# Economic distributions and primitive distributions in Industrial Organization and International Trade

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**Abstract**

We link fundamental technological and taste distributions to endogenous economic distributions of prices and firm size (output, profit) in extensions of canonical IO and Trade models. We provide constructive proofs to recover the demand structure, mark-ups, and distributions of cost, price, output and profit from just two distributions (or from demand and one distribution). For CES, all distributions lie in the same family (e.g., the "Pareto circle"). Introducing quality breaks the circle. We extend our general analysis, modeling the technological relation between quality and cost to link two distribution groups (output, profit, and quality-cost; price and cost). The distributions of output, profit, and prices suffice to recover the cost distribution, the demand form, and the quality-cost relation. A continuous logit demand model illustrates: exponential (resp. normal) quality-cost distributions generate Pareto (log-normal) economic size distributions. Pareto prices and profits are reconciled through an appropriate quality-cost relation. We also find long-run equilibrium distributions.

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# 1 Introduction

Distributions of economic variables have attracted the interest of economists at least since Pareto (1896). In industrial organization, firm size (output, sales, or profit) distributions have been analyzed, while different studies have looked at the distribution of prices within an industry. Firm sizes (profitability, say) within industries are wildly asymmetric, and frequently involve a long-tail of smaller firms. The idea of the long tail has recently been invoked prominently in studies of Internet Commerce (Anderson, 2006, Elberse and Oberholzer-Gee, 2006), and particular distributions – mainly the Pareto and log-normal – seem to fit the data well in other areas too (see Head, Mayer, and Thoenig, 2014).

In international trade, recent advances have enabled studying distributions of sales revenues (see, e.g., Eaton, Kortum, and Kramarz, 2011). The distributions of these "economic" variables are (presumably) jointly determined by the fundamental underlying distributions of tastes and technologies. In this paper we determine the links between the various distributions. We link the economic ones to each other and to the primitive distributions and tastes. Moreover, the primitives can be uncovered from the observed economic distributions.

Philosophically, the paper closest (and complementary) to ours is Mrázová, Neary, and Parenti (2016). These authors also study the relations between equilibrium distributions of sales and mark-ups, the primitive productivity distribution, and (a specific) demand form (although they do not include heterogeneous quality). They are mainly interested in when distributions are in the same ("self-reflecting") class (e.g., when both productivity and sales are log-normal or Pareto). They also provide some empirical analysis of log-normal and Pareto distributions.

We start by deploying a general monopolistic competition model with a continuum of firms (see Thisse and Uschev, 2016, for a review of this literature). We first show how the demand function delivers a mark-up function, and then we show our key converse result that the mark-up (or "pass-through" function of Weyl and Fabinger, 2013) determines the form of the demand function. We next engage these results with constructive proofs to show how cost and price distributions suffice to determine the shape of the economic profit and output distributions and the demand form. Along broader lines, we show when and how any two elements (e.g., two distributions) suffice to deliver all the missing pieces.

We next illustrate with the CES representative consumer model, which is widely used in economics in conjunction with monopolistic competition. It is used as a theoretical component in the New Economic Geography and Urban Economics, it is the linchpin of Endogenous Growth Theory, Keynesian underpinnings in Macro, and of course, Industrial Organization. The current most intensive use of the model is in International Trade, following Melitz (2003), where it is at the heart of empirical estimation. The convenience of the model stems from its analytic manipulability. The CES model delivers equilibrium mark-ups proportional to marginal costs, and so delivers market imperfection (imperfect competition) in a simple way without complex market interaction. The standard models in this vein (following Melitz, 2003) assume that firms' unit production costs are heterogeneous.

However, when we apply this model to distributions, if one distribution (such as profit) is Pareto, then the distributions of all the economic variables lie in the Pareto family. This we call the "Pareto circle": more generally, we establish the *CES circle* because the result applies to any distribution family. The CES circle is broken by introducing qualities (as do Baldwin and Harrigan, 2011, and Feenstra and Romalis, 2014 for the Pareto) into the demand model. Doing so delivers two fundamental drivers of equilibrium distributions (instead of just one) – the cost distribution and the quality/cost one. Even if one distribution is Pareto, then others can take different forms. Most notably, the output distribution depends on the cost distribution (as before) but now also on the quality/cost distribution. However, the CES still delivers stringent restrictions on the number of distribution classes with which it can be associated. We therefore next explore alternative demand models.

Allowing for both quality and cost heterogeneity,[1] we show a three-way relation between two groups of distributions and the quality-to-cost relation: knowing one element from any two of these ties down the third. On one leg, we generate the relation between equilibrium profit dispersion, firm outputs, and the fundamental quality-cost distribution. On a second leg, we show the relation between the cost distribution and equilibrium price dispersion. If we know demand, then knowing any one of the distributions on one leg suffices to determine the others on that leg. Moreover, knowing a distribution from each leg allows us to determine what the relation between cost and quality must be on the third leg. If the demand form is not known, then we show that it can be deduced from observing price, output, and profit distributions (and the cost distribution and the relation between costs and quality can also be determined).

We next deploy a logit model of monopolistic competition, which we here develop.[2] The logit is the workhorse model in structural empirical IO. Some useful characterization results are that normally distributed quality-costs induce log-normal distributions for profits, and that an exponential distribution of quality-costs leads to a Pareto distribution for profit. Cost heterogeneity alone cannot induce Pareto distributions for both profits and prices. We show by construction for the logit example that the added dimension of quality (and the associated quality-cost relation) can generate Pareto distributions for both, thus allowing sufficient richness to link diverse distribution types. Finally, we apply our results to a long-run analysis in the spirit of Melitz (2003) with the set of active firms determined endogenously.

[1] Ironically, Chamberlin (1933) is best remembered for his symmetric monopolistic competition analysis. Yet he went to great length to point out that he believed asymmetry to be the norm, and that symmetry was a very restrictive assumption. We model both quality and production cost differences across firms.

[2] The Logit is an attractive alternative framework to the CES. Anderson, de Palma, and Thisse (1992) have shown that the CES can be viewed as a form of Logit model.

## 2 Recovering demand from economic distributions

A continuum of firms produce substitute goods. Each has constant unit production costs, but these differ across firms. With a continuum of firms, each firm effectively faces a monopoly problem where the price choice is independent of the actions of rivals. In this spirit, we allow for a general demand formulation, and show how the primitive distributions feed through to the endogenous economic distributions and variables. Conversely, the derived economic distributions can be reverse engineered to back out the model's primitives.

We first give the demand model, and derive the equilibrium mark-up schedule in Theorem 1 as a function of firm unit cost, $c$. Theorem 2 inverts the mark-ups to deliver both the equilibrium output choices and the form of the demand curve. This analysis constitutes an stand-alone contribution to the theory of monopoly pass-through, extending Weyl and Fabinger (2013) by working from pass-through *back* to implied demands. Our proofs here and beyond are constructive: we derive the relations between distributions and primitives.

Theorem 3 shows how to use price and cost distributions to find the shape of the profit and output distributions and demand form (up to a positive factor). Theorem 4 shows how to invert the (potentially observable) output and profit distributions to find the underlying net inverse demand form (i.e., demand up to a cost shift), and the underlying primitive cost distribution, $F_C(c)$. Theorem 5 does likewise with price and profit distributions as the starting point (again up to a positive constant). Theorem 6 shows that all other distribution pairs tie down all primitives and outcome distributions (including constants). Finally, Theorem 7 shows how knowing the demand form and just one distribution ties down everything else. In Section 4 we will extend this analysis to allow qualities to be also idiosyncratic to firms.[3] Throughout,

[3] Then output and profit distributions determine the equilibrium mark-up distribution, which in turn determines (Theorem 12) the underlying distribution of costs, $F_C(c)$, from the (potentially observable) economic distribution of prices, $F_P(p)$. This last step also enables us to uncover the "bridge" function that relates qualities to costs.

we make explicit the appropriate monotonicity conditions that ensure invertibility (see the analogous discussion in Section 4.1).

We shall assume for the exposition that all distributions are absolutely continuous and strictly increasing. As should become apparent, any gaps in a distribution's support will correspond to gaps in supports of the other distributions; the analysis applies piecewise on the interior of the supports. Likewise, mass-points in the interior of the support pose no problem because they correspond to mass points in the other distributions. The support itself delivers the demand form and the mark-up function.

## 2.1 Demand and mark-ups

**Assumption 1** *Suppose that demand for a firm charging* $p$ *is*

$$y = h(p), \tag{1}$$

*a positive, strictly decreasing, strictly* $(-1)$*-concave, and twice differentiable function.*[4]

We suppress for the present the impact of other firms' actions on demand, which would be expressed as aggregate variables in the individual demand function. Under monopolistic competition with a continuum of firms, each firm's individual action has no measurable impact on the aggregate variables. Because we look at the cross-section relation between equilibrium distributions, the actions of other firms are the same across the comparison, and therefore have no bearing on our results. We return to this when we discuss specific examples. The profit for a firm with per unit cost $c$ is $\pi = (p-c)\,h(p) = mh(m+c)$, where $m = p - c$ is its mark-up.

[4] This is equivalent to $\frac{1}{h(.)}$ strictly convex, and is a minimal condition ensuring a maximum to profit. See Caplin and Nalebuff (1991) and Anderson, de Palma, and Thisse (1992, p.164) for more on $\rho$-concave functions; and Weyl and Fabinger (2013) for the properties of pass-through as a function of demand curvature.

With a continuum of firms (monopolistic competition), the equilibrium mark-up satisfies

$$m = -\frac{h(m+c)}{h'(m+c)}. \tag{2}$$

**Theorem 1** *Under Assumption 1, the equilibrium mark-up, $\mu(c) > 0$ is the unique solution to (2), with $\mu'(c) > -1$ (and $\mu'(c) \geq 0$ if $h(.)$ is log-convex and $\mu'(c) \leq 0$ if $h(.)$ is log-concave). The associated equilibrium demand, $h^*(c) \equiv h(\mu(c) + c)$, is strictly decreasing, as is $\pi^*(c) = \mu(c) h^*(c)$, with $\pi^{*\prime}(c) = -h^*(c)$.*

**Proof.** The solution to (2), denoted $\mu(c)$, is uniquely determined (and strictly positive) when the RHS of (2) has slope less than one, as is implied by $h(.)$ being strictly $(-1)$-concave. Applying the implicit function theorem to (2) shows that

$$\mu'(c) = \frac{-\left(\frac{h(m+c)}{h'(m+c)}\right)'}{1 + \left(\frac{h(m+c)}{h'(m+c)}\right)'} > -1, \tag{3}$$

where the denominator is strictly positive under Assumption 1.[5] The numerator is (weakly) positive for $h$ log-convex and (weakly) negative for $h$ log-concave. Let $h^*(c) = h(\mu(c) + c)$ denote the value of $h(.)$ under the profit-maximizing mark-up. Then, $h^*(c)$ is strictly increasing, as claimed, because

$$dh^*(c)/dc = (\mu'(c) + 1) h'(\mu(c) + c) < 0 \tag{4}$$

and given that $\mu'(c) > -1$. Finally, $\pi^*(c) = \mu(c) h^*(c)$ is strictly decreasing by the envelope theorem applied to $\pi = (p - c) h(p)$, so $\pi^{*\prime}(c) = -h^*(c)$. ■

The only demand function with constant (absolute) mark-up is the exponential (associated to the Logit), which has $h(\cdot)$ log-linear in $p$, and so $\frac{h(m+c)}{h'(m+c)}$ is constant. For $h(\cdot)$ strictly log-concave, $\mu'(c) < 0$, so firms with higher costs have lower mark-ups in the cross-section of firm types (price pass-through is less than 100%). They also have lower equilibrium outputs. When

[5] When $h(u)$ is strictly $(-1)$-concave, then $h(u) h''(u) - 2[h'(u)]^2 < 0$, which rearranges to $\left[\frac{h(u)}{h'(u)}\right]' > -1$.

$h(\cdot)$ is strictly log-convex, the mark-up *increases* with $c$, so cost pass-through is greater than 100%, which is a hall-mark of CES demands, which have constant elasticity and hence constant relative mark-up. Notice that the property $\mu'(c) > -1$ is just the standard property that price never goes down as costs increase.

These properties indicate properties of the price distribution relative to the cost distribution. The price distribution is a *compression* of the cost distribution when $h$ is log-concave, and a *magnification* when $h$ is log-convex, in the simple sense that prices are closer together (or, respectively, farther apart) than costs. The border case (Logit / log-linear demand) has constant mark-ups, so the price distribution mirrors the cost one.

An important special case is when demand is $\rho-$linear (which means that $h^\rho$ is linear). Suppose then that

$$h(.) = (1 + (k - p)\rho)^{1/\rho}, \tag{5}$$

where $k$ is a constant. Then

$$\mu(c) = \frac{1 + \rho(k - c)}{1 + \rho},$$

which is linear in $c$.[6] For $\rho = 1$ demand is linear and the standard property is apparent that mark-ups fall fifty cents on the dollar with cost. Log-linearity is $\rho = 0$ (note that $\lim_{\rho \to 0} h(.) = \exp(k - p)$) and delivers a constant mark-up. For $\rho-$linear demands, equilibrium demand is $h^*(c) = \left(\frac{1+\rho(k-c)}{1+\rho}\right)^{1/\rho}$ and then (see (6) below) $\frac{dh^*(c)/dc}{h^*(c)} = \frac{-1}{1+\rho(k-c)} = -\frac{\mu'(c)+1}{\mu(c)} < 0$.

The next implication of Theorem 1 tells us that we can rely on monotonic relations between variables, which is crucial in twinning distributions (as we do below). The price result follows because $\mu'(c) > -1$ by (3) under A1, so that price strictly increases with cost.

**Corollary 1** *Under Assumption 1, higher costs are associated with higher prices, lower output and lower profit. If demand is strictly log-concave (resp. log-convex), higher cost firms have*

[6] More generally, $\mu'(c) \geq \frac{-\rho}{1+\rho}$ when $h$ is $\rho$-convex and $\mu'(c) \leq \frac{-\rho}{1+\rho}$ when $h$ is $\rho$-concave.

*lower (resp. higher) markups.*

In the log-concave case, low-cost firms use their advantage in both mark-up and output dimensions. Under log-convexity, low-cost firms exploit the opportunity to capitalize on much larger demand by setting small mark-ups. The converse result to Theorem 1 indicates how the mark-up function $\mu(c)$ can be inverted to determine the form of $h^*$ (and hence $h(.)$).

**Theorem 2** *Consider a positive mark-up function $\mu(c)$ for $c \in [\underline{c}, \bar{c}]$ with $\mu'(c) > -1$. Then there exists an equilibrium demand function $h^*(c)$ with $h^{*\prime}(c) < 0$, defined on its support $[\underline{c}, \bar{c}]$ and given by (7), which is unique up to a positive multiplicative factor. The associated primitive demand function $h(\cdot)$, given by (8), satisfies Assumption 1 on its support $[\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$: $h(.)$ is log-convex if $\mu'(c) \geq 0$ and log-concave if $\mu'(c) \leq 0$.*

**Proof.** First note from (2) and (4) that

$$\frac{dh^*(c)/dc}{h^*(c)} = \frac{(\mu'(c)+1)\,h'(\mu(c)+c)}{h(\mu(c)+c)} = -\frac{\mu'(c)+1}{\mu(c)} \equiv g(c) < 0, \tag{6}$$

because $\mu'(c) > -1$ by assumption. Thus $[\ln h^*(c)]' = g(c)$, and so $\ln\left(\frac{h^*(c)}{h^*(\underline{c})}\right) = \int_{\underline{c}}^{c} g(v)\,dv$, or

$$h^*(c) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{c} g(v)\,dv\right), \quad c \geq \underline{c}, \tag{7}$$

which determines $h^*(c)$ up to the positive factor $h^*(\underline{c})$; it is strictly decreasing because $g(c) < 0$.

We can now use $h^*(c)$ to back out the original function $h(m+c)$ via the following steps. First, define $u \equiv \phi(c) = \mu(c) + c$, which is strictly increasing because $\mu'(c) + 1 > 0$, so the inverse function $\phi^{-1}(\cdot)$ is strictly increasing. Now, $h(u) = h^*\left(\phi^{-1}(u)\right)$ and thus the function $h(\cdot)$ is recovered on the support $u \in [\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$ (cf. Theorem 1). Using (7) with $h(u) = h^*\left(\phi^{-1}(u)\right)$,

$$h(u) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{\phi^{-1}(u)} g(v)\,dv\right), \tag{8}$$

and so

$$\frac{h(u)}{h'(u)} = \frac{1}{g\left(\phi^{-1}(u)\right)\left[\phi^{-1}(u)\right]'} = \frac{\phi'(c)}{-\frac{\mu'(c)+1}{\mu(c)}} = -\mu(c),$$

where the middle step follows from (6) with $u = \phi(c)$ and the last step follows because $\phi'(c) = \mu'(c) + 1$. Thus

$$\left[\frac{h(u)}{h'(u)}\right]' = -\frac{\mu'(c)}{\mu'(c)+1} > -1,$$

and so $h(u)$ is strictly $(-1)$-concave (as shown in the previous footnote). Note that $h(.)$ is twice differentiable because $\mu(.)$ was assumed differentiable. ■

The reason that demand is only determined up to a positive factor is simply that multiplying demand by a positive constant does not change the optimal mark-up (when marginal costs are constant, as here). The mark-up function can only determine the demand shape, but not its scale.

The steps above are readily confirmed for the $\rho-$linear example given before Theorem 2. Taking Theorems 1 and 2 together, knowing either $\mu(c)$ or $h(.)$ suffices to determine the other and $h^*(c)$. This constitutes a strong characterization result for monopoly pass-through (see Weyl and Fabinger, 2013, for the state of the art, which deeply engages $\rho$-concave functions).

Notice that the function $h(\cdot)$ is tied down only on the support corresponding to the domain on which we have information about the equilibrium mark-up value in the market. Outside that support, we know only that $h(\cdot)$ must be consistent with the maximizer $\mu(c)$, which restricts the shape of $h(\cdot)$ to be not "too" convex.

## 2.2 Deriving all distributions and demand form from price and cost

In the sequel of this Section, we show how knowledge (or assumptions) on two entities allows us to determine the rest. We start with price and cost distributions, and show how the demand form can be determined from them (using Theorem 2) along with the other distributions. We then do likewise for output and profit distributions. Finally, we show how knowing the demand form and just one distribution uncovers the rest. These theorems make extensive use of the following result.

**Lemma 1** *Consider two distributions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, which are absolutely continuous and strictly increasing on their respective domains. Let $X_1$ and $X_2$ be related by a monotone function $X_1 = g(X_2)$. Then $F_{X_2}(x_2) = F_{X_1}(g(x_2))$ for $g(.)$ increasing, and $F_{X_2}(x_2) = 1 - F_{X_1}(g(x_2))$ for $g(.)$ decreasing.*

**Proof.** For $g(.)$ increasing, $F_{X_1}(x_1) = \Pr(X_1 < x_1) = \Pr(g(X_2) < x_1) = \Pr(X_2 < g^{-1}(x_1)) = F_{X_2}(g^{-1}(x_1))$. Equivalently, $F_{X_2}(x_2) = F_{X_1}(g(x_2))$. For $g(.)$ decreasing, $F_{X_1}(x_1) = \Pr(g(X_2) < x_1) = \Pr(X_2 > g^{-1}(x_1)) = 1 - F_{X_2}(g^{-1}(x_1))$; equivalently, $F_{X_2}(x_2) = 1 - F_{X_1}(g(x_2))$. ■

We will be able thus to determine distributions from each other when there are monotone relations between two variables. Suppose that price and cost distributions, $F_P$ and $F_C$, are known. We show how to find the implied other economic distributions as well as the demand form and mark-up function: we can find all other elements in the market from just the two distributions. This strong result relies on the monotonic relations between all pairs of variables from Corollary 1. We now show how this works. Because price strictly increases with cost, the price and cost distributions are matched: the fraction of firms with costs below some level $c$ equals the fraction of firms with prices below the price charged by a firm with cost $c$. This enables us to back out the corresponding mark-up function $\mu(c)$ and then access Theorem 2.

**Theorem 3** *Let there be a continuum of firms with demand (1) satisfying Assumption 1, but (1) is not directly observed. Assume that the implied distribution of costs, $F_C$ and the corresponding distribution of prices, $F_P$, are known. Then the mark-up function $\mu(c)$ (with $\mu'(c) > 0$) is found from (9), and equilibrium demand is found from (7), up to a positive multiplicative factor, $h^*(\underline{c})$. The output and profit distributions are determined, up to $h^*(\underline{c})$, by (10) and (11).*

**Proof.** By Corollary 1, the price charged by a firm with cost $c$ is a strictly increasing function $p(c)$. Then, invoking Lemma 1, $F_C(c) = F_P(p(c))$, and we can write the price-cost margin, as a function of $c$, as

$$\mu(c) = F_P^{-1}(F_C(c)) - c, \tag{9}$$

with $\mu'(c) > -1$. With the function $\mu(c)$ thus determined, we can invoke Theorem 2 to uncover the equilibrium demand function $h^*(\cdot)$ (up to a positive multiplicative factor) as given by (6) and (7), and the demand function is given from (8). Then, by result (4) we know that output $y = h^*(c)$ is a monotonic decreasing function, and so (by Lemma 1) the fraction of firms with output below $y = h^*(c)$ is the fraction of firms with cost above $c$, so $F_Y(h^*(c)) = 1 - F_C(c)$, or

$$F_Y(y) = 1 - F_C\left(h^{*-1}(y)\right). \tag{10}$$

Finally, by Theorem 1 we know that profit $\pi^*(c) = \mu(c)\, h^*(c)$ is a strictly decreasing function, and so the fraction of firms with profit below $\pi^*(c)$ is the fraction of firms with costs above $c$, so $F_\Pi(\pi^*(c)) = 1 - F_C(c)$, or

$$F_\Pi(\pi) = 1 - F_C\left(\pi^{*-1}(\pi)\right). \tag{11}$$

Both output and profit distributions are determined up to the positive multiplicative factor $h^*(\underline{c})$ in the demand function. ■

The idea behind the result is as follows. Given the first key property that prices rise with costs, we know that the $z\%$ of firms with cost below $c$ are the $z\%$ of firms with an equilibrium price below $p$. This links the mark-up and the cost level, so we can use Theorem 2 to uncover the demand form and equilibrium output of the $z$th percentile firm, due to the second key property that equilibrium output is a decreasing function of cost. We hence uncover the output distribution. The profit distribution then follows immediately from knowing the output and mark-up distributions. The latter two distributions are only determined up to a positive factor because the mark-up function is consistent with any multiple of the demand (under the maintained hypothesis of constant returns to scale).

## 2.3 From output and profit distributions to demand form et al.

We here engage $F_Y$ and $F_\Pi$ to show how to back out the underlying cost distribution ($F_C$), and the implied demand. Before this reverse engineering, we first determine how the primitives $F_C$ and $h(.)$ generate the pertinent economic distributions and mark-ups.

As shown already, $h^*(c)$ and $\mu(c)$ are derived from $h(.)$ via Theorems 1 and 2. Now note

$$F_Y(y) = \Pr(h^*(C) < y) = \Pr\left(C > h^{*-1}(y)\right) = 1 - F_C\left(h^{*-1}(y)\right), \text{ and,analogously,}$$

$$F_\Pi(\pi) = \Pr(\Pi < \pi) = \Pr(\pi^*(C) < \pi) = \Pr\left(C > \pi^{*-1}(\pi)\right) = 1 - F_C\left(\pi^{*-1}(\pi)\right),$$

where we have used Theorem 1 that $h^*(c)$ and $\pi^*(c)$ are strictly decreasing. The converse result tells us how to uncover the primitives from the economic distributions $F_Y$ and $F_\Pi$.

**Theorem 4** *Let there be a continuum of firms with demand (1) satisfying Assumption 1, but (1) is not directly observed. Assume that the implied equilibrium distributions of output, $F_Y$, and profit, $F_\Pi$, are known. The cost distribution, $F_C$, is given by (17), the equilibrium demand is found from (13) and (17), and the mark-up function is found from (14) and (17), up to an additive constant $\bar{c}$ in their supports. The net inverse demand function is tied down.*

**Proof.** We know that $h^*(c)$ is strictly decreasing in $c$, and so too is $\pi^*(c) = \mu(c)h^*(c)$ (by Theorem 1). We hence choose some arbitrary level $z \in (0,1)$ such that

$$1 - F_C(c) = F_Y(y) = F_\Pi(\pi) = z. \tag{12}$$

This means that all firm types with cost levels above $c(z) = F_C^{-1}(1-z)$ are the firms with outputs and profits below $y$ and $\pi$. For this proof, we introduce $z$ as an argument into the various outcome variables to track the dependence of the variables on the level of $z(c) = 1 - F_C(c)$. From (12) we can write $y(z) = F_Y^{-1}(z)$ and demand is

$$h^*(c) = y(z(c)) = F_Y^{-1}(1 - F_C(c)). \tag{13}$$

Because $\pi^*(z) = m(z)\,y(z) = F_\Pi^{-1}(z)$ then

$$m(z(c)) = \frac{F_\Pi^{-1}(z(c))}{F_Y^{-1}(z(c))} = \mu(c), \tag{14}$$

and equilibrium profit is $\pi^*(c) = \mu(c)\,h^*(c) = F_\Pi^{-1}(z(c))$.

It remains to find the relation $z(c)$. From (14), $\mu'(c) = m'(z(c))\,z'(c)$ and similarly $h^{*\prime}(c) = y'(z(c))\,z'(c)$ (where $m'(z(c)) = \frac{dm(z(c))}{dz(c)}$, etc.). The two functions $\mu(c)$ and $h^*(c)$, which are to be determined, satisfy condition (6), which implies

$$\frac{h^{*\prime}(c)}{h^*(c)} = \frac{y'(z(c))\,z'(c)}{y(z(c))} = -\frac{m'(z(c))\,z'(c) + 1}{m(z(c))}.$$

Rearranging the last equality to solve out for $z'(c)$ gives[7]

$$z'(c) = -\frac{y(z(c))}{[m(z(c))\,y(z(c))]'} = -\frac{F_Y^{-1}(z(c))}{\left[F_\Pi^{-1}(z(c))\right]'}. \tag{15}$$

Thus:

$$\int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr = -\int_{\bar{c}}^{c} dv = \bar{c} - c,$$

or $c(z) = \bar{c} - \Psi(z)$, where $\Psi(z)$ is the key transformation between $z$ and $c$ given by

$$\Psi(z) = \int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr. \tag{16}$$

[7] An alternative derivation is to use Theorem 1 to write $\pi^{*\prime}(c) = -h^*(c)$, so the relation between the counter $z$ and the cost level $c$ is $dz/dc = -h^*(c)/[\pi^*(z(c))]'$, which is (15).

Because $\Psi'(z) = \frac{\left[F_{\Pi}^{-1}(z)\right]'}{F_Y^{-1}(z)} > 0$, the required relation between $z$ and $c$ is $z(c) = \Psi^{-1}(\bar{c} - c)$.

Observe that $h(p) = h(\mu(c) + c)$ so that inverse demand is $p = \frac{F_{\Pi}^{-1}(z(c))}{F_Y^{-1}(z(c))} + c = \frac{F_{\Pi}^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))} + c$. This makes clear that a shift up in all costs by $\Delta$ and a corresponding shift up in the inverse demand by $\Delta$ (so the support of the cost distribution shifts up by $\Delta$, i.e., $\bar{c}$ becomes $\bar{c} + \Delta$) keeps both the firm's output choice and mark-up constant so output and profit are not changed. This means that these two distributions can only pin down *net* (inverse) demand.

The distribution of cost is thus given by

$$F_C(c) = 1 - z(c) = 1 - \Psi^{-1}(\bar{c} - c). \tag{17}$$

The remaining unknowns can be backed out now knowing $z(c)$: equilibrium demand is $h^*(c) = F_Y^{-1}(\Psi^{-1}(\bar{c} - c))$ from (13), and the mark-up function is $\mu(c) = \frac{F_{\Pi}^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))}$ from (14). ■

What the Theorem ties down is net demand (inverse demand minus cost): if both demand price and cost shift by the same amount then equilibrium quantity (output) and mark-up are unaffected, so profit is unchanged too. Thus output and profit distributions tie down the shape of the inverse demand and the shape of the other distributions, but not the inverse demand curve height. As we saw above, price and cost distributions alone do not tie down the demand scale. But, as we shall see below, other pairs of distribution combinations fully pinpoint all distributions and demand functions.

We illustrate the theorem above with distributions that generate $\rho$-linear demand.

**Example 1:** *$\rho$-linear demands and uniform cost distribution.*

*Suppose that* $F_Y(y) = \frac{(1+\rho)y^{\rho}-1}{\rho}$, $y \in \left[\frac{1}{(1+\rho)^{1/\rho}}, 1\right]$, *and* $F_{\Pi}(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$, *with* $\rho > -1$.

*Hence* $F_Y^{-1}(z) = \left(\frac{\rho z+1}{1+\rho}\right)^{1/\rho}$ *and* $F_{\Pi}^{-1}(z) = \left(\frac{\rho z+1}{1+\rho}\right)^{(1+\rho)/\rho}$. *By (14), the ratio of these two yields the mark-up,* $m(z) = \frac{\rho z+1}{1+\rho} > 0$. *Because* $\left[F_{\Pi}^{-1}(z)\right]' = \left(\frac{\rho z+1}{1+\rho}\right)^{1/\rho}$, *we can write* $\Psi(z) =$

$\int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr = z$, *and, because* $c(z) = F_C^{-1}(1-z)$, *then* $c(z) = \bar{c} - z$ *(with* $c'(z) = -1$*), so that* $\underline{c} = \bar{c} - 1$. *Now,* $F_C(c) = 1 - \Psi^{-1}(\bar{c} - c) = c - \underline{c}$. *Hence* $\mu(c) = \frac{\rho(\bar{c}-c)+1}{1+\rho}$. *Then* $y(c) = F_Y^{-1}(z(c)) = \left(\frac{\rho(\bar{c}-c)+1}{1+\rho}\right)^{1/\rho}$, *and* $h^*(c) = y(c)$. *We now want to find the associated demand,* $h(p)$. *We use the fact that* $p = \mu(c) + c = \frac{1+c+\rho\bar{c}}{1+\rho}$ *to write* $h(p) = (1 + \rho(\bar{c} - p))^{1/\rho}$, *which is therefore a* $\rho-$*linear demand function (see (5)) with the parameter* $k$ *set at* $k = \bar{c}$, *and* $\rho > -1$ *implies* $h(.)$ *is* $(-1)$*-concave.*

*Note that* $y(\bar{c}) = \left(\frac{1}{1+\rho}\right)^{1/\rho}$, *as verified by the upper bound,* $\bar{c}$, *while the lower bound condition* $\underline{c} = \bar{c} - 1$ *implies that* $y(\underline{c}) = 1$, *so costs are uniformly distributed on* $[\underline{c}, \bar{c}]$. *Lastly,* $\lim_{\rho \to 0} y(c) = \exp(\bar{c} - c)$ *gives the logit equilibrium demand (see Section 5).*

The uniform cost example gives a useful benchmark for some important properties relating cost distribution to profit distribution. For the example above, we have $f_\Pi(\pi) = \pi^{-1/(1+\rho)}$, so that the density of the profit distribution is decreasing, despite the underlying cost distribution that generates it being flat. This property indicates how profit density "piles up" at the low end. The output density shape is also interesting. For linear demand ($\rho = 1$), it is clearly flat – equilibrium quantity is a linear function of cost. For convex demand ($\rho < 1$), it is decreasing, but for concave demand it is *increasing*, despite the property just noted that the profit density is decreasing. This suggests that (for concave demand), a decreasing output density requires an increasing cost density, which *a fortiori* entails a decreasing profit density.

As per Theorem 4, the (output, profit) distribution pair does not tie down the value of the constant. In Theorem 6 below we show which distribution pairs do tie down the full model, and we return to the above example to illustrate. We then show how knowing the demand form plus any distribution ties down everything (Theorem 7.)

## 2.4 The other distribution pairs

We now turn to the information that can be gleaned from knowing the other pairs.

**Theorem 5** *Let there be a continuum of firms with demand (1) satisfying Assumption 1, but (1) is not directly observed. Assume that the corresponding distribution of prices and profits are known. Then the demand function is found from (18), and is determined up to a positive constant,* $h\left(\underline{p}\right)$*; the other distributions are then determined.*

**Proof.** Applying the techniques above (see (12)), first write $1-F_C(c)=1-F_P(p)=F_Y(y)=F_\Pi(\pi)=z$. Then we can write $\pi=F_\Pi^{-1}(1-F_P(p))=h(p)\mu(p)\equiv\tilde{\pi}(p)$, where $\tilde{\pi}(p)$ therefore denotes the relation between the maximized profit level observed and the value of the corresponding maximizing price. Recall from the optimal choice of mark-up that $h(p)$ and $\mu(p)$ are related (see (2)) by $\mu(p)=-h(p)/h'(p)$, and so $\tilde{\pi}(p)=-h^2(p)/h'(p)$. Integrating,

$$h(p)=\frac{1}{\int_{\underline{p}}^{p}\frac{dr}{F_\Pi^{-1}(1-F_P(r))}+k}. \tag{18}$$

This determines the demand form up to the positive constant $k=1/h\left(\underline{p}\right)$ (in the position in the above formula). Then, following the lines of the earlier proofs, the other distributions are determined.[8] ■

Thus, knowing the demand at any one point ties down the whole demand function.

The three preceding Theorems spotlight the distribution pairs that only determine demand up to a constant (in a different position in each case). Perhaps surprisingly, the other three distribution pairs tie down all unknowns.

**Theorem 6** *Let there be a continuum of firms with demand (1) satisfying Assumption 1, but (1) is not directly observed. Knowing any of the pairs of distributions* $\{(F_C,F_Y),(F_C,F_\Pi),(F_P,F_Y)\}$ *suffices to tie down all the primitives and the economic distributions.*

**Proof.** Consider first when $F_C$ and $F_\Pi$ are known. Then, using (12) determines the function $\pi^*(c)=F_\Pi^{-1}(1-F_C(c))$ and hence $h^*(c)=-\pi^{*\prime}(c)=\frac{f_C(c)}{f_\Pi\left(F_\Pi^{-1}(1-F_C(c))\right)}$. This therefore ties

[8] See also Theorem 7 below, where we show that knowing demand and one distribution determines everything.

down the form of the equilibrium demand, and thence the mark-up function $\mu(c) = \frac{\pi^*(c)}{h^*(c)}$, and thence both the demand function and the price distribution. Because the equilibrium demand is also equilibrium output, the output distribution is found too.

Suppose now that $F_C$ and $F_Y$ are known. Hence equilibrium demand is known, and the steps above deliver the other distributions above: the profit distribution then follows directly.

Finally, suppose that $F_P$ and $F_Y$ are known. Because we can then relate quantity demanded to price paid, we know the form of $h(p)$ on its support. With that we can find the mark-up and relate it to output, and hence determine the profit distribution and all the rest. ■

In contrast to the preceding Theorems, all these distribution pairs fully tie down the specification of the underlying economic model. We illustrate with the same parameters as Example 1 that knowing the profit and cost distributions ties down the full model.

**Example 2:** *$\rho$-linear demands from uniform cost distribution.*

*Suppose that it is known that $F_C(c) = c$ for $c \in [0,1]$ and (as above) $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$. We first write $\pi^*(c)$ to find $h^*(c) = -\pi^{*\prime}(c)$. Matching the distribution levels, $1 - c = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$, or $\pi^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{(1+\rho)/\rho}$ and hence $y(c) = h^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{1/\rho}$, so both output and profit are power functions. Then we use $c = 1 - F_Y(y)$ with $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$ to get $\mu(c) = \frac{\pi^*(c)}{h^*(c)} = \left(\frac{\rho(1-c)+1}{1+\rho}\right) = [h^*(c)]^\rho$. Now use $p = \mu(c) + c$ to find $h(p) = (1 + \rho(1-p))^{1/\rho}$ and hence the ($\rho$-linear) demand form is tied down, including the value of the constant ($k = 1$: see (5), and consistent with the specification $\overline{c} = 1$).*

The last distribution pair result of Theorem 6 is particularly simple because knowing price and output effectively traces out the demand curve itself. Indeed, this suggests that knowing the demand curve and one distribution determines all the others. This is confirmed next.

## 2.5 From demand to distributions

Researchers often impose specific demand functions (such as CES, or logit). Here we forge the (potentially testable) empirical links that are imposed by so doing: we show that when a specific functional form is imposed for $h$ (as is done in most of the literature), then all the relevant distributions can be found and related together from just one of them.

**Theorem 7** *Let there be a continuum of firms with known demand (1), satisfying Assumption 1. Knowing any one of the four distributions $F_C$, $F_P$, $F_Y$, and $F_\Pi$ fixes the others.*

**Proof.** Let $p(c)$ denote the equilibrium price for a firm with cost $c$; from (3) we have $\mu'(c) > -1$ so that $p(c)$ is strictly increasing, and define the inverse relation as $c(p)$ strictly increasing. The relation $p(c)$ (and hence its inverse) is known from Theorem 1 when $h(.)$ is known.

If $F_C$ is known, then $F_P(p)$ is determined by $F_P(p) = F_C(c(p))$. Similarly, if $F_P$ is known, then $F_C(c) = F_P(p(c))$. Given $F_C$ and $F_P$ are known, then all the other distributions are known, by Theorem 3 (the constant is already determined by knowledge of $h(.)$).

If $F_Y$ is known, because $h(p)$ is strictly decreasing, then $F_P$ is recovered by $F_P(p) = 1 - F_Y(h(p))$. By the argument of the previous paragraph, $F_C$ is recovered, and hence so is $F_\Pi(\pi) = 1 - F_C(c^{-1}(\pi))$, where $c^{-1}(\pi)$ inverts $\pi^*(c)$.

Finally, suppose that $F_\Pi$ is known. By Theorem 1 we know that profit $\pi^*(c) = \mu(c)\, h^*(c)$ is a strictly decreasing function. Therefore $F_C(c)$ is recovered from $F_C(c) = 1 - F_\Pi(\pi^*(c))$. As per the arguments above, $F_P$ is recovered, and hence so is $F_Y$. ■

The key relation underlying the twinning of distributions is the decreasing relation (see Corollary 1) between cost and output (or profit). The theorem uses this decreasing relation to describe how prices and costs can be determined from output or profit distributions. A specific cost distribution generates a specific output (resp. profit) distribution. Conversely, this output or profit distribution could only have been generated from the initial cost distribution. These links are exploited below in Sections 3 and 5. In particular, for the CES to which we now turn, all distributions are from the same class.

# 3 CES models

A flurry of recent contributions use the CES and variants thereof (e.g., Dhingra and Morrow, 2013, Zhelobodko, Kokovin, Parenti, and Thisse, 2012, Bertoletti and Etro, 2014, etc.). Most noticeably, it has enjoyed a huge spurt in popularity in the new international trade literature.[9]

Here we apply the distributional analysis to the CES. We start with the standard CES monopolistic competition model with heterogeneity only in firms' unit production costs (this is the basic Melitz, 2003, approach). Hence, from Theorem 7, all economic distributions (prices, output, profit, and revenue) are tied down by the cost distribution.

A central distribution in the literature has been the Pareto. We show that all relevant distributions are Pareto if any one is (caveat: for prices and costs it is the distribution of the reciprocal that is Pareto). This result we term the *Pareto circle.* To put this another way, if we posit that the reciprocal of costs is Pareto distributed (equivalently, costs have a power distribution), then so is the reciprocal of prices, and the other variables (output, revenue, and profit) are all Pareto distributed. It is not possible to have (for example) a Pareto distribution for profits and (another) Pareto distribution for prices in the CES model. The Pareto circle cannot be escaped if one element is Pareto. Similar results hold for other distributions, yielding a more general *CES circle.*

Following Baldwin and Harrigan (2011) and Feenstra and Romalis (2014), we therefore introduce a further dimension of heterogeneity, interpreted as "quality." We link the two distributions via a "bridge" function that writes quality as a function of cost. Doing this then enables us to get two linked groups of distributions. In one group are profit and revenue, and in the

[9] Although note that Fajgelbaum, Grossman, and Helpman (2009) take a nested multinomial logit approach.

other are costs and prices, while output forms a convex combination. Our leading example is a bridge function that delivers Pareto distributions in each group. We first develop the analysis for cost heterogeneity alone.

## 3.1 Standard CES model

Several forms of CES representative consumer utility functions are prevalent in the literature. We nest these into one embracing form. The CES representative consumer involves a sub-utility functional for the differentiated product $\chi = \left(\int_\Omega q(\omega)^\rho \, d\omega\right)^{1/\rho}$ with $\rho \in (0,1)$ (with $\rho = 1$ being perfect substitutes, and $\rho \to 0$ being independent demands), and the $q$'s are quantities consumed of the differentiated variants. Common forms of representative consumer formulation are (i) Melitz model (see also Dinghra and Morrow, 2013), where $U = \chi$ so there is only one sector); (ii) the classic Dixit-Stiglitz (1977) case much used in earlier trade theory, $U = \chi q_0^\eta$ with $\eta > 0$, where $q_0$ is consumption in an outside sector; (iii) $U = \ln \chi + q_0$, which constitutes a quasi-linear form (with no income effects) and so constitutes a partial equilibrium approach (see Anderson and de Palma, 2000). The first two have unit income elasticities, hence their popularity in Trade models. Utility is maximized under the budget constraint $\int_\Omega q(\omega) \, p(\omega) \, d\omega + q_0 \leq I$, where $I$ is income.

The next results are quite standard. For a given set of prices and a set $\Omega$ of active firms (with total mass $M = \|\Omega\|$), Firm $i$'s demand (output) is:

$$h(p_i) = \frac{\Xi(I)}{p_i} \frac{p_i^{\frac{\rho}{\rho-1}}}{\int_{\omega \in \Omega} p(\omega)^{\frac{\rho}{\rho-1}} \, d\omega}, \tag{19}$$

where $\Xi(I)$ is $I$ for case (i), $\frac{I}{1+\eta}$ for case (ii) (which clearly nests case (i) for $\eta = 0$); and 1 for case (iii). In each case, $\Xi(I)$ is the total amount spent on the differentiated commodity. The denominator in (19) represents the aggregate impact of firms' actions on individual demand: under monopolistic competition, each firm's action has no effect on this statistic.

The price solves $\max\limits_{p_i} \frac{(p_i - c_i)}{p_i} p_i^{\frac{\rho}{\rho-1}}$, so $p_i = \frac{c_i}{\rho}$, and the Lerner index is $\frac{p_i - c_i}{p_i} = (1-\rho)$. Given such pricing, Firm $i$'s equilibrium output is

$$h^*(c_i) = \rho \Xi(I) \frac{c_i^{\frac{\rho}{\rho-1}-1}}{D_C}, \tag{20}$$

where $D_C = M \int c(u)^{\frac{\rho}{\rho-1}} f_C(u)\, du$, and $f_C(.)$ is the unit cost density. Firm $i$'s equilibrium profit is proportional to its sales revenue, $r_i = p_i h^*(p_i)$, with $r_i = \Xi(I) \frac{c_i^{\frac{\rho}{\rho-1}}}{D_C}$, so $\pi_i^* = (1-\rho) r_i$.

We can now tie together the various equilibrium distributions with the help of the following straightforward result, which tells us how distributions are modified by powers and multiplicative transformations. These transformations relate profit, revenue, output, price reciprocal $(1/p)$, and cost reciprocal $(1/c)$ in the CES model. The latter is *productivity* in Mrázová, Neary, and Parenti (2016).

**Lemma 2** *(Transformation) Let $F_X(x)$ be the CDF of a random variable $X$. Then the CDF of $kX^\theta$ with $k > 0$ is $F_{kX^\theta}(x) = F_X\left[\left(\frac{x}{k}\right)^{\frac{1}{\theta}}\right]$ for $\theta > 0$, and $F_{kX^\theta}(x) = 1 - F_X\left[\left(\frac{x}{k}\right)^{\frac{1}{\theta}}\right]$ for $\theta < 0$.*

For example, power distributions beget power distributions under positive power transforms and Pareto distributions under negative power transforms. Furthermore, normal distributions beget normal distributions in both cases, due to the symmetry of the normal distribution, etc. We refer to pairs of distributions with the same functional forms but different parameters as being in the same *class* (e.g., Pareto, power, normal distributions are all classes).

**Proposition 1** *(CES circle) For the CES, the distributions of profit, revenue, output, price reciprocal and cost reciprocal are all in the same class.*

**Proof.** From the analysis above, all these variables for the CES involve positive power transformation and/or multiplication by positive constants. Profit is proportional to revenue; price is proportional to cost, and likewise for their reciprocals. From (20), equilibrium output, $h^*(c_i)$, is related to the cost reciprocal, $1/c_i$, by a positive power and a positive factor. The other relations follow directly. ■

In particular, if any one of these distributions is Pareto (resp. power), then they all are Pareto (resp. power) class, although they have different parameters. Similarly, if one is normal (resp. log-normal) then all are normal (resp. log-normal). This result we term the *CES-circle.* It means that the standard CES model with cost heterogeneity alone cannot deliver (say) Pareto distributions for *both* profit and prices. Indeed, if profit is Pareto distributed, then price must follow a power distribution. We next introduce quality heterogeneity to break the CES-circle.

### 3.2 CES quality-enhanced model

To now extend the model to allow for quality differences across products, we rewrite the sub-utility functional as $\chi = \left(\int_{\Omega} z(\omega)^{\rho} d\omega\right)^{1/\rho}$ with $\rho \in (0,1)$ and interpreting $z(\omega) = v(\omega) q(\omega)$ as the quality-adjusted consumption (see Baldwin and Harrigan, 2011, and Feenstra and Romalis, 2014). The corresponding demands are:

$$h(p_i, \hat{p}_i) = \frac{\Xi(I)}{p_i} \frac{\hat{p}_i^{\frac{\rho}{\rho-1}}}{\int_{\omega \in \Omega} \hat{p}(\omega)^{\frac{\rho}{\rho-1}} d\omega}, \tag{21}$$

where we have defined $\hat{p}_i = p_i/v_i$ which is interpreted as the price per unit of "quality" and $\Xi(I)$ is as above for the three different cases (the amount spent on the differentiated commodity). The key feature of (21) is that $p_i$ enters both with and without quality in the denominator. The standard model (19) ensues when all the $v$'s are the same.

With a continuum of firms (as per the usual monopolistic competition set-up), Firm $i$'s equilibrium price solves $\max_{p_i} \frac{(p_i - c_i)}{p_i} \hat{p}_i^{\frac{\rho}{\rho-1}}$ so the pricing solution $p_i = \frac{c_i}{\rho}$ still holds. Hence, using $x_i = v_i/c_i$,[10] which we refer to as quality/cost, all firms set the same proportional mark-up, and

[10] This is quality/cost whereas the logit model below has quality-cost.

the equilibrium profit is

$$\pi_i = (1-\rho)\,\Xi(I)\,\frac{x_i^{\frac{\rho}{1-\rho}}}{\int_{\omega\in\Omega} x(\omega)^{\frac{\rho}{1-\rho}}\,d\omega} = (1-\rho)\,r_i. \tag{22}$$

Equilibrium profit is still a fraction $(1-\rho)$ of revenue. This implies that profit, sales revenue, and quality/costs distributions are in the same class.[11] There at most two distribution classes.[12] Price and cost distributions are still in the same class as each other, but reciprocal costs and profits are not necessarily in the same class. How the cost and profit distributions are linked is determined by the relation between cost and quality. A functional relation between cost and quality/cost ties down the bridging relation, and the distributions on the "other" side. For what follows, we define two distributions as in the *same* class if they have the same functional form. One distribution is the *inverse* of another one if it is the survival function of the other distribution.

#### 3.2.1 Constant elasticity bridging function

A central example of a bridging function is $x = c^{\gamma}$ so that quality/cost is increasing with cost (so quality rises faster than cost) if $\gamma > 0$ and it is decreasing if $\gamma < 0$. The latter case is embodied in the standard CES model above where $\gamma = -1$ and so "better" firms are those with lower costs. The former case effectively corresponds to Feenstra and Romalis (2014).[13] The advantage of the constant elasticity bridging function is that it allows us to deploy results (Lemma 2) on applying power transforms to random variables.

Because profits are proportional to $x_i^{\frac{\rho}{1-\rho}}$ (see (22)), they are proportional to $c_i^{\gamma\frac{\rho}{1-\rho}}$. Hence if $\gamma > 0$ profits are in the same distribution class as costs. So then too are sales revenues and

[11] Profits are increasing in $x$ so that firms would like this as large as possible. We can link cost and quality through a type of production (or "bridge") function and have (heterogeneous) firms choose their $x$. More anon.

[12] We see in the next Section that an alternative way to introduce quality gives a broader set of distributions.

[13] Along the same lines as Feenstra and Romalis (2014), we can let $v = l^{\alpha}$ be the quality produced at cost $wl+\phi$ with $\phi$ a firm-specific productivity shock, where $l$ is labor input, $w$ is the wage, and $\alpha \in (0,1)$. Maximizing $x = l^{\alpha}/(wl+\phi)$ gives the optimized value relation between cost and quality as $x = \left(\frac{\alpha}{w}\right)^{\alpha} c^{\alpha-1}$ and so the bridge function takes a power form. Here it is decreasing (and depends on the fundamental via $x = \phi^{\alpha-1}$).

quality-costs (see (20)). But if $\gamma < 0$, profits, revenues and quality-costs are in the "opposite" (or "inverse") class - this is the generalization of the earlier standard CES result. Prices, of course, are in the same class as costs, but output is more intricate because it draws its influences from both sides. Indeed, output is proportional to $x_i^{\frac{\rho}{1-\rho}}/c_i$ (see (21)) which equals $c_i^{\gamma\frac{\rho}{1-\rho}-1}$ under the constant elasticity formulation. This implies that for $\gamma > \left(\frac{1-\rho}{\rho}\right)$ the output and cost distributions are in the same class, while otherwise they are in inverse classes. A summarizing statement:

**Proposition 2** *(Breaking the CES circle) Consider the quality-enhanced CES model of monopolistic competition with $x = c^\gamma$. Then:*

*i) the equilibrium price distribution mirrors the unit cost distribution;*

*ii) equilibrium profits, sales revenue, and quality/cost are in the same distribution class as unit costs for $\gamma > 0$ and in the inverse class for $\gamma < 0$;*

*iii) equilibrium output is in the inverse distribution class from unit cost for $\gamma < \left(\frac{1-\rho}{\rho}\right)$, and in the same distribution class for $\gamma > \left(\frac{1-\rho}{\rho}\right)$.*

Note that inverse distributions take the same form for symmetric distributions such as the Normal, so then all distributions belong to the same class – once a normal, always a normal.

Take the example of a Pareto distribution for costs. First, prices are also Pareto distributed. Second, profits, revenue, and quality/cost are Pareto distributed for $\gamma > 0$ and power distributed for $\gamma < 0$ (they are independent of cost if $\gamma = 0$). Third, output is power distributed for $\gamma < \left(\frac{1-\rho}{\rho}\right)$, and Pareto distributed for $\gamma > \left(\frac{1-\rho}{\rho}\right)$.[14] Hence, *we resolve the puzzle of getting Pareto distributions for both prices and profits by including the appropriate bridge function.*

Proposition 2(ii) indicates that quality/cost and profits are in the same distribution class. For example, suppose that the distribution of quality/costs is Pareto: $F_X(x) = 1 - \left(\frac{\underline{x}}{x}\right)^\lambda$ and

[14] If costs are power distributed, Pareto and power are reversed in the above statements.

assume that $\lambda\frac{1-\rho}{\rho} > 1$. Then the size distribution of profit is Pareto with tail parameter $\alpha_\Pi = \lambda\frac{1-\rho}{\rho}$. The well-known claimed empirical regularity "80-20" rule (that the top 20% of firms account for 80% of sales) corresponds to a value $\alpha_\Pi$ of 1.161. Our result is that the profit tail parameter is the confluence of a preference parameter and a quality/cost distribution one.[15]

The CES is special in many respects, even with quality introduced as above. First, the CES still involves only two distributions (and one is the inverse class of the other for the constant elasticity bridging function). Also, prices are independent of qualities, but cost increases are passed on at over 100% (because $p_i = \frac{c_i}{\rho}$). We next return to the general demand function of Section 2, and introduce quality slightly differently, and then deploy a general bridging function between quality and cost. Then we specialize to the Logit case.

# 4 The quality-cost model

We now introduce heterogeneous qualities into the original demand model (1), and, with an eye to the "bridging" function we just illustrated for the CES, we will want to know the relation between quality and cost that we can uncover from the demand form and economic and/or primitive distributions. We start by delivering some quality pass-through results, which hold some independent interest by extending the concept of cost pass-through to quality pass-through.

**Assumption 2** *Suppose that demand for firm with quality $v$ charging price $p$ is*

$$y = \hat{h}(v - p), \tag{23}$$

*a positive, strictly increasing, strictly $(-1)$-concave, twice differentiable function.*

This is analogous to Assumption 1 for concavity properties, except demand now increases in its argument (now quality-price instead of price before). We will also want to distinguish at

[15] Although why they yield the same constant across settings remains intriguing.

later junctures between different degrees of concavity. To this end, we introduce the variant:

**Assumption 2′** *$\hat{h}(v-p)$ is a positive, strictly increasing, strictly log-concave, and twice differentiable function.*

Profit for a firm of quality $v$ with cost $c$ charging price $p$ is $\pi = (p-c)\hat{h}(v-p) = m\hat{h}(x-m)$, where $m = p-c$ is its mark-up, and $x = v-c$ is its *quality-cost* (to be read as quality minus cost).[16] Its equilibrium mark-up satisfies

$$m = \frac{\hat{h}(x-m)}{\hat{h}'(x-m)}. \tag{24}$$

Letting $\mu(x)$ denote the mark-up (with $\mu'(x) < 1$ when $\hat{h}(.)$ is strictly $(-1)$-concave), and $\hat{h}^*(x) = \hat{h}(x-\mu(x))$ the demand under the maximizing mark-up, we have $\hat{h}^{*\prime}(x) > 0$ and

$$\frac{d\hat{h}^*(x)/dx}{\hat{h}^*(x)} = \frac{1-\mu'(x)}{\mu(x)} \equiv \hat{g}(x) > 0. \tag{25}$$

The next Theorems extend Theorems 1 and 2 to allow for heterogeneous qualities: proofs follow the same lines.

**Theorem 8** *Under Assumption 2, the equilibrium mark-up, $\mu(x)$ is the unique solution to (24), with $\mu'(x) < 1$: $\mu'(x) \geq 0$ if $h(.)$ is log-concave and $\mu'(x) \leq 0$ if $h(.)$ is log-convex. The associated equilibrium demand, $\hat{h}^*(x) \equiv \hat{h}(x-\mu(x))$, is strictly increasing, as is $\hat{\pi}^*(x) = \mu(x)\hat{h}^*(x)$, with $\pi^{*\prime}(x) = h^*(x) > 0$.*

The implication is that "better" firms use their advantage to leverage equilibrium output; when demand is strictly log-concave they also extract higher mark-ups:

**Corollary 2** *Under Assumption 2, higher quality-costs are associated with higher output and consequently with higher profit; under Assumption 2′ they also have higher markups.*

[16] By the envelope theorem, the maximized value, $\pi_i^*(x_i)$ is increasing in $x_i$: see also the next Theorem.

Conversely, if demand is strictly log-convex, firms set *lower* mark-ups and exploit the convexity of demand for their larger profits. The converse result to the previous Theorem is:

**Theorem 9** *Consider a mark-up function $\mu(x)$ for $x \in [\underline{x}, \bar{x}]$ with $\mu'(x) < 1$. Then there exists an equilibrium demand function $\hat{h}^*(\cdot)$ defined on its support $[\underline{x}, \bar{x}]$, which is unique up to a positive factor, and associated primitive demand function $\hat{h}(\cdot)$, which satisfies Assumption 2 on its support $[\underline{x} - \mu(\underline{x}), \bar{x} - \mu(\bar{x})]$; it satisfies 2' if $\mu'(x) > 0$.*[17]

Positive quality pass-through (which is equivalent to cost pass-through below 100%) is associated to log-concave demand: for $\hat{h}(\cdot)$ strictly log-concave, $\mu'(x) > 0$, so firms with higher quality-costs have higher mark-ups and outputs in the cross-section of firm types (see Corollary 2). Log-linearity (the Logit case) has constant mark-ups. When $\hat{h}(\cdot)$ is strictly log-convex, the mark-up decreases with $x$: this is analogous to a cost pass-through greater than 100%.[18] Notice that the property $\mu'(x) < 1$ is just the property that price never goes down as costs increase. As before, the function $\hat{h}(\cdot)$ is tied down only on the support corresponding to the domain on which we have information about the equilibrium value in the market.

In the sequel of this section, we first show in Theorem 10 how knowledge of the demand function and two other specific pieces suffices to uncover the missing ones. This is analogous to Theorem 7, except now with more informational requirements because of the extra quality dimension. We then show in Theorem 11 how output and profit distributions uncover the demand form (up to a positive shift). This is analogous to Theorem 4. However, we are now unable to recover the whole system because we have a further unknown dimension (quality). We then look (in the following sub-section) to results on finding the full model, analogous to a broader version of Theorem 4. We proceed in two steps. Although output and profit

[17] They are given by $\hat{h}^*(x) = \hat{h}^*(\underline{x}) \exp\left(\int_{\underline{x}}^{x} \hat{g}(v)\, dv\right)$, $x \geq \underline{x}$, which determines $\hat{h}^*(x)$ up to a positive factor, and $\hat{h}(u) = \hat{h}^*(\underline{x}) \exp\left(\int_{\underline{x}}^{\phi^{-1}(u)} \hat{g}(v)\, dv\right)$ with $\hat{g}(v)$ given in (25) and here $u = \phi(x) = x - \mu(x)$.

[18] We can have quality rise and mark-up go down immensely near the $(-1)$-concave limit: think too of cost pass-through; with a demand 1/p then a zero cost gives a price of zero, but a small cost gives an infinite price.

distributions suffice to find the quality-cost distribution and the demand form (Theorem 11), we need to then know the price distribution to find the quality-cost relation and the cost distribution. This is done in Theorem 12.

## 4.1 From demand form to distributions: Two Legs and one Bridge

Suppose first that the functional form of $\hat{h}(.)$ is known (or imposed by the modeler). We now show that knowing one distribution from $F_Y$, $F_\Pi$, and $F_X$ tells us what the others must be (see also Theorem 7). For example, suppose $F_X$ is known along with $\hat{h}(.)$ (which delivers $\hat{h}^*(x)$ and $\mu(x)$ by Theorem 8). Then we can derive (using Theorem 8 that $\hat{h}^*(x)$ and $\hat{\pi}^*(x)$ are strictly increasing) the output distribution:

$$F_Y(y) = \Pr(Y < y) = \Pr\left(\hat{h}^*(X) < y\right) = \Pr\left(X < \hat{h}^{*-1}(y)\right) = F_X\left(\hat{h}^{*-1}(y)\right),$$

and, analogously, for profit, $F_\Pi(\pi) = \Pr(\hat{\pi}^*(X) < \pi) = F_X\left(\hat{\pi}^{*-1}(\pi)\right)$. By analogous arguments, knowing $F_Y$ or $F_\Pi$ along with $\hat{h}(.)$ determines the other two distributions.

Thus the distributions of quality-cost, output, and profit are determined from any one of them. The other pair of interest are the price and the cost distributions. The link between any of the former distributions and either of the latter two is determined by the relation between costs and quality-costs. This section draws together these relations, and shows how the link between distributions can be determined. Conversely, knowing the relation between cost and quality-cost and one distribution enables us to tie down the other distributions.

We proceed by describing the two separate groups of distributions (the two "legs") and how they are linked (the "bridge").

**Leg #1: Quality-cost, output, or profit.** As shown above, knowledge of any one of these distributions ties down the other two.

**Leg #2: Cost (or price).**

**Bridge: Quality-cost or cost.** We link the distributions across the two legs by postulating a functional relation between tastes and technology, and so we link quality-cost from the first leg with cost (or sometimes price) from the second leg.

Suppose then that it is known that $x = \beta(c)$. Then we can determine the relation between quality, $v$, and cost as $v = \beta(c) + c$. Several cases are possible. Normally, one might expect that quality should increase with cost, so $\beta'(c) > -1$. Otherwise, quality might be increasing or decreasing in $c$ or non-monotonic. A hump-shaped relation represents highest quality-costs for intermediate cost levels (see Figure 2 below). We can either treat $\beta(c)$ as a datum to determine other relations, or else we can infer it from a seed distribution from each distribution leg. We can also view both $x$ and $c$ as determined by the firm, depending on the firm's type. In this case the bridge function traces the relation between optimal choices.[19]

We now synthesize the relations between the different groups of relations.

**Theorem 10** *Assume that demand, $\hat{h}(.)$, in (23) is known and satisfies A2. If one element is known from two of the three following groups, then all elements are known:*

*i) a distribution of quality-cost, profit, or output ($F_X$,$F_Y$,$F_\Pi$);*

*ii) the cost distribution ($F_C$);*

*iii) a strictly monotone relation between $x$ and $c$.*

*If $\beta'(c)\,\mu'(x) > 0$, then the price distribution ($F_P$) can be added to leg (ii).*

**Proof.** Given that $\hat{h}(.)$ is known, then $\mu(x)$, $\hat{h}^*(x) = y$, and $\pi^*(x)$ are determined by Theorem 8. If $x = \beta(c)$ is strictly increasing in $c$ then "better" products are those with *higher* costs; they thus have higher outputs and profits. This all means that we have a sufficient set of monotonicity relations, and we can write a common distribution level as $z = F_C(c) = F_X(x) = F_Y(y) =$

[19]Because higher $x$ gives higher equilibrium profit, then each firm wants to maximize $x$ under whatever technological transformation it faces. For example, following the lines of Feenstra and Romalis (2014), we could assume a production function of the form $v = l^\alpha \theta$ with corresponding cost $wl$, where $l$ is labor input, $w$ is the wage, $\alpha \in (0, 1)$, and $\theta$ is a firm-specific productivity parameter. Then we get a linear bridge function $x = \frac{1-\alpha}{\alpha}c$. This one shows up in the last example of this section.

$F_{\Pi}(\pi)$. Similarly, if $x = \beta(c)$ is strictly decreasing, then "better" products are those with *lower* costs. Then the common distribution level satisfies $z = 1 - F_C(c) = F_X(x) = F_Y(y) = F_{\Pi}(\pi)$.

If we know $\beta(c)$ (i.e., Leg (iii)), then all of the other relations ($F_C$,$F_X$,$F_Y$,$F_{\Pi}$) are determined through knowing any one of them, by logic analogous to that in Theorem 7.

Suppose now (iii) is not known. If $F_X$ is known, we can determine $F_Y = F_X\left(\hat{h}^{*-1}(y)\right)$ and $F_{\Pi} = F_X\left(\hat{\pi}^{*-1}(y)\right)$ from the relations at the start of the proof. Likewise, knowing $F_Y$ or $F_{\Pi}$ determines the other distributions on that leg (as argued at the start of Section 4.1). Via the mark-up function, knowledge of $F_C(c)$ determines the bridge (iii). However, knowing just $F_C(c)$ cannot be used to infer any relation on leg (i) (even if the price distribution were known).

Finally, if $\hat{h}(.)$ is strictly log-concave, then $\mu'(x) > 0$, and, if $\beta'(c) > 0$, then prices necessarily increase with costs. We then further know that the relation $z = F_C(c) = F_P(p)$ holds, so knowing either the cost or price distribution (on Leg (ii)) ties down all distributions. Alternatively, if $\hat{h}(.)$ is strictly log-convex, then $\mu'(x) < 0$; if $\beta'(c) < 0$, then prices necessarily increase with costs. We then know that $z = 1 - F_C(c) = 1 - F_P(p)$, and hence knowing either distribution on leg (ii) tells all. ■

In Theorem 10, Assumption 2′ (that $\hat{h}(.)$ in (23) be strictly log-concave) along with $\beta(.)$ strictly increasing ensures that the appropriate monotonicity conditions hold for us to be able to apply Lemma 1 and determine the price distribution too. Similar price-cost monotonicity also prevails when $\hat{h}(.)$ is strictly log-convex (so that $\mu'(x) < 0$) and $\beta(.)$ is strictly decreasing.

The construction of $F_C(c)$ from $F_X(x)$ and increasing $\beta(c)$ is shown in Figure 1. In the upper right panel we have the "seed" distribution $F_X(x)$, and below it is $\beta^{-1}(x)$. Values of $x$ map into values of $c$ via the relation $\beta^{-1}(x)$ in the lower right panel and hence through the lower left panel into values of $c$ in the upper left panel, where the corresponding value from $F_X(x)$ therefore yields the desired value of $F_C(c)$. The Figure also shows the converse constructions.

INSERT FIGURE 1: Relation between (increasing) cost and quality-cost.

Now consider when the higher quality-cost products are at the lower end of the cost spectrum. This is an important case because it corresponds to the extant literature à la Melitz (2003), which entertains only cost differences.[20] Quality-costs decrease with cost, which entails a reversal of the ordering of products. Similar to Figure 1, a decreasing $\beta$ function can be constructed from $F_C(c)$ and $F_X(x)$.

We next consider the case where $\beta(c)$ is increasing from $\underline{c}$ to $\hat{c}$ and decreasing from $\hat{c}$ to $\bar{c}$. Quality rises faster than cost at first, and then rises slower or even falls (if $\beta' < -1$). This case involves highest quality-cost (and hence highest output and profit, by Theorem 1) for middling cost levels. The cumulative quality-cost distribution is derived from the two pieces. Suppose that $\beta(c) < \beta(\bar{c})$ for $c \in [\underline{c}, \tilde{c})$ (and so $\beta(\tilde{c}) = \beta(\bar{c})$). Then $F_X(x)$ is derived from $F_C(c)$ via $F_X(x) = F_C\left(\beta^{-1}(x)\right)$ for $x \in [\beta(\underline{c}), \beta(\tilde{c}))$. Higher $x$ values can come from either the increasing or decreasing part of $\beta$, and we need to sum the two contributions.

Define $\beta_U^{-1}(x)$ as the inverse function for $\beta$ increasing (i.e., corresponding to $c < \hat{c}$) and $\beta_D^{-1}(x)$ as the inverse function for $\beta$ decreasing (i.e., corresponding to $c > \hat{c}$). Then for $x \in [\beta(\tilde{c}), \beta(\hat{c})]$, $F_X(x)$ is given as the sum of the contributions from the two parts, as per the second line below. Summarizing:

$$F_X(x) = \begin{cases} F_C\left(\beta^{-1}(x)\right) & \text{for } x \in [\beta(\underline{c}), \beta(\tilde{c})) \\ F_C\left(\beta_U^{-1}(x)\right) + 1 - F_C\left(\beta_D^{-1}(x)\right) & \text{for } x \in [\beta(\tilde{c}), \beta(\hat{c})]. \end{cases}$$

Notice indeed that $F_X(x)$ is increasing, with a kink up at $\beta(\tilde{c})$, and that $F_X(\beta(\hat{c})) = 1$. The $\beta(.)$ function used above is illustrated in Figure 2, where $F_C(c) = c$ for $c \in [0,1]$, and $\beta(c) = c\left(\frac{4}{3} - c\right)$ for $c \in [0,1]$, so that $\tilde{c} = \frac{1}{3}$ and $\hat{c} = \frac{2}{3}$. Then $F_X(x) = \frac{2}{3} - \sqrt{\frac{4}{9} - x}$ for $x \in [0, \frac{1}{3}]$ and $F_X(x) = 1 - 2\sqrt{\frac{4}{9} - x}$ for $x \in [\frac{1}{3}, \frac{4}{9}]$.

INSERT FIGURE 2: Hump relation between cost and quality-cost.

[20] We situate this on its home ground in the CES model in Section 3 (Section 5.4 considers the endogenous set of product quality-costs for logit).

## 4.2 Deriving demand form from output and profit distributions

Now suppose demand is not known. We first engage the output and profit distributions ($F_Y$ and $F_\Pi$) to show how to back out the underlying quality-cost distribution ($F_X$), and the implied demand. The proof of the next Theorem follows the same lines as Theorem 4.

**Theorem 11** *Let there be a continuum of firms with demand (23) satisfying Assumption 2, but is not directly observed. Assume that the equilibrium distributions of output and profit are known. The quality-cost distribution, equilibrium demand and the mark-up function are given by (27)-(29), up to an additive constant $\underline{x}$ in their supports.*

**Proof.** By Theorem 8, $y = \hat{h}^*(x)$ is strictly increasing in $x$, and so too is $\hat{\pi}^*(x) = \mu(x)\hat{h}^*(x)$. Choose some arbitrary level $z \in (0,1)$ such that

$$F_X(x) = F_Y(y) = F_\Pi(\pi) = z. \tag{26}$$

Firm types with quality-cost levels below $x = F_X^{-1}(z)$ are those with outputs and profits below $y$ and $\pi$. Then $y(z) = F_Y^{-1}(z)$ and demand is

$$\hat{h}^*(x) = y(z(x)) = F_Y^{-1}(F_X(x)). \tag{27}$$

Because $\pi^*(z) = m(z)\, y(z) = F_\Pi^{-1}(z)$ then

$$m(z(x)) = \frac{F_\Pi^{-1}(z(x))}{F_Y^{-1}(z(x))} = \mu(x), \tag{28}$$

and equilibrium profit is $\pi^*(x) = \mu(x)\hat{h}^*(x) = F_\Pi^{-1}(z(x))$, with $z(x) = \Psi^{-1}(x - \underline{x})$, where $\Psi(z)$ is given by (16). Using (26),

$$F_X(x) = z(x) = \Psi^{-1}(x - \underline{x}), \tag{29}$$

and the remaining unknowns can now be backed out knowing $z(x)$. ■

As per our earlier discussion for the case of equal qualities, the demand position is determined up to a positive shift in inverse demand (i.e., net inverse demand is tied down). Demand is tied down once $\underline{x}$ is known.

### 4.2.1 From price distribution to all

If we also have the price distribution, $F_P(p)$ then we can furthermore back out the cost distribution, $F_C(c)$, and the quality-cost relation $\beta(c)$ under some conditions. The steps are as follows. First, determine the mark-up distribution, $F_M(m)$, from the mark-up relation $\mu(x)$ and the quality-cost distribution, $F_X(x)$. Then, use $F_M(m)$ with the price distribution to uncover the underlying cost distribution, $F_C(c)$. Matching this with $F_X(x)$ uncovers the relation between cost and quality-cost (the function $\beta(c)$ from Section 4.1).

In the sequel we shall consider the special case of strictly log-concave $\hat{h}(.)$, which implies (and is implied by) $\mu'(x) \in (0,1)$. Knowledge of $F_Y(y)$ and $F_\Pi(\pi)$ determines $\mu(x)$ and $F_X(x)$ from Theorem 11. Then we can derive the mark-up distribution, $F_M(m)$, from the relation:

$$F_M(m) = \Pr(M < m) = \Pr(\mu(X) < m) = F_X\left(\mu^{-1}(m)\right), \qquad (30)$$

where we have used that $\mu'(x) > 0$.[21] With the mark-up function $F_M(m)$ thus determined, suppose that $F_P(p)$ is known. To uncover $F_C(c)$, it suffices that costs and mark-ups move together, so that higher costs also entail higher prices. Then we can match distributions by writing $F_P(p) = F_C(c) = F_M(m) = z$, with $p - c = m$, and we can uncover $F_C(c)$ through the relation

$$F_C^{-1}(z) = F_P^{-1}(z) - F_M^{-1}(z). \qquad (31)$$

With $F_C(c)$ thus determined, the final primitive to find is $\beta(c)$. We proceed as we did in

[21] Notice that there are implied properties on the resulting mark-up distribution function. Because $F_X(x) = F_M(\mu(x))$ then $\mu'(x) < 1$ implies $f_M(\mu(x)) > f_X(x)$. Note also that $\mu'(x) = \frac{f_X(x)}{f_M(\mu(x))} > 0$, as desired for a strictly log-concave $\hat{h}(.)$.

Section 4.1: given that $\beta'(c) > 0$, then $F_C(c) = F_X(x) = z$ (see Figure 1), and

$$x = \beta(c) = F_X^{-1}(F_C(c)). \tag{32}$$

The results above are summarized as follows.

**Theorem 12** *Let there be a continuum of firms with quality-cost increasing with cost and demand (23) satisfying Assumption 2′, but neither relation is directly observed. Assume that the distributions of price, output, and profit are known. Then the quality-cost distribution (up to $\underline{x}$), equilibrium demand, the mark-up function, the price and cost distributions, and the quality-cost bridge are given by (27)-(32)).*

The next example shows (for a uniform distribution of both output and prices, and square-root profits) how to use our results to find the other distributions, demand, and the bridge. It also clarifies the position and role of the additive constant $\underline{x}$. Note this constant is analogous to the one found in the set-up without quality heterogeneity (Theorem 4).

**Example 3: Price, output, and profit distributions known.**

*Suppose we know that $F_P(p) = k_p\left(p - \underline{p}\right)$ with $k_p < 2$; $F_Y(y) = 2ky - a$ for $y \in \left[\frac{a}{2k}, \frac{b}{2k}\right]$ with $0 < a < b$ $(= a+1)$ and $k > 0$; and $F_\Pi(\pi) = 2\sqrt{k\pi} - a$ for $\pi \in \left[\frac{a^2}{4k}, \frac{b^2}{4k}\right]$.*

*Then $F_\Pi^{-1}(z) = \frac{1}{k}\left(\frac{z+a}{2}\right)^2$ and $F_Y^{-1}(z) = \frac{z+a}{2k}$; the relation between $z$ and $x$ is given by (16) as $z = \Psi(z) = x - \underline{x}$, so $z = \Psi^{-1}(x - \underline{x}) = F_X(x) = x - \underline{x}$, and hence $F_X(x)$ is uniform on $[\underline{x}, \underline{x}+1]$. Then from (28), $\mu(x) = (x - \underline{x} + a)/2$, so $\mu'(x) > 0$ as stipulated. Then $\hat{h}^*(x) = (x - \underline{x} + a)/2k$ (using (27)), and the associated demand function is linear: $\hat{h}(v-p) = \hat{h}(x-m) = (x - \underline{x} + a - m)/k$ (from Theorem 9). Because $F_M(m) = F_X(\mu^{-1}(m))$, then $F_M(m) = 2m - a$, for $m \in \left[\frac{a}{2}, \frac{b}{2}\right]$.*

*Now, using (31), $F_C^{-1}(z) = F_P^{-1}(z) - F_M^{-1}(z)$. This gives $F_C^{-1}(z) = \left(\frac{z}{k_p} + \underline{p}\right) - \left(\frac{z+a}{2}\right) = z\left(\frac{1}{k_p} - \frac{1}{2}\right) + \underline{c}$ (whence we determine $\underline{c} = \underline{p} - \frac{a}{2}$). Hence we have a uniform cost distribution,*

$F_C(c) = \left(\frac{2k_p}{2-k_p}\right)(c - \underline{c})$.

*We can now determine the bridge function from (32) as* $x = \beta(c) = F_X^{-1}(F_C(c))$. *Hence* $\beta(c) = \left(\frac{2k_p}{2-k_p}\right)(c-\underline{c}) + \underline{x}$*; here* $\beta'(c) > 0$*, as stipulated, given the restriction* $k_p < 2$.[22]

Analogous results to those in Theorem 12 hold for $\beta'(c) < 0$ and $\mu'(x) < 0$, which corresponds to strictly log-convex demand (although recall we still require demand to be $(-1)$-concave to guarantee a unique maximum to firms' profit functions). If $\mu(x)$ is non-monotone, the primitives cannot be backed out on their full support, analogous to the discussion in Section 4.1. Note that $\mu'(x) = 0$ corresponds to the Logit case, which we analyze in more detail next.

# 5 The Logit model of monopolistic competition

We here derive specific results for the Logit model with quality-cost heterogeneity and a continuum of active firms, using a log-linear demand form. Total demand is normalized to 1, so output for Firm $i$ is a Logit function of active firms' qualities and prices:

$$y_i = \hat{h}(v_i - p_i) = \frac{\exp\left(\frac{v_i - p_i}{\mu}\right)}{\int_{\omega\in\Omega} \exp\left(\frac{v(\omega)-p(\omega)}{\mu}\right) d\omega + \exp\left(\frac{v_0}{\mu}\right)}, \quad i \in \Omega, \tag{33}$$

where $\mu > 0$ measures the degree of product heterogeneity and $v_0 \in (-\infty, \infty)$ measures the attractivity of the outside option (which could also be a competitive sector). We thus adapt the continuous Logit model (see Ben-Akiva and Watanada, 1981) to monopolistic competition.[23]

As before, the (gross) profit for Firm $i$ is $\pi_i = (p_i - c_i)\, y_i$, $i \in \Omega$. Since there is a continuum of firms, the own-demand derivative is $\frac{dy_i}{dp_i} = \frac{-y_i}{\mu}$, $i \in \Omega$, so that $\frac{d\pi_i}{dp_i} = y_i\left[1 - \frac{(p_i - c_i)}{\mu}\right]$, $i \in \Omega$: the term inside the square brackets is strictly decreasing in $p_i$, so the profit function is strictly

[22] Such a linear bridge function can arise from endogenous quality-cost choices of heterogeneous firms: see the example in Section 4.1.

[23] Anderson et al. (1992) show that logit demands can be generated from an entropic representative consumer utility function as well as the traditional discrete choice theoretic root (see McFadden, 1978).

quasi-concave and the profit-maximizing price of Firm $i$ is[24]

$$p_i = c_i + \mu, \quad i \in \Omega. \tag{34}$$

As we showed earlier for log-linear demand, the absolute mark-up is the same for all firms.[25] The corresponding equilibrium outputs are

$$y_i = \frac{\exp\left(\frac{x_i}{\mu}\right)}{\int_{\omega\in\Omega} \exp\left(\frac{x(\omega)}{\mu}\right) d\omega + \mathcal{V}_0}, \quad i \in \Omega, \tag{35}$$

where $\mathcal{V}_0 \equiv \exp\left(\frac{v_0}{\mu} + 1\right) \geq 0$, and recall that $x(\omega) = v(\omega) - c(\omega)$ is a one-dimensional parameterization of quality-cost. (35) verifies the output ranking over firms seen before: $y_i > y_j$ if and only if $x_i > x_j$, for $i, j \in \Omega$. Equilibrium (gross) profit is $\mu y_i$, $i \in \Omega$, so outputs and profits are fully characterized by quality-cost levels, yielding the following special case of Theorem 8:

**Proposition 3** *In the Logit Monopolistic Competition model, all firms set the same absolute mark-up, $\mu$. Higher quality-cost entails higher equilibrium output and profit.*

As seen in Section 4.1, insofar as higher qualities also bear higher costs then they are also higher priced, but output and profit may well be highest for medium-quality products.

## 5.1 Quality-cost, output, and profit distributions

Recall that the distribution of quality-cost is $F_X(x) = \Pr(X < x)$, with density $f_X(\cdot)$ and support $[\underline{x}, \infty)$. The corresponding distribution of equilibrium output, $F_Y(y)$, and the relation between $x$ and $y$ is[26]

$$y = \frac{1}{D}\exp\left(\frac{x}{\mu}\right), \quad y \geq \underline{y} = \frac{1}{D}\exp\left(\frac{\underline{x}}{\mu}\right), \tag{36}$$

[24] For oligopoly with $n$ firms, the equilibrium prices are (implicit) solutions to $p_i = c_i + \frac{\mu}{1-y_i}$, $i = 1...n$. Under symmetry, $p = c + \frac{\mu n}{n-1}$, which converges to $c + \mu$ as $n \to \infty$ (Anderson, de Palma, and Thisse, 1992, Ch.7).

[25] The CES model gives a constant *relative* mark-up property, $p_i^* = c_i(1 + \mu)$, regardless of quality (see Section 3). The similarity between the Logit and CES is not fortuitous: $\mu$ is related to $\rho$ in CES models by $\mu = \frac{1-\rho}{\rho}$. Both models can be construed as sharing their individual discrete choice roots (Anderson et al. 1992).

[26] Here all firms are active. Section 5.4 introduces fixed costs to render endogenous the set of active firms.

where we assume henceforth that $f_X(\cdot)$ ensures the output denominator $D$ (which is the aggregate variable) is finite, as is true for any finite support and the examples below:

$$D = M \int_{u \geq \underline{x}} \exp(u/\mu) f_X(u)\, du + \mathcal{V}_0, \tag{37}$$

and $M = \|\Omega\|$ is the total mass of firms. Equilibrium (gross) profit is $\pi = \mu y \geq \underline{\pi} = \mu \underline{y}$. The following uses results from Section 4.1.

**Proposition 4** *For the Logit Monopolistic Competition model, the quality-cost distribution, $F_X(x)$, generates the equilibrium output distribution $F_Y(y) = F_X(\mu \ln(yD))$ and the equilibrium profit distribution $F_\Pi(\pi) = F_X(\mu \ln(\pi D/\mu))$, where $D$ is given by (37). Conversely, $F_X(x)$ can be derived from the equilibrium output distribution as $F_Y\left(\frac{1}{D_Y}\exp\left(\frac{x}{\mu}\right)\right)$ with $D_Y = \frac{\mathcal{V}_0}{(1-My_{av})}$, or from the equilibrium profit distribution as $F_\Pi\left(\frac{1}{D_\Pi}\exp\left(\frac{x}{\mu}\right)\right)$ with $D_\Pi = \frac{\mathcal{V}_0}{(\mu - M\pi_{av})}$, where $y_{av}$ and $\pi_{av}$ denote average output and profit, respectively.*

(Proof in online Appendix 2).

Finally, the distribution of costs $F_C(c)$ and the distribution of prices $F_P(p)$ are related by the mark-up shift, so $F_C(c) = F_P(c+\mu)$, with $\underline{p} = \underline{c} + \mu$. Conversely, knowing the price distribution ties down the cost distribution when the mark-up level, $\mu$, is known. Note some special case results. First, there is no price dispersion if and only if there is no cost dispersion. Second, there is no profit dispersion if and only if there is no quality-cost dispersion: then there is only cost dispersion, which price dispersion mirrors. The "classic" symmetry assumption often analyzed in the literature since Chamberlin (1933) has neither cost nor profit dispersion.

## 5.2 Specific distributions

We derive the equilibrium profit distributions for the Logit model.[27] Proofs are in Appendix 2.

[27] The reverse relations hold by Proposition 4. Equilibrium output distributions are analogous (as $\pi = \mu y$).

The *normal* distribution is perhaps the most natural primitive assumption to take for quality-costs. Then profit $\Pi \in (0, \infty)$ is log-normally distributed. The log-normal has sometimes been fitted to firm size distribution (see Cabral and Mata, 2003, for a well-cited study of Portuguese firms). Note that a truncated normal begets a truncated log-normal (which is therefore important once we consider free entry equilibria below).

The simplest text-book case is the *uniform* distribution. Then the equilibrium profit $\Pi$ has distribution $F_{\Pi}(\pi) = \mu \ln\left(\frac{\pi D}{\mu}\right)$ and its density is unit elastic. A *truncated Pareto* distribution leads to a truncated Log-Pareto for profit (or output).

At a simplistic level, Proposition 4 indicates that we just need to find the log-distribution of the seed distribution. However, we still need to match parameters, as done in Appendix 2 for the examples, and we also need to find the corresponding expression for $D$ and ensure it is defined. Notice too that the methods described above work for more general demands under monopolistic competition (see Section 2).

The most successful function to fit the distribution of firm size has been the Pareto. We reverse-engineer using Proposition 4 to find the distribution of quality-cost. This gives:

**Proposition 5** *Let quality-cost be exponentially distributed:* $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$, $\lambda > 0$, $\underline{x} > 0$, $x \in [\underline{x}, \infty)$, *with* $\lambda\mu > 1$. *Then equilibrium output and profit are Pareto distributed:* $F_Y(y) = 1 - \left(\frac{\underline{y}}{y}\right)^{\alpha_y}$ *and* $F_{\Pi}(\pi) = 1 - \left(\frac{\underline{\pi}}{\pi}\right)^{\alpha_\pi}$, *where* $\alpha_y = \alpha_\pi = \lambda\mu > 1$. *Conversely, a Pareto distribution for equilibrium output or profit can only be generated by an exponential distribution of quality-costs.*

Thus the shape parameter, $\alpha_y = \alpha_\pi$, for the endogenous economic distributions depends just on the product of the taste heterogeneity and the technology shape parameter.[28]

If the price distribution follows the Pareto form (suggested as empirically viable) $F_P(p) =$

[28] $D$ is bounded if $\mu > 1/\lambda$: which requires that taste heterogeneity exceeds average quality-cost.

$1 - \left(\frac{\underline{p}}{p}\right)^{\alpha_p}$, with $p \in [\underline{p}, \infty)$ and $\alpha_p > 1$, the corresponding cost distribution is also Pareto:

$$F_C(c) = 1 - \left(\frac{\underline{c} + \mu}{c + \mu}\right)^{\alpha_p}, \quad c \in [\underline{c}, \infty), \ \alpha_p > 1. \tag{38}$$

Suppose for illustration that prices are Pareto distributed so that $F_C(c)$ is given by (38). If there were no quality heterogeneity, then we would find a power distribution for quality-cost,[29] which would therefore be inconsistent with the required exponential function predicated in Proposition 5. It is the extra relation that we are afforded via $\beta(c)$ that decouples the allowable distributions, and therefore can enable us to fit (for example) both Pareto prices and profits. The next Proposition gives the underlying $\beta(c)$ function.

**Proposition 6** *Let $F_P(p)$ be Pareto distributed with shape parameter $\alpha_p$ and let $F_\Pi(\pi)$ be Pareto distributed with shape parameter $\alpha_\pi > 1$ and $\underline{\pi} < \frac{\mu}{M}\frac{\alpha_\pi - 1}{\alpha_\pi}$, and suppose that $x = \beta(c)$ is an increasing function. Then $\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right)$, where $\beta(\underline{c}) = -\mu\ln\left[\frac{1}{\mathcal{V}_0}\left(\frac{\mu}{\underline{\pi}} - \frac{\alpha_\pi M}{\alpha_\pi - 1}\right)\right]$.*

**Proof.** Let $F_X(x)$ be the exponential function given in Proposition 5. Because $\beta(\cdot)$ is increasing, $F_C(c) = F_X(\beta(c)) = 1 - \exp[-\lambda(\beta(c) - \beta(\underline{c}))]$, $\lambda = \frac{\alpha_\pi}{\mu} > 0$, $\underline{c} > 0$, $c \in [\underline{c}, \infty)$. A Pareto price distribution with shape parameter $\alpha_p$ delivers the cost distribution (38). Equating these two expressions gives

$$\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right), \quad \beta(c) \in [\beta(\underline{c}), \infty].$$

Thus $\beta'(c) > 0$ so that valuations rise faster than costs. The lower bound of the distribution $\beta(\underline{c}) = \underline{x}$ is given from Proposition 5 and is given in the Proposition statement. ■

Thus we can close the loop and deliver a model consistent with Pareto distributions (for example) for both profit and price, once we allow for a quality-cost relation. Figure 1 above is based on just such a logit example (with parameters $\alpha_p = \alpha_\pi = \mu = 2$, $\beta(\underline{c}) = 0$, $\underline{c} = 0$, and

[29] I.e., calling the common quality level $\bar{v}$, $F_X(x) = \left(\frac{\bar{v} - \underline{x} + \mu}{\bar{v} - x + \mu}\right)^{\alpha_p}$, $x \in (-\infty, \underline{x}]$.

$\underline{x} = 1$). So far, we have assumed the population of firm types is fixed: we next use the logit as a vehicle to endogenize the surviving form types when there are entry costs.

## 5.3 Comparative statics of distributions

So far in the paper we have considered cross-section properties of distributions and how they can be uncovered. Although it is not our main theme, we can also give some indication of comparative static properties across equilibria. How do distributions change with underlying preference and technology changes? This is a more tricky question because the values of the aggregate variables in demand, which we were able to suppress in the cross section analysis because they do not change, are now endogenously determined. While a full treatment is beyond our current scope, we can give some pointers (and deliver results) for the current Logit model, and determine how economic distributions change with fundamentals.

We here therefore briefly consider the comparative static properties for the Logit model. Because we are dealing with distributions, the natural way of doing so is to engage first order stochastic dominance (fosd). Proofs for this sub-section are in the CEPR Discussion Paper.

**Proposition 7** *A fosd increase or a mean-preserving spread in quality-cost raises mean output and mean profit, and strictly so if the market is not fully covered (i.e., if $\mathcal{V}_0 > 0$).*

While moving up quality-cost mass will move up output mass ceteris paribus, it also increases competition for all the other firms (a $D$ effect), which ceteris paribus reduces their output. Mean output does not necessarily rise if mean quality-cost rises.[30]

Because the relation between output and profit distributions does not involve $D$, a fosd increase in output implies an increase in profit, and vice versa. However, a fosd increase in quality-cost does not necessarily lead to a fosd increase in output. Suppose for example

[30] Mean output rises with a mean-preserving spread but then if the mean of quality-cost is reduced slightly, mean output can still go up overall, so the two means can move in opposite directions.

that the increase in quality-cost is small for low quality-costs, but large for high ones. Then competition is intensified (an increase in $D$), and output at the bottom end goes down, while rising at the top end. So then there can be a rotation of $F_Y(.)$ (in the sense of Johnson and Myatt, 2006) without fosd (a similar rotation is delivered in Proposition 8 below). Nevertheless, specific examples do deliver stronger relations, as we show in Section 5.2.

We next determine how taste parameters feed through into the endogenous economic distributions.

**Proposition 8** *A more attractive outside option ($\mathcal{V}_0$) fosd decreases outputs and profits. More product differentiation ($\mu$) fosd increases outputs and profits for low quality-cost and fosd decreases them for high quality-cost; a lower profit implies a firm has a lower output.*

The first result is quite obvious, but the impact of higher product heterogeneity is more subtle.[31] When $\mu$ goes up, weak (low quality-cost) firms are helped and good ones are hurt. The intuition is as follows. With little product differentiation, consumers tend to buy the best quality-cost products. With more product differentiation (which increases the mark-up), consumers tend to buy more of the low quality-cost goods (which have lower outputs) and less of the high quality-cost goods (which have higher outputs). Hence, higher $\mu$ evens out demands across options. The fact that output may decrease and profit increase with $\mu$ follows because $\pi_i = \mu y_i$. Thus it can happen that doubling $\mu$ does not double the profit of the top quality-cost firms, but it may more than double the profit of the lowest quality-cost firms. Whether high or low qualities are most profitable depends on whether quality-costs rise or fall with quality.

[31] The proof proceeds by showing that the derivative of consumer surplus is given by the Shannon (1948) measure of information (entropy), which is positive.

## 5.4 Long run Logit

Here we develop the long-run analysis of the logit model following recent directions in Trade models, and emphasize the shape of the equilibrium distributions that ensue. A fuller (more general) analysis along the lines of the previous section would follow similar lines, but here we aim for simplicity. We assume in the groove of Melitz (2003) that firms first pay a cost $K_1$ to get a quality-cost draw, then they pay $K_2$ to actively produce. We solve backwards.[32] To put in play market size effects, we introduce market size (number of consumers) $N$ (which was normalized to 1 in the analysis so far).

For a given mass, $M$, of firms that have paid $K_1$, equilibrium involves all sufficiently good types paying the subsequent fixed cost $K_2$. The firm of type $\tilde{x}$ just covers its cost, $K_2$, and $\tilde{x} \geq \underline{x}$ if $K_2$ is low enough. All types $x \geq \tilde{x}$ will produce (because profits increase in $x$ by Proposition 3). The gross profit of firm with quality-cost $x$ is now

$$\pi(x, \tilde{x}) = \mu N \frac{\exp\left(\frac{x}{\mu}\right)}{D(M, \tilde{x})},$$

where $D(M, \tilde{x}) \equiv M \int_{u \geq \tilde{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du + \mathcal{V}_0$, $\underline{x} \leq \tilde{x}$ (see (37)). $D(M, \tilde{x})$ is decreasing in $\tilde{x}$ so that the profit of the marginal firm, $\pi(\hat{x}, \hat{x})$ is increasing in $\hat{x}$. Hence, as long as $\pi(\underline{x}, \underline{x}) < K_2$, there is a unique cut-off value $\hat{x}$ such that

$$\pi(\hat{x}, \hat{x}) = K_2. \tag{39}$$

This is the case we consider: otherwise, all firms enter, and all make strictly positive profits.

Once a firm has paid the cost $K_1$ to get a draw, it has a probability $1 - F_X(\hat{x})$ to get a good enough draw, and to be active. The mass of potentially active firms, $M$, is determined at

[32] If $X$ is observed by firms before entry, only the first part of the analysis should be retained.

the first step via the zero-profit condition:

$$\mu N \frac{\int_{u\geq\hat{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du}{D(M,\hat{x})} - K_2 \int_{u\geq\hat{x}} f_X(u)\, du = K_1. \tag{40}$$

The first term is the expected gross profit of a firm which has paid the entry cost $K_1$ to get a draw. The second is the fixed (continuation) cost, to be paid by all firms with a draw of at least $\hat{x}$. Inserting (39) in (40) gives

$$\int_{u\geq\hat{x}} \left(\exp\left(\frac{u-\hat{x}}{\mu}\right) - 1\right) f_X(u)\, du = \frac{K_1}{K_2}. \tag{41}$$

The LHS is monotonically decreasing in $\hat{x}$ so there is a *unique* solution for $\hat{x}$ that depends only on the parameters in (41) – it is independent of market size $N$ and of $\mathcal{V}_0$ (Bertoletti and Etro, 2014, show a similar neutrality result). The condition for an interior solution is that

$$\int_{u\geq\underline{x}} \left(\exp\left(\frac{u-\underline{x}}{\mu}\right) - 1\right) f_X(u)\, du < \frac{K_1}{K_2}. \tag{42}$$

Otherwise, all firms are in the market. Given the solution for $\hat{x}$, we can then determine $M$ from (39) with $D(M,\hat{x})$ and defining $x^c = \max\{\hat{x}, \underline{x}\}$:

$$M = \frac{\frac{\mu N}{K_2} \exp\left(\frac{x^c}{\mu}\right) - \mathcal{V}_0}{\int_{u\geq x^c} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du}. \tag{43}$$

Therefore $M > 0$ if

$$Z^c \equiv \frac{\mu N}{K_2} \exp\left(\frac{x^c}{\mu}\right) > \mathcal{V}_0. \tag{44}$$

Otherwise, there is no entry ($M = 0$). Note that condition (44) depends on both $K_1$ and $K_2$ since $\hat{x}$ depends on $K_1/K_2$.

**Proposition 9** *(Logit, long-run) Consider the Logit Monopolistic Competition model, with cost $K_1$ to get a quality-cost draw, and cost $K_2$ to actively produce. Some firms enter if (44) holds;*

*some of these entrants do not produce if (42) holds. Then the solution satisfies (39) and (43).*

This solution parallels that for the Melitz (2003) model.[33] Some comparative static properties readily follow. The elasticity of $M$ with respect to $N$ is (using (44)) $\left(1 - \frac{\mathcal{V}_0}{Z^c}\right)^{-1} > 1$. Hence, if the market is covered ($\mathcal{V}_0 = 0$), the number of firms is proportional to market size. Otherwise, firm numbers *more than double* (see also Melitz, 2003: Bertoletti and Etro, 2016, show the opposite case can arise when there are income effects).

Thus, the long-run uniquely determines $\hat{x}$ and $M$. From these, the long-run distribution of quality-cost is $\tilde{F}_X(x) = \frac{F_X(x) - F_X(\hat{x})}{1 - F_X(\hat{x})}$ for $x \in [\hat{x}, \bar{x}]$, and then Theorems 4 and 10 hold. Moreover, the inheritance properties of the key distributions still apply. In particular, if $F_X$ is an exponential distribution for $F_X$ then so is $\tilde{F}_X$ ($\tilde{F}_X = 1 - \exp(-\lambda(x - \hat{x}))$). Thus profits and outputs are Pareto, and then we can link to the cost and price distributions as we did before: *the size distribution of output and profit is Pareto, with shape parameter* $\lambda\mu$.

# 6 Conclusions

The basic ideas here are simple. Market performance depends on the economic fundamentals of tastes and technologies, and how these interact in the market-place.[34] The fundamental distribution of tastes and technologies feeds through the economic process to generate the endogenous distribution of economic variables, such as prices, outputs, and profits. Invoking the monopolistic competition assumption delivers a straight feed-through from fundamental distributions to performance distributions.

The CES model has been the workhorse model of monopolistic competition with asymmetric firms. In the CES formulation, the assumed distribution of productivity (cost reciprocal) is also the equilibrium distribution of outputs, profits, etc.: Pareto begets Pareto, and, as we

[33] We can readily include a second threshold for exporting firms in an international trade context.

[34] Firm size distributions came to the fore in Chris Anderson's (2006) work on the Long Tail of internet sales.

show the same holds for any other distribution. This cycle is broken by allowing for quality heterogeneity in the CES. Quality and cost differences are especially interesting for empirical work and studying asymmetric firms, but the CES formulation is still quite restrictive.

The Logit model gives some similar properties, while differing on others. For example, the simple CES has constant percentage mark-ups while the Logit has constant absolute mark-ups.[35] The Logit can be deployed for similar purposes as the CES, and has an established pedigree in its micro-economic underpinnings. It has a strong econometric backdrop which is at the heart of much of the structural empirical industrial organization revolution. For Logit, a normal quality-cost distribution leads to a log-normal distribution of firm size, and an exponential quality-cost distribution generates a Pareto distribution. But, our main cross-sectional analysis extends far beyond the restrictive IIA property of Logit and CES.

Indeed, our main results in the heart of the paper show how to back out the demand form from distributions. Surprisingly, this can be done just from profit and price distributions (for example) if firms differ only by production costs. Much richer configurations are delivered from engaging quality differentiation too. Still though the demand form can be deduced from profit and output distributions, and hence, with the additional knowledge of the price distribution, all the primitives of the model can be recovered. To do this, though, we relied on there being a one-dimensional underlying relationship between quality and costs, which we also showed how to recover. One direction for future research is to consider a multi-dimensional relationship.

When the demand form is known (or imposed), then if there is only cost heterogeneity it suffices to know only one distribution to tie down them all. Allowing too for quality heterogeneity yields a much richer palate of possibilities for distribution patterns. In that case, we showed that knowing one element from two of three groups delivers everything. This analysis shows prices and costs (one group) to be in a different orbit from output, profit, and quality

[35] The latter property is perhaps quite descriptive for cinema movies, DVDs, and CDs.

cost (the other distribution group), and they are linked through the third relation, which is that between quality and cost.

Another future research direction we only initiated here (in the context of Logit) is the comparative statics of distributions. Hopefully, we have given some pointers how to find how equilibrium distributions change with changes in underlying fundamentals, namely the quality-cost relation and consumer tastes. More can be contributed on understanding the inheritance properties of distributions.

# References


[1] Anderson, Chris (2006). The Long Tail: Why the Future of Business Is Selling Less of More. New York: Hyperion.

[2] Anderson, Simon and André de Palma (2000), "From local to Global Competition," European Economic Review, 44, 423-448.

[3] Anderson, Simon P., André de Palma, and Jacques François Thisse (1992). Discrete Choice Theory of Product Differentiation. Cambridge, MA: MIT Press.

[4] Baldwin, Richard, and James Harrigan (2011). "Zeros, Quality, and Space: Trade Theory and Trade Evidence." American Economic Journal: Microeconomics 3(2), 60-88.

[5] Ben-Akiva, Moshe, and Thawat Watanatada (1981). "Application of a Continuous Spatial Choice Logit Model." Structural Analysis of Discrete Data with Econometric Application. Ed. Charles F. Manski and Daniel McFadden. Cambridge, MA: MIT.

[6] Bertoletti, Paolo, and Federico Etro (2016). "Monopolistic competition when income matters." Economic Journal, forthcoming.

[7] Cabral, Luis , and Jose Mata (2003). "On the evolution of the firm size distribution: Facts and theory." American Economic Review, 93(4), 1075-1090.

[8] Caplin, Andrew, and Barry Nalebuff (1991). "Aggregation and imperfect competition: On the existence of equilibrium." Econometrica, 59(1), 25–59.

[9] Chamberlin, Edward (1933). Theory of Monopolistic Competition. Cambridge, MA: Harvard University Press

[10] Crozet, Matthieu, Keith Head, and Thierry Mayer (2011). "Quality Sorting and Trade Firm-level Evidence for French Wine." Review of Economic Studies, 79(2), 609-644.

[11] Dixit, Avinash K., and J. S. Stiglitz (1977). "Monopolistic Competition and Optimal Product Diversity." American Economic Review, 67(3), 297-308.

[12] Dhingra Swati and John Morrow (2013) Monopolistic competition and optimum product diversity under firm heterogeneity, LSE, mimeo.

[13] Eaton, Jonathan, Samuel Kortum, and Francis Kramarz (2011). "An Anatomy of International Trade from French Firms." Econometrica, 79(5), 1453-1498.

[14] Elberse, Anita, and Felix Oberholzer-Gee (2006). Superstars and underdogs: An examination of the long tail phenomenon in video sales. Harvard Business School.

[15] Fajgelbaum, Pablo D., Gene M. Grossman, and Elhanan Helpman (2011). "Income Distribution, Product Quality, and International Trade." Journal of Political Economy, 119(4), 721-765.

[16] Feenstra, Robert C. and John Romalis (2014). International Prices and Endogenous Quality, Quarterly Journal of Economics, 129(2), 477-527.

[17] Fisk, P. (1961). "The Graduation of Income Distributions." Econometrica, 29(2), 171-185.

[18] Head, Keith, Thierry Mayer, and Mathias Thoenig (2014). "Welfare and Trade without Pareto." American Economic Review 104(5): 310-16.

[19] Johnson, Justin P. and David, P. Myatt (2006). "Multiproduct Cournot oligopoly". The RAND Journal of Economics, 37(3), 583-601.

[20] Mrázová, Monika, J. Peter Neary, and Mathieu Parenti (2016). "Sales and Markup Dispersion: Theory and Empirics," mimeo, University of Oxford.

[21] McFadden, Daniel (1978) "Modelling the Choice of Residential Location." in A. Karlqvist, L. Lundqvist, F. Snickars, and J. Weibull (eds.), Spatial Interaction Theory and Planning

Models, Amsterdam: North Holland, 75-96, Reprinted in J. Quigley (ed.), 1997, The economics of housing, Vol. I, Edward Elgar: London, 531-552.

[22] Melitz, Marc J. (2003). "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity." Econometrica, 71(6), 1695-1725.

[23] Pareto, Vilfredo (1965). "La Courbe de la Répartition de la Richesse" (Original in 1896): Oeuvres Complètes de Vilfredo. Ed. Busino G. Pareto. Geneva: Librairie Droz, 1-5.

[24] Shannon, Claude. E. (1948). "A Mathematical Theory of Communication," Bell System Technical Journal, 27(3), 379–423.

[25] Thisse, Jacques-François, and Philip Ushchev (2016). "Monopolistic competition without apology." Mimeo, Higher School of Economics, St. Petersburg.

[26] Weyl, Glen, and Michal Fabinger (2013). "Pass-Through as an Economic Tool: Principles of Incidence under Imperfect Competition." Journal of Political Economy, 121(3), 528-583.

[27] Zhelobodko, Evgeny, Sergey Kokovin, Mathieu Parenti, and Jacques-François Thisse (2012). "Monopolistic competition: Beyond the constant elasticity of substitution." Econometrica 80(6), 2765-2784.

# Appendix 1

**Proof of Proposition** 5

We first calculate the logit denominator, $D$, from (37), using the exponential CDF $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$ with density $f_X(x) = \lambda \exp(-\lambda(x - \underline{x}))$ and $\lambda > 0$, $\underline{x} > 0$, and $x \in [\underline{x}, \infty)$. Integrating,

$$D = \frac{M\lambda\mu}{\lambda\mu - 1} \exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0,$$

which is positive and bounded for any $\mathcal{V}_0$ under the assumption that $\lambda\mu > 1$. Now, from Proposition 4,

$$F_\Pi(\pi) = 1 - \exp\left(-\lambda\left(\mu \ln\left(\frac{\pi D}{\mu}\right) - \underline{x}\right)\right) = 1 - \left(\frac{\pi D}{\mu}\right)^{-\lambda\mu} \exp(\lambda \underline{x}).$$

The profit, $\underline{\pi}$, of the lowest quality-cost firm solves $F_\Pi(\underline{\pi}) = 0$, and thus verifies the expected property $\underline{\pi} = \frac{\mu}{D} \exp\left(\frac{\underline{x}}{\mu}\right)$. Inserting this value back into $F_\Pi(\pi)$ gives the expression in Proposition 5. The output distribution follows from the profit distribution:

$$F_Y(y) = \Pr(Y < y) = \Pr\left(\frac{\Pi}{\mu} < y\right) = F_\Pi(\mu y) = 1 - \left(\frac{\underline{\pi}}{\mu y}\right)^{\lambda\mu} = 1 - \left(\frac{\underline{y}}{y}\right)^{\lambda\mu},$$

where the lowest output, $\underline{y}$, is associated to the lowest profit, $\underline{\pi} = \mu \underline{y}$.

The last statement follows from Theorem 4: starting with a Pareto distribution for output or profit implies an underlying exponential distribution for quality-cost. The lowest quality-cost is given by the condition $\underline{y} = \frac{1}{D} \exp\left(\frac{\underline{x}}{\mu}\right)$, so

$$\underline{\pi} = \mu \underline{y} = \frac{\mu}{\frac{M\lambda\mu}{\lambda\mu - 1} \exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0} \exp\left(\frac{\underline{x}}{\mu}\right) < \frac{1}{M}\left(\mu - \frac{1}{\lambda}\right). \tag{45}$$

Inverting (45) gives $\underline{x}$.

**Proof of Theorem 10**

Because $F_X(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ and because $\beta^{-1}(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ with $\beta^{-1}(\underline{x}) > 0$. Then $F_C(c) = \Pr(C < c)$ is uniquely defined and continuous and increasing on support $[\underline{c}, \bar{c}]$:

$$F_C(c) = \Pr\left(\beta^{-1}(X) < c\right) = \Pr(X < \beta(c)) = F_X(\beta(c)).$$

The last term is a continuous and increasing function of a continuous and increasing function, so $F_C(c)$ is recovered. Constructing $F_X(x)$ from $F_C(c)$ and $\beta(c)$ is completely analogous.

We now show how to construct a unique increasing $\beta(c)$ from the two distributions: let $F_X(x) = \Pr(X < x)$ and we postulate that there exists a continuous increasing function $\beta(C) = X$ and so $F_X(x) = \Pr(\beta(C) < x) = \Pr\left(C < \beta^{-1}(x)\right)$ which is then equal to $F_C\left(\beta^{-1}(x)\right)$. Now, since $F_X(x) = F_C\left(\beta^{-1}(x)\right)$, then $\beta^{-1}(x) = F_C^{-1}(F_X(x))$, so $\beta(x) = \left[F_C^{-1}(F_X(x))\right]^{-1}$ and $\beta(x) = F_X^{-1}(F_C(x))$. This is clearly increasing and continuous in $x$ as desired.

The claim in the Theorem is shown because $F_X(x)$ can be used to construct the other distributions on its leg, and can be constructed from them; and likewise for $F_C(c)$.

# Appendix 2 Distribution details (NOT FOR PUBLICATION)

**Proof of Proposition** 4

We first seek the distribution of outputs, $F_Y(y) = \Pr(Y < y)$, that is generated from the primitive distribution of quality-cost. First note from (36) that $Y = \frac{\exp\left(\frac{X}{\mu}\right)}{D}$, so:

$$F_Y(y) = \Pr\left(\frac{\exp\left(\frac{X}{\mu}\right)}{D} < y\right) = F_X(\mu \ln(yD)),$$

where $D$ is given by (37). Because equilibrium profit is proportional to output ($\pi = \mu y$), we have a similar relation for the distribution of profit, $F_\Pi(\pi) = \Pr(\Pi < \pi)$:

$$F_\Pi(\pi) = \Pr\left(\mu\frac{\exp\left(\frac{X}{\mu}\right)}{D} < \pi\right) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right),$$

where $D$ is given by (37).

We next prove the converse result. We first determine the distribution of quality-costs consistent with a given observed distribution of output. Suppose that output has a distribution $F_Y(y)$. Applying the increasing transformation $y = \frac{1}{D}\exp\left(\frac{x}{\mu}\right)$, and $Y = \frac{1}{D}\exp\left(\frac{X}{\mu}\right)$, we get:

$$\begin{aligned} F_X(x) &= \Pr\left(\frac{1}{D}\exp\left(\frac{X}{\mu}\right) < \frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right) \\ &= \Pr\left(Y < \frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right) = F_Y\left(\frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right). \end{aligned}$$

However, $D$ is written in terms of $f_X(x)$, and we want to find the distribution solely in terms of $F_Y(y)$: this means writing $D$ in terms of $f_Y(y)$. The corresponding expression, denoted $D_y$ is derived below as (46). Similar reasoning gives the profit expression:

$$F_X(x) = \Pr(X < x) = \Pr\left(\Pi < \frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right) = F_\Pi\left(\frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right),$$

where the expression for $D$ in terms of $f_\Pi(\pi)$ (i.e., $D_\pi$) is given in the Theorem and derived below as (47).

We now show here how to write the function $D$ as a function of $f_Y(.)$ or $f_\pi(.)$.

We first find the value of $D$ in terms of the distribution of $Y$. Recall (37):

$$D = M \int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du + \mathcal{V}_0.$$

Now, $\Pr(x < X) = \Pr(y < Y)$, so $y(x) = \frac{\exp\left(\frac{x}{\mu}\right)}{M \int \exp\left(\frac{u}{\mu}\right) f_X(u) du + \mathcal{V}_0}$; hence $D \int y(u) f_Y(u)\, du = (D - \mathcal{V}_0)/M$, and thus:

$$D_y = \frac{\mathcal{V}_0}{1 - M y_{av}}. \tag{46}$$

The denominator in the last expression is necessarily positive because $M y_{av}$ is total output, which is less than one when $\mathcal{V}_0 > 0$ because the market is not fully covered. Similarly, $F_X(x) = F_\Pi\left(\frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right)$, so that

$$D_\pi = \frac{\mathcal{V}_0}{1 - \frac{M}{\mu}\pi_{av}}, \tag{47}$$

which is now expressed as a function of $f_\Pi(.)$, and where $\pi_{av}$ is average firm profit. (47) is positive because total profit, $M\pi_{av}$, is less than $\mu$ because the market is not fully covered ($\mathcal{V}_0 > 0$).

**Study of specific distributions**

We now derive the distributions in Section 5.2: these involve parameter matching for the distribution examples.

**Normal**: For the normal, $F_X(x) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}} \int_{-\infty}^{x} \exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) du$, where $\tilde{\pi} = 3.1415...$ . From Theorem 4, we have

$$F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right),$$

where $\mu \ln\left(\frac{\pi D}{\mu}\right) \in (-\infty, \infty)$, so

$$F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}} \int_{-\infty}^{\mu \ln\left(\frac{\pi D}{\mu}\right)} \exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) du.$$

Using the change of variable $\Pi = \frac{\mu}{D} \exp\left(\frac{u}{\mu}\right)$ (so $u = \mu \ln\left(\frac{\Pi D}{\mu}\right)$ and $du = \frac{\mu}{\Pi} d\Pi$) we obtain

$$F_{\Pi}(\pi) = \frac{\mu}{\sigma\sqrt{2\tilde{\pi}}} \int_0^{\pi} \exp\left(-\frac{\left(\mu \ln\left(\frac{\Pi D}{\mu}\right) - m\right)^2}{2\sigma^2}\right) \frac{d\Pi}{\Pi},$$

which can be written in a standard form as:

$$F_{\Pi}(\pi) = \frac{1}{\left(\frac{\sigma}{\mu}\right)\sqrt{2\tilde{\pi}}} \int_0^{\pi} \frac{1}{\Pi} \exp\left(-\frac{\left(\ln \Pi - \left(\frac{m}{\mu} - \ln\left(\frac{D}{\mu}\right)\right)\right)^2}{2\left(\frac{\sigma}{\mu}\right)^2}\right) d\Pi.$$

Hence profits are log-normally distributed with parameters $\left[\frac{m}{\mu} - \ln\left(\frac{D}{\mu}\right)\right]$ and $\left(\frac{\sigma}{\mu}\right)$.

Recall: $D = M \int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du + \mathcal{V}_0$. Then:

$$D = \frac{M}{\sigma\sqrt{2\tilde{\pi}}} \int_{-\infty}^{\infty} \exp\left(\frac{x}{\mu}\right) \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right) dx + \mathcal{V}_0.$$

Routine computation shows that:

$$D = M \exp\left(\frac{m}{\mu} + \frac{\sigma^2}{2\mu^2}\right) + \mathcal{V}_0.$$

**Logistic.**

A *logistic* distribution for quality-cost has a CDF given by $F_X(x) = \left(1 + \exp\left(-\frac{x-m}{s}\right)\right)^{-1}$, $x \in (\underline{x}, \infty)$, with mean $m$ and variance $s^2\pi^2/3$. The PDF is similar in shape to the normal, but it has thicker tails (see the discussion in Fisk, 1961, and the comparison with the Weibull distribution). Hence, for $\mu > s$, profit $\Pi \in (0, \infty)$ is log-logistically distributed with parameters $\left(\frac{\mu}{D}\right) \exp\left(\frac{m}{\mu}\right)$ and $\frac{\mu}{s}$:

$$F_{\Pi}(\pi) = \left(1 + \left(\frac{\pi}{\left(\frac{\mu}{D}\right) \exp\left(-\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}\right)^{-1}, \qquad \pi \in [0, \infty).$$

There is no closed form expression for $D$ in this case. However, it can be shown that the condition $\mu > s$ guarantees that the output denominator $D$ exists. The Log-logistic distribution (which provides a one parameter model for survival analysis) is very similar in shape to the log-normal distribution, but it has fatter tails. It has an explicit functional form, in contrast to the Log-normal distribution.

The logistic distribution (with mean $m$ and standard deviation $s\tilde{\pi}/\sqrt{3}$) is given by:

$$F_X(x) = \frac{1}{1+\exp\left(-\frac{x-m}{s}\right)}, x \in (\underline{x}, \infty).$$

From Theorem 4,

$$F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{1+\exp\left(-\frac{\mu\ln\left(\frac{\pi D}{\mu}\right)-m}{s}\right)},$$

where $F_\Pi(0) = 0$ and $F_\Pi(\infty) = 1$. Thus

$$F_\Pi(\pi) = \frac{1}{1+\exp\left(\frac{m}{s}\right)\exp\left(\ln\left(\frac{\pi D}{\mu}\right)^{-\frac{\mu}{s}}\right)} = \frac{1}{1+\left(\frac{\pi}{\left(\frac{\mu}{D}\right)\exp\left(\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}}.$$

Recall the log-logistic distribution is defined as:

$$F^{LL}(x;\alpha,\beta) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}, \quad x > 0.$$

Thus, the parameter matching is:

$$F_\Pi(\pi) = F^{LL}\left(x; \left(\frac{\mu}{D}\right)\exp\left(\frac{m}{\mu}\right), \frac{\mu}{s}\right).$$

We need to check when $D$ converges, i.e., when $\int_{-\infty}^{\infty} \exp\left(\frac{x}{\mu}\right) f_X(x)\, dx$ converges. Because

$$f_X(x) = \frac{1}{s}\frac{\exp\left(-\frac{x-m}{s}\right)}{\left(1+\exp\left(-\frac{x-m}{s}\right)\right)^2},$$

we need to ensure the convergence of the expression

$$\int_{-\infty}^{\infty} \frac{\exp\left(-x\left(\frac{1}{s}-\frac{1}{\mu}\right)\right)}{\left(1+\exp\left(-\frac{x-m}{s}\right)\right)^2} dx.$$

Convergence is guaranteed if and only if $\mu > s$.

**Pareto:** The Pareto distribution is given by: $F_X(x) = \frac{1-\left(\frac{\underline{x}}{x}\right)^{\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}$. From Theorem 4

$$F_\Pi(\pi) = \frac{1-\left(\frac{\underline{x}}{\mu\ln\left(\frac{\pi D}{\mu}\right)}\right)^{\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

Recall that $\pi = \frac{\mu\exp\left(\frac{x}{\mu}\right)}{D}$ or $x = \mu\ln\left(\frac{\pi D}{\mu}\right)$, so that $\underline{x} = \mu\ln\left(\frac{\underline{\pi}D}{\mu}\right)$ and $\overline{x} = \mu\ln\left(\frac{\overline{\pi}D}{\mu}\right)$, so that $F_\Pi(\underline{\pi}) = 0$ and $F_\Pi(\overline{\pi}) = 1$. $D$ is bounded because the quality-cost distribution is bounded.

Consider a log-Pareto distribution with scale parameter $\sigma$ and shape parameters $\gamma$ and $\beta$:

$$F^{LP}(\pi;\gamma,\beta,\sigma) = \frac{1-\left(1+\frac{1}{\beta}\ln\left(1+\frac{\pi-\underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}{1-\left(1+\frac{1}{\beta}\ln\left(1+\frac{\overline{\pi}-\underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}, \qquad \pi > \underline{\pi}.$$

In order to match parameters, of $F_\Pi(\pi)$ with $F^{LP}(\pi;\gamma,\beta,\sigma)$ observe that

$$\mu\ln\left(\frac{\pi D}{\mu}\right) = \mu\ln\left(\frac{\pi D}{\mu}\frac{\underline{\pi}}{\underline{\pi}}\right) = \underline{x} + \mu\ln\left(\frac{\pi}{\underline{\pi}}\right) = \underline{x} + \mu\ln\left(1+\frac{\pi-\underline{\pi}}{\underline{\pi}}\right).$$

Therefore:

$$F_\Pi(\pi) = \frac{1-\underline{x}^{\alpha}\left(\mu\ln\left(\frac{\pi D}{\mu}\right)\right)^{-\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}} = \frac{1-\left(1+\frac{\mu}{\underline{x}}\ln\left(1+\frac{\pi-\underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

Thus $\pi$ obeys a Log-Pareto distribution $F^{LP}\left(\pi;\frac{1}{\alpha},\frac{\underline{x}}{\mu},\underline{\pi}\right)$, i.e., $\gamma = \frac{1}{\alpha}, \beta = \frac{\underline{x}}{\mu}, \sigma = \underline{\pi}$.

It remains to check that the normalization factors are equal. Recall that $\overline{\pi}/\underline{\pi} = \exp\frac{\overline{x}-\underline{x}}{\mu}$.

Using the specification $\gamma = \frac{1}{\alpha}, \beta = \frac{\overline{x}}{\mu}, \sigma = \underline{\pi}$, we get:

$$\begin{aligned} \left(1 + \frac{\mu}{\overline{x}} \ln\left(1 + \frac{\overline{\pi} - \underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} &= \left(1 + \frac{\mu}{\overline{x}} \ln\left(\frac{\overline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} \\ &= \left(1 + \left(\frac{\overline{x} - \underline{x}}{\overline{x}}\right)\right)^{-\alpha} = \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}. \end{aligned}$$

Figure 1: Increasing cost to quality-cost relation.

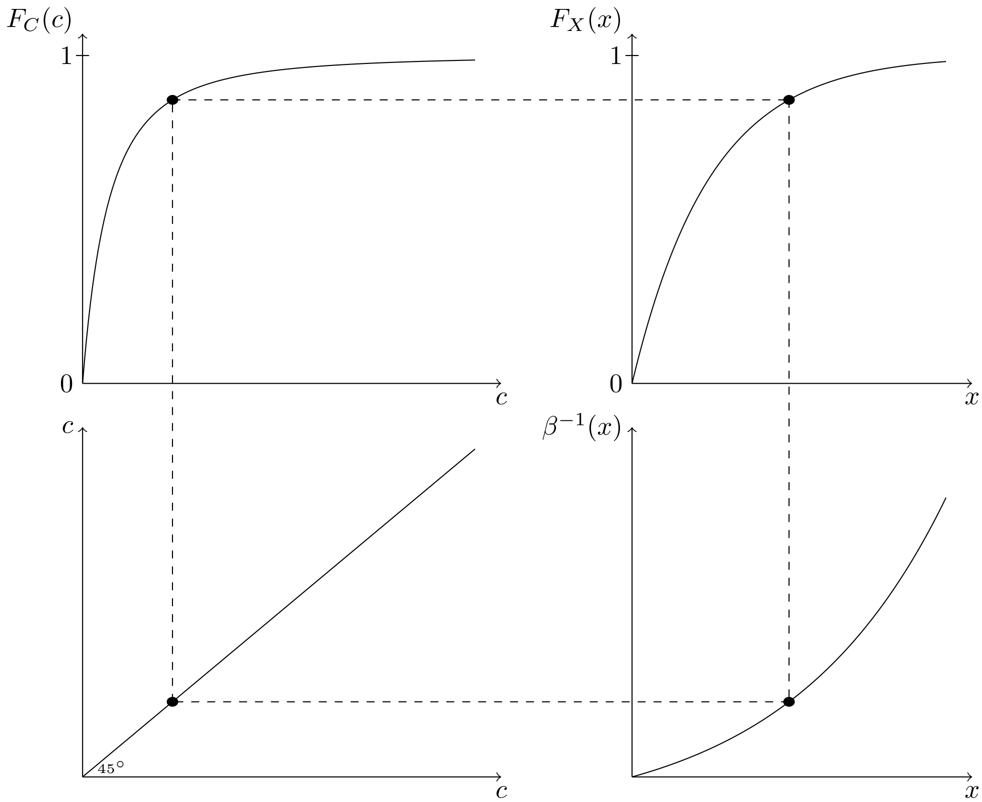

Figure 2: Hump relation between cost and quality-cost.