A Dynamic Model of Crowdfunding *

Kenjiro Asami †

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Abstract

We propose a theoretical model to analyze the dynamics of investment behaviors in the crowdfunding platforms where each investor can observe the aggregate amount of investment pledged before her. When a project has quality uncertainty, information about decisions made by other investors might convey some information about the quality to the successors. Such transmission of information might enable better decisions to be made, but it is also quite likely that information cascade impede better information aggregation. We show that introduction of the popular *all-or-nothing* scheme put out the investors' incentive to herd while the funding goal is yet to be reached. The logic is as follows: larger amount of pledges itself is a good information, but on the other hand, since the goal is close at hand, the project is likely to be funded even when it is of bad quality. After the funding goal is met, the investment behaviors exhibit information cascade, under some condition on the form of population uncertainty. It presents a new issue on the existence of aggregate information cascade by Guarino et al.(2011). We also examine a static version of our model and show that the difference between these models is negligible with diminishing population uncertainty. Finally, we also show that the crowdfunding scheme succeeds in information aggregation to some degree.

1 Introduction

Crowdfunding has become a popular way for small entrepreneurs or households to finance their projects. By posting their project on the crowdfunding websites, entrepreneurs can collect funds from a large audience in which each individual provides a very small amount usually via Internet. Total funding volumes in 2015 were around 34 Billion US$ and is still growing rapidly.

The popularity of the crowdfunding scheme is somewhat puzzling from a traditional view of financing.When investment decisions for a particular project are made by a large number of investors, they usually

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†Graduate School of Economics, University of Tokyo

have less ability to assess and monitor the project and tends to incur duplication costs in these activities. Instead, bank loans are seen to have some advantages both in production of information and managing moral hazard problem. Investors in the market as a whole, however, may have better information about the project than the bankers. In the case of crowdfunding project for a new product for example, as a consumer, each of the investors may have vague information about the quality or future demand for the product. Thus, effective aggregation of such information from a large number of investors potentially enables a better assessment of the project. This leads to the following questions: "How well does the crowdfunding mechanism aggregate information?" and "How severe is the moral hazard problem?"

The main objective of this paper is to answer the first question in a dynamic model of crowdfunding with quality uncertainty. Moreover we analyze the situation where the quality or the value of the project is common but unknown among all investors, which is especially relevant for P2P lending or equity-based crowdfunding[1].

As in most of the crowdfunding campaigns, each project is posted on a website for a certain period of time, during which investors can pledge to invest on the project. Upon visiting the website, investors can observe not only the description of the project but also the amount of money already pledged by other investors at that point. Each investor receives some noisy information about the project without any cost. For simplicity, the timing of decision for each investor is assumed to be random and unobservable to the investor. In addition, we assume that the population size of the investors are uncertain. A popular mechanism called *all-or-nothing* scheme is applied: a project is executed only when a funding goal, which is set in advance, is met within the specified period, otherwise the project is canceled and pledged money is refunded.

When a project has quality uncertainty, information about decisions made by other investors might convey some information about the quality to the successors. Such transmission of information might enable better decisions to be made, but it is also quite likely that information cascade impede better information aggregation. Some empirical studies on *Prosper.com* suggest that investors tend to invest in a well-funded listings [2].

We show that introduction of the popular *all-or-nothing* scheme cancel out the investors' incentive to herd when the funding goal is yet to be reached. Actually, in the equilibria in our model, no matter what the amount of money pledged before her is, the set of best responses to a particular signal is unchanged. Suppose that pledging and abstention are indifferent when you observe a bad signal and only a small fraction of the target is funded. Then they are also indifferent even when the target is close at hand, as far as you observe a bad signal. The key fact for this result is that investors evaluate the project conditional on the event that it is funded. Smaller amount of pledges before you itself is a negative information for the quality, but conditioning on the event that the target will be reached, larger number of investors will invest after

[1] P2P lending accounts for more than 70 % and each of the donation, reward and equity-based crowdfunding accounts for a little less than 10%.

[2] Zhang and Liu(2012) and Herzenstein, Dholakia and Andrews(2010) etc. Note that the mechanism used when the data for these studies are collected vary from the current model which is relevant to our model.

you, which is a good information. This logic is similar to that of Dekel and Piccione(2000) in the voting literature, which showed that any equilibrium in a simultaneous voting game can be realized as a sequential equilibrium where voters completely ignore the predecessors' votes.

On the other hand, investment behaviors after the funding goal is reached basically exhibits an aggregate informational cascade: investors completely ignore their own signal after some threshold (usually the funding goal) is reached. This observation is similar to the aggregate information cascades model by Guarino et al.(2011). The new result in our model is that the aggregate information cascades do not occur in some cases. As we mentioned earlier, we have a population uncertainty in our model. Whether the information cascades occurs or not is governed by the distribution of the population size and increasing hazard rate property is a sufficient condition for the existence.

We also demonstrate that various patterns of investment, including herding behaviors after the accumulated amount of investment gets close to the funding goal, can constitute equilibria. As we mentioned above, herding behaviors are not due to learning. This result indicates that herding behaviors commonly observed in many crowdfunding situations may not be a result of learning nor irrational behaviors.

Finally we evaluate the performance of crowdfunding scheme as a *gatekeeper* of finance: only the projects with high quality should be funded. We show that crowdfunding with *all-or-nothing* scheme partially aggregates investors' information in sense that the decision of the execution of the project is more correct in crowdfunding than using the information held by a single investor.

1.1 Related Literature

Though it is a relatively new form of financing, the growing popularity of crowdfunding is urging many researchers to work on it these days. Agrawal et al.(2014) and Bellflamme et al.(2015) survey some costs and benefits of crowdfunding. On the one hand crowdfunding has a potential to aggregate information about the demand or the quality of the projects, on the other hand it may suffer from moral hazard problem. In the common value environment, it may also suffer from information cascades in a dynamic procedure.

Most of the recent works such as Strauz(2017) and Ellman and Hurkens(2015) focuses on reward-based crowdfunding and explores the efficient or optimal mechanisms for the entrepreneur. Strauz(2017) characterizes the efficient mechanism under treat of moral hazard problem and demand uncertainty, which is achieved by all-or-nothing scheme with deferred payment. Ellman and Hurkens(2015) considers the optimal mechanism for the entrepreneurs taking price discrimination into account. Chang(2016) shares the motivation similar to these papers but his analysis is on the common value environment with a continuum of investors. He showed that *all-or-nothing* mechanism is more preferable to the entrepreneur than the flexible mechanism where the money you pledged will never be refunded even when the project is canceled. Hakenes and Schlegel(2014) studies a finite-agent common value model with endogenous information acquisition. They find that in the optimal campaign firms set the loan rate and funding goal too low so that too many players buy the signals, which is a loss in welfare. However, crowdfunding campaign enables more good projects to receive funding than standard debt financing.

All of these papers above abstract from the dynamic nature of the crowdfunding process commonly observed in reality and investigate static mechanisms only. Our study abstracts from the entrepreneur or borrower's behaviors but stands out in that it theoretically examines the dynamics of investors' behaviors in the crowdfunding model with purely common value environment.

Our work, especially for the analysis of investment behaviors after the target is reached, is related to the theory of *informational cascades*, which explores the situation where many players sequentially make decisions observing the predecessor's decisions.

In this kind of situations, as we mentioned earlier, it is often focused on that players tend to choose the same action as the predecessors, or *herd behavior*. *Information cascade* is a situation where players always imitate their predecessor's action no matter what signals they receive. In information cascades, by definition, a player's action conveys completely no information to the successors. Therefore, if informational cascades occur, most of the information held by the investors fails to exploited and incorrect decision might prevail in the society.

In the classical models of observational learning like Banerjee(1992) or Bikhchandani, Hirshleifer and Welch(1992), investors can perfectly observe the actions made by the predecessors. In this strand of literature, conditions on signal structures for informational cascades are explored (Smith and Sorensen(2000)).

Another strand of literature develops the social learning model in various dimensions. For example, Callander and Horner(2009) explores the case where only the number of players who took each actions, not the whole sequence, are observable. Herrera and Horner(2013) considers a model where only one type of action, say investment, is observable to the successors. Guarino, Harmgart and Huck(2011), which partly inspired our paper, considers a model where players can only observe the number of investment decisions made before themselves. Though there are only finite number of players in their model, its limit case where the number of players goes to infinity is approximated by our model with diminishing population uncertainty. The crucial difference from their work is that when the distribution of the population lacks *increasing hazard rate property*, it might be that there does not occur information cascades.

There are some empirical papers on the herding behaviors in several crowdfunding platforms.

For the P2P lending or equity-based crowdfunding, which is most relevant to our common value model, Herzenstein, Dholakia and Andrews(2010) and Zhang and Liu(2012) investigate the herding behaviors in *Prosper.com*, one of the largest platforms for P2P lending. These papers point out that well-funded borrower listings tend to attract more funding, which is a result of investors' active observational learning. [3] Thus, we can say that the possibility of herding is a big issue that cannot be ignored. For reward-based crowdfunding, Kuppuswamy and Bayus(2017,2018) conducted an empirical investigation on *Kickstarter*, one of the largest crowdfunding platforms for startups. They report that a Kickstarter project whose funding is close to the

[3] These studies actually does not correspond to our model since they used the data collected before 2010 when Prosper.com was adopting the auction mechanism. Zhang and Chen(2017) also reports the existence of herding behavior in *Renrendai*, a P2P lending platform with fixed loan rate model.

target tends to attract more pledges. They argue that this goal-gradient effect is due to people's preference for making an impact on the outcome rather than learning. Although not all of these observed investment patterns are explained in our model, it provides a basis for the discussion on this topic.

2 Model

There are two states of the world $\omega \in \Omega \equiv \{G, B\}$, where $G(B)$ denotes the case where the quality of the project is good(bad). There is a mass N of investors each of whom are supposed to choose binary action $a \in A \equiv \{I, NI\}$ where I denotes pledging to invest and NI denotes abstention. Before making an investment decision, she privately receives a binary signal $s \in S \equiv \{g, b\}$ about the true state ω. The accuracy of the signal is denoted by $p > 1/2$, that is $\Pr(g|G) = \Pr(b|B) = p$ and $\Pr(b|G) = \Pr(g|B) = 1-p$. The private signals are independent across investors conditional on the state.

The population size N of the investors is drawn from a distribution F with density $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $0 < \mathbb{E}[N] < \infty$. For simplicity, we assume that F is a full support distribution. Without loss of generality, we can normalize F so that $\mathbb{E}[N] = 1$. Additionally we assume that it is independent of the state ω. Each investor doesn't know the population when she makes a decision beyond its distribution.

One thing to note is that the distribution of the population size perceived by the participants is different from f [4]. To understand this, we first consider a truncation of F to an interval $[0,n]$ for each $n \in \mathbb{N}$, i.e., $F_n(x) = \frac{F(x)}{F(n)} \mathbb{1}_{\{x \leq n\}}$. There are potential players of mass n, each of whom is recruited with probability $\frac{N'}{n}$ where N' is distributed according to F_n. Let t denote an auxiliary random variable which is uniformly distributed over $[0,n]$. Given that a player obtained the opportunity to participate in the game, her belief over the population size N' being below $x(<n)$ is;

$$\Pr(N' \leq x | N' \geq t) = \frac{\Pr(t \leq N' \leq x)}{\Pr(N' \geq t)} = \frac{\int_0^x (F_n(x) - F_n(t))dt}{\int_0^\infty (1 - F_n(t))dt}$$

We can show the pointwise convergence of F_n to F. We can also show that [5]

$$\lim_{n\to\infty} \frac{\int_0^x (F_n(x) - F_n(t))dt}{\int_0^\infty (1 - F_n(t))dt} = \int_0^x (F(x) - F(t))dt \equiv F(x|rec)$$

We assume that the cumulative distribution function of N, perceived by a player who is recruited as an investor, is $F(N|rec)$, where rec denotes the event that the investor is recruited. It can be shown that $F(\cdot|rec)$ first order stochastically dominates F [6]. Differentiating with respect to N yields the density of N conditional

[4] See Myerson(1998) for this discussion, though his model assumes finite number of players.

[5] See Lemma 4 in Appendix for its proof.

[6] Define $Q(x) \equiv F(x) - \int_0^x (F(x) - F(t))dt = (1-x)F(x) + \int_0^x F(t)dt$. Then $Q(0) = 0$, $\lim_{x\to\infty} Q(x) = 0$, and $Q'(x) = (1-x)f(x)$ implies that $Q(x) \geq 0$ for all x.

on the event that the investor is recruited;

$$f(x|rec) = xf(x)$$

Random population size is meant to capture an aspect of the reality: the number of investors who visit the website may vary while the duration of the project to be posted on the website is fixed. This situation can be translated into the model where the duration of the posting is random while investors visit the website at a constant rate. The game starts at time $t = 0$ and ends at $t = N$, where the deadline N is randomly determined by density f. Mass N of investors are uniformly located on a line $[0,N]$, and we call the investor located at $t \in [0,N]$ by investor t. Each investor can observe the amount of investment pledged before her, but cannot observe her own index t.

The project is executed only when the amount of pledges (defined later) reach the funding goal k^*, otherwise it is canceled. If canceled, all the investors get the payoff of zero. Once the goal is met those who chose I pays r and gets the payoff of 1 only when $\omega = G$. Let e denote the event that the project is executed and c denote the event that the project is canceled. In our model, we abstract from the incentive for the borrowers' side, i.e., k^* and r are assumed to be exogenously given while it is often the case that they are set by the borrowers' side. Actually in Prosper.com the interest rate is determined by the platform for each listing.

Thus an investor's strategy is a function $\sigma : \{g,b\} \times [0,\infty) \to [0,1]$, where $\sigma(s,k)$ denotes the probability of choosing I after observing signal s and accumulated amount of pledges k.

Given σ and ω, accumulated amount of pledges at time t, $k_\omega(t)$, are defined as the solutions to the following Cauchy problems;

$$\begin{aligned} \frac{dk_G}{dt}(t) &= p\sigma(g,k_G(t)) + (1-p)\sigma(b,k_G(t)), \;\; k_G(0) = 0 \\ \frac{dk_B}{dt}(t) &= p\sigma(b,k_B(t)) + (1-p)\sigma(g,k_B(t)), \;\; k_B(0) = 0 \end{aligned} \tag{1}$$

We restrict our attention to the class of strategies such that $\sigma(s,\cdot)$ is Lebesgue measurable. Proposition 1 ensures the existence and the uniqueness of the Cauchy problem (1).

Proposition 1. Suppose that $\sigma(g,\cdot)$ and $\sigma(b,\cdot)$ are Lebesgue measurable. Then each of the Cauchy problems (1) has a unique solution.

Proof. Cid, Heikkila and Pouso(2006) provides us a sufficient condition for the existence and uniqueness of the Caratheodory type solution for a class of initial value problem(IVP). The IVP in question is of the following form;

$$\frac{dx}{dt}(t) = f(t,x(t)), \;\; x(t_0) = x_0 \in \mathbb{R}. \tag{2}$$

where $f : I \times J \to \mathbb{R}$, $I = [t_0, t_0 + T]$, $J = [x_0, x_0 + R]$, $T > 0$ and $R > 0$. I.e., we want to find an absolutely continuous function x which takes the value x_0 at t_0 and satisfies the differential equation in (2) at almost every t. Cid et al(2006) provides several sets of sufficient conditions for the uniqueness and existence of the solutions for the IPV.

Now we define an auxiliary function $\tilde{\sigma}$ as follows;

$$\tilde{\sigma}(s,k) = \begin{cases} 1 & \text{if} \quad k \geq \inf\{k | \sigma(g,k) + \sigma(b,k) = 0\} \equiv k^{DOWN} \\ \sigma(s,k) & \text{otherwise} \end{cases}$$

Note that Lebesgue measurability of $\sigma(g,\cdot)$ and $\sigma(b,\cdot)$ is preserved for $\tilde{\sigma}$'s and $\tilde{\sigma}(g,k) + \tilde{\sigma}(b,k) > 0$ for all $k \in \mathbb{R}_+$. These facts implies that the conditions in Theorem 3 in Cid et al.(2006) to be satisfied, which ensures the uniqueness and the existence of the solution to the auxiliary problem below.

$$\begin{aligned} \frac{d\tilde{k}_G}{dt}(t) &= p\tilde{\sigma}(g,\tilde{k}_G(t)) + (1-p)\tilde{\sigma}(b,\tilde{k}_G(t)), \;\; \tilde{k}_G(0) = 0 \\ \frac{d\tilde{k}_B}{dt}(t) &= p\tilde{\sigma}(b,\tilde{k}_B(t)) + (1-p)\tilde{\sigma}(g,\tilde{k}_B(t)), \;\; \tilde{k}_B(0) = 0 \end{aligned} \tag{3}$$

Then we have the unique solution to our original problem (1);

$$k_\omega(t) = \min\{\tilde{k}_\omega(t), k^{DOWN}\}$$

□

3 Equilibrium Analysis

The dynamic crowdfunding model defined above is described by four parameters (π, p, r, k^*) and we call such game as $\Gamma(\pi, p, r, k^*)$ or just Γ. In this section, we investigate how the investment behaviors in the equilibria look like.

First we define the equilibrium concept for the game Γ.

Investor's Payoff We assume that the investors are risk neutral. Then the expected payoff from pledging observing the signal s and the accumulated level k is;

$$U(s|r,k,\sigma) = (1-r)\Pr(G,e|s,k,rec,\sigma) - r\Pr(B,e|s,k,rec,\sigma)$$

Definition 1. An action profile, σ, is a Bayesian Nash Equilibrium in Γ if and only if for $s \in \{g, b\}$,

$$U(s|r,k,\sigma) > 0 \Rightarrow \sigma(s,k) = 1;$$
$$U(s|r,k,\sigma) < 0 \Rightarrow \sigma(s,k) = 0$$

Since each investor's payoff is not affected by the others' decisions made after the goal is met, our analysis can be divided into two parts: the behavior of the investors when the funding goal is yet to be reached (3.1) and the behaviors after it is reached (3.2). Here we define $\bar{\Gamma}$ as the truncated form of Γ which ends immediately after the funding goal is reached: $\bar{\Gamma}$ is totally the same as Γ except that the payoff function is set to be 0 if $k \geq k^*$. Then the first part can be reduced to the equilibrium analysis of the game $\bar{\Gamma}$.

Remark In Prosper.com, the borrowing project is closed after the funding target is achieved. Therefore this game corresponds to $\bar{\Gamma}$.

Remark If we set $k^* = 0$, then it reduces to a model of aggregate social learning similar to that of Guarino et al.(2011).

3.1 Investment Behaviors when the Goal is Yet to be Met

In this section, we conduct an analysis of the game $\bar{\Gamma}$.

First of all, by definition of equilibrium, it is obvious that if the project is never executed under σ, $U(s|r,k,\sigma)$ is always zero and thus I and NI are indifferent.

Corollary 1. Strategy profile σ under which the project is executed with probability 0 constitutes a Bayesian Nash equilibrium.

Next we focus on the equilibria where $\sigma(g,k) + \sigma(b,k) > 0$ for all $k \in [0, k^*)$, otherwise the project is never executed. Then, k_ω are strictly increasing. Thus we can take inverse of them, which is hereafter referred to as t_ω. Using the Bayes rule, the followings hold for $k \in [0, k^*)$;

$$\Pr(G,e|b,rec,k,\sigma) = \frac{\Pr(G,b,e|rec,k,\sigma)}{\Pr(b|rec,k,\sigma)}$$
$$\begin{aligned}\Pr(b|rec,k,\sigma) &= (1-p)\Pr(G|rec,k,\sigma) + p\Pr(B|rec,k,\sigma) \\ &= \pi(1-p)\frac{g(t_G(k)|rec)}{h(k)} + (1-\pi)p\frac{g(t_B(k)|rec)}{h(k)}\end{aligned}$$
$$\begin{aligned}\Pr(G,b,e|rec,k,\sigma) &= \Pr(G,b,N \geq X_G^\sigma|rec,k,\sigma) \\ &= \Pr(G|rec,k,\sigma)\Pr(b,N \geq X_G^\sigma|G,k,rec,\sigma) \\ &= \pi(1-p)\Pr(N \geq X_G^\sigma|t_G(k),rec)\frac{g(t_G(k)|rec)}{h(k)}\end{aligned}$$

where $X^\sigma_\omega = \inf\{t | k_\omega(t) \geq k^*\}$, $g(\cdot|rec)$ denotes the density function of each investor's index t and $h(k) \equiv \pi g(t_G(k)|rec) + (1-\pi) g(t_B(k)|rec)$. Formally, the function g is calculated as follows;

$$g(t|rec) = \int_0^\infty \frac{1}{N} \mathbb{1}_{\{t \leq N\}} f(N|rec) dN = \int_0^\infty \mathbb{1}_{\{t \leq N\}} f(N) dN = 1 - F(t)$$

Then the density of N given t is calculated as follows;

$$\begin{aligned} f(N|t, rec) &= g(t|N, rec) \frac{f(N|rec)}{g(t|rec)} \\ &= \mathbb{1}_{\{t \leq N\}} \frac{f(N|rec)}{N g(t|rec)} = \mathbb{1}_{\{t \leq N\}} \frac{f(N)}{g(t|rec)} \end{aligned}$$

Thus,

$$\begin{aligned} \Pr(N \geq X^\sigma_G | t_G(k), rec) g(t|rec) &= \int_{X^\sigma_G}^\infty f(N) dN \quad (\because t_G(k) \leq X^\sigma_G) \\ &= 1 - F(X^\sigma_G) \end{aligned}$$

Therefore, $U(s|r, k, \sigma)$ can be rewritten as;

$$\begin{aligned} U(g|r, k, \sigma) &= \frac{\pi p(1-r)(1 - F(X^\sigma_G)) - (1-\pi)(1-p) r (1 - F(X^\sigma_B))}{\pi p (1 - F(t_G(k))) + (1-\pi)(1-p)(1 - F(t_B(k)))} \\ U(b|r, k, \sigma) &= \frac{\pi(1-p)(1-r)(1 - F(X^\sigma_G)) - (1-\pi) p r (1 - F(X^\sigma_B))}{\pi(1-p)(1 - F(t_G(k))) + (1-\pi) p (1 - F(t_B(k)))} \end{aligned} \tag{4}$$

Then we immediately obtain the following lemma.

Lemma 1 (Monotonicity)**.** Suppose that σ constitutes a Bayesian Nash Equilibrium in $\bar{\Gamma}$. Then, unless $1 - F(X^\sigma_G) = 1 - F(X^\sigma_B) = 0$,

1. $\exists k \ \ s.t. \ \ \sigma(b, k) > 0 \Rightarrow \forall k \ \ \sigma(g, k) = 1$;
2. $\exists k \ \ s.t. \ \ \sigma(g, k) < 1 \Rightarrow \forall k \ \ \sigma(b, k) = 0$;
3. $1 - F(X^\sigma_G) \geq 1 - F(X^\sigma_B) > 0$.

Proof. The following inequality together with the definition of equilibrium implies the first two parts.

$$\begin{aligned} & D_g(k) U(g|r, k, \sigma) - D_b(k) U(b|r, k, \sigma) \\ =& \{\pi p(1-r)(1 - F(X^\sigma_G)) - (1-\pi)(1-p) r (1 - F(X^\sigma_B))\} \\ -& \{\pi(1-p)(1-r)(1 - F(X^\sigma_G)) - (1-\pi) p r (1 - F(X^\sigma_B))\} \\ =& (2p-1)\{\pi(1-r)(1 - F(X^\sigma_G)) + (1-\pi) r (1 - F(X^\sigma_B))\} > 0, \end{aligned}$$

where

$$D_g(k) = \pi p(1-F(t_G(k))) + (1-\pi)(1-p)(1-F(t_B(k))) > 0;$$
$$D_b(k) = \pi(1-p)(1-F(t_G(k))) + (1-\pi)p(1-F(t_B(k))) > 0.$$

The first two statements above directly imply that $k_G(t) \geq k_B(t)$ and therefore $X_G^\sigma \leq X_B^\sigma$. Thus, $1-F(X_G^\sigma) \geq 1-F(X_B^\sigma)$. Next, suppose that $1-F(X_G^\sigma) > 1-F(X_B^\sigma) = 0$ holds. Then $U(s|r,k,\sigma) > 0$ for both $s \in \{g,b\}$, which calls for $1-F(X_G^\sigma) = 1-F(X_B^\sigma)$. Hence the third part is also verified. □

This lemma indicates that if an investor has an incentive to choose I for some value of accumulated amount of investment k even though she observes a bad signal, then she never abstains when she observes a good signal. (The counterpart of this statement also holds.)

Corollary 1, Lemma 1 and equations (4) yield our first theorem that characterizes the equilibrium in the dynamic crowdfunding model.

Theorem 1. A strategy profile σ constitutes an equilibrium in $\bar{\Gamma}$ if and only if one of the following conditions holds;

1. $1-F(X_G^\sigma) = 1-F(X_B^\sigma) = 0$;
2. $\forall k \in [0,k^*);\ \sigma(g,k) = 1, \sigma(b,k) = 1$ and $\dfrac{p}{1-p} < \dfrac{\pi}{1-\pi}\dfrac{1-r}{r}$;
3. $\forall k \in [0,k^*);\ \sigma(g,k) = 1, \sigma(b,k) \geq 0$ and $\dfrac{\pi}{1-\pi}\dfrac{1-r}{r}\dfrac{1-F(X_G^\sigma)}{1-F(X_B^\sigma)} = \dfrac{p}{1-p}$;
4. $\forall k \in [0,k^*);\ \sigma(g,k) = 1, \sigma(b,k) = 0$ and $\dfrac{1-p}{p} < \dfrac{\pi}{1-\pi}\dfrac{1-r}{r}\dfrac{1-F(\frac{k^*}{p})}{1-F(\frac{k^*}{1-p})} < \dfrac{p}{1-p}$;
5. $\forall k \in [0,k^*);\ \sigma(g,k) \leq 1, \sigma(b,k) = 0$ and $\dfrac{1-p}{p} = \dfrac{\pi}{1-\pi}\dfrac{1-r}{r}\dfrac{1-F(X_G^\sigma)}{1-F(X_B^\sigma)}$.

The surprising point of this theorem is that even though equilibrium strategy σ can depend on k, the set of best responses to a particular observation (k,s) is actually invariant across all $k \in [0,k^*)$. For example, suppose that I and NI are indifferent when you observe k' and b (condition No.3 in Theorem 1). Then I and NI are also indifferent when you observe any k'' and b.

The key fact for this result is that investors evaluate the project conditional on the event that it is funded. Smaller amount of pledges before you itself is a negative information for the quality, but on the event that the target will be reached, larger number of investors will invest after you, which is a good information. These two effects cancel out each other. This logic is similar to that of Dekel and Piccione(2000) in the voting literature, which showed that any equilibrium in a simultaneous voting game can be realized as a sequential equilibrium where voters completely ignore the predecessors' votes. In the voting case, voting behavior is governed by her expectation over the state of the world given that her vote is pivotal.

Therefore, under any equilibrium strategies, investors do not change their behavior due to learning from the accumulated amount of pledges k.

To illustrate above idea, we compute the equilibria when the population N follows the exponential distribution.

Example 1. Consider the case where the population N follows the exponential distribution: $F(N) = 1 - e^{-N}$. Further we assume $\pi = r = \frac{1}{2}$.

Proposition 2. (i) If $k^* > \frac{p(1-p)}{2p-1}\log(\frac{p}{1-p})$,

(i-a) there exists an equilibrium with history independent strategy σ such that $\sigma(g,k) = 1$ and $\sigma(b,k) = \beta$ for all $k \in [0,k^*)$;

(i-b) there is $\tilde{k}$ such that for any Borel set $B \subset [0,k^*]$ such that $m(B) = \tilde{k}$, history dependent strategy σ such that $\sigma(g,k) = 1$ and $\sigma(b,k) = \mathbb{1}_B$ constitutes an equilibrium.

(ii) If $k^* \leq \frac{p(1-p)}{2p-1}\log(\frac{p}{1-p})$, there exists an equilibrium σ such that $\sigma(g,k) = 1$ and $\sigma(b,k) = 0$ for all $k \in [0,k^*)$.

Proof. (i-a) For a strategy σ such that $\sigma(g,k) = 1$ and $\sigma(b,k) = \beta$ for all $k \in [0,k^*)$, define $b(\beta)$ as

$$b(\beta) = \frac{1 - F(\frac{k^*}{p+(1-p)\beta})}{1 - F(\frac{k^*}{p\beta+(1-p)})} = \exp\left(\frac{k^*}{p\beta + (1-p)} - \frac{k^*}{p + (1-p)\beta}\right)$$

Then $b(1) = 1$ and $b(0) > \frac{p}{1-p}$. Since $b(\cdot)$ is continuous, there is a β^* such that $b(\beta^*) = \frac{p}{1-p}$.

(i-b) Let $\tilde{k} = k^* - \frac{p(1-p)}{2p-1}\log(\frac{p}{1-p})$. Then,

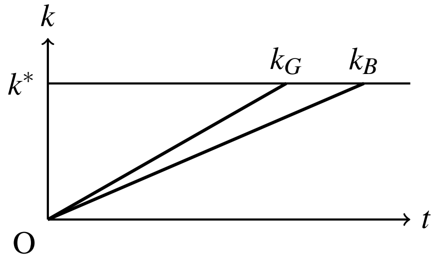
$$X_G^\sigma = k^* + \frac{(1-p)^2}{2p-1}\log(\frac{p}{1-p}), \;\; X_B^\sigma = k^* + \frac{p^2}{2p-1}\log(\frac{p}{1-p})$$

and therefore

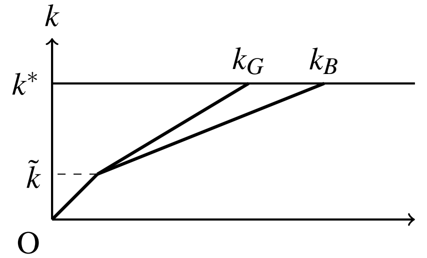
$$\frac{1 - F(X_G^\sigma)}{1 - F(X_B^\sigma)} = \frac{p}{1-p}.$$

□

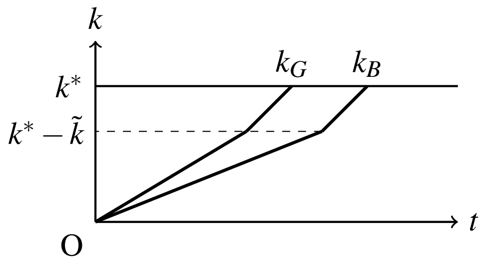
The two equilibria we constructed in Proposition 2 (i) satisfies the condition No.3 in Theorem 1, and the equilibrium in (ii) satisfies condition No.4. As for (i), there exist multiple equilibria and some of which are described as graphs of k_G and k_B below. In (a), reaction to the same signal does not depend on k, while in (b-1) to (b-3) they always invest regardless of the signal when k is in some regions. It seems that some investors discard information since they learn from k in (b-2) but it is actually not the case. What they only concern is the ratio of the probability of the goal to be met under G and B.



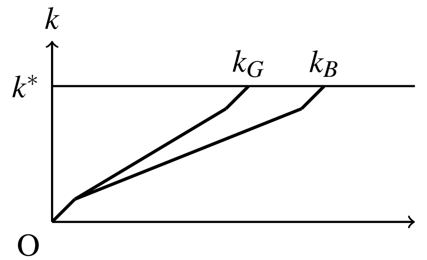
(a) History independent



(b-1) Boom at the beginning



(b-2) Boom at the end



(b-3) Boom at the end and the beginning

3.2 Investment Behaviors after the Goal is Met

We next investigate the investment behaviors after the funding goal is met. Since investors know that the investment project is already funded, they don't have to take into account the decisions by their successors to infer its quality. This feature is shared by the standard model of observational learning.

Here we introduce the concept of aggregate information cascade, a notion similar to that of Guarino et al.(2011).

Definition 2. An aggregate information cascade occurs when there is a critical value of k after which all agents choose the same action independently of their signals. Especially we say that it is an aggregate up(down) cascade if there is a critical value of k after which all agents choose I (NI) independently of their signals.

This definition can be rephrased in terms of σ. Aggregate up cascade occurs under σ if there exists $k^+ \in R_+$ such that $\sigma(g,k) = \sigma(b,k) = 1$ for all $k \geq k^+$. Aggregate down cascades occurs under σ if there exists $k^- \in R_+$ such that $\sigma(g,k^-) = \sigma(b,k^-) = 0$. We define $k^{UP} = \inf\{k^+ | \sigma(g,k) = \sigma(b,k) = 1 \;\; \forall k \geq k^+\}$ and $k^{DOWN} = \inf\{k | \sigma(g,k) + \sigma(b,k) = 0\}$.

Using this concept, Corollary 1 can be rephrased as follows.

Corollary 2. Aggregate down cascade with $k^{DOWN} \in [0, k^*)$ can be supported as some equilibria.

Now we calculate the expected utility from pledging observing signal s and accumulated level $k \geq k^*$.

$$U(s|r,k,\sigma) = (1-r)\Pr(G|s,k,rec,\sigma) - r\Pr(B|s,k,rec,\sigma)$$

We define $t_\omega^{DOWN} \equiv \inf\{t|k_\omega(t) = k^{DOWN}\}$. Then k_ω is strictly increasing in $[0, t_\omega^{DOWN})$, so that its inverse function t_ω can be defined on $[0, k^{DOWN})$. Using the Bayes rule, the followings hold;

$$\Pr(G|b,rec,k,\sigma) = \frac{\Pr(G,b|rec,k,\sigma)}{\Pr(b|rec,k,\sigma)}$$

$$\Pr(b|rec,k,\sigma) = (1-p)\Pr(G|rec,k,\sigma) + p\Pr(B|rec,k,\sigma)$$

$$= \begin{cases} \pi(1-p)\dfrac{g(t_G(k)|rec)}{h(k)} + (1-\pi)p\dfrac{g(t_B(k)|rec)}{h(k)} & \text{if} \quad k < k^{DOWN} \\ \\ \pi(1-p)\dfrac{1-G(t_G^{DOWN}|rec)}{\Pr(k^{DOWN}|rec,\sigma)} + (1-\pi)p\dfrac{1-G(t_B^{DOWN}|rec)}{\Pr(k^{DOWN}|rec,\sigma)} & \text{if} \quad k = k^{DOWN} \end{cases}$$

$$\Pr(G,b|rec,k,\sigma) = \Pr(G,b|rec,k,\sigma) = \Pr(G|rec,k,\sigma)\Pr(b|G,k,rec,\sigma)$$

$$= \begin{cases} \pi(1-p)\dfrac{g(t_G(k)|rec)}{h(k)} = \pi(1-p)\dfrac{1-F(t_G(k))}{h(k)} & \text{if} \quad k < k^{DOWN} \\ \\ \pi(1-p)\dfrac{1-G(t_G^{DOWN}|rec)}{\Pr(k^{DOWN}|rec,\sigma)} & \text{if} \quad k = k^{DOWN} \end{cases}$$

where $1 - G(x|rec) = \int_x^\infty g(t|rec)dt = \int_x^\infty (t-x)f(t)dt$.
Therefore, $U(s|r,k,\sigma)$ can be rewritten as;

$$U(g|r,k,\sigma) = \begin{cases} \dfrac{\pi p(1-r)(1-F(t_G(k))) - (1-\pi)(1-p)r(1-F(t_B(k)))}{\pi p(1-F(t_G(k))) + (1-\pi)(1-p)(1-F(t_B(k)))} & \text{if} \quad k < k^{DOWN} \\ \\ \dfrac{\pi p(1-r)(1-G(t_G^{DOWN}|rec)) - (1-\pi)(1-p)r(1-G(t_B^{DOWN}|rec))}{\pi p(1-G(t_G^{DOWN}|rec)) + (1-\pi)(1-p)(1-G(t_B^{DOWN}|rec))} & \text{if} \quad k = k^{DOWN} \end{cases}$$

$$U(b|r,k,\sigma) = \begin{cases} \dfrac{\pi(1-p)(1-r)(1-F(t_G(k))) - (1-\pi)pr(1-F(t_B(k)))}{\pi(1-p)(1-F(t_G(k))) + (1-\pi)p(1-F(t_B(k)))} & \text{if} \quad k < k^{DOWN} \\ \\ \dfrac{\pi(1-p)(1-r)(1-G(t_G^{DOWN}|rec)) - (1-\pi)pr(1-G(t_B^{DOWN}|rec))}{\pi(1-p)(1-G(t_G^{DOWN}|rec)) + (1-\pi)p(1-G(t_B^{DOWN}|rec))} & \text{if} \quad k = k^{DOWN} \end{cases}$$

Then the following lemma can be proved in exactly the same way as Lemma 1.

Lemma 2 (Monotonicity)**.** Suppose that σ constitutes a Bayesian Nash Equilibrium in Γ and $k^{DOWN} > k^*$.

1. $\forall k \in [k^*, k^{DOWN})$; $\sigma(b,k) > 0 \Rightarrow \sigma(g,k) = 1$;
2. $\forall k \in [k^*, k^{DOWN})$; $\sigma(g,k) < 1 \Rightarrow \sigma(b,k) = 0$;
3. $k_G(t) \geq k_B(t)$, $t_G(k) \leq t_B(k)$ and $t_G^{DOWN} \leq t_B^{DOWN}$.

The following theorem follows the discussions above.

Theorem 2. A strategy profile σ constitutes a Bayesian Nash equilibrium in Γ if and only if

1. σ constitutes an equilibrium in $\bar{\Gamma}$;
2. If $k^* \leq k^{DOWN}$, for $k \in [k^*, k^{DOWN})$,

$$\frac{1-p}{p} \leq \frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-F(t_G(k))}{1-F(t_B(k))} \quad \text{and} \quad \sigma(g,k) > 0.$$

   Moreover,

$$\frac{p}{1-p} < \frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-F(t_G(k))}{1-F(t_B(k))} \Rightarrow \sigma(b,k) = 1.$$

3. If $k^* \leq k^{DOWN} < \infty$,

$$\frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-G(t_G^{DOWN}|rec)}{1-G(t_B^{DOWN}|rec)} \leq \frac{1-p}{p}.$$

Proposition 3. If $\frac{\pi}{1-\pi}\frac{1-r}{r} > \frac{p}{1-p}$, aggregate up cascades occur with $k^{UP} = 0$. If $\frac{1-p}{p} < \frac{\pi}{1-\pi}\frac{1-r}{r}$, aggregate down cascade with $k^{DOWN} \in [k^*, \infty)$ never occurs.

This proposition says two things: (i) if the signal is so weak that an investor would choose I when the only information available was a bad signal, then aggregate up cascade occurs immediately after the game starts; (ii) if the signal is not so weak that she would follow the good signal if no other information is available, aggregate down cascades never occurs.

Proposition 4. Suppose that F has the nondecreasing hazard rate property. In addition, suppose that there exists an equilibrium σ such that $\frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-F(t_G(k^{**}))}{1-F(t_B(k^{**}))} = \frac{p}{1-p}$ holds for some $k^{**} \geq k^*$. Then, $\tilde{\sigma}$ defined below constitutes an equilibrium that exhibits an aggregate up cascade with threshold $k^{UP} \leq k^{**}$.

$$\tilde{\sigma}(s,k) = \begin{cases} \sigma(s,k) & \text{if} \quad k \in [0, k^{**}) \\ 1 & \text{if} \quad k \geq k^{**} \end{cases}$$

Proof. Define a function $B : [k^*, \infty) \to \mathbb{R}_+$ as

$$B(k) = \frac{1-F(t_G(k))}{1-F(t_B(k))}.$$

Then,

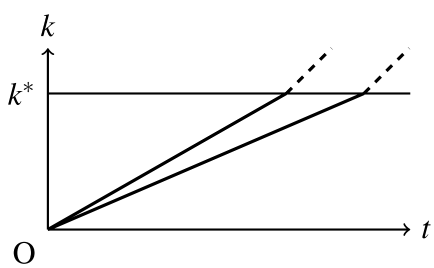
$$
\begin{aligned}
sgn(B'(k)) &= sgn\left(f(t_B(k))[1-F(t_G(k))]t'_B(k) - f(t_G(k))[1-F(t_B(k))]t'_G(k)\right) \\
&= sgn\left(\frac{\lambda(t_B(k))}{\lambda(t_G(k))} - \frac{t'_G(k)}{t'_B(k)}\right) = sgn\left(\frac{\lambda(t_B(k))}{\lambda(t_G(k))} - \frac{p\sigma(b,k)+(1-p)\sigma(g,k)}{p\sigma(g,k)+(1-p)\sigma(b,k)}\right)
\end{aligned}
$$

Lemma 2 and the nondecreasing hazard rate assumption implies that $\frac{\lambda(t_B(k))}{\lambda(t_G(k))} \geq 1$ and $\frac{p\sigma(b,k)+(1-p)\sigma(g,k)}{p\sigma(g,k)+(1-p)\sigma(b,k)} \in [\frac{1-p}{p}, 1]$. Thus $B'(k) \geq 0$ and therefore $\frac{p}{1-p} \leq \frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-F(t_G(k))}{1-F(t_B(k))}$ for any $k \geq k^{**}$. □

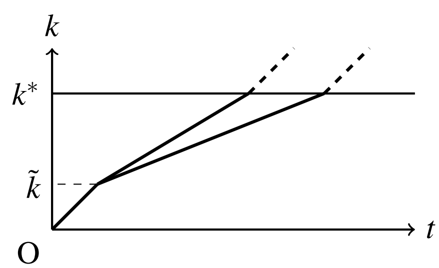
The above proposition claims that if the distribution of the population size satisfies a regularity condition, then once the belief over the good state gets high enough then the aggregate up information cascades starts from that point. The threshold is actually reached at the point that the goal is met when the condition No.3 in Theorem 1 holds, which implies that aggregate up cascades immediately starts right after the goal is met.

Corollary 3. Suppose that F has the nondecreasing hazard rate property. Fix any σ which constitutes an equilibrium in $\bar{\Gamma}$ such that $\sigma(b,k) > 0$ for some $k \in [0,k^*)$. Then σ^{UP} defined below constitutes an equilibrium in Γ.

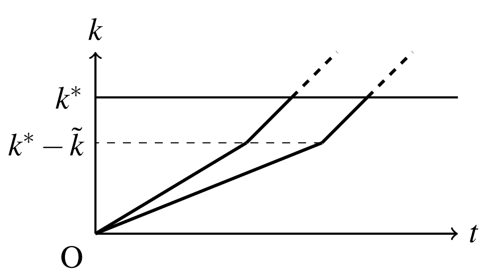
$$
\sigma^{UP}(s,k) = \begin{cases} \sigma(s,k) & \text{if} \quad k \in [0,k^*) \\ 1 & \text{if} \quad k \geq k^* \end{cases}
$$



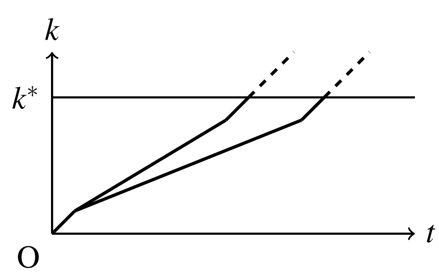
(a') History independent



(b'-1) Boom at the beginning



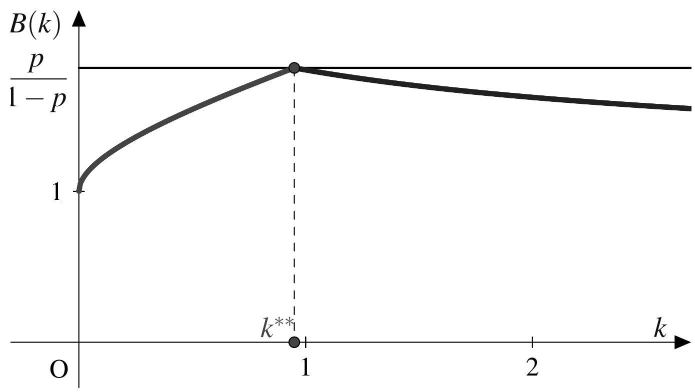
(b'-2) Boom at the end



(b'-3) Boom at the end and the beginning

Example 2. Consider the case where the population N follows the exponential distribution: $F(N) = 1 - e^{-N}$. Further we assume $\pi = r = \frac{1}{2}$. Note that F has a constant hazard rate property. Then Proposition 2 and Corollary 3 implies that, under the condition in Proposition 2 (i), investors chooses I regardless of their own signal after the funding goal is met.

Example 3. We give an example which highlight the role of nondecreasing hazard rate property. Suppose that the population N follows the Weibull distribution with shape parameter $m = 0.5$ and mean 1: $F(N) = 1 - \exp(-(\Gamma(3)N)^{0.5})$. This exhibits decreasing hazard rate property. Further we assume $\pi = r = \frac{1}{2}$ and $k^* = 0$.



The figure shows the graph of $B(k)$ defined in the proof of Proposition 3 for σ such that $\sigma(g,k) = 1$ and $\sigma(b,k) = 0$ for $k < k^{**}$ and $\sigma(g,k) = \sigma(b,k) = 1$ for $k \geq k^{**}$, where $B(k^{**}) = \frac{p}{1-p}$ and $p = 0.65$. We can see from this figure that under σ, an investor who observes bad signal b and $k > k^{**}$ has no incentive to choose I. Therefore σ violates the equilibrium condition, which shows that an aggregate up cascades with $k^{UP} = k^{**}$ does not occur.

Remark Unlike the model of Guarino et al.(2011), we proposed a model of aggregate social learning where information cascades do not occur. This implies that the form of distribution over the population size has an important role in the possibility of aggregate information cascade.

4 Comparison to the Static Model

To highlight the role of dynamics in the model, we build a static model where all the investors simultaneously decides whether or not to invest in the crowdfunding project.

Like the baseline model in Section 2, there are two states of the world $\omega \in \Omega \equiv \{G, B\}$, where $G(B)$ denotes the case where the quality of the project is good(bad). There is a mass N of investors each of whom are supposed to choose binary action $a \in A \equiv \{I, NI\}$ where I denotes pledging to invest and NI denotes abstention. Before making an investment decision, she privately receives a binary signal $s \in S \equiv \{g, b\}$ about the true state ω with accuracy p. The private signals are independent across investors conditional on the state. The population size N of the investors is drawn from a distribution F with density $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $0 < \mathbb{E}[N] = 1$.

The project is executed only when the mass of investors who choose I exceeds k^*. Unlike the baseline model, investors choose their actions simultaneously. Thus, an investor's strategy is a function $\sigma : \{g, b\} \to [0,1]$, where $\sigma(s)$ denotes the probability of choosing I after observing signal s. Given σ, the mass of investors who choose I is defined as $\{p\sigma(g) + (1-p)\sigma(b)\}N$ when $\omega = G$ and $\{p\sigma(b) + (1-p)\sigma(g)\}N$ when $\omega = B$.

We call this static game as $\hat{\Gamma}$.

Investor's Payoff Expected payoff from pledging observing the signal s is;

$$U(s|r,\sigma) \equiv (1-r)\Pr(G,e|s,rec,\sigma) - r\Pr(B,e|s,rec,\sigma).$$

Definition 3. An action profile, σ, is a Bayesian Nash Equilibrium in $\hat{\Gamma}$ if for $s \in \{g, b\}$,

$$U(s|r,\sigma) > 0 \Rightarrow \sigma(s) = 1;$$
$$U(s|r,\sigma) < 0 \Rightarrow \sigma(s) = 0$$

Using the Bayes rule, the followings hold;

$$\Pr(G,e|b,rec,\sigma) = \frac{\Pr(G,b,e|rec,\sigma)}{\Pr(b|rec,\sigma)} = \frac{\Pr(G,b,e|rec,\sigma)}{\Pr(b)}$$
$$\Pr(G,b,e|rec,\sigma) = \pi(1-p)\Pr(e|G,b,rec,\sigma)$$
$$= \pi(1-p)\Pr(N \geq X_G^\sigma|rec)$$

where X_ω^σ denotes the minimum number of investors recruited necessary for the target to be reached given σ and ω. More formally, for a strategy σ s.t. $(\sigma(g), \sigma(b)) = (\gamma, \beta)$ and $\gamma + \beta > 0$,

$$X_G^\sigma = \frac{k^*}{p\gamma + (1-p)\beta}, \quad X_B^\sigma = \frac{k^*}{(1-p)\gamma + p\beta}$$

If $\gamma = \beta = 0$, let $X_G = X_B = \infty$. Likewise, $\Pr(B,b,e|rec,\sigma) = (1-\pi)p\Pr(N \geq X_B^\sigma|rec)$.

Thus, $U(s|r,\sigma)$ can be rewritten as follows;

$$U(g|r,\sigma) = \{\pi p(1-r)\Pr(N \geq X_G^\sigma|rec) - (1-\pi)(1-p)r\Pr(N \geq X_B^\sigma|rec)\}/\Pr(g)$$
$$U(b|r,\sigma) = \{\pi(1-p)(1-r)\Pr(N \geq X_G^\sigma|rec) - (1-\pi)pr\Pr(N \geq X_B^\sigma|rec)\}/\Pr(b) \tag{5}$$

Then close observation shows the following facts.

Corollary 4. There is a trivial equilibrium with no investment.

Proof. Consider a strategy σ^0 such that $(\sigma^0(g), \sigma^0(b)) = (0,0)$. Then $U(g|r,\sigma^0) = U(b|r,\sigma^0) = 0$, which means that N and NI are indifferent. □

Next we focus on the equilibria where the project can be executed with positive probability.

Lemma 3 (Monotonicity)**.** Unless $\Pr(N \geq X_G^\sigma|rec) = \Pr(N \geq X_B^\sigma|rec) = 0$,

1. $\beta > 0 \Rightarrow \gamma = 1$
2. $\gamma < 1 \Rightarrow \beta = 0$
3. $\Pr(N \geq X_G^\sigma|rec) \geq \Pr(N \geq X_B^\sigma|rec) > 0$

Proof. For the first part, we only show the proof for the statement $\beta > 0 \Rightarrow \gamma = 1$. Unless $\Pr(N \geq X_G^\sigma|rec) = \Pr(N \geq X_B^\sigma|rec) = 0$,

$$\begin{aligned}
&\Pr(g)U(g|r,\sigma) - \Pr(b)U(b|r,\sigma) \\
&= \{\pi p(1-r)\Pr(N \geq X_G^\sigma|rec) - (1-\pi)(1-p)r\Pr(N \geq X_B^\sigma|rec)\} \\
&- \{\pi(1-p)(1-r)\Pr(N \geq X_G^\sigma|rec) - (1-\pi)pr\Pr(N \geq X_B^\sigma|rec)\} \\
&= (2p-1)\{\pi(1-r)\Pr(N \geq X_G^\sigma|rec) + (1-\pi)r\Pr(N \geq X_B^\sigma|rec)\} > 0.
\end{aligned} \tag{6}$$

By the definition of the equilibrium, $\beta > 0$ implies $U(b|r,\sigma) \geq 0$. Then by (6), $U(g|r,\sigma) > 0$ must be satisfied, which implies that $\gamma = 1$ by the equilibrium condition.

$X_G^\sigma \leq X_B^\sigma$ and $\Pr(N \geq X_G^\sigma|rec) \geq \Pr(N \geq X_B^\sigma|rec) > 0$ are derived directly from the definition of $X_\omega(\sigma)$. □

Corollary 4, Lemma 3 and equations (5) yield the characterization of the equilibrium in the static crowdfunding model $\hat{\Gamma}$.

Proposition 5. The strategy profile σ is a Bayesian Nash equilibrium in $\hat{\Gamma}$ if and only if one of the followings holds;

1. $\sigma(g) = \sigma(b) = 0$;
2. $\sigma(g) = 1, \sigma(b) = 1$ and $\dfrac{p}{1-p} < \dfrac{\pi}{1-\pi}\dfrac{1-r}{r}$;
3. $\sigma(g) = 1, \sigma(b) \geq 0$ and $\dfrac{\pi}{1-\pi}\dfrac{1-r}{r}\dfrac{1-F(X_G^\sigma|rec)}{1-F(X_B^\sigma|rec)} = \dfrac{p}{1-p}$;
4. $\sigma(g) = 1, \sigma(b) = 0$ and $\dfrac{1-p}{p} < \dfrac{\pi}{1-\pi}\dfrac{1-r}{r}\dfrac{1-F(X_G^\sigma|rec)}{1-F(X_B^\sigma|rec)} < \dfrac{p}{1-p}$;
5. $\sigma(g) \leq 1, \sigma(b) = 0$ and $\dfrac{1-p}{p} = \dfrac{\pi}{1-\pi}\dfrac{1-r}{r}\dfrac{1-F(X_G^\sigma|rec)}{1-F(X_B^\sigma|rec)}$.

If you compare this proposition with Theorem 1, you will find that the similar sets of conditions are offered except that in Proposition 5, (i) the strategy cannot depend on k and (ii) $1-F(X^{\sigma}_{\omega})$ is replaced with $1-F(X^{\sigma}_{\omega}|rec)$. As to the second point, in other words, the probability of the event that the goal is met under each state is evaluated by ex ante distribution in Theorem 1, while evaluated by the distribution given that the player is recruited in Proposition 5.

Although informal, we believe that this difference is negligible when we consider the case where the population size is almost deterministic. Consider a sequence of distribution functions $\{F_n\}$ such that $\int_0^{\infty} N dF_n = 1$ and

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{if} \quad x > 1 \\ 1/2 & \text{if} \quad x = 1 \\ 0 & \text{if} \quad 0 \leq x < 1 \end{cases}$$

Next we consider $\{F_n(\cdot|rec)\}$, where $F_n(\cdot|rec)$ is defined as $F_n(x|rec) = \int_0^x F_n(x) - F_n(t)dt$ for all x. Then

$$\lim_{n\to\infty} F_n(x|rec) = \begin{cases} 1 & \text{if} \quad x > 1 \\ 1/2 & \text{if} \quad x = 1 \\ 0 & \text{if} \quad 0 \leq x < 1 \end{cases}$$

That is, the limits of F_n and $F_n(\cdot|rec)$ coincide.

5 Welfare Analysis

In this section, we discuss how well the crowdfunding mechanism aggregate the information. In the standard information cascade model like Banerjee(1992), the probability of correct decisions to be made does not get better no matter how many players there are, as far as they sequentially make their decisions. Recall that in the aggregate informational cascades model in Guarino et al.(2011), although there is a certain level of accumulated investment until which investors follow their own signal, aggregation of information completely fail in the sense that ex ante each investor makes the right decision with probability p, which is the same as the accuracy of just one signal.

For the evaluation of our crowdfunding model, we use the following formula:

$$\begin{aligned} W &= \pi(1-r)\Pr(e|G) - (1-\pi)r\Pr(e|B) \\ &= \pi(1-r)(1-F(X^{\sigma}_{G})) - (1-\pi)r(1-F(X^{\sigma}_{B})) \end{aligned}$$

Notice that this welfare criterion is nothing to do with the investment behaviors after the funding goal is met.

Under the trivial equilibria with zero probability of achievement, which always exists by Corollary 1, it is obvious that $W = 0$. Next we consider equilibria with positive probability of achievement, using Theorem 1. When $\frac{p}{1-p} < \frac{\pi}{1-\pi}\frac{1-r}{r}$, there is an equilibrium where investors always invest regardless of their own signal. In such a case, $W = (\pi - r)(1 - F(X_G^{\sigma}))(> 0)$. For the remainder we focus on the case of $\frac{p}{1-p} \geq \frac{\pi}{1-\pi}\frac{1-r}{r}$, where there may exist some equilibria such that investment decisions depends on signals. Under such equilibria, W can be rewritten as follows;

$$W = \pi(1-r)\left(1 - \frac{1}{c(\sigma)}\right)(1 - F(X_G^{\sigma}))$$

where

$$c(\sigma) \equiv \frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-F(X_G^{\sigma})}{1-F(X_B^{\sigma})} \in \left[\frac{1-p}{p}, \frac{p}{1-p}\right]$$

We evaluate the welfare when the population size N is almost deterministic by constructing a sequence of games $\{\bar{\Gamma}_n\}_{n=0}^{\infty}$ where $\bar{\Gamma}_n$ is equipped with F_n that satisfies the following:

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{if} \quad x > 1 \\ 1/2 & \text{if} \quad x = 1 \\ 0 & \text{if} \quad 0 \leq x < 1 \end{cases}$$

Proposition 6. If $k^* \in (1-p, 1)$, there exists a sequence $\{\sigma_n\}_{n=0}^{\infty}$ such that σ_n constitutes an equilibrium in $\bar{\Gamma}_n$, $F_n(X_G^{\sigma_n}) \to 0$ and $c(\sigma_n) \to \frac{p}{1-p}$. The sequence $\{W_n\}_{n=1}^{\infty}$ such that $W_n = \pi(1-r)\left(1 - \frac{1}{c(\sigma_n)}\right)\left(1 - F(X_G^{\sigma_n})\right)$ converges to $\dfrac{2p-1}{p}\pi(1-r) > 0$.

Proof. Define $\beta^* \equiv \max\{\dfrac{k^* - p}{1-p}, 0\}$. Then $\frac{k^*}{p+(1-p)\beta^*} \leq 1$ and $\frac{k^*}{(1-p)+p\beta^*} > 1$. By definition of $\{F_n\}$, for any $\varepsilon > 0$ there exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$,

$$1 - F_n\left(\frac{k^*}{p + (1-p)\beta^*}\right) > \frac{1}{2} - \varepsilon$$
$$1 - F_n\left(\frac{k^*}{(1-p) + p\beta^*}\right) < \varepsilon$$

which implies that

$$\frac{1 - F_n\left(\frac{k^*}{p+(1-p)\beta^*}\right)}{1 - F_n\left(\frac{k^*}{(1-p)+p\beta^*}\right)} > \frac{1}{2\varepsilon} - 1.$$

Therefore, there exists $\tilde{n} \in \mathbb{N}$ such that for all $n \geq \tilde{n}$,

$$\frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-F_n\left(\frac{k^*}{p+(1-p)\beta^*}\right)}{1-F_n\left(\frac{k^*}{(1-p)+p\beta^*}\right)} < \frac{p}{1-p} < \frac{\pi}{1-\pi}\frac{1-r}{r}.$$

Since F_n is continuous by assumption, there exist $\beta_n \in (\beta^*, 1)$ such that $\frac{\pi}{1-\pi}\frac{1-r}{r}\frac{1-F_n\left(\frac{k^*}{p+(1-p)\beta_n}\right)}{1-F_n\left(\frac{k^*}{(1-p)+p\beta_n}\right)} = \frac{p}{1-p}$. Define a sequence of strategies such that for $k \in [0, k^*)$

$$\sigma_n(g,k) = p + (1-p)\beta_n$$
$$\sigma_n(b,k) = (1-p) + p\beta_n.$$

Then by Theorem 1, σ_n constitutes an equilibrium for $\bar{\Gamma}_n$. Moreover $\beta_n \in (\beta^*, 1)$ implies that $1 - F_n\left(\frac{k^*}{p+(1-p)\beta_n}\right) > 1 - \varepsilon$. Therefore $1 - F_n\left(\frac{k^*}{p+(1-p)\beta_n}\right)$ goes to 1 as $n \to \infty$. □

We let the welfare level in the limit $\frac{2p-1}{p}\pi(1-r)$ denoted by W^*. If you have to decide whether to execute the project on the basis of just one signal with accuracy p, the welfare level W_0 is calculated as follows;

$$W_0 = \begin{cases} \pi - r & \text{if } \frac{p}{1-p} < \frac{\pi}{1-\pi}\frac{1-r}{r} \\ (1-r)\pi p - r(1-\pi)(1-p) & \text{if } \frac{1-p}{p} \leq \frac{\pi}{1-\pi}\frac{1-r}{r} \leq \frac{p}{1-p} \\ 0 & \text{if } \frac{1-p}{p} > \frac{\pi}{1-\pi}\frac{1-r}{r} \end{cases}$$

Note that for $\frac{\pi}{1-\pi}\frac{1-r}{r} \leq \frac{p}{1-p}$,

$$\begin{aligned} W^* - W_0 &\geq \frac{2p-1}{p}\pi(1-r) - \{(1-r)\pi p - r(1-\pi)(1-p)\} \\ &= \left(\frac{1-p}{p}\right)((1-\pi)pr - \pi(1-p)(1-r)) \geq 0 \end{aligned}$$

since we are assuming that $\frac{p}{1-p} \geq \frac{\pi}{1-\pi}\frac{1-r}{r}$. Thus, decisions to execute or not made through the crowdfunding mechanism is better than the decision made with just one signal.

Remark This gain does not come from the dynamics but from the all-or-nothing scheme, since in principle the set of equilibria in the dynamic model and those in the static model coincides when we focus on the case of diminishing population uncertainty. However, the gain from dynamics can be captured if we also evaluate the fact that the investors herds with stronger information for the good state after the funding goal is met.

Remark Since there is a continuum of investors, we could perfectly detect the true state if we could directly observe all the signals they receive. Thus, the welfare level in the first best is $\bar{W} = \pi(1-r)$ which is strictly larger than W^*. This loss in welfare comes from the free-riding effects among the investors: suppose that all other investors follow their own signals, then investors with bad information have a strong incentive to invest, since the project quality is more likely to be good conditional on the event that the project is funded. This effect is present in both the static and dynamic model.

6 Extensions

Since our model is very preliminary and captures limited aspects of the reality, there are several directions to be extended.

First, we can consider the model where entrepreneurs may set the price (interest rate) r or the funding goal k^*. Moreover we can proceed to the mechanism design problem which explicitly takes dynamics into account. Moral hazard problems might also be taken into account. This direction of studies can be viewed as a sophisticated version of Chang(2016) and Ellman and Hurkens(2015).

Second, we can also explore different state space and signal structures such as continuous state space and signal space, which is used in Chang(2016). Like in the standard information cascade model, it might be that the signal structures determine some properties of investment behaviors. We believe it still holds, however, that learning from current k does not essentially affect investors' behaviors before the funding goal is met. We can also consider the case where there are some agents with no information but we presume that it won't basically alter the results.

Third, we can investigate the model of crowdfunding with private value environment, which is more relevant for reward-based crowdfunding. Though we don't have to care about learning problems about the quality in this case, we have to care about free-riding problem because once the project is funded it is usually the case that we can buy the product at the retail stage at lower risk. Thus, we think that it is also an important aspect to be investigated regarding the dynamics of the crowdfunding procedures. Psychological factors might also be taken into account. Kuppuswamy and Bayus(2017,2018) conducted an empirical investigation on *Kickstarter*, one of the largest crowdfunding platforms for startups. They report that a Kickstarter project whose funding is close to the target tends to attract more pledges. They argue that this goal-gradient effect is due to people's preference for making an impact on the outcome rather than learning.

Finally, in order to make it more realistic, it is worthwhile to consider the model where information about the timing of their arrival is observable to the investors or investors can wait for some periods.

7 Conclusion

We have introduced a model of dynamic crowdfunding where a continuum of investors sequentially make their investment decisions. Assuming common value environment, which is especially relevant for equity-based crowdfunding or P2P lending, we analyze the dynamics of investment behaviors in this model. The question we answered in this paper is: "How well does the crowdfunding mechanism aggregate information?"

The greatest concern is whether or not information cascades occur. The popular all-or-nothing scheme prevent the information cascades to occur while the funding goal is yet to be reached. Learning from the observation of how much money has already been pledged does not alter the optimal reactions to their own information. The key fact for this result is that investors evaluate the project conditional on the event that it is funded. Smaller amount of pledges before you is a negative information for the quality, but on the event that the target will be reached, larger number of investors will invest after you, which is a good information. These two effects cancel out each other.

We also demonstrate that various patterns of investment (ex. momentum at the beginning, end, or both) can constitute equilibria. As we mentioned above, herding behaviors are not due to learning. This result indicates that herding behaviors commonly observed in many crowdfunding situations may not be a result of learning nor irrational behaviors.

On the other hand, an information cascade occurs after the funding goal is met under some regularity condition. The analysis for this part provides a new aspect to the aggregate information cascade model a la Guarino et al.(2011). We pointed out the population uncertainty introduced in our model has a key role in determining the existence of information cascades: if the distribution of the population size lacks the increasing hazard rate property, aggregate information cascade may not occur.

Finally we evaluate the performance of crowdfunding scheme as a *gatekeeper* of finance: only the projects with high quality should be funded. We show that crowdfunding with *all-or-nothing* scheme partially aggregates investors' information in the sense that the decision of the execution of the project is more correct in crowdfunding than using the information held by a single investor.

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8 Appendix

Lemma 4. F_n pointwise converges to F. Furthermore,

$$\lim_{n\to\infty} \frac{\int_0^x (F_n(x) - F_n(t))dt}{\int_0^\infty (1 - F_n(t))dt} = \int_0^x (F(x) - F(t))dt \equiv F(x|rec)$$

Proof. (First part) Recall that by definition, $F_n(x) = \frac{F(x)}{F(n)} \mathbb{1}_{\{x \leq n\}}$. Since F is a distribution function, $\lim_{n\to\infty} F(n) = 1$. Second, fixing any x, $\lim_{n\to\infty} \mathbb{1}_{\{x\leq n\}} = 1$. Thus, for any $x \in \mathbb{R}_+$, $\lim_{n\to\infty} F_n(x) = F(x)$.

(Second part) First, we show that $\lim_{n\to\infty} \int_0^\infty (1 - F_n(t))dt = \int_0^\infty (1 - F(t))dt$. By definition, $1 - F_n(t) \leq 1 - F(t)$ for all t, n. Further, $\int_0^\infty (1 - F(t))dt = 1 < \infty$. Thus, dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_0^\infty (1 - F_n(t))dt = \int_0^\infty \lim_{n\to\infty} (1 - F_n(t))dt = \int_0^\infty (1 - F(t))dt.$$

The same kind of argument applies to the numerator, so that for any x,

$$\lim_{n\to\infty} \int_0^x (F_n(x) - F_n(t))dt = \int_0^x (F(x) - F(t))dt.$$

This ends the proof. □