Savage in the Market[1]

Federico Echenique
Caltech

Kota Saito
Caltech

January 30, 2015

Abstract

We develop a behavioral axiomatic characterization of Subjective Expected Utility (SEU) under risk aversion. Given is an individual agent's behavior in the market: assume a finite collection of asset purchases with corresponding prices. We show that such behavior satisfies a "revealed preference axiom" if and only if there exists a SEU model (a subjective probability over states and a concave utility function over money) that accounts for the given asset purchases.

1. INTRODUCTION

The main result of this paper gives a revealed preference characterization of risk-averse subjective expected utility. Our contribution is to provide a necessary and sufficient condition for an agent's market behavior to be consistent with risk-averse subjective expected utility (SEU).

The meaning of SEU for a preference relation has been well understood since Savage (1954), but the meaning of SEU for agents' behavior in the market has been unknown until now. Risk-averse SEU is widely used by economists to describe agents' market behavior, and the new understanding of risk-averse SEU provided by our paper is hopefully useful for both theoretical and empirical purposes.

Our paper follows the revealed preference tradition in economics. Samuelson (1938) and Houthakker (1950) describe the market behaviors that are consistent with utility maximization. They show that a behavior is consistent with utility maximization if and only if it satisfies the strong axiom of revealed preference. We show that there is an analogous revealed preference axiom for risk-averse SEU. A behavior is consistent with risk-averse SEU if and only if it satisfies the "strong axiom of revealed subjective expected utility (SARSEU)." (In the following, we write SEU to mean risk-averse SEU when there is no potential for confusion.)

The motivation for our exercise is twofold. In the first place, there is a theoretical payoff from understanding the behavioral counterpart to a theory. In the case of SEU, we believe that SARSEU gives meaning to the assumption of SEU in a market context. The second motivation for the exercise is that SARSEU can be used to test for SEU in actual data. We discuss each of these motivations in turn.

SARSEU gives meaning to the assumption of SEU in a market context. We can, for example, use SARSEU to understand how SEU differs from maxmin expected utility (Section 6). The difference between the SEU and maxmin utility representations is obvious, but the difference in the behaviors captured by each model is much harder to grasp. In fact, we show that SEU and maxmin expected utility are indistinguishable in some situations. In a similar vein, we can use SARSEU to understand the behavioral differences between SEU and probabilistic sophistication (Section 5). Finally, SARSEU helps us understand how SEU restricts behavior over and above what is captured by the more general model of state-dependent utility (Section 4). The online appendix discusses additional theoretical implications.

Our results allow one to test SEU non-parametrically in an important economic decision-making environment, namely that of choices in financial markets. The test does not only dictate what to look for in the data (SARSEU), but it also suggests experimental designs. The syntax of SARSEU may not immediately lend itself to a practical test, but there are two efficient algorithms for checking the axiom. One of them is based on linearized "Afriat inequalities" (see Lemma 7 of Appendix A). The other is implicit in Proposition 2. SARSEU is on the same computational standing as the strong axiom of revealed preference.

Next, we describe data one can use to test SARSEU. There are experiments of decision-making under uncertainty where subjects make financial decisions, such as Hey and Pace (2014), Ahn et al. (2014), or Bossaerts et al. (2010). Hey and Pace, and Ahn et. al. test SEU parametrically: they assume a specific functional form. A nonparametric test, such as SARSEU, seems useful because it frees the analysis from such assumptions. Bossaerts et al. (2010) do not test SEU itself; they test an implication of SEU on equilibrium prices and portfolio choices.

The paper by Hey and Pace fits our framework very well. They focus on the explanatory power of SEU relative to various other models, but they do not test how well SEU fits the data. Our test, in contrast, would evaluate goodness of fit, and in addition be free of parametric assumptions.

The experiments by Ahn et al. and Bossaerts et al. do not fit the setup in our paper because they assume that the probability of one state is known. In an extension of our results to a generalization of SEU (see Appendix B), we show how a version of SARSEU characterizes expected utility when the probabilities of some states are objective and known. Hence the results in our paper are readily applicable to the data from Ahn et al. and Bossaerts et al. We discuss this application further in Appendix B.

SARSEU is not only useful to testing SEU with existing experimental data, but it also guides the design of new experiments. In particular, SARSEU suggests how one should choose the parameters of the design (prices and budgets) so as to evaluate SEU. For example, in a setting with two states, one could choose each of the configurations described in Section 3.1 to evaluate where violations of SEU come from: state-dependent utility or probabilistic sophistication.

Related literature.   The closest precedent to our paper is the important work of Epstein (2000). Epstein's setup is the same as ours; in particular, he assumes data on

2

state-contingent asset purchases, and that probabilities are subjective and unobserved but stable. We differ in that he focuses attention on pure probabilistic sophistication (with no assumptions on risk aversion), while our paper is on risk-averse SEU. Epstein presents a necessary condition for market behavior to be consistent with probabilistic sophistication. Given that the model of probabilistic sophistication is more general than SEU, one expects that the two axioms may be related: Indeed we show in Section 5 that Epstein's necessary condition can be obtained as a special case of SARSEU. We also present an example of data that are consistent with a risk averse probabilistically sophisticated agent, but that violate SARSEU.

Polisson and Quah (2013) develops tests for models of decision under risk and uncertainty, including SEU (without the requirement of risk aversion). They develop a general approach by which testing a model amounts to solving a system of (nonlinear) Afriat inequalities. See also Bayer et al. (2012), who study different models of ambiguity by way of Afriat inequalities. Non-linear Afriat inequalities can be problematic because there is no known efficient algorithm for deciding if they have a solution.

Another strain of related work deals with objective expected utility, assuming observable probabilities. The papers by Green and Srivastava (1986), Varian (1983), Varian (1988), and Kubler et al. (2014) characterize the datasets that are consistent with objective expected utility theory. Datasets in these papers are just like ours, but with the added information of probabilities over states. Green and Srivastava allow for the consumption of many goods in each state, while we focus on monetary payoffs. Varian's and Green and Srivastava's characterization is in the form of Afriat inequalities; Kubler et. al. improve on these by presenting a revealed preference axiom. We discuss the relation between their axiom and SARSEU in the online appendix.

The syntax of SARSEU is similar to the main axiom in Fudenberg et al. (2014), and in other works on additively separable utility.

2. Subjective Expected Utility

Let S be a finite set of *states*. We occasionally use S to denote the number $|S|$ of states. Let $\Delta_{++} = \{\mu \in \mathbf{R}_{++}^{S} | \sum_{s=1}^{S} \mu_s = 1\}$ denote the set of strictly positive probability measures on S. In our model, the objects of choice are state-contingent monetary payoffs, or *monetary acts*. A monetary act is a vector in $\mathbf{R}_{+}^{S}$.

We use the following notational conventions: For vectors $x, y \in \mathbf{R}^{n}$, $x \leq y$ means that

$x_i \leq y_i$ for all $i = 1, \ldots, n$; $x < y$ means that $x \leq y$ and $x \neq y$; and $x \ll y$ means that $x_i < y_i$ for all $i = 1, \ldots, n$. The set of all $x \in \mathbf{R}^n$ with $0 \leq x$ is denoted by $\mathbf{R}^n_+$ and the set of all $x \in \mathbf{R}^n$ with $0 \ll x$ is denoted by $\mathbf{R}^n_{++}$.

Definition 1. *A* dataset *is a finite collection of pairs* $(x, p) \in \mathbf{R}^S_+ \times \mathbf{R}^S_{++}$.

The interpretation of a dataset $(x^k, p^k)_{k=1}^K$ is that it describes K purchases of a state-contingent payoff x^k at some given vector of prices p^k, and income $p^k \cdot x^k$.

A subjective expected utility (SEU) model is specified by a subjective probability $\mu \in \Delta_{++}$ and a utility function over money $u : \mathbf{R}_+ \to \mathbf{R}$. An SEU maximizing agent solves the problem

$$\max_{x \in B(p,I)} \sum_{s \in S} \mu_s u(x_s) \tag{1}$$

when faced with prices $p \in \mathbf{R}^S_{++}$ and income $I > 0$. The set $B(p, I) = \{y \in \mathbf{R}^S_+ : p \cdot y \leq I\}$ is the *budget set* defined by p and I.

A dataset is our notion of observable behavior. The meaning of SEU as an assumption, is the behaviors that are *as if* they were generated by an SEU maximizing agent. We call such behaviors *SEU rational.*

Definition 2. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* subjective expected utility rational *(SEU rational) if there is* $\mu \in \Delta_{++}$ *and a concave and strictly increasing function* $u : \mathbf{R}_+ \to \mathbf{R}$ *such that, for all* k,

$$y \in B(p^k, p^k \cdot x^k) \Rightarrow \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x^k_s).$$

Three remarks are in order. Firstly, we restrict attention to concave (i.e., risk-averse) utility, and our results will have nothing to say about the non-concave case. In second place, we assume that the relevant budget for the kth observation is $B(p^k, p^k \cdot x^k)$. Implicit is the assumption that $p^k \cdot x^k$ is the relevant income for this problem. This assumption is somewhat unavoidable, and standard procedure in revealed preference theory. Thirdly, we should emphasize that there is in our model only one good (which we think of as money) in each state. The problem with many goods is interesting, but beyond the methods developed in the present paper (see Remark 4).

3. A Characterization of SEU Rational Data

In this section we introduce the axiom for SEU rationality and state our main result. We start by deriving, or calculating, the axiom in a specific instance. In this derivation,

we assume (for ease of exposition) that u is differentiable. In general, however, an SEU rational dataset may not be rationalizable using a differentiable u; see Remark 3 below.

The first-order conditions for SEU maximization (1) are:

$$\mu_s u'(x_s) = \lambda p_s. \tag{2}$$

The first-order conditions involve three unobservables: subjective probability μ_s, marginal utilities $u'(x_s)$ and Lagrange multipliers λ.

3.1. The 2×2 case: $K = 2$ and $S = 2$

We illustrate our analysis with a discussion of the 2×2 case, the case when there are two states and two observations. In the 2×2 case we can easily see that SEU has *two kinds* of implications, and, as we explain in Sections 4 and 5, each kind is derived from a different qualitative feature of SEU.

Let us impose the first-order conditions (2) on a dataset. Let $(x^{k_1}, p^{k_1}), (x^{k_2}, p^{k_2})$ be a dataset with $K = 2$ and $S = 2$. For the dataset to be SEU rational there must exist $\mu \in \Delta_{++}$, $(\lambda^k)_{k=k_1,k_2}$ and a concave function u such that each observation in the dataset satisfies the first order conditions (2). That is,

$$\mu_s u'(x_s^k) = \lambda^k p_s^k, \tag{3}$$

for $s = s_1, s_2$, and $k = k_1, k_2$.

Equation (3) involves the observed x and p, as well as the unobservables u', λ, and μ. One is free to choose (subject to some constraints) the unobservables to satisfy Equation (3). We can understand the implications of Equation (3) by considering situations in which the unobservable λ and μ cancel out:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_1}^{k_2})} \frac{u'(x_{s_2}^{k_2})}{u'(x_{s_2}^{k_1})} = \frac{\mu_{s_1} u'(x_{s_1}^{k_1})}{\mu_{s_1} u'(x_{s_1}^{k_2})} \frac{\mu_{s_2} u'(x_{s_2}^{k_2})}{\mu_{s_2} u'(x_{s_2}^{k_1})} = \frac{\lambda^{k_1} p_{s_1}^{k_1}}{\lambda^{k_2} p_{s_1}^{k_2}} \frac{\lambda^{k_2} p_{s_2}^{k_2}}{\lambda^{k_1} p_{s_2}^{k_1}} = \frac{p_{s_1}^{k_1}}{p_{s_1}^{k_2}} \frac{p_{s_2}^{k_2}}{p_{s_2}^{k_1}} \tag{4}$$

Equation (4) is obtained by dividing first order conditions to eliminate terms involving μ and λ: this allows us to constrain the observable variables, x and p. There are two situations of interest.

Suppose first that $x_{s_1}^{k_1} > x_{s_1}^{k_2}$ and that $x_{s_2}^{k_2} > x_{s_2}^{k_1}$. The concavity of u implies then that $u'(x_{s_1}^{k_1}) \leq u'(x_{s_1}^{k_2})$ and $u'(x_{s_2}^{k_2}) \leq u'(x_{s_2}^{k_1})$. This means that the left hand side of Equation (4) is smaller than 1. Thus:

$$x_{s_1}^{k_1} > x_{s_1}^{k_2} \text{ and } x_{s_2}^{k_2} > x_{s_2}^{k_1} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_1}^{k_2}} \frac{p_{s_2}^{k_2}}{p_{s_2}^{k_1}} \leq 1. \tag{5}$$

In second place, suppose that $x_{s_1}^{k_1} > x_{s_2}^{k_1}$ while $x_{s_2}^{k_2} > x_{s_1}^{k_2}$ (so the bundles x^{k_1} and x^{k_2} are on opposite sides of the 45 degree line in $\mathbf{R}^2$). The concavity of u implies that $u'(x_{s_1}^{k_1}) \leq u'(x_{s_2}^{k_1})$ and $u'(x_{s_2}^{k_2}) \leq u'(x_{s_1}^{k_2})$. The far-left of Equation (4) is then smaller than 1. Thus:

$$x_{s_1}^{k_1} > x_{s_2}^{k_1} \text{ and } x_{s_2}^{k_2} > x_{s_1}^{k_2} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}} \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1. \tag{6}$$

Requirements (5) and (6) are implications of risk-averse SEU for a dataset when $S = 2$ and $K = 2$. We shall see that they are *all* the implications of risk-averse SEU in this case, and that they capture distinct qualitative components of SEU (Sections 4 and 5).

3.2. General K and S

We now turn to the general setup, and to our main result. First, we shall derive the axiom by proceeding along the lines suggested above in Section 3.1: Using the first-order conditions (2), the SEU-rationality of a dataset requires that

$$\frac{u'(x_{s'}^{k'})}{u'(x_s^k)} = \frac{\mu_s}{\mu_{s'}} \frac{\lambda^{k'}}{\lambda^k} \frac{p_{s'}^{k'}}{p_s^k}.$$

The concavity of u implies something about the left-hand side of this equation when $x_{s'}^{k'} > x_s^k$, but the right-hand side is complicated by the presence of unobservable Lagrange multipliers and subjective probabilities. So we choose pairs $(x_s^k, x_{s'}^{k'})$ with $x_s^k > x_{s'}^{k'}$ such that subjective probabilities and Lagrange multipliers cancel out. For example, consider

$$x_{s_1}^{k_1} > x_{s_2}^{k_2}, x_{s_2}^{k_3} > x_{s_3}^{k_1}, \text{ and } x_{s_3}^{k_2} > x_{s_1}^{k_3}.$$

By manipulating the first-order conditions we obtain that:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_2})} \cdot \frac{u'(x_{s_3}^{k_3})}{u'(x_{s_3}^{k_1})} \cdot \frac{u'(x_{s_2}^{k_2})}{u'(x_{s_3}^{k_3})} = \left(\frac{\mu_{s_2}}{\mu_{s_1}} \frac{\lambda^{k_1}}{\lambda^{k_2}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \right) \cdot \left(\frac{\mu_{s_3}}{\mu_{s_2}} \frac{\lambda^{k_3}}{\lambda^{k_1}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \right) \cdot \left(\frac{\mu_{s_1}}{\mu_{s_3}} \frac{\lambda^{k_2}}{\lambda^{k_3}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}} \right) = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}$$

Notice that the pairs $(x_{s_1}^{k_1}, x_{s_2}^{k_2})$, $(x_{s_2}^{k_3}, x_{s_3}^{k_1})$, and $(x_{s_3}^{k_2}, x_{s_1}^{k_3})$ have been chosen so that the subjective probabilities μ_s appear in the nominator as many times as in the denominator, and the same for λ^k; hence these terms cancel out. Such "canceling out" motivates conditions (2) and (3) in the axiom below.

Now the concavity of u and the assumption that $x_{s_1}^{k_1} > x_{s_2}^{k_2}$, $x_{s_2}^{k_3} > x_{s_3}^{k_1}$, and $x_{s_3}^{k_2} > x_{s_1}^{k_3}$ imply that the product of the prices $\frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}$ cannot exceed 1. Thus, we obtain an implication of SEU on prices, an observable entity.

6

In general, the assumption of SEU rationality requires that, for any collection of sequences as above, appropriately chosen so that subjective probabilities and Lagrange multipliers will cancel out, the product of the ratio of prices cannot exceed 1. Formally:

Strong Axiom of Revealed Subjective Utility (SARSEU): *For any sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

1. *$x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ for all i;*

2. *each s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right);*

3. *each k appears as k_i (on the left of the pair) the same number of times it appears as k_i' (on the right):*

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

Theorem 1. *A dataset is SEU rational if and only if it satisfies SARSEU.*

It is worth noting that the syntax of SARSEU is similar to that of the main axiom in Kubler et al. (2014), with "risk-neutral" prices playing the role of prices in the model with objective probabilities. The relation between the two is discussed further in Appendix B. We conclude the section with some remarks on Theorem 1. The proof is in Section A.

Remark 1. In the 2×2 case of Section 3.1, Requirements (5) and (6) are equivalent to SARSEU.

Remark 2. The proof of Theorem 1 is in Section A. It relies on setting up a system of inequalities from the first-order conditions of an SEU agent's maximization problem. This is similar to the approach in Afriat (1967), and in many other subsequent studies of revealed preference. The difference is that our system is nonlinear, and must be linearized. A crucial step in the proof is an approximation result, which is complicated by the fact that the unknown subjective probabilities, Lagrange multipliers, and marginal utilities, all take values in non-compact sets.

Remark 3. Under the following assumption on the dataset:

$$x_s^k \neq x_{s'}^{k'} \text{ if } (k, s) \neq (k', s'),$$

SARSEU implies SEU rationality using a *smooth* rationalizing u. This condition on the dataset plays a similar role to the assumption used by Chiappori and Rochet (1987) to obtain a smooth utility using the Afriat construction.

Remark 4. In our framework we assume choices of monetary acts, which means that consumption in each state is one-dimensional. Our results are not easily applicable to the multidimensional setting, essentially because concavity is in general equivalent to cyclic monotonicity of supergradients, which we cannot deal with in our approach. In the one-dimensional case, concavity requires only that supergradients are monotone. The condition that some unknown function is monotone is preserved by a monotonic transformation of the function, but this is not true of cyclic monotonicity. If one sets up the multidimensional problem as we have done, then one loses the property of cyclic monotonicity when linearizing the system.

Finally, it is not obvious from the syntax of SARSEU that one can verify whether a particular datasets satisfies SARSEU in finitely many steps. We show that, not only is SARSEU decidable in finitely many steps, but there is in fact an efficient algorithm that decides whether a dataset satisfies SARSEU.

Proposition 2. *There is an efficient algorithm that decides whether a dataset satisfies SARSEU.*[1]

We provide a direct proof of Proposition 2 in Section C. Proposition 2 can also be seen as a result of Lemma 7 together with the linearization in the proof of Theorem 1. The resulting linear system can be decided by using linear programming.

4. STATE-DEPENDENT UTILITY

SEU asserts, among other things, the existence of a (concave) state-dependent utility (i.e., an additively separable utility across states). SEU requires more, of course, but it is interesting to compare SEU with the weaker theory of state-dependent utility. We shall trace the assumption of state-dependent utility to a particular weakening on SARSEU.

The state-dependent utility model says that an agent maximizes $U^{sd}(x) = \sum_{s \in S} u_s(x_s)$, an additively separable utility, for some collection $(u_s)_{s \in S}$ of concave and strictly increasing state-dependent functions, $u_s : \mathbf{R}_+ \to \mathbf{R}$. We should emphasize that here, as in the rest of the paper, we restrict attention to concave utility.

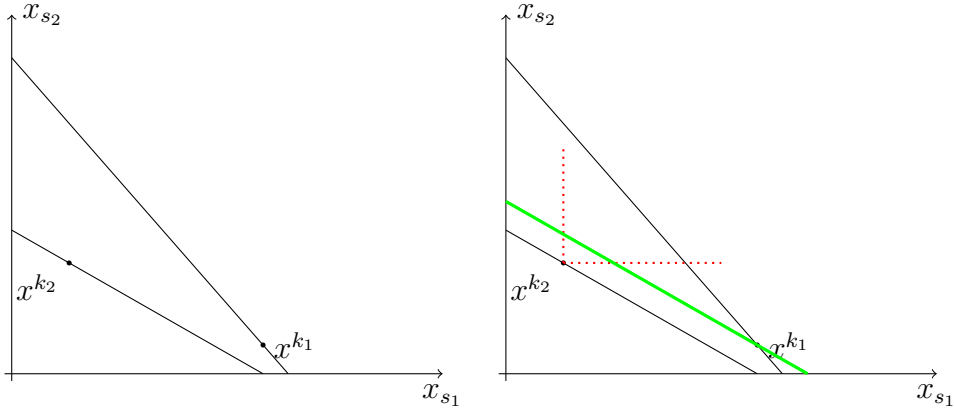[1]Efficient means that the algorithm runs in polynomial time.

Figure 1: A violation of Requirement (5)

4.1. The 2×2 case: $K = 2$ and $S = 2$

We argued in Section 3.1 that requirements (5) and (6) are necessary for SEU rationality. It turns out that (5) alone captures state-dependent utility.

A dataset that violates Requirement (5) can be visualized on the left of Figure 1. The figure depicts choices with $x_{s_1}^{k_1} > x_{s_1}^{k_2}$ and $x_{s_2}^{k_2} > x_{s_2}^{k_2}$, but where Requirement (5) is violated. Figure 1 presents a geometrical argument for why such a dataset is not SEU rational. Suppose, towards a contradiction, that the dataset is SEU rational. Since the rationalizing function u is concave, it is easy to see that optimal choices must be increasing in the level of income (the demand function of a risk-averse SEU agent is normal). At the right of Figure 1, we include a budget with the same relative prices as when x^{k_2} was chosen, but where the income is larger. The larger income is such that the budget line passes through x^{k_1}. Since her demand is normal, the agent's choice on the larger (green) budget line must be larger than at x^{k_2}. The choice must lie in the line segment on the green budget line that consists of bundles larger than x^{k_2}. But such a choice would violate the weak axiom of revealed preference (WARP). Hence the (counterfactual) choice implied by SEU at the green budget line would be inconsistent with utility maximization, contradicting the assumption of SEU rationality. It is useful to emphasize that Requirement (5) is a strengthening of WARP, something we shall return to below.

So SEU rationality implies (5), which is a strengthening of WARP. Now we argue that state-dependent utility implies (5) as well. To see this, suppose that the agent maximizes $u_{s_1}(\cdot) + u_{s_2}(\cdot)$, where u_{s_i} is concave. As in Section 3.1, assume that u_{s_i} is differentiable.

9

Then, $x_{s_1}^{k_1} > x_{s_1}^{k_2}$ and $x_{s_2}^{k_2} > x_{s_2}^{k_1}$ imply that $u_{s_1}'(x_{s_1}^{k_1}) \leq u_{s_1}'(x_{s_1}^{k_2})$ and $u_{s_2}'(x_{s_2}^{k_2}) \leq u_{s_2}'(x_{s_2}^{k_1})$. The first order conditions are $u_s'(x_s^k) = \lambda^k p_s^k$; hence

$$\frac{p_{s_1}^{k_2} \, p_{s_2}^{k_1}}{p_{s_1}^{k_1} \, p_{s_2}^{k_2}} = \frac{u_{s_1}'(x_{s_1}^{k_2}) \, u_{s_2}'(x_{s_2}^{k_1})}{u_{s_1}'(x_{s_1}^{k_1}) \, u_{s_2}'(x_{s_2}^{k_2})} \leq 1.$$

Indeed, this is Requirement (5).

4.2. General K and S

A dataset $(x^k, p^k)_{k=1}^K$ is *state-dependent utility (SDU) rational* if there is an additively separable function U^{sd} such that, for all k,

$$y \in B(p^k, p^k \cdot x^k) \Rightarrow U^{sd}(y) \leq U^{sd}(x^k).$$

In the 2×2 case, we have seen that Requirement (5) is necessary for rationalization by a state-dependent utility. More generally, there is a natural weakening of SARSEU that captures rationalization by a state-dependent utility. This weakening is strong enough to imply WARP. Concretely, if one substitutes condition (2) in SARSEU with the statement $s_i = s_i'$ the resulting axiom characterizes SDU rationality:

Strong Axiom of Revealed State-dependent Utility (SARSDU): *For any sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

1. *$x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ for all i;*

2. *$s_i = s_i'$.*

3. *each k appears as k_i (on the left of the pair) the same number of times it appears as k_i' (on the right):*

The product of prices satisfies that

$$\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

It should be obvious that SARSEU implies SARSDU. SARSDU is equivalent to SDU rationality.

Theorem 3. *A dataset is SDU rational if and only if it satisfies SARSDU.*

The proof that SARSDU is necessary for state-dependent utility is simple, and follows along the lines developed in Section 3. The proof of sufficiency is similar to the proof used for the characterization of SEU, and is omitted.

Note that, in the 2×2 case, SARSDU and Requirement (5) are equivalent. Hence, (5) characterizes state-dependent utility in the 2×2 case.

By the theorem above, we know that SARSDU implies the weak axiom of revealed preference (WARP), but it may be useful to present a direct proof of the fact that SARSDU implies WARP.

Definition 3. *A dataset $(x^k, p^k)_{k=1}^K$ satisfies WARP if there is no k and k' such that $p^k \cdot x^k \geq p^k \cdot x^{k'}$ and $p^{k'} \cdot x^{k'} > p^{k'} \cdot x^k$.*

Proposition 4. *If a dataset satisfies SARSDU, then it satisfies WARP.*

Proof. Suppose, towards a contradiction, that a dataset $(x^k, p^k)_{k=1}^K$ satisfies SARSDU but that it violates WARP. Then there are k and k' such that $p^k \cdot x^k \geq p^k \cdot x^{k'}$ and $p^{k'} \cdot x^{k'} > p^{k'} \cdot x^k$. It cannot be the case that $x_s^k \geq x_s^{k'}$ for all s, so the set $S_1 = \{s : x_s^k < x_s^{k'}\}$ is nonempty. Choose $s^* \in S_1$ such that

$$\frac{p_{s^*}^{k'}}{p_{s^*}^{k}} \geq \frac{p_s^{k'}}{p_s^{k}} \text{ for all } s \in S_1.$$

Now, $p^k \cdot x^k \geq p^k \cdot x^{k'}$ implies that

$$(x_{s^*}^k - x_{s^*}^{k'}) \geq \frac{-1}{p_{s^*}^{k}} \sum_{s \neq s^*} p_s^k (x_s^k - x_s^{k'}).$$

We also have that $p^{k'} \cdot x^{k'} > p^{k'} \cdot x^k$, so

$$0 > \sum_{s \neq s^*} p_s^{k'}(x_s^k - x_s^{k'}) + p_{s^*}^{k'}(x_{s^*}^k - x_{s^*}^{k'})$$

$$\geq \sum_{s \neq s^*} p_s^{k'}(x_s^k - x_s^{k'}) + \frac{-p_{s^*}^{k'}}{p_{s^*}^{k}} \sum_{s \neq s^*} p_s^k (x_s^k - x_s^{k'})$$

$$= \underbrace{\sum_{s \notin S_1} p_s^{k'}(1 - \frac{p_{s^*}^{k'} p_s^k}{p_{s^*}^{k} p_s^{k'}})(x_s^k - x_s^{k'})}_{A} + \underbrace{\sum_{s \in S_1 \setminus \{s^*\}} p_s^{k'}(1 - \frac{p_{s^*}^{k'} p_s^k}{p_{s^*}^{k} p_s^{k'}})(x_s^k - x_s^{k'})}_{B}.$$

We shall prove that $A \geq 0$ and that $B \geq 0$, which will yield the desired contradiction.

11

For all $s \notin S_1$ we have that $(x_s^k - x_s^{k'}) \geq 0$. Then SARSDU implies that

$$\frac{p_{s*}^{k'} p_s^k}{p_{s*}^k p_s^{k'}} \leq 1,$$

as $x_{s*}^k < x_{s*}^{k'}$ so that the sequence $\{(x_{s*}^{k'}, x_{s*}^k), (x_s^k, x_s^{k'})\}$ satisfies (1), (2), and (3) in SARSDU. Hence $A \geq 0$.

Now consider B. By definition of s^*, we have that $\frac{p_{s*}^{k'} p_s^k}{p_{s*}^k p_s^{k'}} \geq 1$ for all $s \in S_1$. Then, $(x_s^k - x_s^{k'}) < 0$ implies that

$$\left(1 - \frac{p_{s*}^{k'} p_s^k}{p_{s*}^k p_s^{k'}}\right)(x_s^k - x_s^{k'}) \geq 0,$$

for all $s \in S_1$. Hence $B \geq 0$. $\qquad\square$

We make use of these results in the online appendix, where we show how SARSDU and SARSEU rule out violations of Savage's axioms. The online appendix also includes a condition on the data under which SEU and SDU are observationally equivalent.

5. Probabilistic Sophistication

We have looked at the aspects of SARSEU that capture the existence of an additively separable representation. SEU also affirms the existence of a unique subjective probability measure guiding the agent's choices. We now turn to the behavioral counterpart of the existence of such a probability. We do not have a characterization of probability sophistication. In this section, we simply observe how SARSEU and Requirement (6) are related to the existence of a subjective probability.[2] We also show how SARSEU is related to Epstein's necessary condition for probability sophistication.

5.1. The 2×2 case: $K = 2$ and $S = 2$

Consider, as before, the 2×2 case: $S = 2$ and $K = 2$. We argued in Section 3.1 that Requirements (5) and (6) are necessary for SEU rationality, and in Section 4 that Requirement (5) captures the existence of a state-dependent representation. We now show that Requirement (6) results from imposing a unique subjective probability guiding the agent's choices.

---

[2]In the online appendix, we also show that Requirement (6) rules out violations of Savage's P4, which captures the existence of a subjective probability (Machina and Schmeidler (1992)).
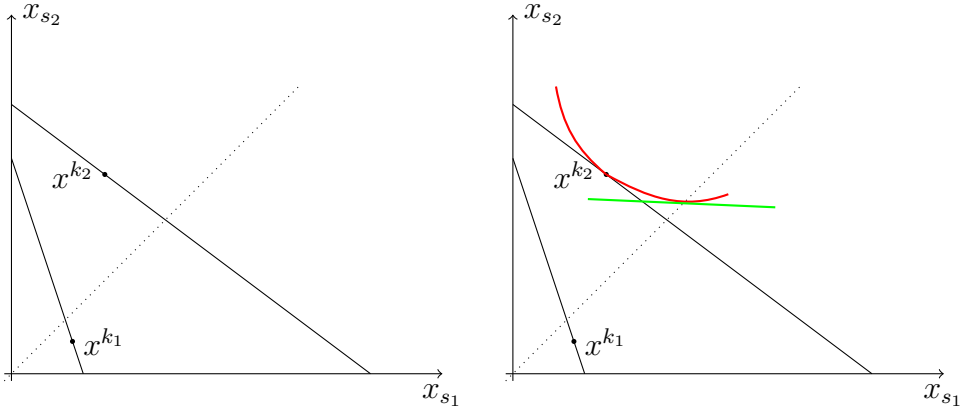
Figure 2: Violation of Requirement (6)

Figure 2 exhibits a dataset that violates Requirement (6). We have drawn the indifference curve of the agent when choosing x^{k_2}. Recall that the marginal rate of substitution (MRS) is $\mu_{s_1} u'(x_{s_1})/\mu_{s_2} u'(x_{s_2})$. At the point where the indifference curve crosses the 45 degree line (dotted), one can read the agent's subjective probability off the indifference curve because $u'(x_{s_1})/u'(x_{s_2}) = 1$, and therefore the MRS equals μ_{s_1}/μ_{s_2}. So the tangent line to the indifference curve at the 45 degree line describes the subjective probability. It is then clear that this tangent line (depicted in green in the figure) must be *flatter* than the budget line at which x^{k_2} was chosen. On the other hand, the same reasoning reveals that the subjective probability must define a *steeper* line than the budget line at which x^{k_1} was chosen. This is a contradiction, as the latter budget line is steeper than the former.

5.2. General K and S

In the following, we focus instead on the relation with probabilistic sophistication, namely the relation between SARSEU and the axiom in Epstein (2000). Epstein studies the implications of probabilistic sophistication for consumption datasets. He considers the same kind of economic environment as we do, and the same notion of a dataset. He focuses on probabilistic sophistication instead of SEU, and importantly does not assume risk aversion. Epstein shows that a dataset is inconsistent with probabilistic sophistication if there exist $s, t \in S$ and $k, \hat{k} \in K$ such that (i) $p_s^k \geq p_t^k$ and $p_s^{\hat{k}} \leq p_t^{\hat{k}}$, with at least one strict inequality; and (ii) $x_s^k > x_t^k$ and $x_s^{\hat{k}} < x_t^{\hat{k}}$.

Of course, an SEU rational agent is probabilistically sophisticated. Indeed, our next result establishes that a violation of Epstein's condition implies a violation of SARSEU.
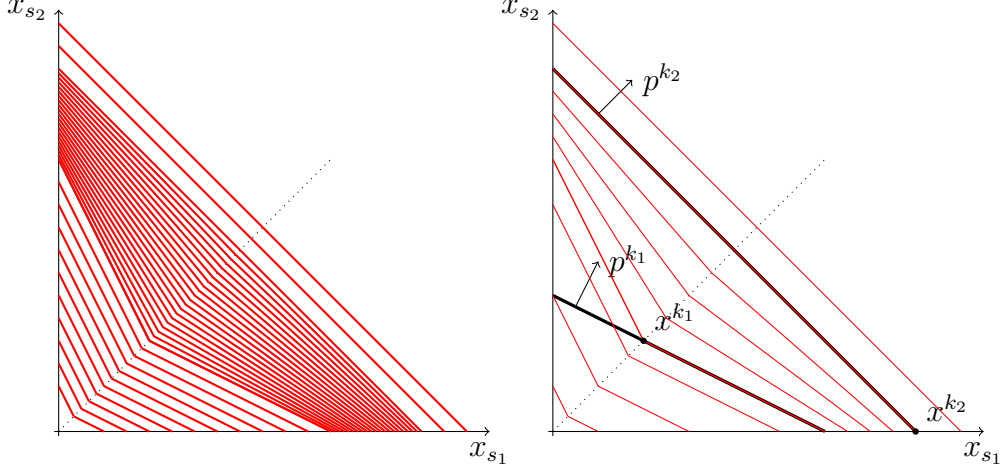
13

Figure 3: Probabilistically sophisticated violation of SARSEU

Proposition 5. *If a dataset* $(x^k, p^k)_{k=1}^K$ *satisfies SARSEU, then (i) and (ii) cannot both hold for some* $s, t \in S, k, \hat{k} \in K$

Proof. Suppose that $s, t \in S, k, \hat{k} \in K$ are such that (ii) holds. Then $\{(x_s^k, x_t^k), (x_t^{\hat{k}}, x_s^{\hat{k}}))\}$ satisfies the conditions in SARSEU. Hence, SARSEU requires that $\dfrac{p_s^k}{p_t^k} \dfrac{p_t^{\hat{k}}}{p_s^{\hat{k}}} \leq 1$, so that $p_s^k \leq p_t^k$ or $p_s^{\hat{k}} \geq p_t^{\hat{k}}$. Hence, (i) is violated. $\qquad \square$

Proposition 5 raises the issue of whether SEU and probabilistic sophistication are distinguishable. In the following, we show that we can indeed distinguish the two models: We present an example of a dataset that violates SARSEU, but that is consistent with a risk-averse probabilistically sophisticated agent. Hence the weakening in going from SEU to probabilistic sophistication has empirical content.

Let $S = \{s_1, s_2\}$. We define a dataset as follows. Let $x^{k_1} = (2, 2)$, $p^{k_1} = (1, 2)$, $x^{k_2} = (8, 0)$, and $p^{k_2} = (1, 1)$. It is clear that the dataset violates SARSEU: $x_{s_1}^{k_2} > x_{s_1}^{k_1}$ and $x_{s_2}^{k_1} > x_{s_2}^{k_2}$ while

$$\frac{p_{s_1}^{k_2}}{p_{s_1}^{k_1}} \frac{p_{s_2}^{k_1}}{p_{s_2}^{k_2}} = 2 > 1.$$

Observe, moreover, that the dataset specifically violates Requirement (5), not (6). Figure 3 illustrates the dataset.

We shall now argue that this dataset is rationalizable by a risk-averse probabilistically sophisticated agent. Fix $\mu \in \Delta_{++}$ with $\mu_1 = \mu_2 = 1/2$, a uniform probability over S. Any

vector $x \in \mathbf{R}_+^2$ induces the probability distribution on $\mathbf{R}_+$ given by x_1 with probability $1/2$ and x_2 with probability $1/2$. Let Π be the set of all uniform probability measures on $\mathbf{R}_+$ with support having cardinality smaller than or equal to 2.

We shall define a function $V : \Pi \to \mathbf{R}$ that represents a probabilistically sophisticated preference, and for which the choices in the dataset are optimal.[3] We construct a monotone increasing and quasiconcave $h : \mathbf{R}_+^2 \to \mathbf{R}_+$, and then define $V(\pi) = h(\bar{x}^\pi, \underline{x}^\pi)$, where $\underline{x}^\pi$ is the smallest point in the support of π, and $\bar{x}^\pi$ is the largest. As a consequence of the monotonicity of h, V represents a probabilistically sophisticated preference. The preference is also risk-averse.

The function h is constructed so that $(x_1, x_2) \mapsto h(\max\{x_1, x_2\}, \min\{x_1, x_2\})$ has the map of indifference curves illustrated on the left in Figure 3. There are two important features of the indifference curves drawn in the figure. The first is that indifference curves exhibit a convex preference, which ensures that the agent will be risk-averse. The second is that indifference curves become "less convex" as one moves up and to the right in the figure. As a result, the line that is normal to p^{k_1} supports the indifference curve through x^{k_1}, while the line that is normal to p^{k_2} supports the indifference curve through x^{k_2}. It is clear from Figure 3 that the construction rationalizes the choices in the dataset.

6.   MAXMIN EXPECTED UTILITY

In this section, we demonstrate one use of our main result to study the differences between SEU and maxmin expected utility. We show that SEU and maxmin are behaviorally indistinguishable in the 2×2 case, but distinguishable more generally.

The maxmin SEU model, first axiomatized by Gilboa and Schmeidler (1989), posits that an agent maximizes

$$\min_{\mu \in M} \sum_{s \in S} \mu_s u(x_s),$$

where M is a closed and convex set of probabilities. A dataset $(x^k, p^k)_{k=1}^K$ is *maxmin expected utility rational* if there is a closed and convex set $M \subseteq \Delta_{++}$ and a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ such that, for all k,

$$y \in B(p^k, p^k \cdot x^k) \Rightarrow \min_{\mu \in M} \sum_{s \in S} \mu_s u(y_s) \le \min_{\mu \in M} \sum_{s \in S} \mu_s u(x_s^k).$$

---

[3]It is easy to show that there is a probabilistically sophisticated weak order $\succeq$ defined on the set of all probability measures on $\mathbf{R}_+^2$ with finite support, such that V represents $\succeq$ on Π. The details of the example are technical and left to the online appendix.

Note that we restrict attention to *risk-averse* maxmin expected utility.

Proposition 6. *Let $S = K = 2$. Then a dataset is maxmin expected utility rational if and only if it is SEU rational.*

The proof of Proposition 6 is in the online appendix. The result in Proposition 6 does not, however, extend beyond the case of two observations. In the online appendix, we provide an example of data from a (risk-averse) maxmin expected utility agent that violates SARSEU.

A.   PROOF OF THEOREM 1

We first give three preliminary and auxiliary results. Lemma 7 provides nonlinear Afriat inequalities for the problem at hand. A version of this lemma appears, for example, in Green and Srivastava (1986), Varian (1983), or Bayer et al. (2012). Lemmas 8 and 9 are versions of the theorem of the alternative.

Lemma 7. *Let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent:*

1. *$(x^k, p^k)_{k=1}^K$ is SEU rational.*

2. *There are strictly positive numbers v_s^k, λ^k, μ_s, for $s = 1, \ldots, S$ and $k = 1, \ldots, K$, such that*

$$\mu_s v_s^k = \lambda^k p_s^k, \quad x_s^k > x_{s'}^{k'} \Rightarrow v_s^k \le v_{s'}^{k'}. \tag{7}$$

Proof. We shall prove that (1) implies (2). Let $(x^k, p^k)_{k=1}^K$ be SEU rational. Let $\mu \in \mathbf{R}_{++}^S$ and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of SEU rational dataset. Then (see, for example, Theorem 28.3 of Rockafellar (1997)), there are numbers $\lambda^k \ge 0$, $k = 1, \ldots, K$ such that if we let

$$v_s^k = \frac{\lambda^k p_s^k}{\mu_s}$$

then $v_s^k \in \partial u(x_s^k)$ if $x_s^k > 0$, and there is $\underline{w} \in \partial u(x_s^k)$ with $v_s^k \ge \underline{w}$ if $x_s^k = 0$ (here we have $\partial u(0) \ne \emptyset$). In fact, since u is strictly increasing, it is easy to see that $\lambda^k > 0$, and therefore $v_s^k > 0$.

By the concavity of u, and the consequent monotonicity of $\partial u(x_s^k)$ (see Theorem 24.8 of Rockafellar (1997)), if $x_s^k > x_{s'}^{k'} > 0$, $v_s^k \in \partial u(x_s^k)$, and $v_{s'}^{k'} \in \partial u(x_{s'}^{k'})$, then $v_s^k \le v_{s'}^{k'}$. If $x_s^k > x_{s'}^{k'} = 0$, then $\underline{w} \in \partial u(x_{s'}^{k'})$ with $v_{s'}^{k'} \ge \underline{w}$. So $v_s^k \le \underline{w} \le v_{s'}^{k'}$.

In second place, we show that (2) implies (1). Suppose that the numbers v_s^k, λ^k, μ_s, for $s = 1, \ldots, S$ and $k = 1, \ldots, K$ are as in (2). Enumerate the elements of $\mathcal{X}$ in increasing order: $y_1 < y_2 < \ldots < y_n$. Let $\underline{y}_i = \min\{v_s^k : x_s^k = y_i\}$ and $\bar{y}_i = \max\{v_s^k : x_s^k = y_i\}$. Let $z_i = (y_i + y_{i+1})/2$, $i = 1, \ldots, n-1$; $z_0 = 0$, and $z_n = y_n + 1$. Let f be a correspondence defined as follows:

$$
f(z) = \begin{cases} [\underline{y}_i, \bar{y}_i] & \text{if } z = y_i, \\ \max\{\bar{y}_i : z < y_i\} & \text{if } y_n > z \text{ and } \forall i (z \neq y_i), \\ \underline{y}_n/2 & \text{if } y_n < z. \end{cases} \tag{8}
$$

Then, by the assumptions placed on v_s^k, and by construction of f, $y < y'$, $v \in f(y)$ and $v' \in f(y')$ imply that $v' \leq v$. Then the correspondence f is monotone, and there exists a concave function u for which $\partial u = f$ (see e.g. Theorem 24.8 of Rockafellar (1997)). Given that $v_s^k > 0$ all the elements in the range of f are positive, and therefore u is a strictly increasing function.

Finally, for all (k, s), $\lambda^k p_s^k / \mu_s = v_s^k \in \partial u(v_s^k)$ and therefore the first-order conditions to a maximum choice of x hold at x_s^k. Since u is concave the first-order conditions are sufficient. The dataset is therefore SEU rational. $\qquad\square$

We shall use the following lemma, which is a version of the Theorem of the Alternative. This is Theorem 1.6.1 in Stoer and Witzgall (1970). We shall use it here in the cases where F is either the real or the rational number field.

Lemma 8. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong to a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{F}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, $E \cdot u \gg 0$.*

2. *There is $\theta \in \mathbf{F}^r$, $\eta \in \mathbf{F}^l$, and $\pi \in \mathbf{F}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

The next lemma is a direct consequence of Lemma 8: see Lemma 12 in Chambers and Echenique (2014) for a proof.

Lemma 9. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E are rational numbers. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{R}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, and $E \cdot u \gg 0$.*

2. *There is $\theta \in \mathbf{Q}^r$, $\eta \in \mathbf{Q}^l$, and $\pi \in \mathbf{Q}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

A.1. Necessity

Lemma 10. *If a dataset $(x^k, p^k)_{k=1}^K$ is SEU rational, then it satisfies SARSEU.*

Proof. Let $(x^k, p^k)_{k=1}^K$ be SEU rational, and let $\mu \in \Delta_{++}$ and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of SEU rational. By Lemma 7, there exists a strictly positive solution v_s^k, λ^k, μ_s to the system in Statement (2) of Lemma 7 with $v_s^k \in \partial u(x_s^k)$ when $x_s^k > 0$, and $v_s^k \geq \underline{w} \in \partial u(x_s^k)$ when $x_s^k = 0$.

Let $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ be a sequence satisfying the three conditions in SARSEU. Then $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$. Suppose that $x_{s_i'}^{k_i'} > 0$. Then, $v_{s_i}^{k_i} \in \partial u(x_{s_i}^{k_i})$ and $v_{s_i'}^{k_i'} \in \partial u(x_{s_i'}^{k_i'})$. By the concavity of u, it follows that $\lambda^{k_i} \mu_{s_i'} p_{s_i}^{k_i} \leq \lambda^{k_i'} \mu_{s_i} p_{s_i'}^{k_i'}$ (see Theorem 24.8 of Rockafellar (1997)). Similarly, if $x_{s_i'}^{k_i'} = 0$, then $v_{s_i}^{k_i} \in \partial u(x_{s_i}^{k_i})$ and $v_{s_i'}^{k_i'} \geq \underline{w} \in \partial u(x_{s_i'}^{k_i'})$. So $\lambda^{k_i} \mu_{s_i'} p_{s_i}^{k_i} \leq \lambda^{k_i'} \mu_{s_i} p_{s_i'}^{k_i'}$ Therefore,

$$1 \geq \prod_{i=1}^n \frac{\lambda^{k_i} \mu_{s_i'} p_{s_i}^{k_i}}{\lambda^{k_i'} \mu_{s_i} p_{s_i'}^{k_i'}} = \prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}},$$

as the sequence satisfies Conditions (2) and (3) of SARSEU; and hence the numbers λ^k and μ_s appear the same number of times in the denominator as in the numerator of this product. $\qquad\square$

A.2. Sufficiency

We proceed to prove the sufficiency direction. An outline of the argument is as follows. We know from Lemma 7 that it suffices to find a solution to the Afriat inequalities (actually first order conditions), written as statement (2) in the lemma. So we set up the problem to find a solution to a system of linear inequalities obtained from using logarithms to linearize the Afriat inequalities in Lemma 7.

Lemma 11 establishes that SARSEU is sufficient for SEU rationality when the logarithms of the prices are rational numbers. The role of rational logarithms comes from our use of a version the theorem of the alternative (see Lemma 9): when there is no solution to the linearized Afriat inequalities, then the existence of a rational solution to the dual

system of inequalities implies a violation of SARSEU. The bulk of the proof goes into constructing a violation of SARSEU from a given solution to the dual.

The next step in the proof (Lemma 12) establishes that we can approximate any dataset satisfying SARSEU with a dataset for which the logarithms of prices are rational, and for which SARSEU is satisfied. This step is crucial, and somewhat delicate. One might have tried to obtain a solution to the Afriat inequalities for "perturbed" systems (with prices that are rational after taking logs), and then considered the limit. This does not work because the solutions to our systems of inequalities are in a non-compact space. It is not clear how to establish that the limits exist and are well-behaved. Lemma 12 avoids the problem.

Finally, Lemma 13 establishes the result by using another version of the theorem of the alternative, stated as Lemma 8 above.

The statement of the lemmas follow. The rest of the paper is devoted to the proof of these lemmas.

Lemma 11. *Let dataset* $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Suppose that* $\log(p_s^k) \in \mathbf{Q}$ *for all* k *and* s. *Then there are numbers* v_s^k, λ^k, μ_s, *for* $s = 1, \ldots, S$ *and* $k = 1, \ldots, K$ *satisfying* (7) *in Lemma 7.*

Lemma 12. *Let dataset* $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Then for all positive numbers* $\bar{\varepsilon}$, *there exists* $q_s^k \in [p_s^k - \bar{\varepsilon}, p_s^k]$ *for all* $s \in S$ *and* $k \in K$ *such that* $\log q_s^k \in \mathbf{Q}$ *and the dataset* $(x^k, q^k)_{k=1}^k$ *satisfy SARSEU.*

Lemma 13. *Let dataset* $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Then there are numbers* v_s^k, λ^k, μ_s, *for* $s = 1, \ldots, S$ *and* $k = 1, \ldots, K$ *satisfying* (7) *in Lemma 7.*

A.2.1. Proof of Lemma 11

We linearize the equation in System (7) of Lemma 7. The result is:

$$\log v_s^k + \log \mu_s - \log \lambda^k - \log p_s^k = 0, \tag{9}$$

$$x_s^k > x_{s'}^{k'} \Rightarrow \log v_s^k \leq \log v_{s'}^{k'}. \tag{10}$$

In the system comprised by (9) and (10), the unknowns are the real numbers $\log v_s^k$, $\log \mu_s$, $\log \lambda^k$, for $k = 1, \ldots, K$ and $s = 1, \ldots, S$.

First, we are going to write the system of inequalities (9) and (10) in matrix form.

We shall define a matrix A such that there are positive numbers v_s^k, λ^k, μ_s the logs of which satisfy Equation (9) if and only if there is a solution $u \in \mathbf{R}^{K \times S + K + S + 1}$ to the system of equations

$$A \cdot u = 0,$$

and for which the last component of u is strictly positive.

Let A be a matrix with $K \times S$ rows and $K \times S + S + K + 1$ columns, defined as follows: We have one row for every pair (k, s); one column for every pair (k, s); one column for every s; one column for each k; and one last column. In the row corresponding to (k, s) the matrix has zeroes everywhere with the following exceptions: it has a 1 in the column for (k, s); it has a 1 in the column for s; it has a -1 in the column for k; and $-\log p_s^k$ in the very last column.

Matrix A looks as follows:

	(1,1)	⋯	(k,s)	⋯	(K,S)	1	⋯	s	⋯	S	1	⋯	k	⋯	K	p
(1,1)	1	⋯	0	⋯	0	1	⋯	0	⋯	0	-1	⋯	0	⋯	0	$-\log p_1^1$
⋮	⋮		⋮		⋮	⋮		⋮		⋮	⋮		⋮		⋮	⋮
(k,s)	0	⋯	1	⋯	0	0	⋯	1	⋯	0	0	⋯	-1	⋯	0	$-\log p_s^k$
⋮	⋮		⋮		⋮	⋮		⋮		⋮	⋮		⋮		⋮	⋮
(K,S)	0	⋯	0	⋯	1	0	⋯	0	⋯	1	0	⋯	0	⋯	-1	$-\log p_S^K$

Consider the system $A \cdot u = 0$. If there are numbers solving Equation (9), then these define a solution $u \in \mathbf{R}^{K \times S + S + K + 1}$ for which the last component is 1. If, on the other hand, there is a solution $u \in \mathbf{R}^{K \times S + S + K + 1}$ to the system $A \cdot u = 0$ in which the last component $(u_{K \times S + S + K + 1})$ is strictly positive, then by dividing through by the last component of u we obtain numbers that solve Equation (9).

In second place, we write the system of inequalities (10) in matrix form. There is one row in B for each pair (k, s) and (k', s') for which $x_s^k > x_{s'}^{k'}$. In the row corresponding to $x_s^k > x_{s'}^{k'}$ we have zeroes everywhere with the exception of a -1 in the column for (k, s) and a 1 in the column for (k', s'). Let $\mathcal{B}$ be the number of rows of B.

In third place, we have a matrix E that captures the requirement that the last component of a solution be strictly positive. The matrix E has a single row and $K \times S + S + K + 1$ columns. It has zeroes everywhere except for 1 in the last column.

To sum up, there is a solution to system (9) and (10) if and only if there is a vector

20

$u \in \mathbf{R}^{K \times S + S + K + 1}$ that solves the following system of equations and linear inequalities

$$S1: \begin{cases} A \cdot u = 0, \\ B \cdot u \geq 0, \\ E \cdot u \gg 0. \end{cases}$$

Note that $E \cdot u$ is a scalar, so the last inequality is the same as $E \cdot u > 0$.

The entries of A, B, and E are either 0, 1 or -1, with the exception of the last column of A. Under the hypotheses of the lemma we are proving, the last column consists of rational numbers. By Lemma 9, then, there is such a solution u to $S1$ if and only if there is no vector $(\theta, \eta, \pi) \in \mathbf{Q}^{K \times S + \mathcal{B} + 1}$ that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

In the following, we shall prove that the non-existence of a solution u implies that the dataset must violate SARSEU. Suppose then that there is no solution u and let (θ, η, π) be a rational vector as above, solving system $S2$.

By multiplying (θ, η, π) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, π) to be integer vectors.

Henceforth, we use the following notational convention: For a matrix D with $K \times S + S + K + 1$ columns, write D_1 for the submatrix of D corresponding to the first $K \times S$ columns; let D_2 be the submatrix corresponding to the following S columns; D_3 correspond to the next K columns; and D_4 to the last column. Thus, $D = [D_1|D_2|D_3|D_4]$.

Claim 14. *(i) $\theta \cdot A_1 + \eta \cdot B_1 = 0$; (ii) $\theta \cdot A_2 = 0$; (iii) $\theta \cdot A_3 = 0$; and (iv) $\theta \cdot A_4 + \pi \cdot E_4 = 0$.*

Proof. Since $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$, then $\theta \cdot A_i + \eta \cdot B_i + \pi \cdot E_i = 0$ for all $i = 1, \dots, 4$. Moreover, since B_2, B_3, B_4, E_1, E_2, and E_3 are zero matrices, we obtain the claim. $\square$

We transform the matrices A and B using θ and η. Define a matrix A^* from A by letting A^* have the same number of columns as A and including: (i) θ_r copies of the rth row when $\theta_r > 0$; (ii) omitting row r when $\theta_r = 0$; and (ii) θ_r copies of the rth row multiplied by -1 when $\theta_r < 0$. We refer to rows that are copies of some r with $\theta_r > 0$ as *original* rows, and to those that are copies of some r with $\theta_r < 0$ as *converted* rows.

21

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Claim 15. *For any (k, s), all the entries in the column for (k, s) in A_1^* are of the same sign.*

Proof. By definition of A, the column for (k, s) will have zero in all its entries with the exception of the row for (k, s). In A^*, for each (k, s), there are three mutually exclusive possibilities: the row for (k, s) in A can (i) not appear in A^*, (ii) it can appear as original, or (iii) it can appear as converted. This shows the claim. $\square$

Claim 16. *There exists a sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ of pairs that satisfies Condition (1) in SARSEU.*

Proof. Define $\mathcal{X} = \{x_s^k | k \in K, \ s \in S\}$. We define such a sequence by induction. Let $B^1 = B^*$. Given B^i, define B^{i+1} as follows: Denote by $>^i$ the binary relation on $\mathcal{X}$ defined by $z >^i z'$ if $z > z'$ and there is at least one copy of a row corresponding to $z > z'$ in B^i: there is at least one pair (k, s) and (k', s') for which (1) $x_s^k > x_{s'}^{k'}$, (2) $z = x_s^k$ and $z' = x_{s'}^{k'}$, and (3) the row corresponding to $x_s^k > x_{s'}^{k'}$ in B had strictly positive weight in η.

The binary relation $>^i$ cannot exhibit cycles because $>^i \subseteq >$. There is therefore at least one sequence $z_1^i, \ldots z_{L_i}^i$ in $\mathcal{X}$ such that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \ldots, L_i - 1$ and with the property that there is no $z \in \mathcal{X}$ with $z >^i z_1^i$ or $z_{L_i}^i >^i z$.

Observe that B^i has at least one row corresponding to $z_j^i >^i z_{j+1}^i$, for all $j = 1, \ldots, L_i - 1$. Let the matrix B^{i+1} be defined as the matrix obtained from B^i by omitting one copy of a row corresponding to $z_j^i > z_{j+1}^i$, for each $j = 1, \ldots L_i - 1$

The matrix B^{i+1} has strictly fewer rows than B^i. There is therefore n^* for which B^{n^*+1} would have no rows. The matrix B^{n^*} has rows, and the procedure of omitting rows from B^{n^*} will remove all rows of B^{n^*}.

Define a sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ of pairs by letting $x_{s_i}^{k_i} = z_1^i$ and $x_{s_i'}^{k_i'} = z_{L_i}^i$. Note that, as a result, $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ for all i. Therefore the sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ of pairs satisfies Condition (1) in SARSEU. $\square$

We shall use the sequence $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n^*}$ of pairs as our candidate violation of SARSEU.

Consider a sequence of matrices A^i, $i = 1, \ldots, n^*$ defined as follows. Let $A^1 = A^*$, $B^1 = B^*$, and $C^1 = \begin{bmatrix} A^1 \\ B^1 \end{bmatrix}$. Observe that the rows of C^1 add to the null vector by Claim 14.

We shall proceed by induction. Suppose that A^i has been defined, and that the rows of $C^i = \begin{bmatrix} A^i \\ B^i \end{bmatrix}$ add to the null vector.

Recall the definition of the sequence $x_{s_i}^{k_i} = z_1^i > \ldots > z_{L_i}^i = x_{s_i'}^{k_i'}$. There is no $z \in \mathcal{X}$ with $z >^i z_1^i$ or $z_{L_i}^i >^i z$, so in order for the rows of C^i to add to zero there must be a -1 in A_1^i in the column corresponding to (k_i', s_i') and a 1 in A_1^i in the column corresponding to (k_i, s_i). Let r_i be a row in A^i corresponding to (k_i, s_i), and r_i' be a row corresponding to (k_i', s_i'). The existence of a -1 in A_1^i in the column corresponding to (k_i', s_i'), and a 1 in A_1^i in the column corresponding to (k_i, s_i), ensures that r_i and r_i' exist. Note that the row r_i' is a converted row while r_i is original. Let A^{i+1} be defined from A^i by deleting the two rows, r_i and r_i'.

Claim 17. *The sum of r_i, r_i', and the rows of B^i which are deleted when forming B^{i+1} (corresponding to the pairs $z_j^i > z_{j+1}^i$, $j = 1, \ldots, L_i - 1$) add to the null vector.*

Proof. Recall that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \ldots, L_i - 1$. So when we add rows corresponding to $z_j^i >^i z_{j+1}^i$ and $z_{j+1}^i >^i z_{j+2}^i$, then the entries in the column for (k, s) with $x_s^k = z_{j+1}^i$ cancel out and the sum is zero in that entry. Thus, when we add the rows of B^i that are not in B^{i+1} we obtain a vector that is 0 everywhere except the columns corresponding to z_1^i and $z_{L_i}^i$. This vector cancels out with $r_i + r_i'$, by definition of r_i and r_i'. $\qquad\square$

Claim 18. *The matrix A^* can be partitioned into pairs of rows (r_i, r_i'), in which the rows r_i' are converted and the rows r_i are original.*

Proof. For each i, A^{i+1} differs from A^i in that the rows r_i and r_i' are removed from A^i to form A^{i+1}. We shall prove that A^* is composed of the $2n^*$ rows r_i, r_i'.

First note that since the rows of C^i add up to the null vector, and A^{i+1} and B^{i+1} are obtained from A^i and B^i by removing a collection of rows that add up to zero, then the rows of C^{i+1} must add up to zero as well.

We now show that the process stops after n^* steps: all the rows in C^{n^*} are removed by the procedure described above. By way of contradiction, suppose that there exist rows left after removing r_{n^*} and r_{n^*}'. Then, by the argument above, the rows of the matrix C^{n^*+1} must add to the null vector. If there are rows left, then the matrix C^{n^*+1} is well defined.

By definition of the sequence B^i, however, B^{n^*+1} is an empty matrix. Hence, rows remaining in $A_1^{n^*+1}$ must add up to zero. By Claim 15, the entries of a column (k, s) of A^*

23

are always of the same sign. Moreover, each row of A^* has a non-zero element in the first $K \times S$ columns. Therefore, no subset of the columns of A_1^* can sum to the null vector. $\square$

Claim 19. *(i) For any k and s, if $(k_i, s_i) = (k, s)$ for some i, then the row corresponding to (k, s) appears as original in A^*. Similarly, if $(k'_i, s'_i) = (k, s)$ for some i, then the row corresponding to (k, s) appears converted in A^*.*
(ii) If the row corresponding to (k, s) appears as original in A^*, then there is some i with $(k_i, s_i) = (k, s)$. Similarly, if the row corresponding to (k, s) appears converted in A^*, then there is i with $(k'_i, s'_i) = (k, s)$.

Proof. (i) is true by definition of $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})$. (ii) is immediate from Claim 18 because if the row corresponding to (k, s) appears original in A^* then it equals r_i for some i, and then $(k_i, s_i) = (k, s)$. Similarly when the row appears converted. $\square$

Claim 20. *The sequence $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^{n^*}$ satisfies Conditions (2) and (3) in SARSEU.*

Proof. By Claim 14 (ii), the rows of A_2^* add up to zero. Therefore, the number of times that s appears in an original row equals the number of times that it appears in a converted row. By Claim 19, then, the number of times s appears as s_i equals the number of times it appears as s'_i. Therefore Condition (2) in the axiom is satisfied.

Similarly, by Claim 14 (iii), the rows of A_3^* add to the null vector. Therefore, the number of times that k appears in an original row equals the number of times that it appears in a converted row. By Claim 19, then, the number of times that k appears as k_i equals the number of times it appears as k'_i. Therefore Condition (3) in the axiom is satisfied. $\square$

Finally, in the following, we show that $\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s'_i}^{k'_i}} > 1$, which finishes the proof of Lemma 11 as the sequence $(x_{s_i}^{k_i}, x_{s'_i}^{k'_i})_{i=1}^{n^*}$ would then exhibit a violation of SARSEU.

Claim 21. $\prod_{i=1}^{n^*} \frac{p_{s_i}^{k_i}}{p_{s'_i}^{k'_i}} > 1.$

Proof. By Claim 14 (iv) and the fact that the submatrix E_4 equals the scalar 1, we obtain

$$0 = \theta \cdot A_4 + \pi E_4 = (\sum_{i=1}^{n^*} (r_i + r'_i))_4 + \pi,$$

24

where $(\sum_{i=1}^{n^*}(r_i + r_i'))_4$ is the (scalar) sum of the entries of A_4^*. Recall that $-\log p_{s_i}^{k_i}$ is the last entry of row r_i and that $\log p_{s_i'}^{k_i'}$ is the last entry of row r_i', as r_i' is converted and r_i original. Therefore the sum of the rows of A_4^* are $\sum_{i=1}^{n^*}\log(p_{s_i'}^{k_i'}/p_{s_i}^{k_i})$. Then, $\sum_{i=1}^{n^*}\log(p_{s_i'}^{k_i'}/p_{s_i}^{k_i}) = -\pi < 0$. Thus $\prod_{i=1}^{n^*}\frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} > 1$. $\qquad\square$

A.2.2. Proof of Lemma 12

For each sequence $\sigma = (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ that satisfies Conditions (1), (2), and (3) in SARSEU, we define a vector $t_\sigma \in \mathbf{N}^{K^2 S^2}$. For each pair $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})$, we shall identify the pair with $((k_i, s_i), (k_i', s_i'))$. Let $t_\sigma((k, s), (k', s'))$ be the number of times that the pair $(x_s^k, x_{s'}^{k'})$ appears in the sequence σ. One can then describe the satisfaction of SARSEU by means of the vectors t_σ. Define

$$T = \left\{ t_\sigma \in \mathbf{N}^{K^2 S^2} | \sigma \text{ satisfies Conditions (1), (2), (3) in SARSEU} \right\}.$$

Observe that the set T depends only on $(x^k)_{k=1}^K$ in the dataset $(x^k, p^k)_{k=1}^K$. It does not depend on prices.

For each $((k, s), (k', s'))$ such that $x_s^k > x_{s'}^{k'}$, define $\hat{\delta}((k, s), (k', s')) = \log\left(\frac{p_s^k}{p_{s'}^{k'}}\right)$. And define $\hat{\delta}((k, s), (k', s')) = 0$ when $x_s^k \leq x_{s'}^{k'}$. Then, $\hat{\delta}$ is a $K^2 S^2$-dimensional real-valued vector. If $\sigma = (x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$, then

$$\hat{\delta} \cdot t_\sigma = \sum_{((k,s),(k',s'))\in(KS)^2} \hat{\delta}((k, s), (k', s'))t_\sigma((k, s), (k', s')) = \log\left(\prod_{i=1}^n \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}}\right).$$

So the dataset satisfies SARSEU if and only if $\hat{\delta} \cdot t \leq 0$ for all $t \in T$.

Enumerate the elements in $\mathcal{X}$ in increasing order: $y_1 < y_2 < \cdots < y_N$. And fix an arbitrary $\underline{\xi} \in (0, 1)$. We shall construct by induction a sequence $\{(\varepsilon_s^k(n))\}_{n=1}^N$, where $\varepsilon_s^k(n)$ is defined for all (k, s) with $x_s^k = y_n$.

By the denseness of the rational numbers, and the continuity of the exponential function, for each (k, s) such that $x_s^k = y_1$, there exists a positive number $\varepsilon_s^k(1)$ such that $\log(p_s^k \varepsilon_s^k(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon_s^k(1) < 1$. Let $\varepsilon(1) = \min\{\varepsilon_s^k(1)|x_s^k = y_1\}$.

In second place, for each (k, s) such that $x_s^k = y_2$, there exists a positive $\varepsilon_s^k(2)$ such that $\log(p_s^k \varepsilon_s^k(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon_s^k(2) < \varepsilon(1)$. Let $\varepsilon(2) = \min\{\varepsilon_s^k(2)|x_s^k = y_2\}$.

25

In third place, and reasoning by induction, suppose that $\varepsilon(n)$ has been defined and that $\underline{\xi} < \varepsilon(n)$. For each (k, s) such that $x_s^k = y_{n+1}$, let $\varepsilon_s^k(n+1) > 0$ be such that $\log(p_s^k \varepsilon_s^k(n+1)) \in \mathbf{Q}$, and $\underline{\xi} < \varepsilon_s^k(n+1) < \varepsilon(n)$. Let $\varepsilon(n+1) = \min\{\varepsilon_s^k(n+1) | x_s^k = y_n\}$.

This defines the sequence $(\varepsilon_s^k(n))$ by induction. Note that $\varepsilon_s^k(n+1)/\varepsilon(n) < 1$ for all n. Let $\bar{\xi} < 1$ be such that $\varepsilon_s^k(n+1)/\varepsilon(n) < \bar{\xi}$.

For each $k \in K$ and $s \in S$, let $q_s^k = p_s^k \varepsilon_s^k(n)$, where n is such that $x_s^k = y_n$. We claim that the dataset $(x^k, q^k)_{k=1}^K$ satisfies SARSEU. Let δ^* be defined from $(q^k)_{k=1}^K$ in the same manner as $\hat{\delta}$ was defined from $(p^k)_{k=1}^K$.

For each pair $((k, s), (k', s'))$ with $x_s^k > x_{s'}^{k'}$, if n and m are such that $x_s^k = y_n$ and $x_{s'}^{k'} = y_m$, then $n > m$. By definition of ε,

$$\frac{\varepsilon_s^k(n)}{\varepsilon_{s'}^{k'}(m)} < \frac{\varepsilon_s^k(n)}{\varepsilon(m)} < \bar{\xi} < 1.$$

Hence,

$$\delta^*((k, s), (k', s')) = \log \frac{p_s^k \varepsilon_s^k(n)}{p_{s'}^{k'} \varepsilon_{s'}^{k'}(m)} < \log \frac{p_s^k}{p_{s'}^{k'}} + \log \bar{\xi} < \log \frac{p_s^k}{p_{s'}^{k'}} = \hat{\delta}(x_s^k, x_{s'}^{k'}).$$

Thus, for all $t \in T$, $\delta^* \cdot t \leq \hat{\delta} \cdot t \leq 0$, as $t \geq 0$ and the dataset $(x^k, p^k)_{k=1}^K$ satisfies SARSEU. Thus the dataset $(x^k, q^k)_{k=1}^K$ satisfies SARSEU. Finally, note that $\underline{\xi} < \varepsilon_s^k(n) < 1$ for all n and each $k \in K, s \in S$. So that by choosing $\underline{\xi}$ close enough to 1 we can take the prices (q^k) to be as close to (p^k) as desired.

A.2.3. Proof of Lemma 13

Consider the system comprised by (9) and (10) in the proof of Lemma 11. Let A, B, and E be constructed from the dataset as in the proof of Lemma 11. The difference with respect to Lemma 11 is that now the entries of A_4 may not be rational. Note that the entries of E, B, and A_i, $i = 1, 2, 3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (9) and (10). Then, by the argument in the proof of Lemma 11 there is no solution to System $S1$. Lemma 8 with $\mathbf{F} = \mathbf{R}$ implies that there is a real vector (θ, η, π) such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$ and $\eta \geq 0, \pi > 0$. Recall that $B_4 = 0$ and $E_4 = 1$, so we obtain that $\theta \cdot A_4 + \pi = 0$.

Let $(q^k)_{k=1}^K$ be vectors of prices such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies SARSEU and $\log q_s^k \in \mathbf{Q}$ for all k and s. (Such $(q^k)_{k=1}^K$ exists by Lemma 12.) Construct matrices A', B', and E' from this dataset in the same way as A, B, and E is constructed in the proof

of Lemma 11. Note that only the prices are different in (x^k, q^k) compared to (x^k, p^k). So $E' = E$, $B' = B$ and $A_i' = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A_4' may be different from A_4.

By Lemma 12, we can choose prices q^k such that $|\theta \cdot A_4' - \theta \cdot A_4| < \pi/2$. We have shown that $\theta \cdot A_4 = -\pi$, so the choice of prices q^k guarantees that $\theta \cdot A_4' < 0$. Let $\pi' = -\theta \cdot A_4' > 0$.

Note that $\theta \cdot A_i' + \eta \cdot B_i' + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A_i' = A_i$, $B_i' = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $B_4 = 0$ so $\theta \cdot A_4' + \eta \cdot B_4' + \pi' E_4 = \theta \cdot A_4' + \pi' = 0$. We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution to $S2$ for matrices A', B', and E'.

Lemma 8 then implies that there is no solution to $S1$ for matrices A', B', and E'. So there is no solution to the system comprised by (9) and (10) in the proof of Lemma 11. However, this contradicts Lemma 11 because the dataset (x^k, q^k) satisfies SARSEU and $\log q_s^k \in \mathbf{Q}$ for all $k = 1, \ldots K$ and $s = 1, \ldots, S$.

B. Subjective–Objective Expected Utility

We turn to an environment in which a subset of states have known probabilities. Let $S^* \subseteq S$ be a set of states, and assume given μ_s^*, the probability of state s for $s \in S^*$.

We allow for the two extreme cases: $S^* = S$ when all states are objective and we are in the setup of Green and Srivastava (1986), Varian (1983), and Kubler et al. (2014), or $S^* = \emptyset$, which is the situation in the body of our paper. The case when S^* is a singleton is studied experimentally by Ahn et al. (2014) and Bossaerts et al. (2010).

Definition 4. *A dataset $(x^k, p^k)_{k=1}^K$ is* subjective–objective expected utility rational *(SOEU rational) if there is $\mu \in \Delta_{++}$, $\eta > 0$, and a concave and strictly increasing function $u : \mathbf{R}_+ \to \mathbf{R}$ such that for all $s \in S^*$ $\mu_s = \eta \mu_s^*$ and for all $k \in K$,*

$$y \in B(p^k, p^k \cdot x^k) \Rightarrow \sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x_s^k).$$

In the definition above, η is a parameter that captures the difference in how the agent treats objective and subjective probabilities. Note that, since η is constant, relative objective probabilities (the ratio of the probability of one state in S^* to another) is unaffected by η. The presence of η has the result of, in studies with a single objective state (as in Ahn et al. and Bossaerts et al.), rendering the objective state subjective.

In studies of objective expected utility, a crucial aspect of the dataset are the price-probability ratios, or "risk neutral prices," defined as follows: for $k \in K$ and $s \in S^*$,

$\rho_s^k = \dfrac{p_s^k}{\mu_s^*}$. Let $r_s^k = p_s^k$ if $s \notin S^*$ and $r_s^k = \rho_s^k$ if $s \in S^*$. The following modification of SARSEU characterizes SOEU rationality.

Strong Axiom of Revealed Subjective–Objective Expected Utility (SARSOEU): *For any sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ *for all i;*

2. *for each $s \notin S^*$, s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right);*

3. *each k appears as k_i (on the left of the pair) the same number of times it appears as k_i' (on the right):*

Then $\prod_{i=1}^n (r_{s_i}^{k_i}/r_{s_i'}^{k_i'}) \le 1$.

Note that SARSEU is a special case of SARSOEU when $S^* = \emptyset$, and when S^* is a singleton (as in Ahn et al. and Bossaerts et al.).

Theorem 22. *A dataset is SOEU rational if and only if it satisfies SARSOEU.*

For completeness, we write out the SARSOEU for the case when $S^* = S$.

Strong Axiom of Revealed Objective Expected Utility (SAROEU): *For any sequence of pairs $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ in which*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$ *for all i;*

2. *each k appears in k_i (on the left of the pair) the same number of times it appears in k_i' (on the right):*

The product of price-probability ratios satisfies that $\prod_{i=1}^n (\rho_{s_i}^{k_i}/\rho_{s_i'}^{k_i'}) \le 1$.

The proof of Theorem 22 with additional discussions are in the online appendix.

C.   Proof of Proposition 2

Let $\Sigma = \{(k, s, k', s') \in K \times S \times K \times S : x_s^k > x_{s'}^{k'}\}$, and let $\delta \in \mathbf{R}^\Sigma$ be defined by $\delta_\sigma = (\log p_s^k - \log p_{s'}^{k'})$. Define a $(K + S) \times |\Sigma|$ matrix G as follows. G has a row for each $k \in K$ and for each $s \in S$, and G has a column for each $\sigma \in \Sigma$. The entry for row $\hat{k} \in K$

and column $\sigma = (k, s, k', s')$ is 1 if $\hat{k} = k$, it is -1 if $\hat{k} = k'$, and it is zero otherwise. The entry for row $\hat{s} \in S$ and column $\sigma = (k, s, k', s')$ is 1 if $\hat{s} = s$, it is -1 if $\hat{s} = s'$, and it is zero otherwise.

Note that every sequence $(x^{k_i}_{s_i}, x^{k'_i}_{s'_i})^n_{i=1}$ in the conditions of SARSEU can be identified with a vector $t \in \mathbf{Z}^\Sigma_+$ such that $t \cdot \delta > 0$ and $G \cdot t = 0$.

Consider the following statements,

$$\exists t \in \mathbf{Z}^\Sigma_+ \text{ s.t. } G \cdot t = 0 \text{ and } t \cdot \delta > 0, \tag{11}$$

$$\exists t \in \mathbf{Q}^\Sigma_+ \text{ s.t. } G \cdot t = 0 \text{ and } t \cdot \delta > 0, \tag{12}$$

$$\exists t \in \mathbf{R}^\Sigma_+ \text{ s.t. } G \cdot t = 0 \text{ and } t \cdot \delta > 0, \tag{13}$$

$$\exists t \in [0, N]^\Sigma \text{ s.t. } G \cdot t = 0 \text{ and } t \cdot \delta > 0, \tag{14}$$

where $N > 0$ can be chosen arbitrarily. We show that: $(11) \Leftrightarrow (12) \Leftrightarrow (13) \Leftrightarrow (14)$. The proof follows because there are efficient algorithms to decide (14) (see, e.g. Chapter 8 in Papadimitriou and Steiglitz (1998)).

That $(11) \Leftrightarrow (12)$ and $(13) \Leftrightarrow (14)$ is true because if $t \cdot \delta > 0$ and $G \cdot t = 0$, then for any scalar λ, $(\lambda t) \cdot \delta > 0$ and $G \cdot (\lambda t) = 0$.

To show that $(12) \Leftrightarrow (13)$ we proceed as follows. Obviously $(12) \Rightarrow (13)$; we focus on the converse. Note that the entries of G are rational numbers (in fact they are 1, -1 or 0). Then one can show that the null space of the linear transformation defined by G, namely $\Omega = \{t \in \mathbf{R}^\Sigma : G \cdot t = 0\}$, has a rational basis $(q_h)^H_{h=1}$. Suppose that (13) is true, and let $t^* \in \mathbf{R}^\Sigma_+$ be such that $G \cdot t^* = 0$ and $t^* \cdot \delta > 0$. Then $t^* = \sum^H_{h=1} \alpha_h q_h$ for some coefficients $(\alpha_h)^H_{h=1}$. The linear function $(\alpha'_h)^H_{h=1} \mapsto \sum^H_{h=1} \alpha'_h q_h$ is continuous and onto Ω. For any neighborhood B of t^* in Ω, B intersects the strictly positive orthant in Ω, which is open in Ω. Therefore there are rational α'_h such that $\sum^H_{h=1} \alpha'_h q_h \geq 0$ and $(\alpha'_h)^H_{h=1}$ can be taken arbitrarily close to $(\alpha_h)^H_{h=1}$. Since $t^* \cdot \delta > 0$ we can take $(\alpha'_h)^H_{h=1}$ to be rational and such that $\sum^H_{h=1} \alpha'_h q_h \geq 0$ and $\delta \cdot \sum^H_{h=1} \alpha'_h q_h > 0$. Letting $t = \sum^H_{h=1} \alpha'_h q_h$ establishes (12).

References

AFRIAT, S. N. (1967): "The Construction of Utility Functions from Expenditure Data," *International Economic Review*, 8, 67–77.

AHN, D., S. CHOI, D. GALE, AND S. KARIV (2014): "Estimating ambiguity aversion in a portfolio choice experiment," *Quantitative Economics*, 5, 195–223.

BAYER, R.-C., S. BOSE, M. POLISSON, AND L. RENOU (2012): "Ambiguity revealed," Mimeo, University of Adelaide.

BOSSAERTS, P., P. GHIRARDATO, S. GUARNASCHELLI, AND W. R. ZAME (2010): "Ambiguity in asset markets: Theory and experiment," *Review of Financial Studies*, 23, 1325–1359.

CHAMBERS, C. P. AND F. ECHENIQUE (2014): "On the consistency of data with bargaining theories," *Theoretical Economics*, 9, 137–162.

CHIAPPORI, P.-A. AND J.-C. ROCHET (1987): "Revealed preferences and differentiable demand," *Econometrica*, 687–691.

EPSTEIN, L. G. (2000): "Are probabilities used in markets?" *Journal of Economic Theory*, 91, 86–90.

FUDENBERG, D., R. IIJIMA, AND T. STRZALECKI (2014): "Stochastic Choice and Revealed Perturbed Utility," Mimeo, Harvard University.

GILBOA, I. AND D. SCHMEIDLER (1989): "Maxmin expected utility with non-unique prior," *Journal of mathematical economics*, 18, 141–153.

GREEN, R. C. AND S. SRIVASTAVA (1986): "Expected Utility Maximization and Demand Behavior," *Journal of Economic Theory*, 38, 313–323.

HEY, J. AND N. PACE (2014): "The explanatory and predictive power of non two-stage-probability theories of decision making under ambiguity," *Journal of Risk and Uncertainty*, 49, 1–29.

HOUTHAKKER, H. (1950): "Revealed preference and the utility function," *Economica*, 17, 159–174.

KUBLER, F., L. SELDEN, AND X. WEI (2014): "Asset Demand Based Tests of Expected Utility Maximization," *American Economic Review*, 104, 3459–80.

MACHINA, M. J. AND D. SCHMEIDLER (1992): "A More Robust Definition of Subjective Probability," *Econometrica*, 745–780.

PAPADIMITRIOU, C. H. AND K. STEIGLITZ (1998): *Combinatorial optimization: algorithms and complexity*, Courier Dover Publications.

POLISSON, M. AND J. QUAH (2013): "Revealed preference tests under risk and uncertainty," Working Paper 13/24 University of Leicester.

ROCKAFELLAR, R. T. (1997): *Convex analysis*, Princeton university press.

SAMUELSON, P. (1938): "A note on the pure theory of consumer's behaviour," *Economica*, 5, 61–71.

SAVAGE, L. J. (1954): *The Foundations of Statistics*, New York: Wiley.

STOER, J. AND C. WITZGALL (1970): *Convexity and optimization in finite dimensions*, Springer-Verlag Berlin.

VARIAN, H. R. (1983): "Nonparametric Tests of Models of Investor Behavior," *Journal of Financial and Quantitative Analysis*, 18, 269–278.

——— (1988): "Estimating Risk Aversion from Arrow-Debreu Portfolio Choice," *Econometrica*, 56, 973–979.