High-Frequency Trading Arms Race under National Market System : Welfare Analysis under CLOB and FBA

By Kazuyuki Higashi*

Budish,Cramton,Shim(2015)(BCS) develop the tractable Glosten-Milgrom(GM) model. They assume the specific probability process and introduce the cost of high frequency trading into Glosten-Milgrom model. My research makes a toy model of the security exchange market competition under National Market System in US, using this tractable GM model. First, I analyze the fictionless markets. Under Continuous Limit Order Book(CLOB), which is the most popular trading system, I endogenize bid ask spread by combining the tractable Glosten-Milgrom model and price competition under National Market System between Security Exchange Markets. This equilibrium bid ask spread is increasing over the number of the security exchange markets. So severe competition between security exchange markets leads to welfare loss by the arms race for investing on high frequency trading(HFT) technologies. BCS also argues that Frequent Batch Auction(FBA) reduces this kind of welfare loss. I also shows that if the number of security exchange markets who adopt FBA is larger than 2, then no tradings occur in CLOB markets. That is, FBA dominates the CLOB if the security exchange markets prefer to the situation that the tradings occur in its own markets. Second, I add the friction into the competition among security exchange markets. I show that the equilibrium bid ask spread is decreasing over the number of security exchange markets. However, the welfare loss is still increasing over the number of security exchange markets except that the friction is extremely high. This implies that the small bid ask spread cannot be used as the benchmark for the efficient market. With friction, the bid ask spread is increasing over the number of the security exchange markets. Without friction, the bid ask spread is decreasing. In both cases, the welfare loss is increasing except such a special case. That is, under HFT arms race, the fragmentation leads to welfare loss.

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I. Introduction

High frequency trading (HFT) changes the fundamental of the market. HFT trading based on computer algorithms are faster than trading by human. Speed is the key factor in order to get the profits under the current market system, which is continuous limit order book (CLOB) where the trade occurs whenever the demand and supply crosses. Limit order means that every traders bid the pair of price and quantity of the security which they want to buy or sell. For example, when the price of Toyota stock is 100(bid)-101(ask) dollars in NASDAQ, and 101(bid)-102(ask) dollars in NYSE, then traders can get the profits by buying this stock in NASDAQ and selling it in NYSE before the market maker reflects this price change. The winner of this arbitrage is only who can trade the fastest. So HFT traders invest in HFT technology. For example, these HFT technology includes useful software and hardware like optical cable and collocation service provided by the security exchanges. However, this kind of investment is a kind of prisoner's dilemma. Such investment is needed for HFT traders to win, but does not improve efficiency of society. This welfare loss is pointed out by Budish, Cramton and Shim (2015) (BCS model).

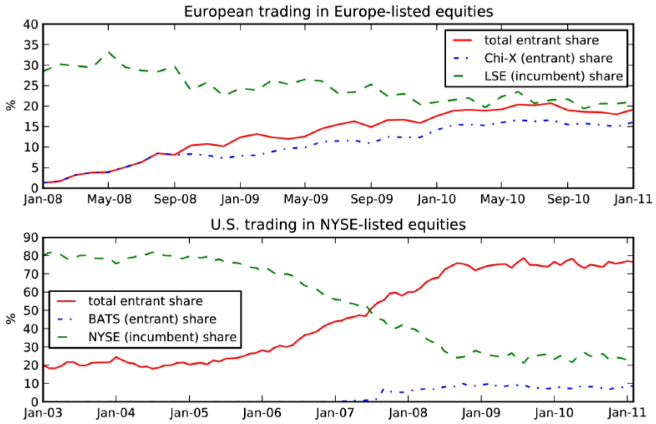
According to Glosten and Milgrom (1985) (GM model), the origin of the bid ask spread comes from the existence of the informed traders. Market maker loses the profits via trading with the informed traders. In order to compensate for such loss, they set positive bid ask spread to extract the surplus from the noise traders. BCS model rewrites this GM model by assuming the specific probability process and introducing the cost of high frequency trading into GM model. In their model, the origin of the bid ask spread comes from the existence of the stale quote snipers who submit orders before market maker changes the orders when the fundamental value of the price of the stocks changes. Such behavior of HFT traders is called sniping. Market maker sets positive bid ask spread to compensate for the loss of being sniped. And they also describe the HFT traders have to invest in HFT technologies by prisoner's dilemma, which leads to the welfare loss. Their model reflects the real market situation because both market makers and stale quotes snipers are HFT traders in real economy (Lewis (2014)).

I use BCS model to analyze the market fragmentation under National Market System in US. National Market System provides the order protection rule as follows. Every orders are transferred into the security exchange markets whose bid ask spread is the lowest among every security exchange markets. This order protection rules leads to severe competition among security exchange markets. There are more than 50 security exchange markets in US ((O'Hara, 2015). According to Menkveld (2011), the shares of the NYSE and NASDAQ are decreasing through recent years (See Figure 1). In contrast, entrants' share is increasing, which implies market fragmentation goes on. First, I analyze the friction-less market like the Bertrand competition. I show that the more security exchange markets leads to the more welfare loss. The mechanism is that the more security exchange markets split the trading needs of investors from which market makers extract surplus.

In addition, the more security exchange markets mean the more arbitrage opportunities, which gives the stale quote snipers the sniping opportunities. In order to compensate the loss of revenue and being sniped, the market maker of each market increases the bid ask spread, which is empirically supported by Baldauf and Mollner (2017*b*). Through this mechanism, the welfare loss becomes larger as the number of security exchange markets increases. So there are too many security exchange markets in US. I also show the possibility that the frequent batch auction (FBA) will surpass the CLOB. According to Budish, Cramton and Shim (2014), Budish, Cramton and Shim (2015), if the trading is done by FBA under sufficiently large discrete time, FBA can stop HFT arms race and the bid ask spread becomes 0. Using this property of FBA and considering the competition among security exchange markets, I show that if at least two security exchange markets adopt FBA, then we can stop arms races. Finally, I introduce the friction into the model like Salop (1979). In this case, the bid ask spread is decreasing over the number of security exchange markets by the competition effect, which is empirically supported by O'Hara and Mao (2011). O'Hara and Mao (2011) is the empirical research in US markets under National Market System. So, in reality, US markets has friction. We can interpret this friction originates from the speed of the order arrived in U.S. Securities and Exchange Commission (SEC). I also shows that the welfare loss from the investment of HFT technology is increasing over the number of the security exchange markets. In addition, total welfare loss is increasing except that the friction is extremely high. This implies that the bid ask spread might not be used as the benchmark for market efficiency.

The related literature are as follows. Budish, Cramton and Shim (2015) open up new possibilities with high tractability of GM model and also proposed new solution(FBA) against HFT arms race. Baldauf and Mollner (2017*a*) and Haas and Zoican (2016) are extension of BCS model. Baldauf and Mollner (2017*a*) show that trade-off between the research intensity of fundamental value and the size of bid ask spread. Haas and Zoican (2016) point out the possibility of larger bid ask spread under FBA than that under CLOB. Another paper Pagnotta and Phillippon (2017) discusses the effect of trading speed and price protection on the welfare. They use search theoretic model (Lagos and Rocheteau, 2009) and show that greater fragmentation and faster speed may lead to less welfare. My work compliments the these literature. My contribution is to prove that the fragmentation under strict order protection rule like National Market System leads to welfare loss, and point out the potential popularity of frequent batch auction.This popularity of FBA is discussed in the seminar slide (Budish, Cramton and Shim, 2017), which is a independent work of mine. I hope the regulators of all worlds sincerely discuss the adoption of FBA.

The reminder of my paper as follows. Section 2 presents the model without friction, Section 3 gives the analysis about the bid ask spread and welfare loss by speed race. Section 4 points out the potential popularity of FBA. Section 5 presents the model without friction. Section 6 concludes the paper.



source: Barclays Capital Equity Research and Federation of European Securities Exchanges

FIGURE 1. MARKET FRAGMENTATION (MENKVELD, 2011)

II. The model without friction

The baseline model is BCS model. But I introduce the new players, security exchange markets. There are $N \geq 2$ security exchange markets who adopt the system of continuous limit order book (CLOB) where the trade occurs whenever the demand and supply crosses. Limit order means that every traders bid the pair of price and quantity of the security which they want to buy or sell . In this popular CLOB, the selling orders less than the crossing equilibrium price and the buying orders larger than equilibrium price are traded immediately when the demand and supply balances in this market. There are single type of security in this world. There are many traders in each market. The number of traders are endogenized later. Only one of traders in each markets can become the market maker who submit bid and ask orders anytime. The market maker is determined by Bertrand competition. So only one traders become market maker who submit best bid-ask spread, which means that the market makers who submit the lowest bid and highest ask. Among traders they do the homogeneous goods Bertrand competition. So the liquidity providers gain zero profit in equilibrium. I normalize the cost of trading as 0 if he don't invest in the HFT technology. If a trader invest in HFT technology whose cost is C_{speed}, his order reach $\delta > 0$ milliseconds faster than a trader who doesn't invest in HFT technology. Slow traders, who don't invest in HFT technology, reach his order to security exchange after δ_{slow} milliseconds. Fast traders, who invest in HFT technology, reach his order to security exchange after δ_{fast} milliseconds. I assume $\delta_{slow} > \delta_{fast}$. So $\delta = \delta_{slow} - \delta_{fast}$. This technology is regarded as the cost of high frequency data conection, the collocation facilities, and the algorithm development. Also the investors who has liquidity needs of trading visit the markets following the Poisson process λ_I^n, which I regard it as the demand function of market n. To sum up, there are three types of players(security exchange markets, traders, investors).

The action of each players are defined as follows. Each security exchange markets set trading fees f^n (n is the notation of the security exchange markets.). Each traders and investors can exchange only one unit of security. Market makers have to set bid and ask in anytime and they always sell and buy the security when the orders happen. I denote the bid ask spread as s^n. Traders snipe the markets makers when the price of the security change. It means the traders can buy and sell the security at the price before the market makers reflects the change of the fundamental value. The fundamental value y_t of security change happens following the Poisson process λ_J. I also assume the change of the fundamental value follows $+\sigma, -\sigma$ with equal probabilities. This seemingly strict assumption is the same as the related literature like Baldauf and Mollner (2017*b*), Haas and Zoican (2016), Pagnotta and Phillippon (2017). And I also assume σ is sufficiently large as sniping behavior is profitable. This seemingly strict assumption is the same as the related literature like Baldauf and Mollner (2017*b*), Haas and Zoican (2016), Pagnotta and Phillippon (2017). I denote the absolute value of change as $J = |j|$. Investors always buy or sell based on their liquidity needs. After her trade is finished, she leave form the exchange market. I also assume the probability of the need of investors to buy and sell is the same ($\lambda_I^b = \lambda_I^s, \lambda_I^b + \lambda_I^s = \lambda_I$). This relation holds in each markets($\lambda_I^{nb} = \lambda_I^{ns}, \lambda_I^{nb} + \lambda_I^{ns} = \lambda_I^n$).

The utility functions of the players are defined as follows. Each players maximize utility per unit time. The utility of security exchange market $n \in \{1, ...N\}$ is

$$U_n^{market}(f^n) = 2f^n[\lambda_I^n + \lambda_J^n] \tag{1}$$

The utility function of traders (he) are

$$U^{trader} = y_t - p_t^n - f^n - C_{speed}\mathbb{1}\{c = 1\} \quad \text{(when he buys the security.)} \tag{2}$$

$$U^{trader} = p_t^n - y_t - f^n - C_{speed}\mathbb{1}\{c = 1\} \quad \text{(when he sells the security.)} \tag{3}$$

$c \in \{0, 1\}$. $c = 1$ means that he invests in HFT technology. $c = 0$ means that he doesn't invest in HFT technology. So the traders' action is the bid and ask and the investment decision. The utility function of investors (she) are

$$U^{investors} = v + y_{t'} - p_{t'}^n - d(t' - t) - f^n \quad \text{(when her liquidity need is to buy.)} \tag{4}$$

(5) $U^{investors} = v + p^n_{t'} - y_{t'} - d(t'-t) - f^n$ (when her liquidity need is to sell.)

So, the action of the investors is the bid or ask based on her liquidity needs. v is the sufficiently large private value for liquidity needs. So I assume $v \geq \sigma$. t' denotes the time the trade happens. t is the time she visits the market. $d(t)$ is the increasing function of waiting for trading. Price follows the martingale by assumption of price jump. So the investors trade immediately when she visits the markets by avoiding paying the delay cost of trading. After trading, the investor leave from the exchange market.

National Market System (NMS) provides the order protection rule as follows. Every orders are transferred into the security exchange markets whose bid ask spread is the lowest among every security exchange markets. So I assume the demand function of investors for each markets as follows.

$$\lambda^n_I = \begin{cases} \frac{\lambda_I}{|m|} & s^n = \underset{1 \leq l \leq N}{\arg\min}\, s^l, m = \{k | s^k \in \{\underset{1 \leq l \leq N}{\arg\min}\, s^l\}\} \\ 0 & \exists j \neq n, s^n > s^j \end{cases} \tag{6}$$

,which implies the markets splits the whole needs of the investors based on the bid ask spreads of each markets. $|m|$ is the number of the security exchange market whose bid ask spread is the lowest. Jump is also influenced by the bid ask spread as follows.

$$\lambda^n_J = \begin{cases} N\frac{\lambda_J}{|m|} & s^n = \underset{1 \leq l \leq N}{\arg\min}\, s^l, m = \{k | s^k \in \{\underset{1 \leq l \leq N}{\arg\min}\, s^l\}\} \\ 0 & \exists j \neq n, s^n > s^j \end{cases} \tag{7}$$

The change of price occurs at λ_J. However the order for arbitrage sniping is not demand for trade, but homogeneous shock. So, Jump occurs in every market. But the trading for the sniping occurs based on the bid ask spreads of each markets. This paper can successfully derive the analytical solution of the trading fee of security exchange under National Market System.

The time flow of this game is as follows.

1) The exchange markets choose the fee f simultaneously.
2) The other players start to trade. Then bid ask spread is determined in the equilibrium.

III. Analysis

A. *Equilibrium Bid Ask Spread Under NMS*

I will solve this problem by backward induction. First, in the second stage of the game, at using the zero profit condition of liquidity provider, I characterize the bid ask spread as a function of trading fee and λ^n_I, λ^n_J. Second, the Bertrand competition among security exchange markets starts under National Market System. So, optimal fee is equal zero.

Second stage is the competition among traders. The zero profit condition of the traders is as follows. L^n is the number of traders who enter the markets n. In sniping behavior, the security is allocated with equal probability among traders who submit orders.

$$\lambda_I^n(\frac{s^n}{2} - f^n) - \lambda_J^n(\sigma - \frac{s^n}{2} + f^n)\frac{L^n - 1}{L^n} = C_{speed} \tag{8}$$

The first term of left hand side is the profit of the market maker. The second term of left hand side is the loss of the market maker by being snipe by the snipers. The right hand sides is the cost of investment on HFT technology. I show later that investing on the HFT technology is the dominant strategy of the traders. The zero profit condition of the snipers is

$$\lambda_J^n(\sigma - \frac{s^n}{2} + f^n)\frac{1}{L^n} = C_{speed} \tag{9}$$

The left hand side means the expected payoff of the snipers. The right hand side is the cost of HFT technology.

Because for traders becoming the market maker and becoming the snipers are indifferent, we get,

$$\lambda_I^n(\frac{s^n}{2} - f^n) = \lambda_J^n(\sigma - \frac{s^n}{2} + f^n) \tag{10}$$

LEMMA 1: *Equation (10) gives the equilibrium bid ask spread given f^n and $\lambda_I^n > 0, \lambda_J^n > 0$*

$$s^{n*} = \frac{2[\sigma\lambda_J^n + (\lambda_I^n + \lambda_j^n)f^n]}{\lambda_I^n + \lambda_J^n} \tag{11}$$

PROOF:

The equation (10) gives

$$(\lambda_I^n + \lambda_J^n)\frac{s}{2} = \sigma\lambda_J^n + (\lambda_I^n + \lambda_j^n)f^n \tag{12}$$

which gives the equation (11). (Q.E.D)

When $\lambda_I^n > 0, \lambda_J^n > 0$ is not satisfied, then no trade happens in this market n. So we can ignore such a trivial case.

Now I come back to the first stage. Using the definition of λ_I^n, λ_J^n and the objective function of the security exchange markets (1), the optimal fee for all markets is

$$f = 0 \tag{13}$$

So, the optimal symmetric bid ask spread given $f = 0$ is

$$s^* = \frac{2\sigma\lambda_J}{\frac{\lambda_I}{N} + \lambda_J} \tag{14}$$

This number is irrelevant of $|m|$. So, the market maker cannot undercut anymore from this equilibrium spread to attract more traders and investors. However, if the order protection rule of National Market System doesn't exist and the perfect Bertrand competition take effect only on investors, then the equilibrium bid ask spread is

$$s^* = \frac{2\sigma\lambda_J}{\frac{\lambda_I}{|m|} + \lambda_J} \tag{15}$$

Under the perfect Bertrand competition without order protection rule, we need to assume the symmetric strategy in order to guarantee the uniqueness of the bid ask spread.

THEOREM 1: *If N security exchange markets exist, then the optimal bid ask spread is* $s* = \frac{2\sigma\lambda_J}{\frac{\lambda_I}{N}+\lambda_J}$

- s^* *is increasing over* N
- s^* *is increasing over* λ_J
- s^* *is decreasing over* λ_I
- s^* *is increasing over* σ

PROOF:

$$\frac{\partial s*}{\partial N} = \frac{\frac{\lambda_I}{N^2}}{(\frac{\lambda_I}{N} + \lambda_J)^2} > 0, \quad \frac{\partial s*}{\partial \lambda_J} = \frac{2\sigma[\frac{\lambda_I}{N} + \lambda_J] - 2\lambda_J\sigma}{(\frac{\lambda_I}{N} + \lambda_J)^2} > 0$$

$$\frac{\partial f*}{\partial \lambda_I} = \frac{-\frac{1}{N}}{(\frac{\lambda_I}{N} + \lambda_J)^2} < 0, \quad \frac{\partial f*}{\partial \sigma} = \frac{2\lambda_I}{\frac{\lambda_I}{N} + \lambda_J} > 0$$

(Q.E.D.)

The intuition of this theorem is as follows. The origin of bid ask spread is the existence of sniping behavior. By setting positive bid ask spread to extract the surplus of investors, the market makers compensate this kind of loss originated from being sniped. Larger λ_J means the higher frequency of being sniped. That is why the market maker sets the larger bid ask spread. Larger λ_I means the more opportunity to extract surplus from the investors. So, there is less need for the market makers to set higher bid ask spread to compensate by loss from being sniped. Larger N means the small opportunity to extract surplus from the investors, which means the need to set higher bid ask spread. I also note that the equilibrium bid ask comes from the indifference condition for traders of becoming the market makers and snipers. So, the spread is robust to the existence of HFT technology. I mean if there is no HFT technology, then the right hand side of (8)(9) is 0. However the indifference condition is not changed.

B. *Welfare Loss by arms races*

Now I can calculate the welfare loss by HFT arms race. First, I show that the investing on HFT technology is the dominant strategy for traders. The following lemmas correspond to the Proposition 3 (BCS).
The following lemma is robust to binomial jumping process except for the martingale of the price. So, within the proof, I replace $(\sigma - \frac{s^n}{2} + f^n)$ with $Pr(J > \frac{s'}{2} + f^n)E[J - \frac{s'}{2} + f^n | J > \frac{s'}{2} + f^n$, which means that the expected loss of being sniped given the price change is sufficiently large.

LEMMA 2: *If the market maker (he) who doesn't invest on HFT technology, his payoff is negative.*

PROOF:

The investors who doesn't invest on HFT technology cannot become snipers, their order is always too late to snipe the market makers.
If a investor who doesn't invest on HFT technology sets $s' < s^n*$ and becomes the market maker, then

$$\lambda_I^n(\frac{s'}{2} - f^n) - \lambda_J Pr(J > \frac{s'}{2} + f^n)E[J - \frac{s'}{2} + f^n | J > \frac{s'}{2} + f^n]$$
$$< \lambda_I^n(\frac{s^n}{2} - f^n) - \lambda_J Pr(J > \frac{s^n*}{2} + f^n)E[J - \frac{s^n*}{2} + f^n | J > \frac{s^n*}{2} + f^n] = 0$$

$\lambda_I^n(\frac{s}{2} - f^n)$ is increasing over s.$-\lambda_J Pr(J > \frac{s}{2} + f^n)E[J - \frac{s}{2} + f^n | J > \frac{s}{2} + f^n]$ is also increasing over s. These facts give the first inequality. The last equality comes from the zero profit condition.
(Q.E.D)

LEMMA 3: *It is the dominant strategy for the traders to invest on the HFT technology.*

PROOF:

The endogenized number of the traders of each markets is L^n. If only $L^{n\prime} < L^n$ traders invest on HFT technology, the profit of the traders who don't invest on HFT technology and become sniper is zero. By Lemma 3, the profit of the traders who don't invest on HFT technology and become market maker is negative. The profit of the traders who invest on the HFT technology and become market maker is

$$\lambda_I^n(\frac{s^n}{2}-f^n)-\lambda_J Pr(J>\frac{s^n}{2}+f^n)E[J-\frac{s^n}{2}+f^n|J>\frac{s^n}{2}+f^n]\frac{L^{n\prime}-1}{L^{n\prime}}-C_{speed}>0$$

The profit of the traders who invest on the HFT technology and become sniper is

$$\lambda_J Pr(J>\frac{s^n}{2}+f^n)E[J-\frac{s^n}{2}+f^n|J>\frac{s^n}{2}+f^n]\frac{1}{L^{n\prime}}-C_{speed}>0$$

So, there is a incentive to invest on HFT technology.
(Q.E.D)

Combining (8) and (9), we get,

$$\lambda_I^n(\frac{s}{2}-f)=L^{n*}C_{speed} \tag{16}$$

There are N security exchange markets. And in the equilibrium, $\lambda_I^n=\frac{\lambda_I}{N}, f=0$, so the equilibrium welfare loss of the whole market is

$$Loss=NL^{n*}C_{speed}=\frac{\lambda_I\lambda_J\sigma}{\frac{\lambda_I}{N}+\lambda_J} \tag{17}$$

THEOREM 2: *Welfare loss under CLOB is*

- *increasing over N*
- *increasing over* λ_I
- *increasing over* λ_J

PROOF:

$$\frac{\partial(Loss)}{\partial N} = \frac{\frac{\lambda_I}{N^2}}{(\frac{\lambda_I}{N} + \lambda_J)^2} > 0$$
$$\frac{\partial(Loss)}{\partial \lambda_I} = \frac{\lambda_J\sigma(\frac{\lambda_I}{N} + \lambda_J) - \lambda_I\lambda_J\sigma(\frac{1}{N})}{(\frac{\lambda_I}{N} + \lambda_J)^2} > 0$$
$$\frac{\partial(Loss)}{\partial \lambda_J} = \frac{\lambda_I\sigma(\frac{\lambda_I}{N} + \lambda_J) - \lambda_I\lambda_J\sigma}{(\frac{\lambda_I}{N} + \lambda_J)^2} > 0$$
$$\frac{\partial(Loss)}{\partial \sigma} = \frac{\lambda_I\lambda_J}{\frac{\lambda_I}{N} + \lambda_J} > 0$$

(Q.E.D)

IV. FBA will Dominate CLOB under National Market System in US

National Market System is the regulation in US. Under this regulation, every order is transferred into only the market whose best bid ask spread is the lowest among all markets. So this regulation makes the competition among security exchange perfect Bertrand. By the property of the FBA, if there exist at least two FBA security exchange markets, then bid ask spread is 0 in FBA markets if the batch interval is sufficiently large, which comes from the proposition 10 of Budish, Cramton and Shim (2015). Since, under National Market System, every order is transferred into FBA markets, no trade happens in CLOB markets. The definition of FBA is the same as Budish, Cramton and Shim (2015). So, see the Budish, Cramton and Shim (2015). FBA is the trading system where the trades are done in discrete time τ. If we set,

$$\frac{\delta}{\tau}\lambda_J\sigma < C_{speed}$$

,then no HFT technology investment become profitable under the condition that at least two security exchange markets adopt FBA. In this case, the welfare loss is

$$\frac{1}{\tau}\int_0^\tau d(x)\lambda_I dx$$

per unit of time. We cannot compare welfare loss of FBA with that of CLOB because we cannot know the function of delay cost $d(x)$. But the welfare loss of FBA is robust to market fragmentation. So under the condition that there are more than 50 security exchange markets in US, the welfare loss of FBA are likely to be less than that of CLOB.

THEOREM 3: *If there exist at least two FBA security exchange markets, then* *bid ask spread is 0 in FBA markets. Under National Market System, every order is transferred into FBA markets, so no trade happens in CLOB markets.*

PROOF:

By the property of the National Market System, the competition is the same as the perfect Bertrand. So if there exist at least two FBA security exchange markets, then bid ask spread is 0 in FBA markets. Bid ask spread of CLOB is strictly larger than 0 because the market maker set bid ask spread positive in order to compensate for the loss of being sniped. That means there is no trade in CLOB markets. (Q.E.D.)

If, at the first stage of the game, the security exchange market choose both fee and the trading system (CLOB or FBA), the equilibrium fee f=0 means that choosing CLOB or FBA is indifferent, which originates from the zero profit by the property of the Bertrand competition. So, the problem that the FBA has not been adopted is the coordination problem.

LEMMA 4: *There is a subgame perfect equilibrium where every market adopts FBA. If security exchange markets avoid no trading situation in own market, every market adopts FBA, which is unique subgame perfect equilibrium.*

V. Analysis with Friction among Security Exchange Markets

In this section, I introduce the friction into the model like Salop (1979). If the competition among security exchange markets has friction, then the larger number of security exchange markets leads to the less bid ask spread, which is consistent with O'Hara and Mao (2011). However, the welfare loss is already increasing over the number of security exchange markets. This model implies that the small bid ask spread does not mean the efficient market structure. This counter-intuitive result originates from the fact that competition leads to larger investment on HFT technology among traders.

Like Salop (1979), the security exchange markets are located in unit circle. I assume they are located symmetrically. So each distance is $\frac{1}{N}$. I denote the location of the security exchange n as l_n. The utility function with friction of investors (she) who buy the security exchange markets n and their location l_i are

$$U^{investors} = v+y_{t'}-p^n_{t'}-d(t'-t)-f^n-D|l_n-l_i| \quad \text{(when her liquidity need is to buy.)}$$

$$U^{investors} = v+p^n_{t'}-y_{t'}-d(t'-t)-f^n-D|l_n-l_i| \quad \text{(when her liquidity need is to sell.)}$$

I assume l_i follows $U[0,1]$. So, the investors are located uniformly on unit circle. $D > 0$ is the friction between the markets. The interpretation of this model is the friction like distance between the security exchange markets and investors. In US, each security markets are located different places. So , the timing of the reach for investors' order are different. Another interpretation is that the time difference between the foreign exchanges. For example, Nikkei 225 future (ETF of the Japanese companies stocks) are traded in Osaka, Singapore and Chicago.

So this modelling based on Salop (1979) seems to reflect the real world well. This modeling resembles Baldauf and Mollner (2017*b*), which is independent from mine and their work focus only on the bid ask spread. My main analysis focus on not only bid ask spread, but also welfare analysis. Unlike the sections, I assume λ_J is the common over the security exchange.

The security exchange market n captures all investors within distance x given by,

$$\frac{s^n}{2} + Dx \leq \frac{s^{n+1}}{2} + D|\frac{1}{N} - x|$$
$$\frac{s^n}{2} + Dx \leq \frac{s^{n-1}}{2} + D|\frac{1}{N} - x|$$

which implies given symmetric price like $s_{n-1} = s_{n+1} = \overline{s}$,

$$x = \frac{\frac{\overline{s}}{2} - \frac{s^n}{2} + \frac{D}{N}}{2D}$$

So the total demand of security exchange markets n is,

$$2x = \frac{\overline{s} - s^n + \frac{2D}{N}}{2D}$$

So,the zero profit conditions of traders are

$$\lambda_I \frac{\overline{s} - s^n + \frac{2D}{N}}{2D} (\frac{s^n}{2} - f^n) - \lambda_J(\sigma - \frac{s^n}{2} + f^n)\frac{L^n - 1}{L^n} = C_{speed} \tag{18}$$

$$(\sigma - \frac{s^n}{2} + f^n)\frac{1}{L^n} = C_{speed} \tag{19}$$

The (17) is the zero profit condition for market makers. The (18) is the zero profit condition for snipers. Combining these two equations, we get

$$\lambda_I \frac{\overline{s} - s^n + \frac{2D}{N}}{2D} (\frac{s^n}{2} - f^n) = \lambda_J(\sigma - \frac{s^n}{2} + f^n) \tag{20}$$

The equation (19) can be changed into

$$\{\lambda_I \frac{\overline{s} - s^n + \frac{2D}{N}}{2D} \frac{s^n}{2} + \lambda_J\} f^n = \lambda_I [\frac{\overline{s} - s^n + \frac{2D}{N}}{2D}] \frac{s^n}{2} - \lambda_J(\sigma - \frac{s^n}{2}) \tag{21}$$

The left hand side is the profit of security exchange market n. The right hand side is quadratic and concave function of s^n. So, we take FOC over s^n against the left hand side. We can get the symmetric equilibrium bid ask spread if the interior solution exists ,

$$s^* = \frac{\frac{2\lambda_I D}{N} + 2D\lambda_J}{\lambda_I} \tag{22}$$

The equation (21) gives

$$f^n = \frac{\lambda_I[\frac{\bar{s}-s^n+\frac{2D}{N}}{2D}]\frac{s^n}{2} - \lambda_J(\sigma - \frac{s^n}{2})}{\{\lambda_I\frac{\bar{s}-s^n+\frac{2D}{N}}{2D}\frac{s^n}{2} + \lambda_J\}} \tag{23}$$

Using (22), then

$$f^* = \frac{s^*}{2} - \frac{\lambda_J\sigma}{\frac{\lambda_I}{N} + \lambda_J} \tag{24}$$

By the assumption of the non negative condition for f^n,

$$\begin{aligned} f^* &= \frac{s^*}{2} - \frac{\lambda_J\sigma}{\frac{\lambda_I}{N} + \lambda_J} \\ &= \frac{D(\frac{\lambda_I}{N} + \lambda_J)^2 - \lambda_J\sigma}{\frac{\lambda_I}{N} + \lambda_J} \geq 0 \end{aligned}$$

So, the friction is sufficiently large,

$$D \geq \frac{\lambda_J\sigma}{(\frac{\lambda_I}{N} + \lambda_J)^2} \tag{25}$$

, we can get the interior solution. If not,the competition becomes the perfect Bertrand. So, the bid ask spread jump to the equation (14) $s^* = \frac{2\sigma\lambda_J}{\frac{\lambda_I}{N}+\lambda_J}$. The welfare loss from the investment of HFT technology can be derived using (18), (19) and (24).

$$NL^{n*}C_{speed} = N\lambda_I^n(\frac{s^*}{2} - f^*) = \frac{\lambda_I\lambda_J\sigma}{\frac{\lambda_I}{N} + \lambda_J} \tag{26}$$

THEOREM 4: *Under the model with sufficiently large friction*

$$D \geq \frac{\lambda_J\sigma}{(\frac{\lambda_I}{N} + \lambda_J)^2}$$

, the optimal bid ask spread $s^* = \frac{\frac{2\lambda_I D}{N}+2D\lambda_J}{\lambda_I}$ *is*

- *decreasing over* N*.*

- *decreasing over* λ_I
- *increasing over* λ_J
- *increasing over* D

If the friction is small,

$$D \leq \frac{\lambda_J \sigma}{(\frac{\lambda_I}{N} + \lambda_J)^2}$$

,then the optimal bid ask spread is the same as the theorem 1.

PROOF:
The proof is almost same as the former theorems. So I omit the proof. (Q.E.D)
The intuition of this theorem is the same as the theorem 1 in λ_I, λ_J. However, the effect of the number of security exchange is opposite. The friction is a kind of monopolistic. So the larger competition leads to the small bid ask spread. The larger friction D leads to larger monopolistic power and results in the larger bid ask spread.

THEOREM 5: *Under the model with sufficiently large friction,*

$$D \geq \frac{\lambda_J \sigma}{(\frac{\lambda_I}{N} + \lambda_J)^2}$$

the welfare loss from the investment of HFT technology $\frac{\lambda_I \lambda_J \sigma}{\frac{\lambda_I}{N} + \lambda_J}$ *is*

- *increasing over* N*.*
- *increasing over* λ_I
- *increasing over* λ_J
- *increasing over* σ

If the friction is small,

$$D \leq \frac{\lambda_J \sigma}{(\frac{\lambda_I}{N} + \lambda_J)^2}$$

,then the equilibrium welfare loss is the same as the theorem 2.

PROOF:
The proof is obvious.
(Q.E.D.)
The intuition of this theorem is as follows. The larger N leads to the more competition among the security exchange. This leads to the smaller fee. Fee becomes small more drastically than the bid ask spread. So, the difference between half of the bid ask spread and fee becomes larger as the number of the security

exchange market grows. This leads to the more room for traders to invest on HFT technology, which leads to the welfare loss.

If we consider the welfare loss from both the investment of HFT technology and the friction of the market, then welfare loss is

$$\frac{\lambda_I \lambda_J \sigma}{\frac{\lambda_I}{N} + \lambda_J} + \frac{\lambda_I D}{2N} \tag{27}$$

This welfare loss is increasing over $\lambda_I, \lambda_J, \sigma$. If

$$\frac{2N^2 \lambda_J \sigma}{\lambda_I} \geq D$$

then this welfare is still increasing over N. However, if not, welfare loss is decreasing over N. This means the welfare might be improved under the extremely high friction D when the number of security exchange increases. However, N^2 is at least 2500 in US since there are more than 50 security exchanges. So, this extremely high friction case doesn't fit the US situation.

I cannot model the competition among the security exchange markets with different trading system (CLOB, and FBA) because we cannot know the function of delay cost $d(x)$. But the welfare loss of FBA is robust to market fragmentation except the welfare loss from friction. So under the condition that there are more than 50 security exchange markets in US, the welfare loss of FBA are likely to be less than that of CLOB.

VI. Conclusion

There are two policy implications and one important warning against the empirical research.

First,under CLOB and ideal friction-less structure like the situation that National Markets System works well, the bid ask spread and the welfare loss are increasing over the number of security exchange markets. There are already 56 security exchange markets in US. From the theoretical perspective, the regulator should promote the merger of the security exchange markets.

Second, FBA can eliminate the sniping behavior and if at least two security exchanges who adopt FBA, no trading occurs under the security exchanges who adopt CLOB. FBA is the totally new idea. There are some potential risks to provide liquidity. However, after the experiments to check the performance of FBA, if FBA has no troubles, the regulator should promote the adoption of FBA because the reason that the FBA have not been adopted is a kind of coordination problem from my theoretical analysis.

Finally, with friction, the bid ask spread is decreasing over the number of the security exchange markets. However, the welfare loss is still increasing over this number of it in almost all cases. My toy model implies that the bid ask spread might not work as the benchmark for the efficiency of the market structure. Under the popularity of HFT, we need to rethink the strategy of empirical research.

The outcomes of paper depend on the model specification. I assume the probability process and very simple friction. I also assume v is the sufficiently large random variable so that investors always buy or sell, which means we cannot discuss the monopoly case. If we introduce the analysis of private value of traders like Baldauf and Mollner (2017*a*), we cannot say monotonicity over the number of security exchange markets. But my model can shed light on the new theoretical risk of the market fragmentation under the popularity of HFT.

REFERENCES

Baldauf, Markus, and Joshua Mollner. 2017*a*. “High-Frequency Trading and Market Performance.” Mimeo.

Baldauf, Markus, and Joshua Mollner. 2017*b*. “Trading in Fragmented Markets.” Mimeo.

Budish, Eric, Peter Cramton, and John Shim. 2014. “Implementation Details for Frequent Batch Auctions: Slowing Down Markets to the Blink of an Eye.” *American Economic Review*, 104(5): 418–24.

Budish, Eric, Peter Cramton, and John Shim. 2015. “The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response.” *The Quarterly Journal of Economics*, 130(4): 1547–1621.

Budish, Eric, Peter Cramton, and John Shim. 2017. “Will the Market Fix the Market?” Seminar Slides.

Glosten, Lawrence R., and Paul R. Milgrom. 1985. “Bid, ask and transaction prices in a specialist market with heterogeneously informed traders.” *Journal of Financial Economics*, 14(1): 71 – 100.

Haas, Marlene, and Marius Zoican. 2016. “Beyond the Frequency Wall: Speed and Liquidity on Batch Auction Markets.” Mimeo.

Lagos, Ricardo, and Guillaume Rocheteau. 2009. “Liquidity in Asset Markets With Search Frictions.” *Econometrica*, 77(2): 403–426.

Lewis, Michael. 2014. *Flash Boys: A Wall Street Revolt.* New York:Norton.

Menkveld, A.J. 2011. “Electronic trading and market structure.”

O’Hara, Maureen. 2015. “High frequency market microstructure.” *Journal of Financial Economics*, 116(2): 257 – 270.

O’Hara, Maureen, and Ye Mao. 2011. “Is market fragmentation harming market quality?” *Journal of Financial Economics*, 100(3): 459–474.

Pagnotta, Emiliano, and Thomas Phillippon. 2017. "Competing on Speed." Mimeo.

Salop, Steven C. 1979. "Monopolistic Competition with Outside Goods." *Bell Journal of Economics*, 10(1): 141–156.