# Optimal Timing of Decisions: A General Theory Based on Continuation Values

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# A Heuristic Example

## Example (Job Search I)

1. At each $t$, an unemployed worker obtains a wage offer $w_t = w(Z_t)$.
   - $(Z_t)_{t\geq 0}$: the underlying state process.
2. Two choices:
   - accept the offer: work permanently at $w_t$.
   - reject the offer: receive compensation $\tilde{c}_0$ and reconsider next period.
3. **Objective**: A stopping rule maximizing expected total returns.

# Road Map

1. Value function based method
2. Continuation value based method
    - Potential advantages
3. The theory we develop: continuation value based
    - Optimality results
    - Properties of continuation values and optimal policies

# Primitives

- $(Z_n)_{n\geq 0}$: a time homogeneous Markov process

$$Z_n\colon (\Omega, \mathscr{F}, \{\mathscr{F}_n\}_{n\geq 0}, \mathbb{P}) \to (\mathsf{Z}, \mathscr{Z})$$

- $P\colon \mathsf{Z} \times \mathscr{Z} \to [0,1]$ : the stochastic kernel of $(Z_n)_{n\geq 0}$
    - $A \mapsto P(z, A)$ is a probability measure, $\forall z \in \mathsf{Z}$
    - $z \mapsto P(z, A)$ is $\mathscr{Z}$-measurable, $\forall A \in \mathscr{Z}$
- $\mathscr{M}$: the set of stopping times on $\Omega$ w.r.t $\{\mathscr{F}_n\}_{n\geq 0}$
- $r\colon \mathsf{Z} \to \mathbb{R}$ : exit reward function
- $c\colon \mathsf{Z} \to \mathbb{R}$ : flow continuation reward function
- $\beta \in (0,1)$: discount factor

# Value Function Based Method

The value function:

$$v^*(z) := \sup_{\tau \in \mathscr{M}} \mathbb{E}_z \left\{ \sum_{t=0}^{\tau-1} \beta^t c(Z_t) + \beta^\tau r(Z_\tau) \right\}.$$

The Bellman operator:

$$Tv(z) := \max \left\{ r(z), c(z) + \beta \int v(z') P(z, \mathrm{d}z') \right\}.$$

**Theorem (Stokey etc. (1989); Peskir and Shiryaev (2006))**

If $r, c \in bc\mathsf{Z}$ and $P$ is Feller, then $v^*$ solves the Bellman equation:

$$v^*(z) = \max \left\{ r(z), c(z) + \beta \int v^*(z') P(z, \mathrm{d}z') \right\},$$

and $T : (bc\mathsf{Z}, \| \cdot \|) \to (bc\mathsf{Z}, \| \cdot \|)$ is a contraction mapping with unique fixed point $v^*$.

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# Value Function Based Method

## Example (Job search I cont.)

The exit and flow continuation rewards:

$$r(z) := \frac{u(w(z))}{1-\beta} \quad \text{and} \quad c(z) \equiv c_0 := u(\tilde{c}_0).$$

The Bellman operator:

$$Tv(z) := \max\left\{\frac{u(w(z))}{1-\beta},\, c_0 + \beta \int v(z')P(z, \mathrm{d}z')\right\}.$$

If $u$ is bounded and continuous and $P$ is Feller, then

$$v^*(z) = \max\left\{\frac{u(w(z))}{1-\beta},\, c_0 + \beta \int v^*(z')P(z, \mathrm{d}z')\right\},$$

and $T : bcZ \to bcZ$ is a contraction with unique fixed point $v^*$.

# Value Function Based Method: Limitations

1. Unbounded rewards are common:
   - AR(1) state process (unit root possible)

$$Z_{t+1} = b + \rho Z_t + \varepsilon_{t+1}, \quad (\varepsilon_t) \overset{\text{IID}}{\sim} N(0, \sigma^2), \quad \rho \in [-1, 1],$$

   - Log-normal wage process: $w(z) = \mathrm{e}^z$
   - CRRA utility

$$u(w) = \begin{cases} \frac{w^{1-\delta}}{1-\delta}, & \text{if } \delta \geq 0 \text{ and } \delta \neq 1 \\ \ln w, & \text{if } \delta = 1 \end{cases}$$

2. Among others ...

# An Alternative Method

Recall the value function

$$v^*(z) = \max\left\{r(z),\ c(z) + \beta \int v^*(z')P(z, \mathrm{d}z')\right\}. \tag{1}$$

Define the *continuation value function*

$$\psi^*(z) := c(z) + \beta \int v^*(z)P(z, \mathrm{d}z'). \tag{2}$$

Substituting (2) into (1):

$$v^*(z) = \max\{r(z), \psi^*(z)\}. \tag{3}$$

Substituting (3) into (2):

$$\psi^*(z) = c(z) + \beta \int \max\{r(z'), \psi^*(z')\}P(z, \mathrm{d}z').$$

# Continuation Value Based Method

Define the *continuation value operator*, or *Jovanovic operator*

$$Q\psi(z) = c(z) + \beta \int \max\left\{r(z'), \psi(z')\right\} P(z, \mathrm{d}z').$$

**Example (Jovanovic, 1982)**

Firm's decision: stay in or exit the industry?

$$V(x;p) = \pi(x;p) + \beta \int \max\left\{W, V(x';p)\right\} P(x, \mathrm{d}x').$$

- $V(x;p)$: *the expected value of staying in the industry*
- $\pi(x;p)$: the expected profit from the current industry (bounded)
- $W$: expected return in a different industry (a constant)

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# Continuation Value Based Method

## Example (Job Search I cont.)

The Bellman operator

$$Tv(z) = \max \left\{ \frac{u(w(z))}{1-\beta}, c_0 + \beta \int v(z') P(z, \mathrm{d}z') \right\}.$$

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# Continuation Value Based Method

**Potential advantage–1:** smoother fixed points

- Facilitating analysis of the optimal policy.
- Facilitating analysis related to unbounded rewards.
- Computationally less expensive.

# Continuation Value Based Method

## Example (Job Search II)

A job search model with learning:

- The wage process

$$\ln w_t = \theta + \varepsilon_t, \quad (\varepsilon_t)_{t \geq 0} \overset{\text{IID}}{\sim} N(0, \gamma_\varepsilon)$$

- $\theta$: unobservable component, prior belief $N(\mu, \gamma)$
- If the offer is rejected, the agent observes $w'$ and updates belief
- Posterior belief: $\theta | w' \sim N(\mu', \gamma')$
  - $\gamma' = 1/(1/\gamma + 1/\gamma_\varepsilon)$ and $\mu' = \gamma' (\mu/\gamma + \ln w'/\gamma_\varepsilon)$
- $f(w'|\mu, \gamma) = LN(\mu, \gamma + \gamma_\varepsilon)$: the current expectation of the next period wage distribution.

# Continuation Value Based Method

## Example (Job Search II cont.)

The Bellman operator

$$Tv(w,\mu,\gamma) := \max\left\{\frac{u(w)}{1-\beta}, c_0 + \beta\int v(w',\mu',\gamma')f(w'|\mu,\gamma)\,\mathrm{d}w'\right\}$$

and the Jovanovic operator

$$Q\psi(\mu,\gamma) := c_0 + \beta\int \max\left\{\frac{u(w')}{1-\beta}, \psi(\mu',\gamma')\right\} f(w'|\mu,\gamma)\,\mathrm{d}w',$$

where $f(w'|\mu,\gamma) = LN(\mu,\gamma+\gamma_\varepsilon)$.

**Note:**

- 3-dimensional ($T$) v.s. 2-dimensional ($Q$).

# Continuation Value Based Method

**Potential advantage–2:** lower state dimension

- Simplifying challenging problems associated with
  - unbounded rewards
  - parametric continuity
  - differentiability
  - so on ...
- Mitigating the curse of dimensionality.

# What We Do

The first systematic study of optimal timing of decisions, based on continuation value functions and operators.

**Main results:**

- A general optimality theory: bounded or unbounded rewards
- Conditions under which continuation values satisfy
  - (parametric) continuity
  - monotonicity
  - (continuous) differentiability
- Conditions under which threshold policies satisfy
  - (parametric) continuity
  - monotonicity
  - (continuous) differentiability: an expression for the derivative

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# Literature

Individual applications of optimal timing:

- Jovanovic (1982), Posche (2010), Chatterjee and Rossi-Hansberg (2012), Kellogg (2014)

Unbounded dynamic programming:

- The weighted supremum norm theory
  - Boyd (1990), Alvarez and Stokey (1998), Le Van and Vailakis (2005)
- The local contraction theory
  - Rincon-Zapatero and Rodriguez-Palmero (2003), Martins-Da-Rocha and Vailakis (2010)

# Optimality Results

## Assumption 2.1

There exist a $\mathscr{Z}$-measurable function $g : \mathsf{Z} \to \mathbb{R}$ and constants $n \in \mathbb{N}_0$ and $a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

$$\int |r(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2, \tag{4}$$

$$\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4, \tag{5}$$

$$\text{and} \quad \int g(z') P(z, \mathrm{d}z') \leq m g(z) + d. \tag{6}$$

**Note:**

1. If $r$ and $c$ are bounded, then assumption 2.1 holds.
2. True for some $n \in \mathbb{N}_0 \implies$ true for all integer $n' \geq n$.
3. May use $n_1$ in (4), $n_2$ in (5), and $n_1 \neq n_2$.

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# Optimality Results

For $\kappa : \mathsf{Z} \to (0, +\infty)$, the *$\kappa$-weighted supremum norm*:

$$\|f\|_\kappa := \sup_{z \in \mathsf{Z}} \frac{|f(z)|}{\kappa(z)}.$$

Then $(b_\kappa \mathsf{Z}, \| \cdot \|_\kappa)$ is a Banach space, where

$$b_\kappa \mathsf{Z} := \{f \in m\mathscr{Z} : \|f\|_\kappa < \infty\}.$$

Recall the Jovanovic operator:

$$Q\psi(z) := c(z) + \beta \int \max\left\{r(z'), \psi(z')\right\} P(z, \mathrm{d}z').$$

# Optimality Results

## Theorem 2.1

Under assumption 2.1, there exist $m', d' > 0$ such that for $\ell : \mathsf{Z} \to \mathbb{R}$ given by

$$\ell(z) := m' \left( \sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d',$$

1. $Q$ is a contraction mapping on $(b_\ell \mathsf{Z}, \| \cdot \|_\ell)$.
2. The unique fixed point of $Q$ in $b_\ell \mathsf{Z}$ is $\psi^*$.
3. $\sigma^*(z) = \mathbb{1}\{r(z) \geq \psi^*(z)\}$ is an optimal policy.

**Note:**

1. Assumption 2.1 holds for $n = 0 \implies \ell(z) = g(z) + d'$.
2. Assumption 2.1 holds for $n = 1 \implies \ell(z) = m'|c(z)| + g(z) + d'$.

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# Optimality Results

## Example (Job Search I cont.)

Let $w(z) = \mathrm{e}^z$, and the state process

$$Z_{t+1} = b + \rho Z_t + \varepsilon_{t+1}, \quad (\varepsilon_t) \overset{\text{IID}}{\sim} N(0, \sigma^2), \quad \rho \in [-1, 1].$$

The agent's preference

$$u(w) = \begin{cases} \frac{w^{1-\delta}}{1-\delta}, & \text{if } \delta \geq 0 \text{ and } \delta \neq 1 \\ \ln w, & \text{if } \delta = 1 \end{cases}$$

The Jovanovic operator

$$Q\psi(z) = c_0 + \beta \int \max \left\{ \frac{u(w(z'))}{1-\beta}, \psi(z') \right\} f(z'|z) \, \mathrm{d}z'.$$

- $f(z'|z) = N(\rho z + b, \sigma^2)$.

# Optimality Results

## Example (Job Search I cont.)

Consider, e.g., $\delta \geq 0, \delta \neq 1$ and $\rho \in [0, 1)$.

- $r(z) = \mathrm{e}^{(1-\delta)z}/((1-\beta)(1-\delta))$ and $c(z) \equiv c_0$.

**Step 1.** Since $X \sim N(\mu, \sigma^2) \implies \mathbb{E}\, e^{sX} = \mathrm{e}^{s\mu + s^2\sigma^2/2}$ (MGF):

$$\int \mathrm{e}^{(1-\delta)z'} P^t(z, \mathrm{d}z') = b_t\, \underline{e^{\rho^t(1-\delta)z}} \quad (b_t \text{ is constant for fixed } t).$$

**Step 2.** Let $\underline{g(z) = \mathrm{e}^{\rho^n(1-\delta)z}}$ and apply MGF:

$$\int g(z') P(z, \mathrm{d}z') \leq (g(z) + 1)\, \underline{\mathrm{e}^{\rho^n \xi}} \quad (\xi \text{ is a constant}).$$

**Step 3.** Choose $n \in \mathbb{N}_0$ s.t $\beta \mathrm{e}^{\rho^n \xi} < 1$, and let $\underline{m = d = \mathrm{e}^{\rho^n \xi}}$.

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# Optimality Results

**Example (Job Search I cont.)**

- The cases $\rho \in (-1, 0]$, $|\rho| = 1$ and $\delta = 1$ can be treated similarly.

**Note:**

- The local contraction method fails in this case.
  - Unbounded shocks and growth rate of the state process.

**Advantages of assumption 2.1:**

- No further restrictions on the key parameters.
- Exploits the smoothing effect of future transitions.

# Optimality Results

## Example (Job Search II cont.)

The wage process

$$\ln w_t = \theta + \varepsilon_t, \quad (\varepsilon_t)_{t \geq 0} \overset{\text{IID}}{\sim} N(0, \gamma_\varepsilon).$$

The Jovanovic operator

$$Q\psi(\mu, \gamma) := c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\mu', \gamma')\right\} f(w'|\mu, \gamma)\, \mathrm{d}w'$$

- $\gamma' = 1/\left(1/\gamma + 1/\gamma_\varepsilon\right)$ and $\mu' = \gamma'\left(\mu/\gamma + \ln w'/\gamma_\varepsilon\right)$
- $f(w'|\mu, \gamma) = N(\mu, \gamma + \gamma_\varepsilon)$
- CRRA preference

$$u(w) = \begin{cases} \frac{w^{1-\delta}}{1-\delta}, & \text{if } \delta \geq 0 \text{ and } \delta \neq 1 \\ \ln w, & \text{if } \delta = 1 \end{cases}$$

# Optimality Results

## Example (Job Search II cont.)

Consider, e.g., $\delta = 1$.

- $r(w) = \ln w/(1-\beta)$ and $c \equiv c_0$.

**Step 1.** Use $|\ln w| \leq w + 1/w$ and MGF:

$$\int |\ln w'| f(w'|\mu,\gamma)\,\mathrm{d}w' \leq \mathrm{e}^{\gamma_\varepsilon/2}\underline{\left(\mathrm{e}^{\mu+\gamma/2} + \mathrm{e}^{-\mu+\gamma/2}\right)}.$$

**Step 2.** Let $\underline{g(\mu,\gamma) = \mathrm{e}^{\mu+\gamma/2} + \mathrm{e}^{-\mu+\gamma/2}}$, and use MGF:

$$\int g(\mu',\gamma') f(w'|\mu,\gamma)\,\mathrm{d}w' = g(\mu,\gamma).$$

**Step 3.** Let $n = 1$, $m = 1$ and $d = 0$.

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**Step 3.** Let $n = 1$, $m = 1$ and $d = 0$.

# Properties of Continuation Values: Continuity

## Assumption 3.1

(1) The stochastic kernel $P$ is Feller,
(2) $c, r, \ell$, and $z \mapsto \int |r(z')| P(z, \mathrm{d}z'), \int \ell(z') P(z, \mathrm{d}z')$ are continuous.

Recall:

- Assumption 2.1 holds for $n = 0 \implies \ell(z) = g(z) + d'$.
- Assumption 2.1 holds for $n = 1 \implies \ell(z) = m'|c(z)| + g(z) + d'$.

## Proposition 3.1

If assumptions 2.1 and 3.1 hold, then $\psi^*$ is continuous.

# Properties of Continuation Values: Differentiability

**Set up:**

- $\mathsf{Z} = \mathsf{Z}^1 \times \cdots \times \mathsf{Z}^m \subset \mathbb{R}^m$
- $P$ has a density representation $f$:

$$P(z, A) = \int_A f(z'|z) \, \mathrm{d}z' \quad \text{for all } A \in \mathscr{Z}.$$

**Notations:**

- $z = (z^1, ..., z^m) \in \mathsf{Z}$
- $z^{-i} = (z^1, ..., z^{i-1}, z^{i+1}, ..., z^m)$
- $D_i^j h(z) := \frac{\partial^j h(z)}{\partial (z^i)^j}$ and $D_i^j f(z'|z) := \frac{\partial^j f(z'|z)}{\partial (z^i)^j}$

**Assumption 3.3**

$D_i c(z)$ exists for all $z \in \text{int}(\mathsf{Z})$ and $i = 1, ..., m$.

# Properties of Continuation Values: Differentiability

**Assumption 3.4**

$P$ has a density representation $f$, and for $i = 1, ..., m$:

(1) $D_i^2 f(z'|z)$ exists for all $(z, z') \in \text{int}(Z) \times Z$;
(2) $(z, z') \mapsto D_i f(z'|z)$ is continuous;

(3) There are finite solutions of $z^i$ to $D_i^2 f(z'|z) = 0$ (denoted by $z_i^*(z', z^{-i})$), and, for all $z_0 \in \text{int}(Z)$, there exist $\delta > 0$ and a compact subset $A \subset Z$ such that $z' \notin A$ implies $z_i^*(z', z_0^{-i}) \notin B_\delta(z_0^i)$.

**Example (Job Search I cont.)**

We show that assumption 3.4 holds:

- $P(z, A) = \int_A f(z'|z) \, \mathrm{d}z'$, where $f(z'|z) = N(\rho z + b, \sigma^2)$.
- $\frac{\partial^2 f(z'|z)}{\partial z^2} = 0$ has two solutions: $z^*(z') = \frac{z' - b \pm \sigma}{\rho}$.
- $|z'| \to \infty \implies |z^*(z')| \to \infty$.

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# Properties of Continuation Values: Differentiability

**Assumption 3.5**

$k$ is continuous and $\int |k(z')D_i f(z'|z)| \, \mathrm{d}z' < \infty$ for all $z \in \text{int}(Z)$, $k \in \{r, \ell\}$ and $i = 1, \ldots, m$.

**Proposition 3.3 (Differentiability)**

Under assumptions 2.1 and 3.3–3.5, $\psi^*$ is differentiable at interior points, and, for all $z \in \text{int}(Z)$ and $i = 1, \ldots, m$,

$$D_i \psi^*(z) = D_i c(z) + \int \max\left\{r(z'), \psi(z')\right\} D_i f(z'|z) \, \mathrm{d}z'.$$

# Properties of Continuation Values: Differentiability

## Assumption 3.6

For $i = 1, ..., m$, the following conditions hold:

(1) $z \mapsto D_i c(z)$ is continuous on int(Z),

(2) $k$ and $z \mapsto \int |k(z')D_i f(z'|z)| \, \mathrm{d}z'$ are continuous on int(Z) for $k \in \{r, \ell\}$.

## Proposition 3.4 (Continuous Differentiability)

If assumptions 2.1, 3.4 and 3.6 hold, then $z \mapsto D_i \psi^*(z)$ is continuous on int(Z) for $i = 1, ..., m$.

# Properties of Continuation Values: Differentiability

Recall that

$$\ell(z) := m' \left( \sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d'.$$

**Example (Job Search I cont.)**

If $\delta \geq 0$ and $\delta \neq 1$, then $\mathbb{E}_z |r(Z_t)| = a_t \mathrm{e}^{\rho^t (1-\delta) z}$ for some $a_t > 0$, $\forall t \geq 0$.

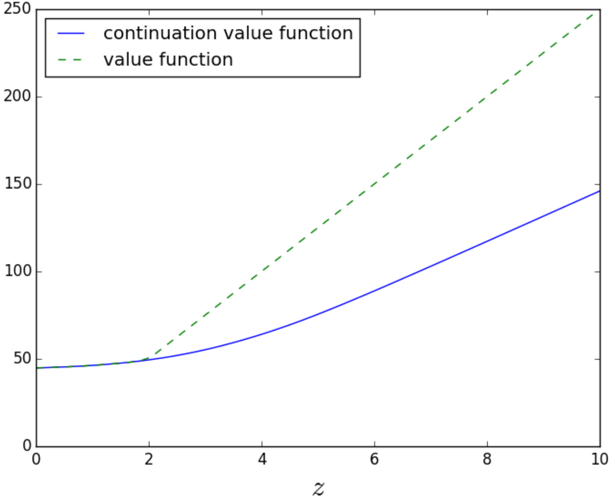
- $f(z'|z) = N(\rho z + b, \sigma^2)$

  $\implies z \mapsto \int \mathrm{e}^{a z'} \left| \frac{\partial f(z'|z)}{\partial z} \right| \mathrm{d}z'$ is continuous for all $a \in \mathbb{R}$.

  $\implies z \mapsto \int \left| r(z') \frac{\partial f(z'|z)}{\partial z} \right| \mathrm{d}z'$, $\int \left| \ell(z') \frac{\partial f(z'|z)}{\partial z} \right| \mathrm{d}z'$ are continuous.

Hence, assumption 3.6 holds, and $\psi^*$ is continuously differentiable.

Simulation: $\beta = 0.96$, $\rho = 0.6$, $\delta = 1$, $b = 0$ and $c = 1$.



# Optimal Policy

**Set up:**

- $\mathsf{Z} \subset \mathbb{R}^m$ with $\mathsf{Z} = \mathsf{X} \times \mathsf{Y} \subset \mathbb{R}^{m_0} \times \mathbb{R}^{m-m_0}$
- $(Z_t)_{t \geq 0} = \{(X_t, Y_t)\}_{t \geq 0}$
- Conditional independence: given $Y_t$, the next period states $(X_{t+1}, Y_{t+1})$ are independent of $X_t$.

Then the stochastic kernel

$$P(z, \mathrm{d}z') = P((x, y), \mathrm{d}(x', y')) = \mathrm{d}\mathbb{F}_y(x', y').$$

- The flow continuation reward $c : \mathsf{Y} \to \mathbb{R}$.

Then the Jovanovic operator

$$Q\psi(y) := c(y) + \beta \int \max\{r(x', y'), \psi(y')\} \, \mathrm{d}\mathbb{F}_y(x', y').$$

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# Optimal Policy

In the next, we consider $m_0 = 1$.

### Assumption 4.1

$r$ is strictly monotone on $\mathsf{X}$. Moreover, for all $y \in \mathsf{Y}$, there exists $x \in \mathsf{X}$ such that $r(x, y) = c(y) + \beta \int v^*(x', y') \, \mathrm{d}\mathbb{F}_y(x', y')$.

- $X_t$: the *threshold state*
- $Y_t$: the *environment*

### The reservation rule property

Under assumption 4.1, there is a *decision threshold* $\bar{x} : \mathsf{Y} \to \mathsf{X}$.

- When $x$ attains $\bar{x}$, the agent is indifferent between stopping and continuing: $r(\bar{x}(y), y) = \psi^*(y)$, for all $y \in \mathsf{Y}$.

# Optimal Policy

**Example (Job Search II cont.)**

The Bellman operator

$$Tv(w,\mu,\gamma) = \max\left\{\frac{u(w)}{1-\beta}, c_0 + \beta \int v(w',\mu',\gamma') f(w'|\mu,\gamma)\,\mathrm{d}w'\right\}.$$

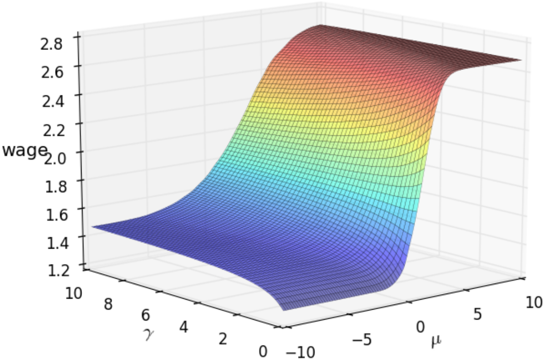
The Jovanovic operator

$$Q\psi(\mu,\gamma) = c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\mu',\gamma')\right\} f(w'|\mu,\gamma)\,\mathrm{d}w'.$$

- $w =: x \in \mathsf{X} := \mathbb{R}_{++}$: threshold state
- $(\mu,\gamma) =: y \in \mathsf{Y} := \mathbb{R} \times \mathbb{R}_{++}$: environment
- $\bar{w} : \mathsf{Y} \to \mathsf{X}$ : the reservation wage

# Optimal Policy: The Reservation Wage

Simulation: $\beta = 0.95$, $\gamma_\varepsilon = 1.0$, $\tilde{c}_0 = 0.6$, $\delta = 3.0$



# Computational Efficiency

**CVI (2-dim) v.s. VFI (3-dim):**

**178 seconds v.s. More than 7 days!**

# Optimal Policy

**Proposition 4.3 (Differentiability of Decision Threshold)**

Let assumptions 2.1, 3.4, 3.6 and 4.1 hold. If $r$ is continuously differentiable on int(Z), then $\bar{x}$ is continuously differentiable on int(Y), with

$$D_i\bar{x}(y) = -\frac{D_i r(\bar{x}(y), y) - D_i\psi^*(y)}{D_x r(\bar{x}(y), y)} \text{ for all } y \in \text{int}(Y) \text{ and } i = 1, \ldots, m.$$

- $r(x, y) - \psi^*(y)$: terminating premium
- $D_i r(\bar{x}(y), y) - D_i\psi^*(y)$: the marginal premium of $y^i$
- $D_x r(\bar{x}(y), y)$: the marginal premium of $x$

# Extension–1: Repeated Sequential Decisions

- At each time $t$, the agent is either active or passive
- Active state: observe $Z_t$, continue or exit?
  - continuation: $c(Z_t)$, remain active at $t+1$
  - exit: $s(Z_t)$, transition to passive at time $t$, return to active with probability $\alpha$ at time $t+1$
- E.g., Arellano (2008)

The value function:

$$v_a^*(z) = \max \left\{ v_p^*(z), c(z) + \beta \int v_a^*(z') P(z, \mathrm{d}z') \right\}$$

$$v_p^*(z) = s(z) + \alpha\beta \int v_a^*(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int v_p^*(z') P(z, \mathrm{d}z')$$

# Extension–2: Sequential Decision with More Choices

- At each period $t$, the agent observes $Z_t$
- Choosing among $N$ alternatives.
  - Alternative $i$: current reward $r_i(Z_t)$, stochastic kernel $P_i(z, \mathrm{d}z')$.
- E.g., Jovanovic (1987), Moscarini and Postel-Vinay (2013)

The value function

$$v^*(z) = \max\{\psi_1^*(z), ..., \psi_N^*(z)\}$$
$$\psi_i^*(z) = r_i(z) + \beta \int v^*(z')P_i(z, \mathrm{d}z')$$

# Conclusions

- Explore hidden advantages of the continuation value based method.
    - Smoothing effect of the future transitions
    - Conditional independence along the transition path
- Develop a general theory of optimal timing of decisions.
- Extend and improve the existing dynamic programming theory of optimal timing of decisions.
    - Analytically and computationally.

Thank you!