Optimal Timing of Decisions: A General Theory Based on Continuation Values[1]

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ABSTRACT. We develop a comprehensive theory of optimal timing of decisions based around continuation values as functions of the state and operators that act on them. Rewards can be bounded or unbounded. One advantage of this approach over standard Bellman equation methods is that continuation value functions are smoother than value functions. Another is that, for a range of problems, the continuation value function exists in a lower dimensional space than the value function. We exploit these advantages to obtain new results and more efficient computation.

Keywords: Continuation values, dynamic programming, optimal timing

1. INTRODUCTION

A large variety of decision making problems involve choosing when to act in the face of risk and uncertainty. Examples include deciding if or when to accept a job offer, exit or enter a market, default on a loan, bring a new product to market, exploit some new technology or business opportunity, or exercise a real or financial option. See, for example, McCall (1970), Jovanovic (1982), Hopenhayn (1992), Dixit and Pindyck (1994), Ericson and Pakes (1995), Peskir and Shiryaev (2006), Arellano (2008), Perla and Tonetti (2014), and Fajgelbaum et al. (2016).

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The most general and robust techniques for solving these kinds of problems revolve around dynamic programming. The standard machinery centers on the Bellman equation, which identifies current value in terms of a trade off between current rewards and the discounted value of future states. The Bellman equation is traditionally solved by framing the solution as a fixed point of the Bellman operator. Standard references include Bellman (1969) and Stokey et al. (1989). Applications to optimal timing can be found in Dixit and Pindyck (1994), Albuquerque and Hopenhayn (2004), Ljungqvist and Sargent (2012), and many other sources.

At the same time, economists have initiated development of an alternative method, based around operations on continuation value functions, that is parallel to the traditional method and yet different in certain asymmetric ways described in detail below. The earliest technically sophisticated example is Jovanovic (1982). In the context of an incumbent firm's exit decision, he studies a contractive operator with a unique fixed point representing the value of staying in the industry for the current period and then behave optimally. Subsequent papers in a similar vein include Burdett and Vishwanath (1988), Gomes et al. (2001), Ljungqvist and Sargent (2008), Lise (2013), Dunne et al. (2013), Moscarini and Postel-Vinay (2013), and Menzio and Trachter (2015).

Most of these works focus on continuation values as a function of the state rather than traditional value functions because they provide sharper economic intuition. For example, in a job search context, the continuation value—the value of rejecting the current offer—is the value of unemployment, and of direct interest for policy. However, there are also deeper advantages associated with this methodology that are not generally recognized or understood.

To explain the key issues, recall that, for a given problem, the value function provides the value of optimally choosing to either act today or wait, given the current environment. The continuation value is the value associated with choosing to wait today and then reoptimize next period, again taking into account the current environment. One key asymmetry arising here is that, if one chooses to wait, then certain aspects of the current environment become irrelevant, and hence need not be considered as arguments to the continuation value function.

To give one example, consider a potential entrant to a market who must consider fixed costs of entry, the evolution of prices, productivity dynamics, and so on. In some settings, certain aspects of the environment will be transitory, while others are persistent. (For example, in Fajgelbaum et al. (2016), prices and beliefs are persistent while fixed costs are transitory.) All relevant state components must be included in the value function, whether persistent or transitory, since all affect the choice of whether to enter or wait today. On the other hand, purely transitory components do not affect continuation values, since, in that scenario, the decision to wait has already been made.

Such asymmetries place the continuation value function in a lower dimensional space than the value function whenever they exist, thereby mitigating the curse of dimensionality.[2] Lower dimensionality can simplify challenging problems associated with unbounded rewards, continuity and differentiability arguments, parametric monotonicity results, and so on. On the computational side, reduction of the state space by one or more dimensions can radically increase computational speed. For example, computation time falls from more than 7 days to less than 3 minutes in a standard job search model considered in section 5.1.

Another asymmetry between value functions and continuation value functions is that the latter are typically smoother. For example, in a job search problem, the value function is usually kinked at the reservation wage. On the other hand, continuation value functions are usually smooth. Their relative smoothness comes from taking expectations over stochastic transitions, since integration is a smoothing operation. Like lower dimensionality, increased smoothness helps on both the

[2]One might imagine that this difference in dimensionality between the two approaches could, in some circumstances, work in the other direction, with the value function existing in a strictly lower dimensional space than the continuation value function. In fact this is not possible. As will be clear from the discussion below, for any decision problem in the broad class that we consider, the dimensionality of the value function is always at least as large.

analytical and the computational side. On the computational side, smoother functions are easier to approximate. On the analytical side, greater smoothness lends itself to sharper results based on derivatives, as elaborated on below.

In summary, economists have pioneered the continuation value function based approach to optimal timing of decisions in the context of specific applications, aiming to obtain tighter intuition and sharper analysis than the traditional approach. However, the method has deeper advantages that have as yet received no systematic exposition. In this paper we undertake a systematic study of optimal timing of decisions based around continuation value functions and the operators that act on them. As well as providing a general optimality theory, we obtain a number of new results on continuation values, optimal policies for timing of actions and threshold values (e.g., reservation wages) as a function of the economic environment.

For example, we provide (i) conditions under which continuation values are (a) continous, (b) monotone, and (c) differentiable as functions of the economic environment, (ii) conditions under which parametric continuity holds (often essential for proofs of existence of recursive equilibria in many-agent environments), and (iii) conditions under which threshold policies are (a) continuous, (b) monotone, and (c) differentiable. In the latter case we derive an expression for the derivative of the threshold relative to other aspects of the economic environment and show how it contributes to economic intuition. The relative smoothness of the continuation value function discussed above is exploited in a number of these findings.

All of the preceding theory is developed in a setting that can accommodate the kinds of unbounded rewards routinely encountered in modeling timing of decisions.[3] This is achieved by building on the approach to unbounded rewards based

[3]For example, many applications include Markov state processes, possibly with unit roots, driving the state, and various common reward functions (e.g., CRRA, CARA and log returns) unbounded (see, e.g, Low et al. (2010), Bagger et al. (2014), Kellogg (2014)). Moreover, many search-theoretic studies model agent's learning behavior (see, e.g., Burdett and Vishwanath (1988), Mitchell (2000), Crawford and Shum (2005), Nagypál (2007), Timoshenko (2015)), and, to work with tractable prior-posterior structures, unbounded state spaces and rewards are routinely used. We show that most of these problems can be handled without difficulty.

on weighted supremum norms pioneered by Boyd (1990) and used in numerous other studies (see, e.g., Alvarez and Stokey (1998) and Le Van and Vailakis (2005)). The underlying idea is to introduce a weighted norm in a space of candidate functions and then establish the contraction property for the relevant operator under this norm. We apply this idea in continuation value function space rather than value function space.[4]

As a second innovation, we exploit n-step transitions of rewards to build weight functions. One benefit of our method is that in many applications, a subset of states are conditionally independent of the future states, so future (even just one step) transitions of the reward functions are defined on a space that is lower dimensional than the state space. As another benefit, for mean-reverting state processes, the initial effect tends to die out as time iterates forward, making the n-step transitions flatter than the original rewards. In the context of optimal timing, the assumptions placed on the primitives in the theory we develop are weaker than those found in existing work framed in terms of traditional dynamic programming arguments.

An alternative line of research on unbounded dynamic programming uses local contractions on increasing sequences of compact subsets (see, e.g., Rincón-Zapatero and Rodríguez-Palmero (2003) or Martins-da Rocha and Vailakis (2010)). This idea exploits the underlying structure of the technological correspondence related to the state process, which, in optimal growth models, provides natural bounds on the growth rate of the state process and, through these bounds, a suitable sequence of compact subsets to construct local contractions. Unfortunately, such structures are missing in most sequential decision settings we study, making the local contraction approach inapplicable.[5]

[4]The proofs of continuity, monotonicity and differentiability mentioned above are more challenging in our unbounded settings. We surmount these difficulties in a variety of ways. For example, the proof of continuity of the continuation value function relies on a generalized Fatou's lemma.

[5]There are also a number of applied studies that treat unbounded problems, such as Poschke (2010), Chatterjee and Rossi-Hansberg (2012)) and Kellogg (2014). In these problems the techniques are specialized to the applications and not readily applicable in other settings.

The paper is structured as follows. Section 2 outlines the method and provides general optimality results. Section 3 discusses the properties of the continuation value function, such as monotonicity and differentiability. Section 4 explores the connections between the continuation value and the optimal policy. Section 5 provides a list of economic applications and compares the computational efficiency of the continuation value approach and traditional approach. Section 6 provides extensions and section 7 concludes. Proofs are deferred to the appendix.

2. Optimality Results

This section studies the optimality results. Prior to this task, we introduce some mathematical techniques and notations used in this paper.

2.1. **Preliminaries.** For $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. If f and g are functions, then $(f \vee g)(x) := f(x) \vee g(x)$. If $(\mathsf{Z}, \mathscr{Z})$ is a measurable space, then $b\mathsf{Z}$ is the set of $\mathscr{Z}$-measurable bounded functions from Z to $\mathbb{R}$, with norm $\|f\| := \sup_{z \in \mathsf{Z}} |f(z)|$. Given a function $\kappa \colon \mathsf{Z} \to [1, \infty)$, the *$\kappa$-weighted supremum norm* of $f \colon \mathsf{Z} \to \mathbb{R}$ is

$$\|f\|_\kappa := \|f/\kappa\| = \sup_{z \in \mathsf{Z}} \frac{|f(z)|}{\kappa(z)}.$$

If $\|f\|_\kappa < \infty$, we say that f is *κ-bounded*. The symbol $b_\kappa \mathsf{Z}$ will denote the set of all functions from Z to $\mathbb{R}$ that are both $\mathscr{Z}$-measurable and κ-bounded. The pair $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$ forms a Banach space (see e.g., Boyd (1990), page 331).

A *stochastic kernel* P on $(\mathsf{Z}, \mathscr{Z})$ is a map $P \colon \mathsf{Z} \times \mathscr{Z} \to [0, 1]$ such that $z \mapsto P(z, B)$ is $\mathscr{Z}$-measurable for each $B \in \mathscr{Z}$ and $B \mapsto P(z, B)$ is a probability measure for each $z \in \mathsf{Z}$. We understand $P(z, B)$ as the probability of a state transition from $z \in \mathsf{Z}$ to $B \in \mathscr{Z}$ in one step. Throughout, we let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For all $n \in \mathbb{N}$, $P^n(z, B) := \int P(z', B) P^{n-1}(z, \mathrm{d}z')$ is the probability of a state transition from z to $B \in \mathscr{Z}$ in n steps. Given a $\mathscr{Z}$-measurable function $h : \mathsf{Z} \to \mathbb{R}$, let

$$(P^n h)(z) =: \mathbb{E}_z h(Z_n) := \int h(z') P^n(z, \mathrm{d}z') \text{ for all } n \in \mathbb{N}_0,$$

where $(P^0 h)(z) =: \mathbb{E}_z h(Z_0) := h(z)$, and we write Ph for short when $n = 1$. We say that the kernel P is *stochastically increasing* if Ph is increasing for all increasing function $h \in b\mathsf{Z}$. When Z is a Borel subset of $\mathbb{R}^m$, a *density kernel* on Z is a measurable map $f : \mathsf{Z} \times \mathsf{Z} \to \mathbb{R}_+$ such that $\int_{\mathsf{Z}} f(z'|z) dz' = 1$ for all $z \in \mathsf{Z}$. We say that the kernel P *has a density representation* if there exists a density kernel f such that $P(z, B) = \int_B f(z'|z) \, dz'$ for all $z \in \mathsf{Z}$ and $B \in \mathscr{Z}$.

2.2. **Set Up.** Let $(Z_n)_{n \geq 0}$ be a Markov process on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in measurable space $(\mathsf{Z}, \mathscr{Z})$. Let P denote the corresponding stochastic kernel. Let $\{\mathscr{F}_n\}_{n \geq 0}$ be a filtration in $\mathscr{F}$ to which $(Z_n)_{n \geq 0}$ is adapted. Let $\mathbb{P}_z$ indicate probability conditioned on $Z_0 = z$, while $\mathbb{E}_z$ is expectation conditioned on the same event. In proofs we take $(\Omega, \mathscr{F})$ to be the canonical sequence space, so that $\Omega = \times_{n=0}^{\infty} \mathsf{Z}$ and $\mathscr{F}$ is the product σ-algebra generated by $\mathscr{Z}$.[6] A random variable τ taking values in $\mathbb{N}_0$ is called a (finite) *stopping time* with respect to the filtration $\{\mathscr{F}_n\}_{n \geq 0}$ if $\mathbb{P}\{\tau < \infty\} = 1$ and $\{\tau \leq n\} \in \mathscr{F}_n$ for all $n \geq 0$. Below, $\tau = n$ has the interpretation of choosing to act at time n. Let $\mathscr{M}$ denote the set of all stopping times on Ω with respect to the filtration $\{\mathscr{F}_n\}_{n \geq 0}$.

Suppose, at each $t \geq 0$, an agent observes Z_t and chooses between stopping (e.g., accepting a job, exercising an option) and continuing. Stopping generates *exit reward* $r(Z_t)$. Continuing yields *flow continuation reward* $c(Z_t)$ and transition to $t+1$, where the agent observes Z_{t+1} and the process repeats. Here $r \colon \mathsf{Z} \to \mathbb{R}$ and $c \colon \mathsf{Z} \to \mathbb{R}$ are measurable functions. Future rewards are discounted at rate $\beta \in (0, 1)$. The value function v^* for this problem is defined at $z \in \mathsf{Z}$ by

$$v^*(z) := \sup_{\tau \in \mathscr{M}} \mathbb{E}_z \left\{ \sum_{t=0}^{\tau-1} \beta^t c(Z_t) + \beta^\tau r(Z_\tau) \right\}. \tag{1}$$

A stopping time $\tau \in \mathscr{M}$ is called an *optimal stopping time* if it attains the supremum in (1). A *policy* is a map σ from Z to $\{0, 1\}$, with 0 indicating the decision to continue and 1 indicating the decision to stop. A policy σ is called *optimal* if $\tau^* := \inf\{t \geq 0 \,|\, \sigma(Z_t) = 1\}$ is an optimal stopping time. The *continuation value function* associated with the sequential decision problem (1) is defined at $z \in \mathsf{Z}$ by

$$\psi^*(z) := c(z) + \beta \int v^*(z') P(z, dz'). \tag{2}$$

[6] For background see section 3.4 of Meyn and Tweedie (2012) or section 8.2 of Stokey et al. (1989).

2.3. **The Continuation Value Operator.** To provide optimality results without insisting that rewards are bounded, we adopt the next assumption:

Assumption 2.1. There exist a $\mathscr{Z}$-measurable function $g \colon \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0$, $m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

$$\max \left\{ \int |r(z')| P^n(z, \mathrm{d}z'), \int |c(z')| P^n(z, \mathrm{d}z') \right\} \leq g(z) \tag{3}$$

and

$$\int g(z') P(z, \mathrm{d}z') \leq m g(z) + d. \tag{4}$$

The interpretation of assumption 2.1 is that both $\mathbb{E}_z |r(Z_n)|$ and $\mathbb{E}_z |c(Z_n)|$ are small relative to some function g such that $\mathbb{E}_z g(Z_t)$ does not grow too fast.[7] Slow growth in $\mathbb{E}_z g(Z_t)$ is imposed by (4), which can be understood as a geometric drift condition (see, e.g., Meyn and Tweedie (2012), chapter 15). Note that if both r and c are bounded, then assumption 2.1 holds for $n := 0$, $g := \|r\| \vee \|c\|$, $m := 1$ and $d := 0$.

Lemma 2.1. *Under assumption 2.1, the value function solves the Bellman equation*

$$v^*(z) = \max \left\{ r(z),\, c(z) + \beta \int v^*(z') P(z, dz') \right\} = \max \left\{ r(z),\, \psi^*(z) \right\}. \tag{5}$$

Using (2) and (5), we obtain the following functional equation

$$\psi^*(z) = c(z) + \int \max \left\{ r(z'), \psi^*(z') \right\} P(z, \mathrm{d}z'). \tag{6}$$

To obtain optimality results concerning ψ^*, we define an operator Q by

$$Q\psi(z) = c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z'). \tag{7}$$

[7] To verify assumption 2.1, it suffices to find a $\mathscr{Z}$-measurable map $g \colon \mathsf{Z} \to \mathbb{R}_+$, constants $n \in \mathbb{N}_0$, $m, d \in \mathbb{R}_+$ with $\beta m < 1$, and $a_1, a_2, a_3, a_4 \in \mathbb{R}_+$ such that $\int |r(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2$, $\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4$ and (4) holds. This fact is used in our applications.

Moreover, if assumption 2.1 holds for some n, then it must hold for all integer $n' > n$. Hence, to satisfy assumption 2.1, it suffices to find a measurable map g and constants $n_1, n_2 \in \mathbb{N}_0$, $m, d \in \mathbb{R}_+$ with $\beta m < 1$ such that $\int |r(z')| P^{n_1}(z, \mathrm{d}z') \leq g(z)$, $\int |c(z')| P^{n_2}(z, \mathrm{d}z') \leq g(z)$ and (4) holds. More general sufficient conditions can be obtained by combining the above facts.

We call Q the *continuation value operator* or *Jovanovic operator*. Recall n, m and d defined in assumption 2.1. Let $m', d' > 0$ such that $m + 2m' > 1$, $\beta(m + 2m') < 1$ and $d' \geq d/(m + 2m' - 1)$. Let the weight function $\ell \colon \mathsf{Z} \to \mathbb{R}$ be defined by

$$\ell(z) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d'. \tag{8}$$

Note that if assumption 2.1 holds for $n = 0$, then $\ell(z) = g(z) + d'$. If it holds for $n = 1$, then $\ell(z) = m'|c(z)| + g(z) + d'$. We have the following optimality result.

Theorem 2.1. *Under assumption 2.1, the following statements are true:*

1. *Q is a contraction mapping on $(b_\ell \mathsf{Z}, \| \cdot \|_\ell)$ of modulus $\beta(m + 2m')$.*
2. *The unique fixed point of Q in $b_\ell \mathsf{Z}$ is ψ^*.*
3. *The policy defined by $\sigma^*(z) = \mathbb{1}\{r(z) \geq \psi^*(z)\}$ is an optimal policy.*

Example 2.1. Consider a job search framework with Markov state dynamics (see, e.g., Jovanovic (1987), Cooper et al. (2007), Ljungqvist and Sargent (2008), Robin (2011), Moscarini and Postel-Vinay (2013), Bagger et al. (2014)). A worker can either accept current wage offer w_t and work permanently at that wage, or reject the offer, receive unemployment compensation $\tilde{c}_0$ and reconsider next period. Let the current offer be a function $w_t = w(Z_t)$ of some idiosyncratic or aggregate state process $(Z_t)_{t \geq 0}$. The exit reward is thus $r(z) = u(w(z))/(1 - \beta)$, the lifetime reward associated with stopping at state z. Here u is a utility function and β is the discount factor. The flow continuation reward is the constant $c_0 := u(\tilde{c}_0)$.

From (7), the Jovanovic operator for this problem is

$$Q\psi(z) = c_0 + \beta \int \max \left\{ \frac{u(w(z'))}{1 - \beta}, \psi(z') \right\} P(z, \mathrm{d}z'). \tag{9}$$

Let $w(z) = \mathrm{e}^z$ and let

$$Z_{t+1} = \rho Z_t + b + \varepsilon_{t+1}, \quad (\varepsilon_t) \overset{\text{IID}}{\sim} N(0, \sigma^2). \tag{10}$$

The state space is $\mathbb{R}$, and P has a density representation $f(z'|z) = N(\rho z + b, \sigma^2)$. Suppose that utility has the CRRA form

$$u(w) = \frac{w^{1-\delta}}{1 - \delta} \quad (\text{if } \delta \geq 0 \text{ and } \delta \neq 1) \quad \text{and} \quad u(w) = \ln w \quad (\text{if } \delta = 1). \tag{11}$$

Consider, for example, $\delta \geq 0$, $\delta \neq 1$ and $\rho \in [0,1)$. In this case, the exit reward is $r(z) = \mathrm{e}^{(1-\delta)z} / ((1-\beta)(1-\delta))$. To verify assumption 2.1, note that

$$\int \mathrm{e}^{(1-\delta)z'} P^t(z, \mathrm{d}z') = a_t \, \mathrm{e}^{\rho^t (1-\delta) z} \text{ for some } a_t > 0. \tag{12}$$

Since $\rho \in [0,1)$, we can select an $n \in \mathbb{N}_0$ such that $\beta \mathrm{e}^{\rho^n \xi} < 1$, where $\xi := \xi_1 + \xi_2$ with $\xi_1 := (1-\delta)b$ and $\xi_2 := (1-\delta)^2 \sigma^2 / 2$. By footnote 7, we can let $g(z) := \mathrm{e}^{\rho^n (1-\delta) z}$. Assumption 2.1 then holds with $m := d := \mathrm{e}^{\rho^n \xi}$, since

$$\int g(z') P(z, \mathrm{d}z') = \mathrm{e}^{\rho^{n+1}(1-\delta)z} \mathrm{e}^{\rho^n \xi_1 + \rho^{2n} \xi_2} \leq \left(\mathrm{e}^{\rho^n (1-\delta) z} + 1 \right) \mathrm{e}^{\rho^n \xi} = m g(z) + d. \tag{13}$$

With ℓ defined as in (8), theorem 2.1 implies that Q is a contraction mapping on $(b_\ell \mathsf{Z}, \| \cdot \|_\ell)$, with unique fixed point ψ^*, the expected value of unemployment. The cases $\rho = 1$, $\delta = 1$ and $\rho \in [-1, 0]$ can be treated using similar methods.[8]

Example 2.2. Let's now add learning to the job search problem, as in, say, McCall (1970), Chalkley (1984), Burdett and Vishwanath (1988), Pries and Rogerson (2005), Nagypál (2007), Ljungqvist and Sargent (2012). The set up is as in example 2.1, except that $(w_t)_{t \geq 0}$ follows

$$\ln w_t = \xi + \varepsilon_t, \quad (\varepsilon_t)_{t \geq 0} \overset{\text{IID}}{\sim} N(0, \gamma_\varepsilon).$$

Here ξ is the unobservable mean of the wage process, over which the worker has prior $\xi \sim N(\mu, \gamma)$. The worker's current estimate of the next period wage distribution is $f(w' | \mu, \gamma) = LN(\mu, \gamma + \gamma_\varepsilon)$. If the current offer is turned down, the worker updates his belief after observing w'. By the Bayes' rule, the posterior satisfies $\xi | w' \sim N(\mu', \gamma')$, where

$$\gamma' = (1/\gamma + 1/\gamma_\varepsilon)^{-1} \quad \text{and} \quad \mu' = \gamma' \left(\mu / \gamma + \ln w' / \gamma_\varepsilon \right). \tag{14}$$

Let the utility of the worker be defined by (11). The state is $z := (w, \mu, \gamma) \in \mathbb{R}_{++} \times \mathbb{R} \times \mathbb{R}_{++} =: \mathsf{Z}$. For any integrable function h, the stochastic kernel P satisfies

$$\int h(z') P(z, \mathrm{d}z') = \int h(w', \mu', \gamma') f(w' | \mu, \gamma) \, \mathrm{d}w', \tag{15}$$

[8] See the working paper version (Ma and Stachurski, 2017) for a detailed proof of all cases.

where μ' and γ' are defined by (14). The Jovanovic operator is

$$Q\psi(\mu,\gamma) = c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\mu',\gamma')\right\} f(w'|\mu,\gamma)\, \mathrm{d}w'. \tag{16}$$

Consider, for example, $\delta \geq 0$ and $\delta \neq 1$. In this case, the reward functions are $c \equiv c_0$ and $r(w) := w^{1-\delta}/((1-\delta)(1-\beta))$. To verify assumption 2.1, since

$$\int w'^{1-\delta} f(w'|\mu,\gamma)\, \mathrm{d}w' = \mathrm{e}^{(1-\delta)^2\gamma_\varepsilon/2} \cdot \mathrm{e}^{(1-\delta)\mu+(1-\delta)^2\gamma/2}, \tag{17}$$

by (15) and footnote 7, we can choose $n := 1$ and $g(\mu,\gamma) := \mathrm{e}^{(1-\delta)\mu+(1-\delta)^2\gamma/2}$. Then assumption 2.1 holds by letting $m := 1$ and $d := 0$, since

$$\mathbb{E}_{\mu,\gamma} g(\mu',\gamma') := \int g(\mu',\gamma') f(w'|\mu,\gamma)\, \mathrm{d}w' = g(\mu,\gamma) = mg(\mu,\gamma) + d. \tag{18}$$

Define ℓ by (8). By theorem 2.1, Q is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$ with unique fixed point ψ^*. The case $\delta = 1$ can be treated similarly.[9]

Remark 2.1. From (16), we see that the continuation value is a function of (μ,γ). On the other hand, current rewards depend on w, so the value function depends on (w,μ,γ). Thus, the former is lower dimensional than the latter.

Example 2.3. Consider an infinite-horizon American call option (see, e.g., Peskir and Shiryaev (2006) or Duffie (2010)) with state process be as in (10) and state space $\mathsf{Z} := \mathbb{R}$. Let $p_t = p(Z_t) = \mathrm{e}^{Z_t}$ be the current price of the underlying asset, and $\gamma > 0$ be the riskless rate of return (i.e., $\beta = \mathrm{e}^{-\gamma}$). With strike price K, the exit reward is $r(z) = (p(z) - K)^+$, while the flow continuation reward is $c \equiv 0$. The Jovanovic operator for the option satisfies

$$Q\psi(z) = \mathrm{e}^{-\gamma} \int \max\{(p(z') - K)^+, \psi(z')\} f(z'|z)\, \mathrm{d}z'.$$

If $\rho \in (-1,1)$, we can let $\xi := |b| + \sigma^2/2$ and fix $n \in \mathbb{N}_0$ such that $\mathrm{e}^{-\gamma+|\rho^n|\xi} < 1$, so assumption 2.1 holds with $g(z) := \mathrm{e}^{\rho^n z} + \mathrm{e}^{-\rho^n z}$ and $m := d := \mathrm{e}^{|\rho^n|\xi}$. (If $\mathrm{e}^{-\gamma+\xi} < 1$, then assumption 2.1 holds with $n = 0$ for all $\rho \in [-1,1]$.)

[9] See the working paper version (Ma and Stachurski, 2017) for a detailed proof of this case.

Example 2.4. Suppose that, in each period, a firm observes an idea with value $Z_t \in \mathsf{Z} := \mathbb{R}_+$ and decides whether to put this idea into productive use or develop it further, by investing in R&D (see, e.g., Jovanovic and Rob (1989), Bental and Peled (1996), Perla and Tonetti (2014)). The first choice gives reward $r(Z_t) = Z_t$. The latter incurs fixed cost $c_0 > 0$. Let the R&D process be governed by the exponential law (with rate parameter $\theta > 0$)

$$F(z'|z) := \mathbb{P}(Z_{t+1} \leq z' | Z_t = z) = 1 - \mathrm{e}^{-\theta(z'-z)} \quad (z' \geq z), \tag{19}$$

Assumption 2.1 is satisfied with $n := 0$, $g(z) := z$, $m := 1$ and $d := 1/\theta$. The Jovanovic operator satisfies

$$Q\psi(z) = -c_0 + \beta \int \max\{z',\ \psi(z')\}\, \mathrm{d}F(z'|z).$$

With ℓ as in (8), Q is a contraction mapping on $b_\ell \mathsf{Z}$ with unique fixed point ψ^*, the expected value of investing in R&D.

Example 2.5. Consider a firm exit problem (see, e.g., Hopenhayn (1992), Ericson and Pakes (1995), Asplund and Nocke (2006), Dinlersoz and Yorukoglu (2012), Coşar et al. (2016)). Suppose that, in each period, an incumbent firm observes a productivity shock a_t, where $a_t = a(Z_t) = \mathrm{e}^{Z_t}$ and $Z_t \in \mathsf{Z} := \mathbb{R}$ obeys (10), and decides whether or not to exit the market next period. A fixed cost $c_f > 0$ is paid each period and the firm's output is $q(a, l) = a l^{\alpha}$, where $\alpha \in (0, 1)$ and l is labor demand. Given output and input prices p and w, the reward functions are $r(z) = c(z) = G a(z)^{\frac{1}{1-\alpha}} - c_f$, where $G = (\alpha p / w)^{\frac{1}{1-\alpha}} (1 - \alpha) w / \alpha$. The Jovanovic operator satisfies

$$Q\psi(z) = \left(G a(z)^{\frac{1}{1-\alpha}} - c_f \right) + \beta \int \max \left\{ G a(z')^{\frac{1}{1-\alpha}} - c_f, \psi(z') \right\} f(z'|z)\, \mathrm{d}z'.$$

For $\rho \in [0, 1)$, choose $n \in \mathbb{N}_0$ such that $\beta \mathrm{e}^{\rho^n \xi} < 1$, where $\xi := \frac{b}{1-\alpha} + \frac{\sigma^2}{2(1-\alpha)^2}$. Then assumption 2.1 holds with $g(z) := \mathrm{e}^{\rho^n z/(1-\alpha)}$ and $m := d := \mathrm{e}^{\rho^n \xi}$. Other parameterizations (such as the unit root case $\rho = 1$) can also be handled—see the working paper (Ma and Stachurski, 2017) for details.

3. Properties of Continuation Values

This section studies some further properties of the continuation value function.

3.1. **Continuity.** We first develop a theory of continuity.

Assumption 3.1. The stochastic kernel P satisfies the Feller property, i.e., P maps bounded continuous functions into bounded continuous functions.

Assumption 3.2. The functions c, r, ℓ, and $z \mapsto \int |r(z')|P(z, \mathrm{d}z')$, $\int \ell(z')P(z, \mathrm{d}z')$ are continuous.

Remark 3.1. Note that if assumption 2.1 holds for $n = 0$ and assumption 3.1 holds, then assumption 3.2 is equivalent to: c, r, g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous.[10] A general sufficient condition for assumption 3.2 is: g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z |r(Z_t)|, \mathbb{E}_z |c(Z_t)|$ are continuous for $t = 0, ..., n$.

Proposition 3.1. *Under assumptions 2.1 and 3.1–3.2, ψ^* is continuous.*

The next result treats the special case when P admits a density representation. The proof is similar to that of proposition 3.1, except that we use lemma 7.2 instead of the generalized Fatou's lemma of Feinberg et al. (2014) to prove continuity in (31). In this case, notably, the continuity of r is not required.

Corollary 3.1. *If assumption 2.1 holds, P admits a density representation $f(z'|z)$ that is continuous in z, and that c, ℓ and $z \mapsto \int |r(z')| f(z'|z) \, \mathrm{d}z'$, $\int \ell(z') f(z'|z) \, \mathrm{d}z'$ are continuous, then ψ^* is continuous.*

Remark 3.2. If the rewards r and c are bounded, assumption 3.1 and the continuity of r and c are sufficient for the continuity of ψ^* (by proposition 3.1). If in addition P has a density representation f, the continuity of c and $z \mapsto f(z'|z)$ (for all $z' \in \mathsf{Z}$) is sufficient for ψ^* to be continuous (by corollary 3.1).

[10] If $n = 0$ in assumption 2.1, then $|r| \leq g$ and $\ell(z) = g(z) + d'$. Since r, g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, theorem 1.1 of Feinberg et al. (2014) implies that $z \mapsto \mathbb{E}_z |r(Z_1)|$ is continuous.

Example 3.1. In the job search model of example 2.1, ψ^* is continuous. Assumption 2.1 holds, as shown. P has a density representation $f(z'|z) = N(\rho z + b, \sigma^2)$ that is continuous in z. Moreover, c, g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z |r(Z_t)|$ is continuous for all $t \in \mathbb{N}$ (see (12)–(13)). Hence, ℓ and $z \mapsto \mathbb{E}_z \ell(Z_1)$ are continuous, and the conditions of corollary 3.1 are satisfied.

Example 3.2. Recall the adaptive search model of example 2.2. Assumption 2.1 holds for $n = 1$, as shown. Assumption 3.1 follows from (15) and lemma 7.2. Moreover, r, c, g, and $(\mu, \gamma) \mapsto \mathbb{E}_{\mu,\gamma}|r(w')|, \mathbb{E}_{\mu,\gamma} g(\mu', \gamma')$ are continuous (see (17)–(18)), where $\mathbb{E}_{\mu,\gamma}|r(w')| := \int |r(w')| f(w'|\mu, \gamma) \, \mathrm{d}w'$. Since $\ell = m'|c| + g + d'$ when $n = 1$, assumption 3.2 holds. By proposition 3.1, ψ^* is continuous.

Example 3.3. Recall the option pricing model of example 2.3. By corollary 3.1, we can show that ψ^* is continuous. The proof is similar to example 3.1, except that we use $|r(z)| \leq \mathrm{e}^z + K$, the continuity of $z \mapsto \int (\mathrm{e}^{z'} + K) f(z'|z) \, \mathrm{d}z'$, and lemma 7.2 to show that $z \mapsto \mathbb{E}_z |r(Z_1)|$ is continuous. The continuity of $z \mapsto \mathbb{E}_z |r(Z_t)|$ (for all $t \in \mathbb{N}$) follows immediately from induction.

Example 3.4. Recall the R&D decision problem of example 2.4. Assumption 2.1 holds for $n = 0$. For all bounded continuous function $h : \mathsf{Z} \to \mathbb{R}$, lemma 7.2 shows that $z \mapsto \int h(z') \, \mathrm{d}F(z'|z)$ is continuous, so assumption 3.1 holds. Moreover, r, c and g are continuous, and $z \mapsto \mathbb{E}_z g(Z_1)$ is continuous since

$$\int |z'| P(z, \mathrm{d}z') = \int_{[z,\infty)} z' \theta \mathrm{e}^{-\theta(z'-z)} \, \mathrm{d}z' = z + 1/\theta.$$

Hence, assumption 3.2 holds. By proposition 3.1, ψ^* is continuous.

Example 3.5. Recall the firm exit model of example 2.5. Through similar analysis to examples 3.1 and 3.3, we can show that ψ^* is continuous.

3.2. **Monotonicity.** We now study monotonicity under the following assumptions.[11]

Assumption 3.3. The flow continuation reward c is increasing, and the function $z \mapsto \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z')$ is increasing for all increasing function $\psi \in b_\ell \mathsf{Z}$.

[11] We mainly treat the monotone increasing case. The monotone decreasing case is similar.

Remark 3.3. If r is increasing and P is stochastically increasing (recall section 2.1), then the second statement of assumption 3.3 holds.

Proposition 3.2. *Under assumptions 2.1 and 3.3, ψ^* is increasing.*

Proof of proposition 3.2. Standard argument shows that $b_\ell i\mathsf{Z}$, the set of increasing functions in $b_\ell \mathsf{Z}$, is a closed subset. To show that ψ^* is increasing, it suffices to verify that $Q(b_\ell i\mathsf{Z}) \subset b_\ell i\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). The assumptions of the proposition guarantee that this is the case. □

Example 3.6. In example 2.1, assumption 2.1 holds. If $\rho \geq 0$, the density kernel $f(z'|z) = N(\rho z + b, \sigma^2)$ is stochastically increasing in z. Since r and c are increasing, assumption 3.3 holds. By proposition 3.2, ψ^* is increasing.

Remark 3.4. Similarly, for the option pricing model of example 2.3 and the firm exit model of example 2.5, if $\rho \geq 0$, then ψ^* is increasing. Moreover, ψ^* is increasing in example 2.4. The details are omitted.

Example 3.7. For the adaptive search model of example 2.2, $r(w)$ is increasing, μ' is increasing in μ, and $f(w'|\mu,\gamma) = N(\mu, \gamma + \gamma_\varepsilon)$ is stochastically increasing in μ, so $\mathbb{E}_{\mu,\gamma}(r(w') \vee \psi(\mu',\gamma'))$ is increasing in μ for all candidate ψ that is increasing in μ. Since $c \equiv c_0$, by proposition 3.2, ψ^* is increasing in μ.

3.3. **Differentiability.** Suppose $\mathsf{Z} \subset \mathbb{R}^m$. For $i = 1, ..., m$, let $\mathsf{Z}^{(i)}$ be the i-th dimension and $\mathsf{Z}^{(-i)}$ the remaining $m-1$ dimensions of Z. A typical element $z \in \mathsf{Z}$ takes form of $z = (z^1, ..., z^m)$. Let $z^{-i} := (z^1, ..., z^{i-1}, z^{i+1}, ..., z^m)$. Given $z_0 \in \mathsf{Z}$ and $\delta > 0$, let $B_\delta(z_0^i) := \{z^i \in \mathsf{Z}^{(i)} : |z^i - z_0^i| < \delta\}$ and $\bar{B}_\delta(z_0^i)$ be its closure.

Given $h : \mathsf{Z} \to \mathbb{R}$, let $D_i h(z) := \partial h(z)/\partial z^i$ and $D_i^2 h(z) := \partial^2 h(z)/\partial (z^i)^2$. For a density kernel f, let $D_i f(z'|z) := \partial f(z'|z)/\partial z^i$ and $D_i^2 f(z'|z) := \partial^2 f(z'|z)/\partial (z^i)^2$. Let $\mu(z) := \int \max\{r(z'), \psi^*(z')\} f(z'|z) \, \mathrm{d}z'$, $\mu_i(z) := \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z) \, \mathrm{d}z'$, and denote $k_1(z) := r(z)$ and $k_2(z) := \ell(z)$.

Assumption 3.4. $D_i c(z)$ exists for all $z \in \mathrm{int}(\mathsf{Z})$ and $i = 1, ..., m$.

Assumption 3.5. P has a density representation f, and, for $i = 1, ..., m$:

(1) $D_i^2 f(z'|z)$ exits for all $(z, z') \in \text{int}(\mathsf{Z}) \times \mathsf{Z}$;
(2) $(z, z') \mapsto D_i f(z'|z)$ is continuous;
(3) There are finite solutions of z^i to $D_i^2 f(z'|z) = 0$ (denoted by $z_i^*(z', z^{-i})$), and, for all $z_0 \in \text{int}(\mathsf{Z})$, there exist $\delta > 0$ and a compact subset $A \subset \mathsf{Z}$ such that $z' \notin A$ implies $z_i^*(z', z_0^{-i}) \notin B_\delta(z_0^i)$.

Remark 3.5. When the state space is unbounded above and below, for example, a sufficient condition for assumption 3.5-(3) is: there are finite solutions of z^i to $D_i^2 f(z'|z) = 0$, and, for all $z_0 \in \text{int}(\mathsf{Z})$, $\|z'\| \to \infty$ implies $|z_i^*(z', z_0^{-i})| \to \infty$.

Assumption 3.6. k_j is continuous and $\int |k_j(z') D_i f(z'|z)| \, \mathrm{d}z' < \infty$ for all $z \in \text{int}(\mathsf{Z})$, $i = 1, ..., m$ and $j = 1, 2$.

The following provides a general result for the differentiability of ψ^*.

Proposition 3.3. *Under assumptions 2.1 and 3.4–3.6, ψ^* is differentiable at interior points, with $D_i \psi^*(z) = D_i c(z) + \mu_i(z)$ for all $z \in \text{int}(\mathsf{Z})$ and $i = 1, ..., m$.*

Proof of proposition 3.3. Fix $z_0 \in \text{int}(\mathsf{Z})$. By assumption 3.5 (2)–(3), there exist $\delta > 0$ and a compact subset $A \subset \mathsf{Z}$ such that $z' \notin A$ implies $z_i^*(z', z_0^{-i}) \notin B_\delta(z_0^i)$, hence $\sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| = |D_i f(z'|z)|_{z^i = z_0^i + \delta} \vee |D_i f(z'|z)|_{z^i = z_0^i - \delta} =: h^\delta(z', z_0)$ for $z' \in A^c$ and $z^{-i} = z_0^{-i}$. By assumption 3.5-(2), there exists $G \in \mathbb{R}_+$, such that for $z^{-i} = z_0^{-i}$,

$$
\begin{aligned}
\sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| &\leq \sup_{z' \in A, z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| \cdot \mathbb{1}(z' \in A) + h^\delta(z', z_0) \cdot \mathbb{1}(z' \in A^c) \\
&\leq G \cdot \mathbb{1}(z' \in A) + \left(|D_i f(z'|z)|_{z^i = z_0^i + \delta} + |D_i f(z'|z)|_{z^i = z_0^i - \delta} \right) \cdot \mathbb{1}(z' \in A^c).
\end{aligned}
$$

Assumption 3.6 then shows that condition (2) of lemma 7.3 holds. By assumption 3.4 and lemma 7.3, $D_i \psi^*(z) = D_i c(z) + \mu_i(z)$, $\forall z \in \text{int}(\mathsf{Z})$, as was to be shown. $\square$

3.4. **Smoothness.** Now we study smoothness (continuous differentiability), an essential property for numerical computation and for exploring optimal policies.

Assumption 3.7. For $i = 1, ..., m$ and $j = 1, 2$, the following conditions hold:

(1) $z \mapsto D_i c(z)$ is continuous on $\mathrm{int}(\mathsf{Z})$;

(2) k_j is continuous, and, $z \mapsto \int |k_j(z') D_i f(z'|z)| \, \mathrm{d}z'$ is continuous on $\mathrm{int}(\mathsf{Z})$.

The next result provides sufficient conditions for smoothness.

Proposition 3.4. *Under assumptions 2.1, 3.5 and 3.7, $z \mapsto D_i \psi^*(z)$ is continuous on $\mathrm{int}(\mathsf{Z})$ for $i = 1, ..., m$.*

Proof of proposition 3.4. Since assumption 3.7 implies assumptions 3.4 and 3.6, by proposition 3.3, $D_i \psi^*(z) = D_i c(z) + \mu_i(z)$ on $\mathrm{int}(\mathsf{Z})$. Since $D_i c(z)$ is continuous by assumption 3.7-(1), to show that ψ^* is continuously differentiable, it remains to verify: $z \mapsto \mu_i(z)$ is continuous on $\mathrm{int}(\mathsf{Z})$. Since $|\psi^*| \leq G\ell$ for some $G \in \mathbb{R}_+$,

$$\left|\max\{r(z'), \psi^*(z')\} D_i f(z'|z)\right| \leq (|r(z')| + G\ell(z')) |D_i f(z'|z)|, \ \forall z', z \in \mathsf{Z}. \tag{20}$$

By assumptions 3.5-(2) and 3.7-(2), both sides of (20) are continuous in z, and $z \mapsto \int [|r(z')| + G\ell(z')] |D_i f(z'|z)| \, \mathrm{d}z'$ is continuous. Lemma 7.2 then implies that $z \mapsto \mu_i(z) = \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z) dz'$ is continuous, as was to be shown. $\square$

Example 3.8. Recall the job search model of example 2.1. For all $a \in \mathbb{R}$, let $h(z, a) := \mathrm{e}^{a(\rho z + b) + a^2 \sigma^2 / 2} / \sqrt{2\pi\sigma^2}$, then the following statements hold:

(a) There are two solutions of z to $\frac{\partial^2 f(z'|z)}{\partial z^2} = 0$: $z^*(z') = \frac{z' - b \pm \sigma}{\rho}$;

(b) $\int \left| \frac{\partial f(z'|z)}{\partial z} \right| \mathrm{d}z' = \frac{|\rho|}{\sigma} \sqrt{\frac{2}{\pi}}$;

(c) $\mathrm{e}^{az'} \left| \frac{\partial f(z'|z)}{\partial z} \right| \leq h(z, a) \exp\left\{ - \frac{[z' - (\rho z + b + a\sigma^2)]^2}{2\sigma^2} \right\} \frac{|\rho z'| + |\rho(\rho z + b)|}{\sigma^2}$;

(d) The two terms on both sides of (c) are continuous in z;

(e) The integration (w.r.t. z') of the right side of (c) is continuous in z.

By remark 3.5 and (a), assumption 3.5-(3) holds. Based on (12), conditions (b)–(e), and lemma 7.2, we can show that assumption 3.7-(2) holds. The other conditions of proposition 3.4 obviously hold. Hence, ψ^* is continuously differentiable.

Example 3.9. Recall the option pricing problem of example 2.3. Similarly as in example 3.8, we can show that ψ^* is continuously differentiable. Figure 1 illustrates.

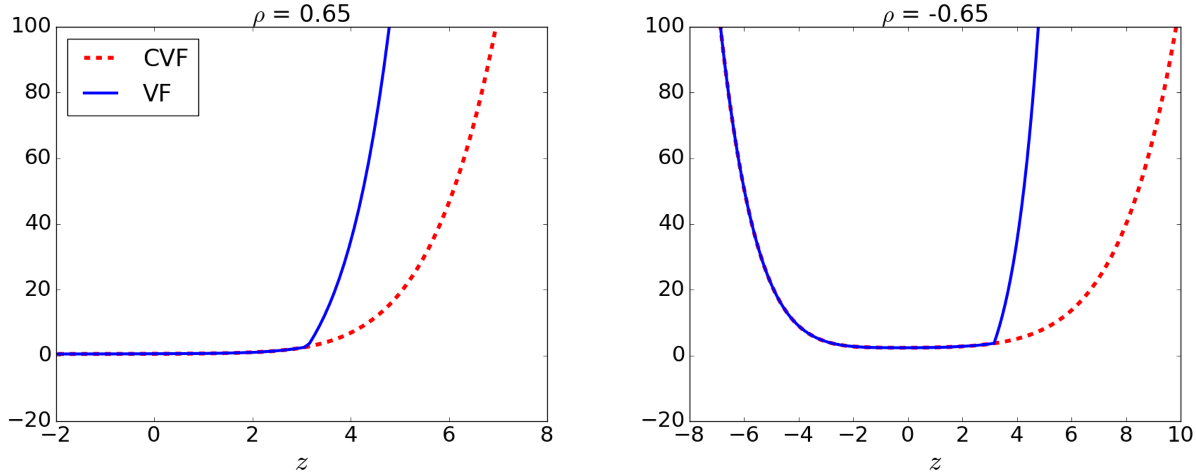

FIGURE 1. Comparison of ψ^* (CVF) and v^* (VF)

We set $\gamma = 0.04$, $K = 20$, $b = -0.2$, $\sigma = 1$, and consider $\rho = \pm 0.65$. While ψ^* is smooth, v^* is kinked at around $z = 3$ in both cases.

Example 3.10. Recall the firm exit model of example 2.5. Through similar analysis to examples 3.8–3.9, we can show that ψ^* is continuously differentiable.

3.5. **Parametric Continuity.** Consider the parameter space $\Theta \subset \mathbb{R}^k$. Let $P_\theta, r_\theta, c_\theta$, v_θ^* and ψ_θ^* denote the stochastic kernel, exit and flow continuation rewards, value and continuation value functions with respect to the parameter $\theta \in \Theta$, respectively. Similarly, let n_θ, m_θ, d_θ and g_θ denote the key elements of assumption 2.1 with respect to θ. Define $n := \sup_{\theta \in \Theta} n_\theta$, $m := \sup_{\theta \in \Theta} m_\theta$ and $d := \sup_{\theta \in \Theta} d_\theta$.

Assumption 3.8. Assumption 2.1 holds at all $\theta \in \Theta$, with $\beta m < 1$ and $n, d < \infty$.

Under this assumption, let $m' > 0$ and $d' > 0$ such that $m + 2m' > 1$, $\beta(m + 2m') < 1$ and $d' \geq d/(m + 2m' - 1)$. Consider $\ell : \mathsf{Z} \times \Theta \to \mathbb{R}$ defined by

$$\ell(z, \theta) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z^\theta |r_\theta(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z^\theta |c_\theta(Z_t)| \right) + g_\theta(z) + d',$$

where $\mathbb{E}_z^\theta$ denotes the conditional expectation with respect to $P_\theta(z, \cdot)$.

Remark 3.6. We implicitly assume that Θ does not include β. However, by letting $\beta \in [0, a]$ and $a \in [0, 1)$, we can incorporate β into Θ. $\beta m < 1$ in assumption 3.8 is then replaced by $am < 1$. All the parametric continuity results of this paper remain true after this change.

Assumption 3.9. $P_\theta(z, \cdot)$ satisfies the Feller property, i.e., $(z, \theta) \mapsto \int h(z') P_\theta(z, \mathrm{d}z')$ is continuous for all bounded continuous function $h : \mathsf{Z} \to \mathbb{R}$.

Assumption 3.10. $(z, \theta) \mapsto c_\theta(z), r_\theta(z), \ell(z, \theta), \int |r_\theta(z')| P_\theta(z, \mathrm{d}z'), \int \ell(z', \theta) P_\theta(z, \mathrm{d}z')$ are continuous.

The following result is a simple extension of proposition 3.1. We omit its proof.

Proposition 3.5. *Under assumptions 3.8–3.10, $(z, \theta) \mapsto \psi^*_\theta(z)$ is continuous.*

Example 3.11. Recall the job search problem of example 2.1. Let $\Theta := [-1, 1] \times A \times B \times C$, where A, B are bounded subsets of $\mathbb{R}_{++}, \mathbb{R}$, respectively, and $C \subset \mathbb{R}$. A typical element of Θ is $\theta = (\rho, \sigma, b, c_0)$. Proposition 3.5 implies that $(\theta, z) \mapsto \psi^*_\theta(z)$ is continuous. The proof is similar to example 3.1.

4. Optimal Policies

This section provides a systematic study of optimal timing of decisions when there are threshold states, and explores the key properties of the optimal policies. We begin in the next section by imposing assumptions under which the optimal policy follows a reservation rule.

4.1. Set Up.

Let the state space Z be a subset of $\mathbb{R}^m$ with $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$, where X is a convex subset of $\mathbb{R}$, Y is a convex subset of $\mathbb{R}^{m-1}$, and the state process $(Z_t)_{t \geq 0}$ takes the form $\{(X_t, Y_t)\}_{t \geq 0}$. Here $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are stochastic processes on X and Y respectively. Assume that $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are conditionally independent, in the sense that conditional on each Y_t, the next period states (X_{t+1}, Y_{t+1}) and X_t are independent. The stochastic kernel $P(z, \mathrm{d}z')$ can then be represented by the conditional distribution of (x', y') on y, denoted as $\mathbb{F}_y(x', y')$, i.e., $P(z, \mathrm{d}z') = P((x, y), \mathrm{d}(x', y')) = \mathrm{d}\mathbb{F}_y(x', y')$.

Assumption 4.1. r is strictly monotone on X. Moreover, for all $y \in \mathsf{Y}$, there exists $x \in \mathsf{X}$ such that $r(x,y) = c(y) + \beta \int v^*(x',y') \, \mathrm{d}\mathbb{F}_y(x',y')$.

With assumption 4.1 in force, we call X_t the *threshold state* and Y_t the *environment state* (or *environment*). We call X the *threshold state space* and Y the *environment space*. Under assumption 4.1, the *reservation rule property* holds: there is a *decision threshold* $\bar{x} : \mathsf{Y} \to \mathsf{X}$ such that when x attains $\bar{x}$, the agent is indifferent between stopping and continuing, i.e., $r(\bar{x}(y),y) = \psi^*(y)$ for all $y \in \mathsf{Y}$. The optimal policy then follows

$$\sigma^*(x,y) = \begin{cases} \mathbb{1}\{x \geq \bar{x}(y)\}, & \text{if } r \text{ is strictly increasing in } x \\ \mathbb{1}\{x \leq \bar{x}(y)\}, & \text{if } r \text{ is strictly decreasing in } x \end{cases} \tag{21}$$

4.2. **Results.** The next few results mainly rely on the implicit function theorem. See the working paper version (Ma and Stachurski, 2017) for proofs.

Proposition 4.1. *Suppose either assumptions of proposition 3.1 or of corollary 3.1 (plus the continuity of r) hold, and that assumption 4.1 holds. Then $\bar{x}$ is continuous.*

Regarding parametric continuity, let $\bar{x}_\theta$ be the decision threshold w.r.t. $\theta \in \Theta$.

Proposition 4.2. *Suppose assumptions of proposition 3.5 and assumption 4.1 hold. Then $(y,\theta) \mapsto \bar{x}_\theta(y)$ is continuous.*

A typical element of Y is $y = (y^1, ..., y^{m-1})$. Given $h : \mathsf{Y} \to \mathbb{R}$ and $l : \mathsf{X} \times \mathsf{Y} \to \mathbb{R}$, we define $D_i h(y) := \partial h(y)/\partial y^i$, $D_i l(x,y) := \partial l(x,y)/\partial y^i$ and $D_x l(x,y) := \partial l(x,y)/\partial x$.

Proposition 4.3. *Let assumptions of proposition 3.4 and assumption 4.1 hold. If r is continuously differentiable on* $\mathrm{int}(\mathsf{Z})$*, then $\bar{x}$ is continuously differentiable on* $\mathrm{int}(\mathsf{Y})$*, with*

$$D_i \bar{x}(y) = -\frac{D_i r(\bar{x}(y),y) - D_i \psi^*(y)}{D_x r(\bar{x}(y),y)} \quad \textit{for all } y \in \mathrm{int}(\mathsf{Y}) \textit{ and } i = 1, ..., m.$$

Since $(x,y) \mapsto r(x,y) - \psi^*(y)$ is the premium of terminating the decision process, which is null at the decision threshold, the instantaneous rate of change of $\bar{x}(y)$ with respect to y^i is equivalent to the ratio of the instantaneous rates of change in the premium in response to y^i and x.

The next result considers monotonicity:

Proposition 4.4. *Let assumptions 2.1, 3.3 and 4.1 hold. If* r *is defined on* X *and is strictly increasing, then* $\bar{x}$ *is increasing.*

5. Applications

Let us now look at applications in some more detail, including how the preceding results can be applied and what their implications are.

5.1. **Job Search III.** Recall the adaptive search model of example 2.2 (see also examples 3.2 and 3.7). By lemma 2.1, the value function satisfies

$$v^*(w, \mu, \gamma) = \max \left\{ \frac{u(w)}{1-\beta}, c_0 + \beta \int v^*(w', \mu', \gamma') f(w'|\mu, \gamma) \, \mathrm{d}w' \right\},$$

while the Jovanovic operator is given by (16). This is a threshold state sequential decision problem, with threshold state $x := w \in \mathbb{R}_{++} =: \mathsf{X}$ and environment $y := (\mu, \gamma) \in \mathbb{R} \times \mathbb{R}_{++} =: \mathsf{Y}$. By the intermediate value theorem, assumption 4.1 holds. Hence, the optimal policy is determined by a reservation wage $\bar{w} : \mathsf{Y} \to \mathbb{R}$ such that when $w = \bar{w}(\mu, \gamma)$, the worker is indifferent between accepting and rejecting the offer. Since all the assumptions of proposition 3.1 hold (see example 3.2), by proposition 4.1, $\bar{w}$ is continuous. Since ψ^* is increasing in μ (see example 3.7), by proposition 4.4, $\bar{w}$ is increasing in μ.

In simulation, we set $\beta = 0.95$, $\gamma_\varepsilon = 1.0$, $\tilde{c}_0 = 0.6$, and consider different levels of risk aversion: $\sigma = 3, 4, 5, 6$. The grid points of (μ, γ) lie in $[-10, 10] \times [10^{-4}, 10]$ with 200 points for the μ grid and 100 points for the γ grid. We obtain ψ^* by linear interpolation, and set ψ^* outside the grid range to its value at the nearby boundary. The integration is computed via Monte Carlo with 1000 draws.[12]

In Figure 2, the reservation wage is increasing in μ, which parallels our analysis. This is natural since a more optimistic agent (higher μ) expects higher offers to be obtained. Moreover, the reservation wage is increasing in γ for given μ of low value, though it is decreasing in γ for given μ of high value. Intuitively, although

[12]Changing the number of Monte Carlo samples, the grid range and grid density produce almost the same results. The same is true for all the later simulations of this paper.

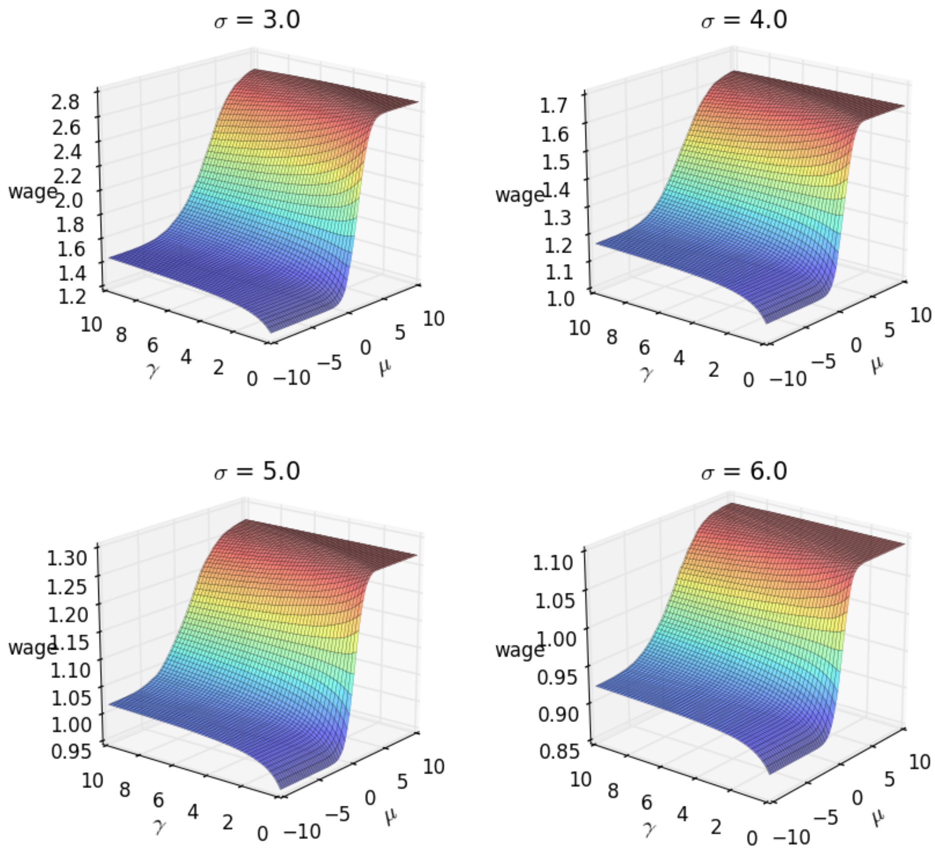

FIGURE 2. The reservation wage

a pessimistic worker (low μ) expects to obtain low offers on average, the downside risks are mitigated because compensation is obtained when the offer is turned down. A higher level of uncertainty (higher γ) provides a better chance of high offers. For an optimistic (high μ) worker, however, the insurance of compensation has less impact. With higher level uncertainty, the risk-averse worker has incentive to enter the labor market at an earlier stage so as to avoid downside risks.

In the computations, we find that computation via value function iteration (VFI) takes more than one week, while continuation value iteration (CVI), being only 2-dimensional, takes 178 seconds.[13]

5.2. **Firm Entry.** Consider a firm entry problem in the style of Fajgelbaum et al. (2016). Each period, an investment cost $f_t > 0$ is observed, where $\{f_t\} \stackrel{\text{IID}}{\sim} h$ with finite mean. The firm then decides whether to incur this cost and enter the market to earn a stochastic dividend x_t or wait and reconsider. Let $x_t = \xi_t + \varepsilon_t^x$, $\{\varepsilon_t^x\} \stackrel{\text{IID}}{\sim} N(0, \gamma_x)$, where ξ_t and ε_t^x are respectively a persistent and a transient component, with

$$\xi_t = \rho \xi_{t-1} + \varepsilon_t^{\xi}, \quad \{\varepsilon_t^{\xi}\} \stackrel{\text{IID}}{\sim} N(0, \gamma_{\xi}).$$

A public signal y_{t+1} is released at the end of each period t, where $y_t = \xi_t + \varepsilon_t^y$, $\{\varepsilon_t^y\} \stackrel{\text{IID}}{\sim} N(0, \gamma_y)$. The firm has prior $\xi \sim N(\mu, \gamma)$ that is updated after observing y' if the firm chooses to wait. The posterior satisfies $\xi | y' \sim N(\mu', \gamma')$, with

$$\gamma' = \left[1/\gamma + \rho^2/(\gamma_{\xi} + \gamma_y)\right]^{-1} \quad \text{and} \quad \mu' = \gamma' \left[\mu/\gamma + \rho y'/(\gamma_{\xi} + \gamma_y)\right]. \tag{22}$$

The firm has utility $u(x) = \left(1 - \mathrm{e}^{-ax}\right)/a$, where $a > 0$ is the coefficient of absolute risk aversion. The value function satisfies

$$v^*(f, \mu, \gamma) = \max \left\{ \mathbb{E}_{\mu,\gamma}[u(x)] - f,\ \beta \int v^*(f', \mu', \gamma') p(f', y' | \mu, \gamma) \, \mathrm{d}(f', y') \right\},$$

where $p(f', y'|\mu, \gamma) = h(f') l(y'|\mu, \gamma)$ with $l(y'|\mu, \gamma) = N(\rho\mu, \rho^2\gamma + \gamma_{\xi} + \gamma_y)$. The exit reward is $r(f, \mu, \gamma) := \mathbb{E}_{\mu,\gamma}[u(x)] - f = \left(1 - \mathrm{e}^{-a\mu + a^2(\gamma + \gamma_x)/2}\right)/a - f$. This is a threshold state problem, with threshold state $x := f \in \mathbb{R}_{++} =: \mathsf{X}$ and environment $y := (\mu, \gamma) \in \mathbb{R} \times \mathbb{R}_{++} =: \mathsf{Y}$. The Jovanovic operator is

$$Q\psi(\mu, \gamma) = \beta \int \max \left\{ \mathbb{E}_{\mu',\gamma'}[u(x')] - f', \psi(\mu', \gamma') \right\} p(f', y' | \mu, \gamma) \, \mathrm{d}(f', y').$$

[13] We terminate the iteration at a level of precision 10^{-4}. The time of CVI is calculated as the average of the four cases ($\sigma = 3, 4, 5, 6$). Moreover, to implement VFI, we set the grid points of w in $[10^{-4}, 10]$ with 50 points, and combine them with the grid points for μ and γ to run the simulation. Indeed, with this additional state, VFI spends more than one week in each of our four cases. All simulations are processed in a standard Python environment on a laptop with a 2.9 GHz Intel Core i7 and 32GB RAM.

Let $n := 1$, $g(\mu, \gamma) := \mathrm{e}^{-\mu + a^2\gamma/2}$, $m := 1$ and $d := 0$. Define ℓ according to (8). We use $\bar{f} : \mathsf{Y} \to \mathbb{R}$ to denote the reservation cost.

Proposition 5.1. *The following statements are true:*

1. *Q is a contraction mapping on $(b_\ell \mathsf{Y}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.*
2. *The value function $v^*(f, \mu, \gamma) = r(f, \mu, \gamma) \vee \psi^*(\mu, \gamma)$, reservation cost $\bar{f}(\mu, \gamma) = \mathbb{E}_{\mu,\gamma}[u(x)] - \psi^*(\mu, \gamma)$, and optimal policy $\sigma^*(f, \mu, \gamma) = \mathbb{1}\{f \leq \bar{f}(\mu, \gamma)\}$.*
3. *ψ^* and $\bar{f}$ are continuous functions.*
4. *If $\rho \geq 0$, then ψ^* is increasing in μ.*

Remark 5.1. Notably, the first three claims of proposition 5.1 have no restriction on the range of ρ values, the autoregression coefficient of $\{\xi_t\}$.

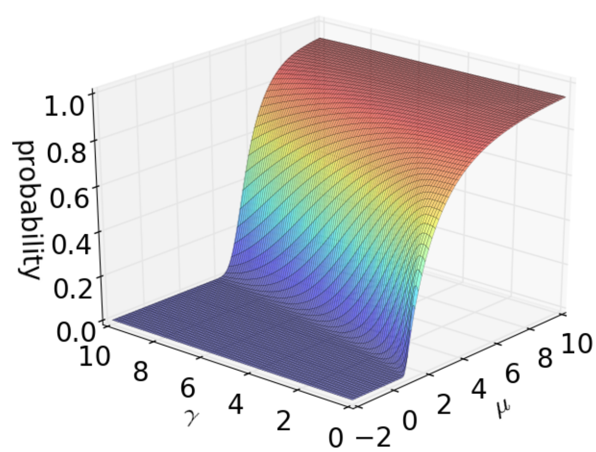

FIGURE 3. The perceived probability of investment

In simulation, we set $\beta = 0.95$, $a = 0.2$, $\gamma_x = 0.1$, $\gamma_y = 0.05$, and $h = LN(0, 0.01)$. Consider $\rho = 1$, $\gamma_\xi = 0$, and let the grid points of (μ, γ) lie in $[-2, 10] \times [10^{-4}, 10]$ with 100 points for each variable. We obtain ψ^* by linear interpolation, and set ψ^* outside the grid range to its value at the nearby boundary. The integration in the operator is computed via Monte Carlo with 1000 draws.

Figure 3 plots the perceived probability of investment, i.e., $\mathbb{P}\left\{f \leq \bar{f}(\mu, \gamma)\right\}$. Naturally, this probability is increasing in μ and decreasing in γ since a more optimistic firm (higher μ) is more likely to invest, and with higher level uncertainty (higher γ) the risk averse firm prefers to delay investment so as to avoid downside risks.[14] In terms of computation time, VFI takes more than one week, while CVI takes only 921 seconds.[15]

5.3. **Job Search IV.** We consider another job search problem. The setup is as in example 2.1, except that the state process follows

$$w_t = \eta_t + \theta_t \xi_t \quad \text{and} \quad \ln \theta_t = \rho \ln \theta_{t-1} + \ln u_t, \tag{23}$$

where $\rho \in [-1, 1]$, $\{\xi_t\} \overset{\text{IID}}{\sim} h$ and $\{\eta_t\} \overset{\text{IID}}{\sim} v$ with finite first moments, and $\{u_t\} \overset{\text{IID}}{\sim} LN(0, \gamma_u)$. Moreover, $\{\xi_t\}$, $\{\eta_t\}$ and $\{u_t\}$ are independent, and $\{\theta_t\}$ is independent of $\{\xi_t\}$ and $\{\eta_t\}$. Similar settings as (23) appear in many search-theoretic and real options studies (see e.g., Gomes et al. (2001), Low et al. (2010), Chatterjee and Eyigungor (2012), Bagger et al. (2014), Kellogg (2014)).

We set $h = LN(0, \gamma_\xi)$ and $v = LN(\mu_\eta, \gamma_\eta)$. In this case, θ_t and ξ_t are persistent and transitory components of income, respectively. η_t can be interpreted as social security, gifts, etc. Recall the utility of the agent defined by (11), the unemployment compensation $\tilde{c}_0 > 0$ and $c_0 := u(\tilde{c}_0)$. The value function of the agent satisfies

$$v^*(w, \theta) = \max\left\{\frac{u(w)}{1-\beta}, c_0 + \beta \int v^*(w', \theta') f(\theta'|\theta) h(\xi') v(\eta') \, \mathrm{d}(\theta', \xi', \eta')\right\},$$

where $w' = \eta' + \theta'\xi'$ and $f(\theta'|\theta) = LN(\rho \ln \theta, \gamma_u)$. The Jovanovic operator is

$$Q\psi(\theta) = c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\theta')\right\} f(\theta'|\theta) h(\xi') v(\eta') \, \mathrm{d}(\theta', \xi', \eta').$$

This is another threshold state problem, with threshold state $x := w \in \mathbb{R}_{++} =: \mathsf{X}$ and environment $y := \theta \in \mathbb{R}_{++} =: \mathsf{Y}$. Let $\bar{w}$ be the reservation wage. Recall the

[14] This result parallels propositions 1–2 of Fajgelbaum et al. (2016).

[15] We terminate the iteration at a level of precision 10^{-4}. For CVI we run the simulation 5 times and calculate the average time. Moreover, to implement VFI, we set the grid points of f in $[10^{-4}, 10]$ with 50 points, and combine them with the grid points for μ and γ to run the simulation.

relative risk aversion coefficient δ in (11). Consider, for example, $\delta \geq 0$, $\delta \neq 1$ and $\rho \in (-1, 1)$. Fix $n \in \mathbb{N}_0$ such that $\beta e^{\rho^{2n}\sigma} < 1$, where $\sigma := (1-\delta)^2 \gamma_u$. Let $g(\theta) := \theta^{(1-\delta)\rho^n} + \theta^{-(1-\delta)\rho^n}$ and $m := d := e^{\rho^{2n}\sigma}$.

Proposition 5.2. *If $\rho \in (-1, 1)$, then the following statements hold:*

1. *Q is a contraction mapping on $(b_\ell \mathsf{Y}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.*
2. *The value function $v^*(w, \theta) = \frac{w^{1-\delta}}{(1-\beta)(1-\delta)} \vee \psi^*(\theta)$, reservation wage $\bar{w}(\theta) = [(1-\beta)(1-\delta)\psi^*(\theta)]^{\frac{1}{1-\delta}}$, and optimal policy $\sigma^*(w, \theta) = \mathbb{1}\{w \geq \bar{w}(\theta)\}$.*
3. *ψ^* and $\bar{w}$ are continuously differentiable, and $\bar{w}'(\theta) = (1-\beta)\bar{w}(\theta)^\delta \psi^{*\prime}(\theta)$.*
4. *If $\rho \geq 0$, then ψ^* and $\bar{w}$ are increasing in θ.*

Remark 5.2. If $\beta e^{(1-\delta)^2 \gamma_u / 2} < 1$, claims 1–3 of proposition 5.2 remain true for $|\rho| = 1$, and claim 4 is true for $\rho = 1$. Moreover, the case $\delta = 1$ can be treated similarly.[16]

Remark 5.3. Since the terminating premium is 0 at the reservation wage, the overall effect of changes in w and θ cancel out. Hence, the rate of change of $\bar{w}$ w.r.t. θ equals the ratio of the marginal premiums of θ and w at the decision threshold, denoted respectively by $\psi^{*\prime}(\theta)$ and $\bar{w}(\theta)^{-\delta}/(1-\beta)$, as documented by claim 3.

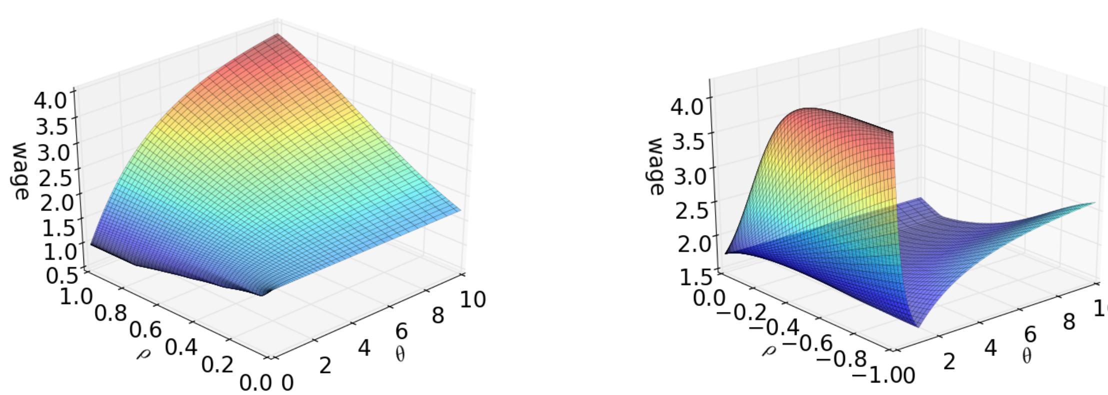

FIGURE 4. The reservation wage

[16] See the working paper version (Ma and Stachurski, 2017) for a detailed proof of this case.

In simulation, we set $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\delta = 2.5$, $\mu_\eta = 0$, $\gamma_\eta = 10^{-6}$, $\gamma_\xi = 5 \times 10^{-4}$, $\gamma_u = 10^{-4}$, and consider parametric class problems of ρ. $\rho \in [0, 1]$ and $\rho \in [-1, 0]$ is treated separately, with 100 grid points in each case. Grid points of θ lie in $[10^{-4}, 10]$ with 200 points, and are scaled to be more dense when θ is smaller. We obtain ψ^* by linear interpolation, and set ψ^* outside the grid range to its value at the boundary. Integration is computed via Monte Carlo with 1000 draws.

When $\rho = 0$, $\{\theta_t\} \overset{\text{IID}}{\sim} LN(0, \gamma_u)$, in which case each realized θ will be forgotten in future stages. As a result, the continuation value is independent of θ, yielding a reservation wage parallel to the θ-axis, as shown in figure 4. When $\rho > 0$, the reservation wage is increasing in θ, which is natural since a higher θ implies a better current situation. Further, since a higher degree of income persistence (higher ρ) prolongs the mean-reverting process, the reservation wage tends to decrease in ρ in bad states ($\theta < 1$) and increase in ρ in good states ($\theta > 1$). When $\rho < 0$, the reservation wage decreases in θ initially and then starts to increase in θ afterwards. Intuitively, a low or a high θ is more favorable than a medium level θ since it allows the agent to take advantage of the countercyclical patterns.

TABLE 1. Time Taken (seconds) under Different Grid Sizes

Time	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9
CVI	0.300	0.295	0.294	0.453	0.450	0.448	0.620	0.618	0.622
VFI	277.33	364.68	527.83	355.92	558.06	870.41	451.06	795.90	1191.43

Note: We set $\rho = 0.75$, $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\delta = 2.5$, $\mu_\eta = 0$, $\gamma_\eta = 10^{-6}$, $\gamma_\xi = 5 \times 10^{-4}$, and $\gamma_u = 10^{-4}$. The grid points for (θ, w) lie in $[10^{-4}, 10]^2$, and the grid sizes for (θ, w) in each test are **Test 1:** $(200, 200)$, **Test 2:** $(200, 300)$, **Test 3:** $(200, 400)$, **Test 4:** $(300, 200)$, **Test 5:** $(300, 300)$, **Test 6:** $(300, 400)$, **Test 7:** $(400, 200)$, **Test 8:** $(400, 300)$, and **Test 9:** $(400, 400)$. For both CVI and VFI, we terminate the iteration at a precision level 10^{-4}. We run the simulation 10 times for CVI, 5 times for VFI, and calculate the average time (in seconds).

With two state variables in the current setting, we can provide a more detailed numerical comparison. Table 1 shows how CVI and VFI perform under different grid sizes. Among test 1–9, CVI is averagely 1317 times faster than VFI. In the best case, CVI is 1943 times faster than VFI (test 6), while in the worst case, CVI is 728 times faster (test 7). Moreover, CVI outperforms VFI more obviously as we increase the grid size: although the speed of VFI drops significantly with increased w grid points, the speed of CVI is not affected (see, e.g., test 1–3); As we increase the grid size of both w and π, there is a slight decrease in the speed of CVI, however, the speed of VFI drops exponentially (see, e.g., test 1, 5 and 9).

6. Extensions

6.1. **Repeated Sequential Decisions.** In many economic models, the choice to stop is not permanent. For example, when a worker accepts a job offer, the resulting job might only be temporary (see, e.g., Ljungqvist and Sargent (2008), Chatterjee and Rossi-Hansberg (2012), Lise (2013), Moscarini and Postel-Vinay (2013), Bagger et al. (2014)). Another example is sovereign default (see, e.g., Arellano (2008), Chatterjee and Eyigungor (2012), Mendoza and Yue (2012), Hatchondo et al. (2016)), where default on sovereign debt leads to a period of exclusion from international financial markets. The exclusion is not permanent, however. With positive probability, the country exits autarky and begins borrowing again.

To put this type of problem in a general setting, suppose that, at date t, an agent is either *active* or *passive*. When active, the agent observes Z_t and chooses whether to continue or to exit. Continuation results in a current reward $c(Z_t)$ and the agent remains active at $t+1$. Exit results in a current reward $s(Z_t)$ and transition to the passive state. From there the agent has no action available, but will return to the active state at $t+1$ and all subsequent period with probability α.

Assumption 6.1. There exist a $\mathscr{Z}$-measurable function $g : \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0$, $m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

(1) $\max \left\{ \int |s(z')| P^n(z, \mathrm{d}z'), \int |c(z')| P^n(z, \mathrm{d}z') \right\} \leq g(z)$;
(2) $\int g(z') P(z, \mathrm{d}z') \leq m g(z) + d$.

Let $v^*(z)$ and $r^*(z)$ be the maximal discounted value starting at $z \in \mathsf{Z}$ in the active and passive state respectively. The following principle of optimality holds.[17]

[17] See the working paper version (Ma and Stachurski, 2017) for the proof of lemma 6.1.

Lemma 6.1. *Under assumption 6.1, v^* and r^* satisfy*

$$v^*(z) = \max\left\{ r^*(z), c(z) + \beta \int v^*(z')P(z, \mathrm{d}z') \right\},$$

$$r^*(z) = s(z) + \alpha\beta \int v^*(z')P(z, \mathrm{d}z') + (1-\alpha)\beta \int r^*(z')P(z, \mathrm{d}z').$$

With $\psi^* := c + \beta P v^*$ we can write $v^* = r^* \vee \psi^*$. By lemma 6.1, we can view ψ^* and r^* as solutions to the functional equations

$$\psi = c + \beta P(r \vee \psi) \quad \text{and} \quad r = s + \alpha\beta P(r \vee \psi) + (1-\alpha)\beta P r. \tag{24}$$

Choose $m', d' > 0$ such that $m + 2m' > 1$, $\beta(m + 2m') < 1$ and $d' \geq \frac{d}{m+2m'-1}$. Consider the weight function $\kappa : \mathsf{Z} \to \mathbb{R}_+$ defined by

$$\kappa(z) := m' \sum_{t=0}^{n-1} \mathbb{E}_z \left[|s(Z_t)| + |c(Z_t)| \right] + g(z) + d'$$

and the product space $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$, where ρ_κ is a metric on $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ defined by

$$\rho_\kappa((\psi, r), (\psi', r')) = \|\psi - \psi'\|_\kappa \vee \|r - r'\|_\kappa.$$

With this metric, $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ inherits the completeness of $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$. Now define the Jovanovic operator L on $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ by

$$L \begin{pmatrix} \psi \\ r \end{pmatrix} = \begin{pmatrix} c + \beta P(r \vee \psi) \\ s + \alpha\beta P(r \vee \psi) + (1-\alpha)\beta P r \end{pmatrix}.$$

Theorem 6.1. *Under assumption 6.1, the following statements hold:*

1. *L is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ of modulus $\beta(m + 2m')$.*
2. *The unique fixed point of L in $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ is $h^* := (\psi^*, r^*)$.*

6.2. **Sequential Decision with More Choices.** In many economic problems, agents face multiple choices in the sequential decision process (see, e.g., Crawford and Shum (2005), Cooper et al. (2007), Vereshchagina and Hopenhayn (2009), Low et al. (2010), Moscarini and Postel-Vinay (2013)). A standard example is on-the-job search, where an employee can choose from quitting the job market, staying in the current job, or searching for a new job (see, e.g., Jovanovic (1987), Bull and

Jovanovic (1988), Gomes et al. (2001)). A key characteristic of this type of problem is that different choices lead to different transition probabilities.

To treat this type of problem generally, suppose that in period t, the agent observes Z_t and makes decisions among N alternatives. A selection of alternative i results in a current reward $r_i(Z_t)$ along with a stochastic kernel P_i. We assume the following.

Assumption 6.2. There exist a $\mathscr{Z}$-measurable function $g : \mathsf{Z} \to \mathbb{R}_+$ and constants $m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$ and $i, j = 1, ..., N$,

$$\int |r_i(z')| P_j(z, \mathrm{d}z') \leq g(z) \quad \text{and} \quad \int g(z') P_i(z, \mathrm{d}z') \leq m g(z) + d.$$

Let v^* be the value function and ψ_i^* be the expected value of alternative i.

Lemma 6.2. *Under assumption 6.2, v^* and $(\psi_i^*)_{i=1}^N$ satisfy*[18]

$$v^*(z) = \max\{\psi_1^*(z), ..., \psi_N^*(z)\} \;\; \textit{and} \;\; \psi_i^*(z) = r_i(z) + \beta \int v^*(z') P_i(z, \mathrm{d}z').$$

By lemma 6.2, ψ_i^* satisfies the functional equation

$$\psi_i^*(z) = r_i(z) + \beta \int \max\{\psi_1^*(z'), ..., \psi_N^*(z')\} P_i(z, \mathrm{d}z'), \quad i = 1, ..., N.$$

Choose $m', d' \in \mathbb{R}_{++}$ such that $\beta(Nm' + m) < 1$ and $d' \geq \frac{d}{Nm'+m-1}$. Consider the weight function $k : \mathsf{Z} \to \mathbb{R}_+$ defined by

$$k(z) := m' \sum_{i=1}^N |r_i(z)| + g(z) + d'.$$

We can show that the product space $\left(\times_{i=1}^N (b_k \mathsf{Z}), \rho_k\right)$ is a complete metric space, where ρ_k is defined by $\rho_k(\psi, \tilde{\psi}) = \vee_{i=1}^N \|\psi_i - \tilde{\psi}_i\|_k$ for all $\psi = (\psi_1, ..., \psi_N)$, $\tilde{\psi} = (\tilde{\psi}_1, ..., \tilde{\psi}_N) \in \times_{i=1}^N (b_k \mathsf{Z})$. The Jovanovic operator on $\left(\times_{i=1}^N (b_k \mathsf{Z}), \rho_k\right)$ is defined by

$$T\psi = T \begin{pmatrix} \psi_1 \\ ... \\ \psi_N \end{pmatrix} = \begin{pmatrix} r_1 + \beta P_1(\psi_1 \vee ... \vee \psi_N) \\ ...... \\ r_N + \beta P_N(\psi_1 \vee ... \vee \psi_N) \end{pmatrix}.$$

The next result is a simple extension of theorem 6.1 and we omit its proof.

[18] We omit the proof, since it is similar to the proof of lemma 6.1.

Theorem 6.2. *Under assumption 6.2, the following statements hold:*

1. *T is a contraction mapping on $\left(\times_{i=1}^{N}(b_k \mathsf{Z}), \rho_k\right)$ of modulus $\beta(Nm' + m)$.*
2. *The unique fixed point of T in $\left(\times_{i=1}^{N}(b_k \mathsf{Z}), \rho_k\right)$ is $\psi^* := (\psi_1^*, ..., \psi_N^*)$.*

7. Conclusion

A general theory of optimal timing of decisions is developed based around continuation values and operators that act on them. Optimality results are provided under general settings, with bounded or unbounded rewards. One advantage of this theory is that, for a range of problems, the continuation value function exists in a lower dimensional space than the value function, which simplifies challenging problems associated with unbounded rewards, continuity and differentiability arguments, parametric monotonicity results, etc. Moreover, lower state dimension can radically enhance numerical efficiency. As another advantage, continuation values are typically smoother than value functions, making them easier to approximate numerically and establish sharp analytical results. This has been exploited to develop a detailed set of continuity, monotonicity and differentiability results.

Appendix A: Some Lemmas

Lemma 7.1. *Under assumption 2.1, there exist $a_1, a_2 \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,*

(1) $|v^*(z)| \leq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + a_1 g(z) + a_2$.
(2) $|\psi^*(z)| \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z |c(Z_t)| + a_1 g(z) + a_2$.

Proof. Without loss of generality, we assume $m \neq 1$ in assumption 2.1. Note that $\mathbb{E}_z |r(Z_n)| \leq g(z)$, $\mathbb{E}_z |c(Z_n)| \leq g(z)$ and $\mathbb{E}_z g(Z_1) \leq mg(z) + d$ for all $z \in \mathsf{Z}$. For all $t \geq 1$, by the Markov property (see, e.g., Meyn and Tweedie (2012), section 3.4.3),

$$\mathbb{E}_z g(Z_t) = \mathbb{E}_z \left[\mathbb{E}_z \left(g(Z_t) | \mathscr{F}_{t-1}\right)\right] = \mathbb{E}_z \left(\mathbb{E}_{Z_{t-1}} g(Z_1)\right)$$
$$\leq \mathbb{E}_z \left(mg(Z_{t-1}) + d\right) = m\mathbb{E}_z g(Z_{t-1}) + d.$$

Induction shows that for all $t \geq 0$,

$$\mathbb{E}_z g(Z_t) \leq m^t g(z) + \frac{1-m^t}{1-m} d. \tag{25}$$

Moreover, for all $t \geq n$, applying the Markov property again shows that

$$\mathbb{E}_z |r(Z_t)| = \mathbb{E}_z \left[\mathbb{E}_z \left(|r(Z_t)| | \mathscr{F}_{t-n} \right) \right] = \mathbb{E}_z \left(\mathbb{E}_{Z_{t-n}} |r(Z_n)| \right) \leq \mathbb{E}_z g(Z_{t-n}).$$

By (25), for all $t \geq n$, we have

$$\mathbb{E}_z |r(Z_t)| \leq m^{t-n} g(z) + \frac{1-m^{t-n}}{1-m} d. \tag{26}$$

Similarly, for all $t \geq n$, we have

$$\mathbb{E}_z |c(Z_t)| \leq \mathbb{E}_z g(Z_{t-n}) \leq m^{t-n} g(z) + \frac{1-m^{t-n}}{1-m} d. \tag{27}$$

Based on (25)–(27), we can show that

$$\begin{aligned} S(z) &:= \sum_{t \geq 1} \beta^t \mathbb{E}_z \left[|r(Z_t)| + |c(Z_t)| \right] \\ &\leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z [|r(Z_t)| + |c(Z_t)|] + \frac{2\beta^n}{1-\beta m} g(z) + \frac{2\beta^{n+1} d}{(1-\beta m)(1-\beta)}. \end{aligned} \tag{28}$$

Since $|v^*| \leq |r| + |c| + S$ and $|\psi^*| \leq |c| + S$, the two claims hold by letting $a_1 := \frac{2\beta^n}{1-\beta m}$ and $a_2 := \frac{2\beta^{n+1} d}{(1-\beta m)(1-\beta)}$. This concludes the proof. □

Let $(X, \mathcal{X}, \nu)$ and $(Y, \mathcal{Y}, u)$ two measure spaces. Lemma 7.2 below can be shown by the Fatou's lemma. The idea of proof is similar to proposition 3.1 below.

Lemma 7.2. *Let $p : Y \times X \to \mathbb{R}$ be a measurable map that is continuous in x. If there exists a measurable map $q : Y \times X \to \mathbb{R}_+$ that is continuous in x with $q(y, x) \geq |p(y, x)|$ for all $(y, x) \in Y \times X$, and that $x \mapsto \int q(y, x) u(\mathrm{d}y)$ is continuous, then the mapping $x \mapsto \int p(y, x) u(\mathrm{d}y)$ is continuous.*

APPENDIX B : MAIN PROOFS

7.1. Proof of Section 2 Results.

Proof of lemma 2.1. By theorem 1.11 of Peskir and Shiryaev (2006), it suffices to show: $\mathbb{E}_z \left(\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty$ for all $z \in \mathsf{Z}$. This is true since

$$\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \leq \sum_{t \geq 0} \beta^t [|r(Z_t)| + |c(Z_t)|] \tag{29}$$

with probability one, and by lemma 7.1 (see (28) in appendix A) and the monotone convergence theorem, the right hand side of (29) is $\mathbb{P}_z$-integrable for all $z \in \mathsf{Z}$. □

Proof of theorem 2.1. To prove claim 1, based on the weighted contraction mapping theorem (see, e.g., Boyd (1990), section 3), it suffices to verify: (a) Q is monotone, i.e., $Q\psi \leq Q\phi$ if $\psi, \phi \in b_\ell \mathsf{Z}$ and $\psi \leq \phi$; (b) $Q0 \in b_\ell \mathsf{Z}$ and $Q\psi$ is $\mathscr{Z}$-measurable for all $\psi \in b_\ell \mathsf{Z}$; and (c) $Q(\psi + a\ell) \leq Q\psi + a\beta(m + 2m')\ell$ for all $a \in \mathbb{R}_+$ and $\psi \in b_\ell \mathsf{Z}$. Obviously, condition (a) holds. By (7)–(8), we have

$$\frac{|(Q0)(z)|}{\ell(z)} \leq \frac{|c(z)|}{\ell(z)} + \beta \int \frac{|r(z')|}{\ell(z)} P(z, dz') \leq (1 + \beta)/m' < \infty$$

for all $z \in \mathsf{Z}$, so $\|Q0\|_\ell < \infty$. The measurability of $Q\psi$ follows immediately from our primitive assumptions. Hence, condition (b) holds. By the Markov property (see, e.g., Meyn and Tweedie (2012), section 3.4.3), we have

$$\int \mathbb{E}_{z'} |r(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |r(Z_{t+1})| \;\text{ and }\; \int \mathbb{E}_{z'} |c(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |c(Z_{t+1})|.$$

Let $h(z) := \sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)|$, then we have

$$\int h(z') P(z, \mathrm{d}z') = \sum_{t=2}^{n} \mathbb{E}_z |r(Z_t)| + \sum_{t=1}^{n} \mathbb{E}_z |c(Z_t)|. \tag{30}$$

By the assumptions on m' and d', we have $m + 2m' > 1$ and $(d + d')/(m + 2m') \leq d'$. Assumption 2.1 and (30) then imply that

$$\begin{aligned}\int \ell(z')P(z, dz') &= m' \left(\sum_{t=2}^{n} \mathbb{E}_z |r(Z_t)| + \sum_{t=1}^{n} \mathbb{E}_z |c(Z_t)| \right) + \int g(z')P(z, dz') + d' \\ &\leq m' \left(\sum_{t=2}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=1}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + (m + 2m')g(z) + d + d' \\ &\leq (m + 2m') \left(\frac{m'}{m + 2m'} h(z) + g(z) + \frac{d + d'}{m + 2m'} \right) \leq (m + 2m')\ell(z).\end{aligned}$$

Hence, for all $\psi \in b_\ell \mathsf{Z}$, $a \in \mathbb{R}_+$ and $z \in \mathsf{Z}$, we have

$$\begin{aligned}Q(\psi + a\ell)(z) &= c(z) + \beta \int \max \left\{ r(z'), \psi(z') + a\ell(z') \right\} P(z, dz') \\ &\leq c(z) + \beta \int \max \left\{ r(z'), \psi(z') \right\} P(z, dz') + a\beta \int \ell(z')P(z, dz') \\ &\leq Q\psi(z) + a\beta(m + 2m')\ell(z).\end{aligned}$$

So condition (c) holds. Claim 1 is verified.

Regarding claim 2, we have shown that ψ^* is a fixed point of Q under assumption 2.1 (see lemma 2.1 and (6)). Moreover, from lemma 7.1 we know that $\psi^* \in b_\ell \mathsf{Z}$. Hence, ψ^* must coincide with the unique fixed point of Q under $b_\ell \mathsf{Z}$.

Finally, by theorem 1.11 of Peskir and Shiryaev (2006), we can show that $\tilde{\tau} := \inf\{t \geq 0 : v^*(Z_t) = r(Z_t)\}$ is an optimal stopping time. Claim 3 then follows from the definition of the optimal policy and the fact that $v^* = r \vee \psi^*$. □

7.2. **Proof of Section 3 Results.**

Proof of proposition 3.1. Let $b_\ell c\mathsf{Z}$ be the set of continuous functions in $b_\ell \mathsf{Z}$. Since ℓ is continuous by assumption 3.2, $b_\ell c\mathsf{Z}$ is a closed subset of $b_\ell \mathsf{Z}$ (see e.g., Boyd (1990), section 3). To show the continuity of ψ^*, it suffices to verify that $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). For fixed $\psi \in b_\ell c\mathsf{Z}$, let $h(z) := \max\{r(z), \psi(z)\}$, then there exists $G \in \mathbb{R}_+$ such that $|h(z)| \leq |r(z)| + G\ell(z) =: \tilde{h}(z)$. By assumption 3.2, $z \mapsto \tilde{h}(z) \pm h(z)$ are nonnegative and continuous. By assumption 3.1 and the generalized Fatou's lemma of Feinberg et al. (2014)

(theorem 1.1), for all sequence $(z_m)_{m\geq 0}$ of Z such that $z_m \to z \in \mathsf{Z}$, we have

$$\int \left(\tilde{h}(z') \pm h(z')\right) P(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \int \left(\tilde{h}(z') \pm h(z')\right) P(z_m, \mathrm{d}z').$$

Since $\lim_{m\to\infty} \int \tilde{h}(z')P(z_m, \mathrm{d}z') = \int \tilde{h}(z')P(z, \mathrm{d}z')$ by assumption 3.2, we have

$$\pm \int h(z')P(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \left(\pm \int h(z')P(z_m, \mathrm{d}z')\right),$$

where we have used the fact that for given sequences $(a_m)_{m\geq 0}$ and $(b_m)_{m\geq 0}$ of $\mathbb{R}$ with $\lim_{m\to\infty} a_m$ exists, we have: $\liminf_{m\to\infty}(a_m + b_m) = \lim_{m\to\infty} a_m + \liminf_{m\to\infty} b_m$. Hence,

$$\limsup_{m\to\infty} \int h(z')P(z_m, \mathrm{d}z') \leq \int h(z')P(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \int h(z')P(z_m, \mathrm{d}z'), \tag{31}$$

i.e., $z \mapsto \int h(z')P(z, \mathrm{d}z')$ is continuous. Since c is continuous by assumption, $Q\psi \in b_\ell c\mathsf{Z}$. Hence, $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$, and ψ^* is continuous, as was to be shown. □

Recall μ, μ_i and k_j defined in the beginning of section 3.3. The next lemma holds.

Lemma 7.3. *Suppose assumption 2.1 holds, and, for $i = 1, ..., m$ and $j = 1, 2$*

(1) P has a density representation f such that $D_i f(z'|z)$ exists, $\forall (z, z') \in \operatorname{int}(\mathsf{Z}) \times \mathsf{Z}$.
(2) For all $z_0 \in \operatorname{int}(\mathsf{Z})$, there exists $\delta > 0$, such that

$$\int |k_j(z')| \sup_{z^i \in \bar{B}_\delta(z_0^i)} \left|D_i f(z'|z)\right| \mathrm{d}z' < \infty \qquad (z^{-i} = z_0^{-i}).$$

Then: $D_i\mu(z) = \mu_i(z)$ for all $z \in \operatorname{int}(\mathsf{Z})$ and $i = 1, ..., m$.

Proof of lemma 7.3. For all $z_0 \in \operatorname{int}(\mathsf{Z})$, let $\{z_n\}$ be an arbitrary sequence of $\operatorname{int}(\mathsf{Z})$ such that $z_n^i \to z_0^i$, $z_n^i \neq z_0^i$ and $z_n^{-i} = z_0^{-i}$ for all $n \in \mathbb{N}$. For the $\delta > 0$ given by (2), there exists $N \in \mathbb{N}$ such that $z_n^i \in \bar{B}_\delta(z_0^i)$ for all $n \geq N$. Holding $z^{-i} = z_0^{-i}$, by the mean value theorem, there exists $\xi^i(z', z_n, z_0) \in \bar{B}_\delta(z_0^i)$ such that

$$|\triangle^i(z', z_n, z_0)| := \left|\frac{f(z'|z_n) - f(z'|z_0)}{z_n^i - z_0^i}\right| = \left|D_i f(z'|z)_{z^i = \xi^i(z', z_n, z_0)}\right| \leq \sup_{z^i \in \bar{B}_\delta(z_0^i)} \left|D_i f(z'|z)\right|.$$

Since in addition $|\psi^*| \leq G\ell$ for some $G \in \mathbb{R}_+$, we have: for all $n \geq N$,

(a) $\left|\max\{r(z'), \psi^*(z')\}\triangle^i(z', z_n, z_0)\right| \leq (|r(z')| + G\ell(z')) \sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)|,$

(b) $\int (|r(z')| + G\ell(z')) \sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| \, dz' < \infty$, and

(c) $\max\{r(z'), \psi^*(z')\} \triangle^i(z', z_n, z_0) \to \max\{r(z'), \psi^*(z')\} D_i f(z'|z_0)$ as $n \to \infty$,

where (b) follows from condition (2). By the dominated convergence theorem,

$$\frac{\mu(z_n) - \mu(z_0)}{z_n^i - z_0^i} = \int \max\{r(z'), \psi^*(z')\} \triangle^i(z', z_n, z_0) \, \mathrm{d}z'$$
$$\to \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z_0) \, \mathrm{d}z' = \mu_i(z_0).$$

Hence, $D_i \mu(z_0) = \mu_i(z_0)$, and the claim of the lemma is verified. □

7.3. **Proof of Section 5 Results.**

Proof of proposition 5.1. Regarding claims 1–2, the exit reward satisfies

$$\left| r(f', \mu', \gamma') \right| \leq 1/a + \left(\mathrm{e}^{a^2 \gamma_x / 2} / a \right) \cdot \mathrm{e}^{-a\mu' + a^2 \gamma' / 2} + f'. \tag{32}$$

Using (22), we can show that

$$\int \mathrm{e}^{-a\mu' + a^2 \gamma' / 2} P(z, \mathrm{d}z') = \int \mathrm{e}^{-a\mu' + a^2 \gamma' / 2} l(y'|\mu, \gamma) \, \mathrm{d}y' = \mathrm{e}^{-a\mu + a^2 \gamma / 2}. \tag{33}$$

Let μ_f denote the mean of $\{f_t\}$. Combining (32)–(33) yields

$$\int \left| r(f', \mu', \gamma') \right| P(z, \mathrm{d}z') \leq (1/a + \mu_f) + \left(\mathrm{e}^{a^2 \gamma_x / 2} / a \right) \cdot \mathrm{e}^{-a\mu + a^2 \gamma / 2}. \tag{34}$$

To verify assumption 2.1, we can let $n := 1$ and $g(\mu, \gamma) := \mathrm{e}^{-a\mu + a^2 \gamma / 2}$ (recall footnote 7). Then assumption 2.1 holds for $m := 1$ and $d := 0$ since (33) implies that $\int g(\mu', \gamma') P(z, \mathrm{d}z') = g(\mu, \gamma)$. Moreover, the intermediate value theorem shows that assumption 4.1 holds. By theorem 2.1 and (21) (section 4.1), claims 1–2 hold.

Regarding claim 3, note that for all bounded continuous function $\tilde{f} : \mathsf{Z} \to \mathbb{R}$, we have: $\int \tilde{f}(z') P(z, \mathrm{d}z') = \int \tilde{f}(f', \mu', \gamma') h(f') l(y'|\mu, \gamma) \, \mathrm{d}(f', y')$. By (22) and lemma 7.2, this function is continuous in (μ, γ), so assumption 3.1 holds. The exit reward r is continuous. By (22), both sides of (32) are continuous in (μ, γ). By (33) or (34), the conditional expectation of the right side of (32) is continuous in (μ, γ). Lemma 7.2 implies that $(\mu, \gamma) \mapsto \mathbb{E}_{\mu,\gamma} |r(Z_1)|$ is continuous. Since in addition g

is continuous and $g(\mu,\gamma) = \mathbb{E}_{\mu,\gamma} g(\mu',\gamma')$ by (33), assumption 3.2 holds. Claim 3 then follows from propositions 3.1 and 4.1.

For claim 4, if $\rho \geq 0$, then l is stochastically increasing in μ. Since r is increasing in μ, by (22), $P(r \vee \psi)$ is increasing in μ for all $\psi \in b_\ell \mathsf{Y}$ increasing in μ, i.e., assumption 3.3 holds. By proposition 3.2, ψ^* is increasing in μ, and claim 4 is verified. □

Proof of proposition 5.2. Regarding claims 1–2, since

$$w'^{1-\delta} = \left(\eta' + \theta'\xi'\right)^{1-\delta} \leq 2\left(\eta'^{1-\delta} + \theta'^{1-\delta}\xi'^{1-\delta}\right), \tag{35}$$

we have

$$\begin{aligned}\int w'^{1-\delta} P(z, \mathrm{d}z') &\leq 2\int \eta'^{1-\delta} v(\eta')\,\mathrm{d}\eta' + 2\int \xi'^{1-\delta} h(\xi')\,\mathrm{d}\xi' \cdot \int \theta'^{1-\delta} f(\theta'|\theta)\,\mathrm{d}\theta' \\ &= 2\mathrm{e}^{(1-\delta)\mu_\eta + (1-\delta)^2\gamma_\eta/2} + 2\mathrm{e}^{(1-\delta)^2(\gamma_\xi+\gamma_u)/2} \cdot \theta^{(1-\delta)\rho}. \end{aligned} \tag{36}$$

Induction shows that for some constants $a_1^{(t)}, a_2^{(t)} > 0$ and all $t \in \mathbb{N}$,

$$\int w'^{1-\delta} P^t(z, \mathrm{d}z') \leq a_1^{(t)} + a_2^{(t)}\theta^{(1-\delta)\rho^t} \leq a_1^{(t)} + a_2^{(t)}\left(\theta^{(1-\delta)\rho^t} + \theta^{-(1-\delta)\rho^t}\right). \tag{37}$$

Define n, g, m and d as in section 5.3, where $g(\theta) := \theta^{(1-\delta)\rho^n} + \theta^{-(1-\delta)\rho^n}$. Then

$$\begin{aligned}\int g(\theta') f(\theta'|\theta)\,\mathrm{d}\theta' &= \left(\mathrm{e}^{(1-\delta)\rho^{n+1}\ln\theta} + \mathrm{e}^{-(1-\delta)\rho^{n+1}\ln\theta}\right)\mathrm{e}^{(1-\delta)^2\rho^{2n}\gamma_u/2} \\ &\leq \left(\mathrm{e}^{(1-\delta)\rho^{n}\ln\theta} + \mathrm{e}^{-(1-\delta)\rho^{n}\ln\theta} + 1\right)\mathrm{e}^{(1-\delta)^2\rho^{2n}\gamma_u/2} \leq mg(\theta) + d. \end{aligned} \tag{38}$$

Hence, assumption 2.1 holds. Assumption 4.1 holds by the intermediate value theorem. Claims 1–2 then follow from theorem 2.1 and (21) (section 4.1).

Regarding claim 3, note that P has a density representation: for all $z \in \mathsf{Z}$ and $B \in \mathscr{Z}$, we have $P(z, B) = \int \mathbb{1}\{(\eta' + \xi'\theta', \theta') \in B\}\, v(\eta')h(\xi')f(\theta'|\theta)\,\mathrm{d}(\eta', \xi', \theta')$. Moreover, it is straightforward to show that $\theta \mapsto f(\theta'|\theta)$ is twice differentiable for all θ', that $(\theta, \theta') \mapsto \partial f(\theta'|\theta)/\partial\theta$ is continuous, and that

$$\partial^2 f(\theta'|\theta)/\partial\theta^2 = 0 \;\text{ has two solutions: }\; \theta = \theta^*(\theta') = \tilde{a}_i\,\mathrm{e}^{\ln\theta'/\rho},\; i = 1, 2$$

where $\tilde{a}_1, \tilde{a}_2 > 0$ are constants. If $\rho > 0$, then $\theta^*(\theta') \to \infty$ as $\theta' \to \infty$ and $\theta^*(\theta') \to 0$ as $\theta' \to 0$. If $\rho < 0$, then $\theta^*(\theta') \to 0$ as $\theta' \to \infty$ and $\theta^*(\theta') \to \infty$ as $\theta' \to 0$.

Hence, assumption 3.5 holds. Based on (35)–(38) and lemma 7.2, we can show that assumption 3.7 holds. Claim 3 then follows from propositions 3.4 and 4.3.

For claim 4, note that $r(w) = r(\eta + \xi\theta) = (\eta + \xi\theta)^{1-\delta}/[(1-\beta)(1-\delta)]$, which is increasing in θ, and when $\rho > 0$, $f(\theta'|\theta)$ is stochastically increasing in θ. Hence, assumption 3.3 holds. By propositions 3.2 and 4.4, ψ^* and $\bar{w}$ are increasing in θ. $\square$

7.4. **Proof of Section 6 Results.**

Proof of theorem 6.1. Regarding claim 1, similar to the proof of theorem 2.1, we have

$$\int \kappa(z')P(z, \mathrm{d}z') \leq (m + 2m')\kappa(z) \tag{39}$$

for all $z \in \mathsf{Z}$. We first show that $L\colon (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa) \to (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all $h := (\psi, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, define the functions $p(z) := c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z')$ and $q(z) := s(z) + \alpha\beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z') + (1-\alpha)\beta \int r(z') P(z, \mathrm{d}z')$. Obviously, p and q are $\mathscr{Z}$-measurable, and there exists $G \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,

$$\frac{|p(z)| \vee |q(z)|}{\kappa(z)} \leq \frac{|c(z)| \vee |s(z)|}{\kappa(z)} + \frac{\beta G \int \kappa(z') P(z, \mathrm{d}z')}{\kappa(z)} \leq \frac{1}{m'} + \beta(m + 2m')G < \infty.$$

These imply $p \in b_\kappa \mathsf{Z}$ and $q \in b_\kappa \mathsf{Z}$, and hence $Lh \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Next, we show that L is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all $h_1 := (\psi_1, r_1)$ and $h_2 := (\psi_2, r_2)$ in $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, we have $\rho_\kappa(Lh_1, Lh_2) = I \vee J$, with $I := \|\beta P(r_1 \vee \psi_1) - \beta P(r_2 \vee \psi_2)\|_\kappa$ and $J := \|\alpha\beta[P(r_1 \vee \psi_1) - P(r_2 \vee \psi_2)] + (1-\alpha)\beta(Pr_1 - Pr_2)\|_\kappa$. For all $z \in \mathsf{Z}$,

$$\begin{aligned} &|P(r_1 \vee \psi_1)(z) - P(r_2 \vee \psi_2)(z)| \leq \int |r_1 \vee \psi_1 - r_2 \vee \psi_2| (z') P(z, \mathrm{d}z') \\ &\leq \int (|\psi_1 - \psi_2| \vee |r_1 - r_2|)(z') P(z, \mathrm{d}z') \leq \rho_\kappa(h_1, h_2) \int \kappa(z') P(z, \mathrm{d}z'), \end{aligned} \tag{40}$$

where the second inequality is due to the elementary fact $|a \vee b - a' \vee b'| \leq |a - a'| \vee |b - b'|$. Combining (39)–(40) implies that $I \leq \beta(m + 2m')\rho_\kappa(h_1, h_2)$. Similar arguments yield $J \leq \beta(m + 2m')\rho_\kappa(h_1, h_2)$. In conclusion, we have

$$\rho_\kappa(Lh_1, Lh_2) = I \vee J \leq \beta(m + 2m')\rho_\kappa(h_1, h_2).$$

Hence, L is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ of modulus $\beta(m + 2m')$, as was to be shown. Claim 1 is verified.

Since ψ^* and r^* solves (24) by lemma 6.1, $h^* := (\psi^*, r^*)$ is indeed a fixed point of L. To prove that claim 2 holds, it remains to show that $h^* \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Since

$$\max\{|r^*(z)|, |\psi^*(z)|\} \leq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|s(Z_t)| + g(Z_t)],$$

this can be proved in a similar way as lemma 7.1. Hence, claim 2 is verified. □

References

ALBUQUERQUE, R. AND H. A. HOPENHAYN (2004): "Optimal lending contracts and firm dynamics," *The Review of Economic Studies*, 71, 285–315.

ALVAREZ, F. AND N. L. STOKEY (1998): "Dynamic programming with homogeneous functions," *Journal of Economic Theory*, 82, 167–189.

ARELLANO, C. (2008): "Default risk and income fluctuations in emerging economies," *The American Economic Review*, 98, 690–712.

ASPLUND, M. AND V. NOCKE (2006): "Firm turnover in imperfectly competitive markets," *The Review of Economic Studies*, 73, 295–327.

BAGGER, J., F. FONTAINE, F. POSTEL-VINAY, AND J.-M. ROBIN (2014): "Tenure, experience, human capital, and wages: A tractable equilibrium search model of wage dynamics," *The American Economic Review*, 104, 1551–1596.

BELLMAN, R. (1969): "A new type of approximation leading to reduction of dimensionality in control processes," *Journal of Mathematical Analysis and Applications*, 27, 454–459.

BENTAL, B. AND D. PELED (1996): "The accumulation of wealth and the cyclical generation of new technologies: A search theoretic approach," *International Economic Review*, 37, 687–718.

BOYD, J. H. (1990): "Recursive utility and the Ramsey problem," *Journal of Economic Theory*, 50, 326–345.

BULL, C. AND B. JOVANOVIC (1988): "Mismatch versus derived-demand shift as causes of labour mobility," *The Review of Economic Studies*, 55, 169–175.

BURDETT, K. AND T. VISHWANATH (1988): "Declining reservation wages and learning," *The Review of Economic Studies*, 55, 655–665.

CHALKLEY, M. (1984): "Adaptive job search and null offers: A model of quantity constrained search," *The Economic Journal*, 94, 148–157.

CHATTERJEE, S. AND B. EYIGUNGOR (2012): "Maturity, indebtedness, and default risk," *The American Economic Review*, 102, 2674–2699.

CHATTERJEE, S. AND E. ROSSI-HANSBERG (2012): "Spinoffs and the Market for Ideas," *International Economic Review*, 53, 53–93.

COOPER, R., J. HALTIWANGER, AND J. L. WILLIS (2007): "Search frictions: Matching aggregate and establishment observations," *Journal of Monetary Economics*, 54, 56–78.

COŞAR, A. K., N. GUNER, AND J. TYBOUT (2016): "Firm dynamics, job turnover, and wage distributions in an open economy," *The American Economic Review*, 106, 625–663.

CRAWFORD, G. S. AND M. SHUM (2005): "Uncertainty and learning in pharmaceutical demand," *Econometrica*, 73, 1137–1173.

DINLERSOZ, E. M. AND M. YORUKOGLU (2012): "Information and industry dynamics," *The American Economic Review*, 102, 884–913.

DIXIT, A. K. AND R. S. PINDYCK (1994): *Investment Under Uncertainty*, Princeton University Press.

DUFFIE, D. (2010): *Dynamic Asset Pricing Theory*, Princeton University Press.

DUNNE, T., S. D. KLIMEK, M. J. ROBERTS, AND D. Y. XU (2013): "Entry, exit, and the determinants of market structure," *The RAND Journal of Economics*, 44, 462–487.

ERICSON, R. AND A. PAKES (1995): "Markov-perfect industry dynamics: A framework for empirical work," *The Review of Economic Studies*, 62, 53–82.

FAJGELBAUM, P., E. SCHAAL, AND M. TASCHEREAU-DUMOUCHEL (2016): "Uncertainty traps," *Forthcoming, The Quarterly Journal of Economics*.

FEINBERG, E. A., P. O. KASYANOV, AND N. V. ZADOIANCHUK (2014): "Fatou's lemma for weakly converging probabilities," *Theory of Probability & Its Applications*, 58, 683–689.

GOMES, J., J. GREENWOOD, AND S. REBELO (2001): "Equilibrium unemployment," *Journal of Monetary Economics*, 48, 109–152.

HATCHONDO, J. C., L. MARTINEZ, AND C. SOSA-PADILLA (2016): "Debt dilution and sovereign default risk," *Journal of Political Economy*, 124, 1383–1422.

HOPENHAYN, H. A. (1992): "Entry, exit, and firm dynamics in long run equilibrium," *Econometrica*, 1127–1150.

JOVANOVIC, B. (1982): "Selection and the evolution of industry," *Econometrica*, 649–670.

——— (1987): "Work, rest, and search: unemployment, turnover, and the cycle," *Journal of Labor Economics*, 131–148.

JOVANOVIC, B. AND R. ROB (1989): "The growth and diffusion of knowledge," *The Review of Economic Studies*, 56, 569–582.

KELLOGG, R. (2014): "The effect of uncertainty on investment: evidence from Texas oil drilling," *The American Economic Review*, 104, 1698–1734.

LE VAN, C. AND Y. VAILAKIS (2005): "Recursive utility and optimal growth with bounded or unbounded returns," *Journal of Economic Theory*, 123, 187–209.

LISE, J. (2013): "On-the-job search and precautionary savings," *The Review of Economic Studies*, 80, 1086–1113.

LJUNGQVIST, L. AND T. J. SARGENT (2008): "Two questions about European unemployment," *Econometrica*, 76, 1–29.

——— (2012): *Recursive Macroeconomic Theory*, MIT Press.

LOW, H., C. MEGHIR, AND L. PISTAFERRI (2010): "Wage risk and employment risk over the life cycle," *The American Economic Review*, 100, 1432–1467.

MA, Q. AND J. STACHURSKI (2017): "Optimal timing of decisions: a general theory based on continuation values," Working Paper on arXiv.org, available at: https://arxiv.org/abs/1703.09832.

MARTINS-DA ROCHA, V. F. AND Y. VAILAKIS (2010): "Existence and uniqueness of a fixed point for local contractions," *Econometrica*, 78, 1127–1141.

MCCALL, J. J. (1970): "Economics of information and job search," *The Quarterly Journal of Economics*, 84, 113–126.

MENDOZA, E. G. AND V. Z. YUE (2012): "A general equilibrium model of sovereign default and business cycles," *The Quarterly Journal of Economics*, 127, 889–946.

MENZIO, G. AND N. TRACHTER (2015): "Equilibrium price dispersion with sequential search," *Journal of Economic Theory*, 160, 188–215.

MEYN, S. P. AND R. L. TWEEDIE (2012): *Markov Chains and Stochastic Stability*, Springer Science & Business Media.

MITCHELL, M. F. (2000): "The scope and organization of production: firm dynamics over the learning curve," *The Rand Journal of Economics*, 180–205.

MOSCARINI, G. AND F. POSTEL-VINAY (2013): "Stochastic search equilibrium," *The Review of Economic Studies*, 80, 1545–1581.

NAGYPÁL, É. (2007): "Learning by doing vs. learning about match quality: Can we tell them apart?" *The Review of Economic Studies*, 74, 537–566.

PERLA, J. AND C. TONETTI (2014): "Equilibrium imitation and growth," *Journal of Political Economy*, 122, 52–76.

PESKIR, G. AND A. SHIRYAEV (2006): *Optimal Stopping and Free-boundary Problems*, Springer.

POSCHKE, M. (2010): "The regulation of entry and aggregate productivity," *The Economic Journal*, 120, 1175–1200.

PRIES, M. AND R. ROGERSON (2005): "Hiring policies, labor market institutions, and labor market flows," *Journal of Political Economy*, 113, 811–839.

RINCÓN-ZAPATERO, J. P. AND C. RODRÍGUEZ-PALMERO (2003): "Existence and uniqueness of solutions to the Bellman equation in the unbounded case," *Econometrica*, 71, 1519–1555.

ROBIN, J.-M. (2011): "On the dynamics of unemployment and wage distributions," *Econometrica*, 79, 1327–1355.

STOKEY, N., R. LUCAS, AND E. PRESCOTT (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press.

TIMOSHENKO, O. A. (2015): "Product switching in a model of learning," *Journal of International Economics*, 95, 233–249.

VERESHCHAGINA, G. AND H. A. HOPENHAYN (2009): "Risk taking by entrepreneurs," *The American Economic Review*, 99, 1808–1830.