_{-i}^{t-1}) - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1})| \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t) \\ & = \frac{1}{2} \sum_{h_{-i}^{t-1}} |\mu_{-i}^{t-1}(h_{-i}^{t-1}) - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1})| \sum_{a_{-i}^t} \sum_{y_{-i}^t} \sigma_{-i}(a_{-i}^t | h_{-i}^{t-1}) q_{-i}(y_{-i}^t | \bar{a}_i, a_{-i}^t) \\ & = \frac{1}{2} \sum_{h_{-i}^{t-1}} |\mu_{-i}^{t-1}(h_{-i}^{t-1}) - \bar{\mu}_{-i}^{t-1}(h_{-i}^{t-1})| \\ & = \|\mu_{-i}^{t-1} - \bar{\mu}_{-i}^{t-1}\|_{TV} \\ & \leq (t-1)\eta \end{aligned}$$

This completes the proof. ■

Lemma A.2 *Suppose i plays \bar{a}_i always. The difference in i 's payoff when others play σ_{-i} versus when they play the non-responsive strategy α_{-i} derived from σ satisfies*

$$|v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\bar{\sigma}_i, \alpha_{-i})| \leq 2 \frac{\delta^2}{1-\delta} \eta \times \|u\|_\infty$$

Proof.

$$\begin{aligned} v_i(\bar{\sigma}_i, \sigma_{-i}) &= (1-\delta) \sum_{t=1}^{\infty} \sum_{h_{-i}^{t-1}} \sum_{a_{-i}} \delta^t u_i(\bar{a}_i, a_{-i}) \sigma_{-i}(a_{-i} | h_{-i}^{t-1}) \bar{\mu}_{-i}(h_{-i}^{t-1}) \\ &= (1-\delta) \sum_{t=1}^{\infty} \delta^t \sum_{h_{-i}^{t-1}} \left(\sum_{a_{-i}} u_i(\bar{a}_i, a_{-i}) \sigma_{-i}(a_{-i} | h_{-i}^{t-1}) \right) \bar{\mu}_{-i}(h_{-i}^{t-1}) \\ &= (1-\delta) \sum_{t=1}^{\infty} \delta^t \sum_{h_{-i}^{t-1}} E[u_i(\bar{a}_i, \sigma_{-i}) | h_{-i}^{t-1}] \bar{\mu}_{-i}(h_{-i}^{t-1}) \end{aligned}$$

Similarly,

$$v_i(\bar{\sigma}_i, \alpha_{-i}) = (1-\delta) \sum_{t=1}^{\infty} \sum_{a_{-i}} \delta^t u_i(\bar{a}_i, a_{-i}) \alpha_{-i}^t(a_{-i})$$

and since

$$\alpha_{-i}^t(a_{-i}) = \sum_{h_{-i}^{t-1}} \sigma_{-i}(a_{-i} | h_{-i}^{t-1}) \mu_{-i}(h_{-i}^{t-1})$$

we have

$$\begin{aligned} v_i(\bar{\sigma}_i, \alpha_{-i}) &= (1 - \delta) \sum_{t=1}^{\infty} \sum_{h_{-i}^{t-1}} \sum_{a_{-i}} \delta^t u_i(\bar{a}_i, a_{-i}) \sigma_{-i}(a_{-i} | h_{-i}^{t-1}) \mu_{-i}(h_{-i}^{t-1}) \\ &= (1 - \delta) \sum_{t=1}^{\infty} \delta^t \sum_{h_{-i}^{t-1}} \left(\sum_{a_{-i}} u_i(\bar{a}_i, a_{-i}) \sigma_{-i}(a_{-i} | h_{-i}^{t-1}) \right) \mu_{-i}(h_{-i}^{t-1}) \\ &= (1 - \delta) \sum_{t=1}^{\infty} \delta^t \sum_{h_{-i}^{t-1}} E[u_i(\bar{a}_i, \sigma_{-i}) | h_{-i}^{t-1}] \mu_{-i}(h_{-i}^{t-1}) \end{aligned}$$

and so

$$\begin{aligned} &|v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\bar{\sigma}_i, \alpha_{-i})| \\ &\leq (1 - \delta) \sum_{t=1}^{\infty} \delta^t \left| \sum_{h_{-i}^{t-1}} E[u_i(\bar{a}_i, \sigma_{-i}) | h_{-i}^{t-1}] (\bar{\mu}_{-i}(h_{-i}^{t-1}) - \mu_{-i}(h_{-i}^{t-1})) \right| \\ &\leq 2(1 - \delta) \sum_{t=1}^{\infty} \delta^t \|\mu_{-i}^{t-1} - \bar{\mu}_{-i}^{t-1}\|_{TV} \times \|u\|_{\infty} \end{aligned}$$

where the second inequality follows from the fact that given any two measures μ and $\bar{\mu}$, $|E_{\mu}[f] - E_{\bar{\mu}}[f]| \leq 2\|\mu - \bar{\mu}\|_{TV} \times \|f\|_{\infty}$ for any f such that the expectations are well-defined (see, for instance, Levin et al., 2009).

Now Lemma A.1 shows that $\|\mu_{-i}^{t-1} - \bar{\mu}_{-i}^{t-1}\|_{TV} \leq (t-1)\eta$. Thus, we obtain²⁰

$$\begin{aligned} |v_i(\bar{\sigma}_i, \sigma_{-i}) - v_i(\bar{\sigma}_i, \alpha_{-i})| &\leq 2(1 - \delta) \sum_{t=1}^{\infty} \delta^t (t-1)\eta \|u\|_{\infty} \\ &= 2 \frac{\delta^2}{1 - \delta} \eta \times \|u\|_{\infty} \end{aligned}$$

■

B Appendix

This appendix contains a general result regarding correlated equilibria that forms the basis of construction of the equilibrium with communication. It says that any

²⁰The bound below is, of course, an overestimate since the TV distance between two measures is always less than or equal to 1.

affiliated distribution is a correlated equilibrium of a game with common interests (identical payoffs).²¹

Definition B.1 Given a game $(A_i, u_i)_{i=1}^n$, $\alpha \in \Delta(A)$ is a correlated equilibrium if for all i , $a_i \in A_i$ such that $\alpha_i(a_i) \equiv \sum_{a_{-i}} \alpha(a_i, a_{-i}) > 0$ and $a'_i \neq a_i$,

$$E[u_i(a_i, a_{-i}) \mid a_i] \geq E[u_i(a'_i, a_{-i}) \mid a_i]$$

It is a strict correlated equilibrium if all the inequalities are strict.

Definition B.2 Suppose that each Y_i is a finite ordered set. Then $q \in \Delta(Y)$ is (pairwise) affiliated if for all i, j , all $y_i < y'_i$ and all $y_j < y'_j$

$$q_{ij}(y_i, y_j) q_{ij}(y'_i, y'_j) \geq q_{ij}(y'_i, y_j) q_{ij}(y_i, y'_j)$$

where q_{ij} is the marginal distribution of q on $Y_i \times Y_j$.

Definition B.3 $q \in \Delta(Y)$ has non-equivalent signals if for all i, j and all $y_i \neq y'_i$, the conditional distributions $q_j(\cdot \mid y_i) \in \Delta(Y_j)$ and $q_j(\cdot \mid y'_i) \in \Delta(Y_j)$ are distinct.

Proposition B.1 Any affiliated $q \in \text{int } \Delta(Y)$ that has non-equivalent signals is a strict correlated equilibrium distribution of a game with common interests (identical payoffs).

Proof. Suppose that for all i , Y_i is a finite, ordered set and suppose $q \in \Delta(Y)$ is an affiliated distribution. Because monotone transforms of affiliated random variables are also affiliated, we can suppose, without loss of generality, that each Y_i is a set of consecutive positive integers, say, $Y_i = \{1, 2, \dots, K_i\}$.

We will exhibit a function $P : Y \rightarrow \mathbb{R}$ such that q is a strict correlated equilibrium of the game $(Y_i, P)_{i=1}^n$.

Given q , define $\phi_i : Y_i \rightarrow \mathbb{R}$ by

$$\phi_i(y_i) = E \left[\sum_{j \neq i} \tilde{y}_j \mid y_i \right]$$

as the expectation (with respect to q) of the sum of the other players' choices, conditional on $\tilde{y}_i = y_i$. Affiliation implies that ϕ_i is non-decreasing and since q has non-equivalent signals, ϕ_i is strictly increasing.

Consider the payoff function $P : Y \rightarrow \mathbb{R}$,

$$P(y) = \frac{1}{2} \sum_{i=1}^n \sum_{j \neq i} y_i y_j - \sum_{i=1}^n \sum_{x_i < y_i} \phi_i(x_i) - \frac{1}{2} \sum_{i=1}^n \phi_i(y_i) \quad (15)$$

²¹The conclusion that the game is one of common interests distinguishes this result from that of Crémer and McLean (1988).

Suppose all players $j \neq i$ follow the recommendation of a mediator who draws $y \in Y$ according to $q \in \Delta(Y)$. When player i receives a recommendation of y_i and chooses y'_i , his expected payoff is²²

$$\begin{aligned} E [P (y'_i, \tilde{y}_{-i}) \mid \tilde{y}_i = y_i] &= y'_i E \left[\sum_{j \neq i} \tilde{y}_j \mid y_i \right] - \sum_{x_i < y'_i} \phi_i (x_i) - \frac{1}{2} \phi_i (y'_i) - C_i (y_i) \\ &= y'_i \phi_i (y_i) - \sum_{x < y'_i} \phi_i (x) - \frac{1}{2} \phi_i (y'_i) - C_i (y_i) \end{aligned}$$

where $C_i (y_i)$ are the terms that depend on the conditional distribution $q (\cdot \mid y_i) \in \Delta (Y_{-i})$ but do *not* depend on player i 's action y'_i . On the other hand, if after receiving a recommendation of y_i , player i follows the recommendation and chooses y_i , his payoff is

$$E [P (y_i, \tilde{y}_{-i}) \mid y_i] = y_i \phi_i (y_i) - \sum_{x < y_i} \phi_i (x) - \frac{1}{2} \phi_i (y_i) - C_i (y_i)$$

Let

$$\Delta = E [P (y_i, \tilde{y}_{-i}) \mid y_i] - E [P (y'_i, \tilde{y}_{-i}) \mid y_i]$$

First, suppose $y'_i < y_i$. Then,

$$\begin{aligned} \Delta &= (y_i - y'_i) \phi_i (y_i) - \sum_{y'_i \leq x < y_i} \phi_i (x) - \frac{1}{2} (\phi_i (y_i) - \phi_i (y'_i)) \\ &= \sum_{y'_i \leq x < y_i} (\phi_i (y_i) - \phi_i (x)) - \frac{1}{2} (\phi_i (y_i) - \phi_i (y'_i)) \\ &= \sum_{y'_i < x < y_i} (\phi_i (y_i) - \phi_i (x)) + \frac{1}{2} (\phi_i (y_i) - \phi_i (y'_i)) \\ &> 0 \end{aligned}$$

since ϕ_i is strictly increasing.

Now suppose $y'_i > y_i$. Then

$$\begin{aligned} \Delta &= -(y'_i - y_i) \phi_i (y_i) + \sum_{y_i < x \leq y'_i} \phi_i (x) + \frac{1}{2} (\phi_i (y'_i) - \phi_i (y_i)) \\ &= \sum_{y_i < x \leq y'_i} (\phi_i (x) - \phi_i (y_i)) + \frac{1}{2} (\phi_i (y'_i) - \phi_i (y_i)) \\ &> 0 \end{aligned}$$

again since ϕ_i is strictly increasing.

Thus, upon receiving the recommendation y_i , player i has a strict incentive to follow the recommendation. So q is a strict correlated equilibrium distribution. ■

²²Note that $\frac{1}{2} \sum_{i \neq j} y_i y_j = y_i \sum_{j \neq i} y_j + \frac{1}{2} \sum_{j \neq k} y_j y_k$ where in the last sum $j \neq i$ and $k \neq i$.

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