

Decentralized College Admissions

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Abstract

We study decentralized colleges admissions in the face of uncertain student preferences. Enrollment uncertainty causes colleges to strategically target their admissions to students overlooked by others. Highly ranked students may receive fewer admissions or suffer from a higher chance of coming up empty—“falling through the cracks”—than those ranked below. When students’ attributes are multidimensional, colleges avoid head-on competition by placing excessive weights on school-specific measures such as essays. Restricting the number of applications or wait-listing alleviates enrollment uncertainty, but the outcomes are inefficient and unfair. A centralized matching via Gale and Shapley’s deferred acceptance algorithm attains efficiency and fairness, but some college may be worse off relative to decentralized matching.

1 Introduction

Centralized matching has gained prominence in economic theory and practice, spurred by successful applications such as medical residency matching and public school choice. Yet, many matching markets remain decentralized notably with college and graduate school admissions. It is often suggested that these markets do not operate well and will therefore benefit from improved coordination or complete centralization, but it is not well understood

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why they remain decentralized and what welfare benefits would be gained by improved coordination. At least part of the problem is the lack of an analytical grasp of decentralized matching markets. A seminal work by [Roth and Xing \(1997\)](#) attributes the problems of such markets to “congestion”—participants are not allowed to make enough offers and acceptances to clear the markets. But, its analytical content, namely the equilibrium and welfare implications of congestion, remains poorly understood. Indeed, we have yet to develop a workhorse model of decentralized matching that could serve as a useful benchmark for comparison with a centralized system.¹

The current paper develops an analytical framework for understanding decentralized matching markets in the context of college admissions. College admissions in countries such as Japan, Korea and the US are organized similarly to decentralized labor markets: Colleges make exploding and binding admission offers to applicants, and the admitted students accept or reject the offers, often within a short window of time. This process provides little opportunity for colleges to learn students’ preferences and adjust their admissions decisions accordingly. Consequently, they often end up enrolling too many or too few students relative to their capacities. For instance, 1,415 freshmen accepted Yale’s invitation to join its incoming class in 1995-96, although the university had aimed for a class of 1,335. At the same year, Princeton also reported 1,100 entering students, the largest in its history. The college had to set up mobile homes in fields and build new dorms to accommodate the students ([Avery, Fairbanks and Zeckhauser, 2003](#)). Undoubtedly, these mistakes are costly for the colleges.²

Controlling yield is particularly challenging for colleges in Korea, since students apply for a department, instead of a college, and each department faces a relatively rigid and low quota. The challenge is not much easier for US counterparts. Applications to US colleges have grown dramatically in recent years, largely due to the introduction of the Common Application—the online platform which allows students to apply to many colleges at negligible costs.³ As a consequence, the average yield rate of four-year colleges in the US has declined significantly over the past decade, from 49% in 2001 to 38% in 2011 (National Association for College Admission Counseling, 2012, [NACAC](#) hereafter).⁴ The declining

¹There are some important exceptions to this characterization. We shall discuss them in detail in Related Literature (see [Section 1.1](#)).

²The cost may also take the form of an explicit penalty imposed on the admitting unit (e.g., department) by the government (as in Korea) or by the college (as in Australia).

³The average number of applications per institution increased 60% between 2002 and 2011; and 79% of Fall 2011 freshmen applied to three or more colleges and 29% of them submitted seven or more applications ([NACAC, 2012](#)).

⁴NACAC reports the *State of College Admission* for each year since 2002, based on data sets including annual counseling trends survey for 2002-2011 and annual admission trends survey for 2002-2011. In the 2011 surveys, 1,928 out of 10,000 secondary schools contacted participated in the former survey, whereas 369 four-year postsecondary institutions out of 1,346 contacted participated in the latter. See [NACAC \(2012\)](#)

rates imply increased uncertainty for colleges.⁵

Importantly, the enrollment uncertainty a college faces is endogenous, depending on the admissions decisions made by other colleges. A student admitted by a college poses a greater uncertainty for its enrollment when she is also admitted by other colleges, since her enrollment depends on the student's (unknown) preference. This interdependent nature of uncertainty introduces a novel strategic interaction among colleges in their admissions decisions. In this paper, we develop a model that captures this feature.

In our baseline model, there are two colleges, each with limited capacity, and a unit mass of students each with a "score" (e.g., high school grade point average (GPA) or Scholastic Aptitude Test (SAT) scores). Students apply to colleges at no cost. Colleges rank students according to their scores, but they do not know students' preferences toward them. This uncertainty takes an aggregate form: The mass of students preferring one college over the other varies with unknown states of nature (in particular, unknown to colleges). Each college incurs a (sufficiently large) constant cost for each enrollment exceeding its capacity. Our baseline model involves a simple time line: Initially, students simultaneously apply to colleges. Each college observes only the scores of those students who apply to it. Then, the two colleges simultaneously offer admissions to sets of students. Finally, students decide on which offer (if any) to accept.

Given that application is costless, students have a (weak) dominant strategy of applying to both colleges. Hence, the focus of the analysis is the colleges' admissions decisions. Our main finding is that the colleges engage in "strategic targeting": In equilibrium, each college avoids good students who are sought after by the other college and admit less attractive students overlooked by the other. The reason is that, for a college, a student receiving a competing offer from the other college presents greater enrollment uncertainty, which adds more capacity cost to the college, than a student who does not receive a competing offer. Thus the college favors the latter type student, all else equal. This incentive may also cause colleges to randomize their admissions offers among students with different scores. Strategic targeting results in an equilibrium that does not have a simple cutoff structure suggested in the existing research, and the effects on the students are non-monotonic with respect to their scores. Students with higher scores may receive admissions fewer colleges or suffer from a higher chance of not being admitted by any (i.e., "falling through the cracks") than students with lower scores. The outcome is unfair in the sense of entailing justified envy—that is, a high-score student may envy a lower-score student—and it fails efficiency in various senses

for detail.

⁵The increased uncertainty also means an increased difficulty in meeting an admission target. Jennifer Delahunty, dean of admissions and financial aid at Kenyon College in Ohio, stated: "Trying to hit those numbers is like trying to hit hot tub when you are skydiving 30,000 feet. I'm going to go to church every day in April" ("In Shifting Era of Admissions, Colleges Sweat," by Kate Zernike, *New York Times*, March 8, 2009).

(to be defined).

We next study the admission problem when students are of multidimensional types. Some measures, such as students' academic performances or system-wide test like SAT, are highly correlated across colleges in their admissions standards, but others measures, such as students' college-specific essays and tests or their extracurricular activities, are less correlated. According to 2011 NACAC admissions trend survey, about 62% of the colleges surveyed place moderate to considerable importance in essays and about 48% of them place moderate to considerable emphasis on extracurricular activities in admissions decisions (NACAC, 2012). Consistent with this, we find that colleges bias their evaluation toward less correlated measures by placing excessive weights on these dimensions in order to avoid head-on competition.

We also study two common ways for colleges to reduce enrollment uncertainty. One is to restrict the number of applications each student can submit. Such restrictions are widely observed; for instance, students in the UK cannot apply to both Cambridge and Oxford, students in Japan can apply to at most two public universities, and students in Korea face a similar restriction. Restrictions are also placed on students applying for early decision or single-choice early action at US colleges. We show that restriction on applications, while reducing uncertainty for colleges, causes students to be strategic in their application decisions, and this entails justified envy and inefficiencies.

Another common method for colleges to cope with enrollment uncertainty is to admit students in sequence, or "wait-listing": Some students are admitted outright and others are placed in a wait list in each of multiple rounds, and students on the wait list are later admitted when some offers are rejected and seats open up. This method is also observed in several countries, including France, Korea and the US. Wait-listing allows colleges to adjust their admission offers based on additional information they learn from the students' acceptance decisions, thereby reducing the uncertainty in enrollment. We show, however, that colleges will still engage in strategic targeting by admitting "non-contiguous" set of students in scores, and the welfare and fairness problems still remain.

Finally, we consider centralized matching via Gale and Shapley's Deferred Acceptance algorithm (DA in short). This eliminates colleges' yield control problem and justified envy completely and attains efficiency. However, it is possible for some colleges to be worse off relative to the decentralized matching. This may explain a possible lack of consensus toward centralization and may underscore why college admissions remain decentralized in many countries.

The paper is organized as follows. [Section 1.1](#) discusses the related literature. After illustrating the main insight via an example in [Section 1.2](#), we introduce our model in [Section 2](#). [Section 3](#) characterizes equilibria and their existence. [Section 3.1](#) discusses welfare and fair-

ness implications of equilibria. [Section 4](#) studies the model of multidimensional admissions criteria. Restricted application and wait-listing are studied in [Section 5](#) and [Section 6](#), respectively, followed by centralized matching via DA in [Section 7](#). [Section 8](#) offers further implications of our findings. Proofs are provided in [Appendix A](#) unless stated otherwise. [Appendix B](#) extends the baseline model to allow for more than two colleges and shows that our analysis in the two-college model carries over to this environment.

1.1 Related Literature

Several papers in the matching literature have studied decentralized matching markets. [Roth and Xing \(1997\)](#) focus on the entry-level market for clinical psychologists in which firms make offers to workers sequentially and workers can either accept, reject or hold the offers within a day. They find that, mainly based on simulations, such a decentralized (but coordinated) market exhibits *congestion*, and the resulting outcome is unstable. [Neiderle and Yariv \(2009\)](#) study a decentralized one-to-one matching market in which firms make offers sequentially through multiple periods. They provide sufficient conditions for such decentralized markets to generate stable outcomes in equilibrium in the presence of market friction (namely, time discounting). [Coles, Kushnir and Neiderle \(2013\)](#) show that introducing a signaling device in a decentralized matching market alleviates congestion and increases welfare in terms of the number of matches and the workers' payoffs.

Similar to these papers, we are concerned with the congestion arising from decentralized matching. Unlike [Roth and Xing \(1997\)](#), however, we study participants' strategic interaction in their admissions game and the welfare and fairness properties of its equilibria. Our framework identifies strategic targeting as an important new implication of decentralized matching. Moreover, the explicit analysis of equilibria permits a clear comparison with the outcome that would arise from a centralized matching.

The college admissions problem has recently received attention in the economics literature. [Chade and Smith \(2006\)](#) study students' application decision as a portfolio choice problem. In [Chade, Lewis and Smith \(forthcoming\)](#), students with heterogeneous abilities make application decisions subject to application costs, and colleges set admission standards based on noisy signals of students' abilities. [Avery and Levin \(2010\)](#) and [Lee \(2009\)](#) study early admissions. Unlike our model, these models have no aggregate uncertainty with respect to students' preferences, so colleges face no enrollment uncertainty in these models. Hence, colleges do not employ strategic targeting; they instead use cutoff strategies.

Some aspects of our equilibrium are related to political lobbying behavior studied by [Lizzeri and Persico \(2001, 2005\)](#). Just as colleges target students in our model, politicians in these models target voters in distributing their favors. Voters are homogenous in their model, whereas students in our model have heterogeneous abilities and preferences. This

heterogeneity leads colleges to choose differing admission rates for different student types. More important, aggregate uncertainty plays a unique role in shaping competition in our model, whereas the rule of splitting the spoils of office (*winner-take-all* versus *proportional* rule) matters in their model.

Our model also shares some similarities with directed search models, such as [Montgomery \(1991\)](#) and [Burdett, Shi and Wright \(2001\)](#). In these studies, each firm (seller) posts a wage (price), and each worker (buyer) decides which job to apply for. Firms have a fixed number of job openings and cannot hire more than the capacity, and workers can only apply to one firm.⁶ Workers’ inability to coordinate their job applications causes “search friction.” Similar to the workers in these models, colleges in our model can be seen to engage in “directed searches” on students. The main differences of our model are that prices play no role in matching and that colleges offer admissions to many (in fact, a continuum of) students with heterogeneous qualities subject to aggregate uncertainty. This leads to strategic targeting, a novel feature of our model.

1.2 Illustrative Example

Before proceeding, we illustrate the idea of strategic targeting via a simple example. Suppose there are only two students, 1 and 2, applying to colleges *A* and *B*. Each college has one seat to fill and faces a prohibitively high cost of having two students. Student *i* has score v_i , $i = 1, 2$, where $0 < v_2 < v_1 < 2v_2$. Each student has an equal probability of preferring either school, which is private information (unknown to the other student and to the colleges). Each college values having student *i* at v_i . The applications are free of cost, and the timing is the same as that explained above.

Given the large cost of over-enrollment, each college admits only one student. Their payoffs are described as follow.

<i>A</i> ’s strategy \ <i>B</i> ’s strategy	Admit 1	Admit 2
Admit 1	$\frac{1}{2}v_1, \frac{1}{2}v_1$	v_1, v_2
Admit 2	v_2, v_1	$\frac{1}{2}v_2, \frac{1}{2}v_2$

This game has a battle of the sexes’ structure (with asymmetric payoffs), so there are two different types of equilibria. First, there are two asymmetric pure-strategy equilibria in which one college admits student 1 and the other admits student 2. There is also a mixed-strategy equilibrium in which each college admits 1 with probability $\gamma := \frac{2v_1 - v_2}{v_1 + v_2} > 1/2$ and admits 2 with probability $1 - \gamma$, where γ is chosen such that the other college is indifferent in whom it admits. Both types of equilibria show the pattern of strategic targeting. In the pure-strategy

⁶See also [Albrecht, Gautier and Vroman \(2006\)](#), [Julien, Kennes and King \(2000\)](#), [Kircher \(2009\)](#), and [Galenianos and Kircher \(2009\)](#) who extend the model to accommodate multiple applications.

equilibria, colleges manage to avoid competition by targeting different students. The mixed-strategy equilibrium also arises from their targeting motive, namely their attempt to avoid students sought after by the other, although it does not result in perfect coordination.

This example, while extremely simple, reveals problems with decentralized matching in terms of welfare and fairness. First, the student with high score (student 1) may go to a less preferred school (in both types of equilibria) even though both colleges prefer that student; that is, justified envy arises. Second, it could be the case that student 1 prefers A and student 2 prefers B , but the former is assigned B and the latter is assigned A , showing that the equilibrium outcome is inefficient among students. Lastly, the mixed-strategy equilibrium is Pareto inefficient because both colleges may admit the same student, and it would be Pareto improving for the unmatched college to match with the other student. The example also shows that not all parties may benefit if the matching were organized centrally. For instance, if DA were used to match students with the colleges, enrollment uncertainty would disappear, and the assignment would be fair and efficient. Yet, a college would be worse off relative to the decentralized matching equilibrium in which it always gets student 1 regardless of her preference.

2 Model

There is a unit mass of students with score v distributed from $\mathcal{V} \equiv [0, 1]$ according to an absolutely continuous distribution $G(\cdot)$. There are two colleges, A and B , each with capacity $\kappa < \frac{1}{2}$. Each college values a student with score v at v and faces a cost $\lambda \geq 1$ for each additional enrollment exceeding the quota.⁷ Each student has a preference over the two colleges, which is private information. A state of nature s is drawn from $[0, 1]$ according to the uniform distribution. In state s , a fraction $\mu(s) \in [0, 1]$ of students prefers A to B (independent on v), where $\mu(\cdot)$ is strictly increasing and continuous in s .⁸

The timing of the game is as follows. First, Nature draws state s (i.e., aggregate uncertainty is realized). Next, all students simultaneously apply to colleges. Each college only observes the scores of those students who apply to it, and based on the scores, colleges simultaneously decide which applicants to admit. Lastly, students accept or reject their offers.

We assume that colleges have no application costs, which makes it weakly dominant for students to apply to both colleges.⁹ Throughout, we focus on a perfect Bayesian equilibrium

⁷The common preference assumption is later relaxed when we allow colleges to consider students' attributes that are imperfectly correlated. See [Section 4](#).

⁸There is no loss of generality to assume the uniform distribution, because for a distribution $F(\cdot)$ of s , we can simply relabel s , and the popularity of a college over the other is captured by $\mu(\cdot)$.

⁹The strategy of applying to both colleges can be made a strictly dominant strategy if students have some

in which students play the weak dominant strategy.

Colleges distribute admissions based on students' scores. Let $\sigma_i : \mathcal{V} \rightarrow [0, 1]$, $i = A, B$, denote college i 's admission strategy specifying the fraction of students with score v it admits. For given $\sigma_i(\cdot)$, let $\mathcal{V}_i := \{v \in [0, 1] \mid \sigma_i(v) > 0\}$ be the types of students college i admits. Let $\mathcal{V}_{AB} := \mathcal{V}_A \cap \mathcal{V}_B$ denote the set of students who receive admissions from both colleges. We call an equilibrium **competitive** if \mathcal{V}_{AB} has a positive measure, and **non-competitive** otherwise.

Consider the students with score v . A fraction $\sigma_i(v)(1 - \sigma_j(v))$ of them, where $i, j = A, B$ and $i \neq j$, is admitted only by college i , and a fraction $\sigma_i(v)\sigma_j(v)$ of them is admitted by both colleges. The former group all accepts i 's admissions, but only a fraction $\mu_i(s)$ of the latter group accepts i 's offer, where $\mu_i(s) = \mu(s)$ if $i = A$ and $\mu_i(s) = 1 - \mu(s)$ if $i = B$. Thus, the mass of students who attend college i in state s , given strategies $\sigma_i(\cdot)$ and $\sigma_j(\cdot)$, is

$$m_i(s) := \int_0^1 \sigma_i(v) (1 - \sigma_j(v) + \mu_i(s)\sigma_j(v)) dG(v). \quad (2.1)$$

Each college enjoys the scores of all enrolled students as its gross payoff and incurs cost λ for each incremental enrollment beyond its capacity. Thus, college i 's ex ante payoff is

$$\pi_i := \mathbb{E} \left[\int_0^1 v \sigma_i(v) (1 - \sigma_j(v) + \mu_i(s)\sigma_j(v)) dG(v) - \lambda \max\{m_i(s) - \kappa, 0\} \right].$$

One immediate observation is that each college's payoff is concave in its own admission strategy; that is, $\pi_i(\eta\sigma_i + (1 - \eta)\sigma'_i) \geq \eta\pi_i(\sigma_i) + (1 - \eta)\pi_i(\sigma'_i)$ for any feasible strategies σ_i and σ'_i and for any $\eta \in [0, 1]$. Therefore, randomizing across distinct σ_i 's is unprofitable for college i . For this reason, any equilibrium is characterized by a pair (σ_A, σ_B) . However, this does not mean that the equilibrium is in pure-strategies; the values of σ_A and σ_B may be strictly interior, in which case the admission strategies would involve randomization.

3 Characterization of Equilibria

We analyze colleges' admission decisions in this section. To this end, we fix any equilibrium (σ_A, σ_B) and explore the properties it must satisfy. Later, we shall prove existence of the equilibria. We begin with the following observations.

Lemma 1. *In any equilibrium (σ_A, σ_B) , the following results hold.*

- (i) $m_A(0) \leq \kappa \leq m_A(1)$ and $m_B(1) \leq \kappa \leq m_B(0)$.
- (ii) $\mathcal{V}_A \cup \mathcal{V}_B$ is an interval with $\sup(\mathcal{V}_A \cup \mathcal{V}_B) = 1$ and $\inf(\mathcal{V}_A \cup \mathcal{V}_B) > 0$.

uncertainty about their scores, which is realistic in case that the scores are either not publicly observable or depend on multiple dimensions of attributes the weighting of which is unknown to the students.

(iii) If the equilibrium is competitive (i.e., \mathcal{V}_{AB} has a positive measure), then there exists a unique $(\hat{s}_A, \hat{s}_B) \in (0, 1)^2$ such that $m_A(\hat{s}_A) = \kappa$ and $m_B(\hat{s}_B) = \kappa$.

(iv) If the equilibrium is non-competitive (i.e., \mathcal{V}_{AB} has zero measure), then $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$. Further, almost every student with $v \geq G^{-1}(1 - 2\kappa)$ receives an admission offer from exactly one college.

Proof. See [Appendix A.1](#). ■

Part (i) of the lemma states that in equilibrium, colleges cannot have strict over-enrollment for all states or strict under-enrollment for all states. If a college over-enrolled in all states, then it would profitably deviate by rejecting some students with $v < 1$ (since $\lambda \geq 1$), and if it under-enrolled in all states, it would profitably deviate by accepting more students. Part (ii) states that the support of types who receive an admission from at least one college must form an interval $[\underline{v}, 1]$, for some $\underline{v} > 0$. Part (iii) states that in a competitive equilibrium, each college will suffer from under-enrollment in some states and over-enrollment in other states. This is intuitive since given (aggregately) uncertain preferences of students, the presence of students receiving admissions from both colleges creates non-trivial enrollment uncertainty. Each college then manages the uncertainty by optimally trading off the cost of over-enrollment in high demand states with the loss from under-enrollment in low demand states. Part (iv) states that in a non-competitive equilibrium, colleges can avoid over- and under-enrollment problems, and almost all top 2κ students receive admissions from only one college. This is also intuitive since the colleges in this case face no enrollment uncertainty; so they will fill their capacities exactly in all states with the top 2κ students.

Fix any equilibrium (σ_A, σ_B) . For ease of notation, let $S_A := \{s \mid s \geq \hat{s}_A\}$ and $S_B := \{s \mid s \leq \hat{s}_B\}$ denote the sets of states that the colleges A and B have over-enrolled, respectively. Note that in a non-competitive equilibrium, $\text{Prob}(s \in S_i) = 0$ for all $i = A, B$, and in a competitive equilibrium, $\text{Prob}(s \in S_A) = 1 - \hat{s}_A$ and $\text{Prob}(s \in S_B) = \hat{s}_B$. It is convenient to rewrite college i 's payoff at the equilibrium as follows:

$$\begin{aligned} \pi_i &= \int_0^1 v \sigma_i(v) (1 - \sigma_j(v) + \mathbb{E}[\mu_i(s)] \sigma_j(v)) dG(v) - \lambda \mathbb{E}[m_i(s) - \kappa \mid s \in S_i] \text{Prob}(s \in S_i) \\ &= \int_0^1 \sigma_i(v) H_i(v, \sigma_j(v)) dG(v) + \lambda \kappa \text{Prob}(s \in S_i) \end{aligned}$$

where

$$H_i(v, \sigma_j(v)) := v(1 - \sigma_j(v) + \mathbb{E}[\mu_i(s)] \sigma_j(v)) - \lambda \text{Prob}(s \in S_i) (1 - \sigma_j(v) + \mathbb{E}[\mu_i(s) \mid s \in S_i] \sigma_j(v))$$

is college i 's marginal payoff from admitting a student with score v for given \hat{s}_i and $\sigma_j(\cdot)$ in equilibrium. (We suppress the dependence of σ on \hat{s}_i unless its role is important.) H_i captures

college i 's local incentive—namely, its benefit from admitting type- v students, holding fixed its opponent's decision and *its own decisions for all other students* at $\sigma_i(\cdot)$.

Lemma 2. *A strategy profile (σ_A, σ_B) is an equilibrium if and only if (i) $H_i(v, \sigma_j(v)) > 0$ implies $\sigma_i(v) = 1$, (ii) $H_i(v, \sigma_j(v)) < 0$ implies $\sigma_i(v) = 0$, (iii) $H_i(v, \sigma_j(v)) = 0$ implies $\sigma_i(v) \in [0, 1]$, where $i, j = A, B$ and $j \neq i$.*

Proof. See [Appendix A.2](#). ■

Lemma 2 characterizes equilibrium based on the sign of H_i : College i admits student v if $H_i(v, \sigma_j(v)) > 0$ and rejects her if $H_i(v, \sigma_j(v)) < 0$. Since this condition only ensures “local” incentive compatibility, there is a concern that a college may profitably deviate “globally” by changing its admission decisions on a mass of students. The lemma assures that no such global deviation is profitable. Hence, a strategy profile satisfying local incentives is indeed an equilibrium.

In what follows, we shall focus on competitive equilibria. Competitive equilibria always exist (see [Theorem 2](#)), whereas non-competitive equilibria can be ruled out if either λ is not too large or κ is not too small (see [Appendix A.3](#)). Even if a noncompetitive equilibrium exists, the characterization provided in [Lemma 1](#)-(iv) is sufficient for our welfare and fairness results, as will be seen later.

Now, for a given competitive equilibrium (σ_A, σ_B) , inspection of colleges' marginal payoffs together with [Lemma 2](#) reveals their admission decisions in more detail. Rewrite

$$H_i(v, \sigma_j(v)) = (1 - \sigma_j(v))(v - \underline{v}_i) + \sigma_j(v) \mathbb{E}[\mu_i(s)](v - \bar{v}_i),$$

where $\underline{v}_i := \lambda \text{Prob}(s \in S_i)$ is college i 's capacity cost of admitting a student who does not receive an offer from college j , and $\bar{v}_i := \lambda \text{Prob}(s \in S_i) \frac{\mathbb{E}[\mu_i(s)|s \in S_i]}{\mathbb{E}[\mu_i(s)]}$ is its cost of admitting a student who receives an offer from j . Recall that a college incurs capacity cost only when it over-enrolls. If the student does not receive an offer from college j , then she accepts i 's admission for sure. Hence, over-enrollment occurs with probability $\text{Prob}(s \in S_i)$, explaining the marginal cost \underline{v}_i . If the student receives an offer from college j , then she accepts i 's offer only when she prefers i to j . Hence, conditional on acceptance, college i over-enrolls with probability $\text{Prob}(s \in S_i) \frac{\mathbb{E}[\mu_i(s)|s \in S_i]}{\mathbb{E}[\mu_i(s)]}$, which explains the marginal cost \bar{v}_i .

Observe that $\bar{v}_i > \underline{v}_i$. This is because while students *without* an offer from the opponent college accept a college's offer independently of the state, students *with* a competing offer are more likely accept a college's offer when it is more popular, i.e., when it over-enrolls. This explains why admitting the latter students is more costly. Hence, a college finds it optimal to favor those who do not have a competing offer over those who have, all else equal.

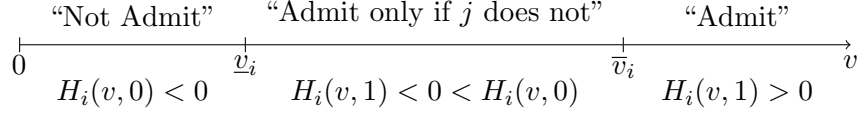


Figure 3.1: College i 's Admission Decision

Remark 1. (Preference for higher yield) It is worth emphasizing that a college disfavors a student with a competing offer, not because of her uncertain preference per se. A college can hedge idiosyncratic preferences uncertainty fully in a large market. The reason why it disfavors such a student is because of the “incidence” of her acceptance: Given the aggregate uncertainty, she is more likely to accept the college’s offer when it over-enrolls, compared with a student without competing offer. This intuition also suggests that a college would favor students with fewer offers, even among those receiving multiple offers, as is seen in [Appendix B](#): Even though all such students exhibit uncertain acceptance decisions, those with fewer competing offers are relatively less likely to accept the college’s offer when it over-enrolls, compared with those with more competing offers. For this reason, colleges in our model exhibit (endogenous) preference for high yield, as they target students with fewer competing offers.

Lemma 3. *In any competitive equilibrium, $H_i(v, x)$, $i = A, B$, is strictly increasing in v for each x . Moreover, for each v , $H_i(v, x)$ satisfies the single crossing property: If $H_i(v, x) \leq 0$, then $H_i(v, x') < 0$ for any $x' > x$.*

Proof. See [Appendix A.4](#). ■

[Lemma 3](#) implies that $H_i(v, \sigma_j(v))$ partitions the students’ types into three intervals, as depicted in [Figure 3.1](#). Since $H_i(v, 1) > 0$ for $v > \bar{v}_i$ and $H_i(v, 0) < 0$ for $v < \underline{v}_i$ (recall H_i is strictly increasing in v), college i admits all students with $v > \bar{v}_i$ even if college j admits all of them and rejects all students with $v < \underline{v}_i$ even if college j rejects all of those students.

For the students with $v \in (\underline{v}_i, \bar{v}_i)$, we have $H_i(v, 1) < 0 < H_i(v, 0)$. This means that college i ’s incentive for admitting these students depends on college j ’s admission decisions toward them. The single crossing property established in [Lemma 3](#) implies that for each v , there exists $\hat{\sigma}_j(v) \in (0, 1)$ such that $H_i(v, x) > 0$ if $x < \hat{\sigma}_j(v)$, $H_i(v, x) < 0$ if $x > \hat{\sigma}_j(v)$, and $H_i(v, \hat{\sigma}_j(v)) = 0$ if $x = \hat{\sigma}_j(v)$. Hence, college i admits (rejects) all students with v if college j admits less (greater) than fraction $\hat{\sigma}_j(v)$ of them and admits any fraction of those students if college j admits exactly fraction $\hat{\sigma}_j(v) \in (0, 1)$ of them. In particular, college i admits all of them if college j does not admit any of them, but does not admit them if college j admits all of them.

Combining the two colleges’ admission decisions leads to the following characterization of equilibria.

Theorem 1. *In any competitive equilibrium, there exist $\underline{v}_i < \bar{v}_i$, $i = A, B$, such that college i admits students with $v > \bar{v}_i$ and $v \in [\underline{v}_i, \underline{v}_j]$ and rejects students with $v < \underline{v}_i$ and $v \in [\bar{v}_j, \bar{v}_i]$, where $j \neq i$. At least one college admits a positive fraction of students with $v \in [\max\{\underline{v}_A, \underline{v}_B\}, \min\{\bar{v}_A, \bar{v}_B\}]$.*

Theorem 1 describes the structure of competitive equilibrium. Note that the characterization does not pin down colleges' admission decisions on students with scores $v \in [\max\{\underline{v}_A, \underline{v}_B\}, \min\{\bar{v}_A, \bar{v}_B\}]$. One can easily construct pure strategy equilibria in which the colleges coordinate to avoid competition.¹⁰ Such a precise coordination seems difficult for the colleges to achieve in practice. For this reason, we focus on **maximally mixed equilibrium** (MME, in short) in which both colleges play mixed strategies for students with score v such that $H_i(v, 1) < 0 < H_i(v, 0)$ for $i = A, B$. The following example shows a MME in which college A is more selective than college B with respect to both low and high cutoffs.

Example 1. Let $v \sim U[0, 1]$, $\lambda = 20$, $\kappa = 0.32$ and $\mu(s) = \frac{1}{2}s + \frac{2}{5}$. Then, there is a competitive equilibrium with

\hat{s}_A	\hat{s}_B	\underline{v}_B	\underline{v}_A	\bar{v}_B	\bar{v}_A
0.97	0.02	0.47	0.59	0.79	0.82

Theorem 1 and the requirement of MME partition the students' types into five regions, as depicted **Figure 3.2** according to **Example 1**.

- **Head-on competition for the top:** These are the students with v such that $H_i(v, 1) > 0$, $i = A, B$. Both colleges admit them since their scores are high enough to justify the enrollment uncertainty arising from head-on competition.
- **The shunning of the second best:** These are the students with v such that $H_i(v, 1) < 0 < H_j(v, 1)$, $i \neq j$. College i finds the students admission-worthy only if they do not receive admissions from college j , but college j finds them admission-worthy even when they receive admissions from college i . This incentive gives “commitment power” to college j (the less selective college, B in **Figure 3.2**). Consequently, the more “selective” college i (A in **Figure 3.2**) totally “shuns” these students.
- **Targeting competition for the middle:** These students have v such that $H_i(v, 1) < 0 < H_i(v, 0)$ for $i = A, B$. Each college i admits a fraction

$$\sigma_i^o(v) := \frac{v - \underline{v}_j}{v - \underline{v}_j + \mathbb{E}[\mu_j(s)](v - \bar{v}_j)}$$

¹⁰That is, some students in $[\max\{\underline{v}_A, \underline{v}_B\}, \min\{\bar{v}_A, \bar{v}_B\}]$ are admitted only by A and the others are admitted only by B .

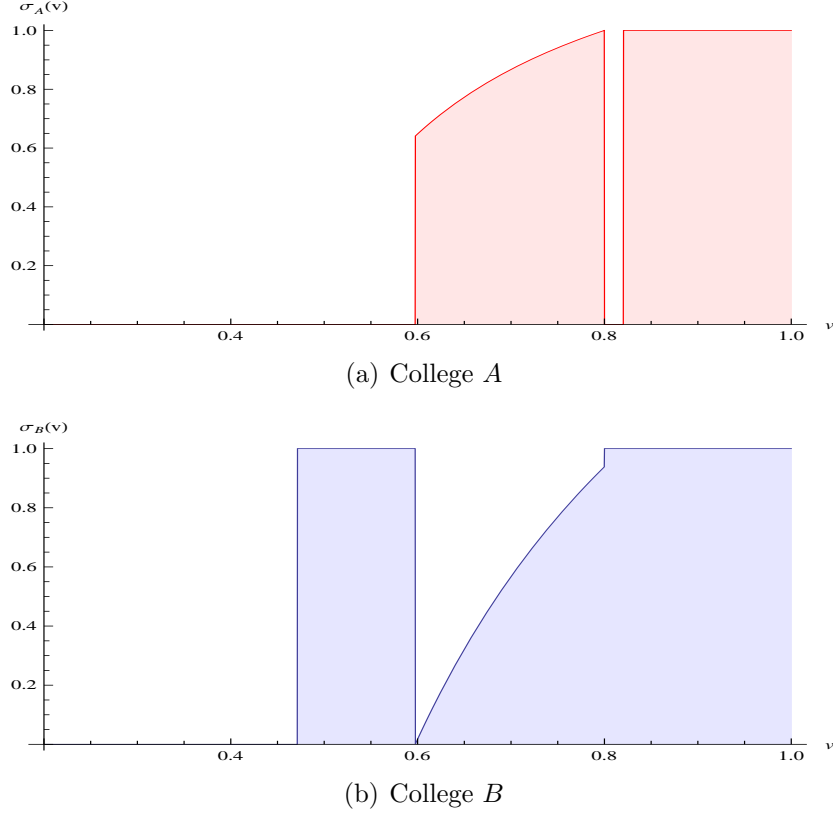


Figure 3.2: Mixed-Strategy Equilibrium

of students with score v , where $\sigma_i^o(v)$ satisfies $H_j(v, \sigma_i^o(v)) = 0$ for $j \neq i$. Given this behavior, college j is indifferent, so it is a best response to admit a fraction $\sigma_j^o(v)$ of students with v , where $\sigma_j^o(v)$ satisfies $H_i(v, \sigma_j^o(v)) = 0$. Observe that $\sigma_i^o(\cdot)$ is increasing.¹¹ This is because higher score students are more valuable, all else equal, so admitting a higher fraction of those students is necessary to keep the opponent college indifferent.

- **Pursuit of the safe:** These students have v such that $H_i(v, 0) < 0 < H_j(v, 0)$, $i \neq j$. Selective college i have no incentive to admit them regardless of what college j does. Hence, these students are “safe” from college j ’s standpoint, and this safety makes them valuable to college j .
- **Rejection of the bottom:** These students have v such that $H_i(v, 0) < 0$ for $i = A, B$, so both colleges reject them.

The equilibrium exhibits a striking sense of “non-monotonicity” in the way students with

¹¹A few other features can be noticed. The strategies involve discrete jumps— $\sigma_A^o(\underline{v}_A) > 0$ and $\sigma_B^o(\bar{v}_B) < 1$ in the figure. The former follows from the fact that $\underline{v}_A > \underline{v}_B$ which implies $H_B(\underline{v}_A, 0) > 0$, and the latter follows from $\bar{v}_A > \bar{v}_B$, which implies $H_A(\bar{v}_B, 1) < 0$.

different scores are treated by the colleges, as opposed to the cutoff strategy equilibrium found by the existing literature (see [Chade, Lewis and Smith, forthcoming](#)). Each college distributes admissions bimodally, rejecting more “middling” students than those above or below. In [Figure 3.2](#), the selective college A rejects the students with $v \in (0.79, 0.82)$; their “curse” stems from being the second-best (thus being highly sought after by the less selective college). Likewise, college B admits the students with $v \in (0, 47, 0.59)$ even though admitting very few students with higher scores, because they are “safe” (i.e., not being sought after by A). These strategies produce non-monotonicity of welfare enjoyed by students with different scores. To be precise, a student enjoys the utility of u from the most preferred college and u' from the less preferred, and zero from non-assignment, where $0 < u' < u$. Then, the following characterization holds:

Corollary 1. *Consider the canonical MME with $\underline{v}_A \neq \underline{v}_B$ and $\bar{v}_A \neq \bar{v}_B$ and $\max\{\underline{v}_A, \underline{v}_B\} < \min\{\bar{v}_A, \bar{v}_B\}$.*

- (i) **“Curse of the Second-Best”:** *The fraction of students receiving admissions from both colleges falls discontinuously as their score v rises at $v = \min\{\bar{v}_A, \bar{v}_B\}$. There is a positive measure of students whose expected utility falls discontinuously at that score.*
- (ii) **“Falling through the Cracks”:** *The fraction of students receiving at least one admission falls discontinuously as their score v rises at $v = \max\{\underline{v}_A, \underline{v}_B\}$. The students’ expected utility falls discontinuously at that score, if u' is sufficiently close to u .*

Proof. Let $\check{v} := \max\{\underline{v}_A, \underline{v}_B\}$ and $\hat{v} := \min\{\bar{v}_A, \bar{v}_B\}$. Part (i) follows since

$$\lim_{v \uparrow \hat{v}} \sigma_A(v) \sigma_B(v) = \min\{\sigma_A(\hat{v}), \sigma_B(\hat{v})\} > 0 = \lim_{v \downarrow \check{v}} \sigma_A(v) \sigma_B(v),$$

where the first equality follows since $\sigma_A(\hat{v}) = 1$ or $\sigma_B(\hat{v}) = 1$. Thus, the expected utility of students who prefer college i with $\sigma_i(\hat{v}) = 1$ falls discontinuously.

Part (ii) follows since

$$\lim_{v \uparrow \check{v}} (1 - (1 - \sigma_A(v))(1 - \sigma_B(v))) = 1 > \lim_{v \downarrow \check{v}} (1 - (1 - \sigma_A(v))(1 - \sigma_B(v))),$$

where the inequality follows from the facts that $\check{v} < \hat{v}$ and $\sigma_i^o(\check{v}) := \frac{\check{v} - \underline{v}_j}{\check{v} - \underline{v}_j + \mathbb{E}[\mu_j(s)](\bar{v}_j - \check{v})} < 1$ for $i = A, B$. The expected utility ranking between students with v slightly below and slightly above \check{v} follows immediately (regardless of their preferences as long as u' is not too smaller than u). ■

Remark 2. (Non-essential attribute as a randomization device) Each college selects students in the middle range of scores at random for admission. In practice, colleges could

use extraneous or nonessential attributes of students for their randomization device. For instance, a college may select students based on extracurricular activities or non-academic performances it may not genuinely care about.¹²

We now state the existence result.

Theorem 2. *There exists a competitive MME.*¹³

Proof. The proof is presented in [Appendix A.5](#). We sketch it here. The proof constructs equilibrium strategies (σ_A, σ_B) with maximal mixing in terms of threshold states (\hat{s}_A, \hat{s}_B) . Since the latter space is Euclidean, we can simply appeal to the Brouwer's fixed point theorem to establish the existence. To begin, fix any candidate threshold states $\hat{s} = (\hat{s}_A, \hat{s}_B)$ for the two colleges. Next, we construct the colleges' mutual best-responses (σ_A, σ_B) at \hat{s} according to the algorithm described earlier. Formally, we set for $i, j = A, B$ and $i \neq j$,

$$\sigma_i(v; \hat{s}) = \begin{cases} 1 & \text{if } H_i(v, 1; \hat{s}) > 0 \\ 0 & \text{if } H_i(v, 1; \hat{s}) < 0, \quad H_j(v, 1; \hat{s}) > 0 \\ \sigma_i^o(v; \hat{s}) & \text{if } H_i(v, 1; \hat{s}) < 0 < H_i(v, 0; \hat{s}), \quad H_j(v, 1; \hat{s}) < 0 < H_j(v, 0; \hat{s}) \\ 1 & \text{if } H_i(v, 0; \hat{s}) > 0, \quad H_j(v, 0; \hat{s}) < 0 \\ 0 & \text{if } H_i(v, 0; \hat{s}) < 0 \end{cases} \quad (3.1)$$

where $\sigma_i^o(\cdot)$ satisfies $H_j(v, \sigma_i^o(v); \hat{s}) = 0$ for $v \in [\max\{\underline{v}_A, \underline{v}_B\}, \min\{\bar{v}_A, \bar{v}_B\}]$.

Since the threshold states (\hat{s}_A, \hat{s}_B) are arbitrary, there is no guarantee that the constructed strategies reproduce them as the correct thresholds. In fact, they will reproduce another possible threshold states $\tilde{s} = (\tilde{s}_A, \tilde{s}_B)$:

$$\tilde{s}_A = \inf \{s \in [0, 1] \mid m_A(s; \hat{s}) - \kappa > 0\} \quad \text{and} \quad \tilde{s}_B = \sup \{s \in [0, 1] \mid m_B(s; \hat{s}) - \kappa > 0\}, \quad (3.2)$$

where m_A and m_B are derived from the formula (2.1).¹⁴

But this process defines a mapping $T : [0, 1]^2 \rightarrow [0, 1]^2$ such that $T(\hat{s}) = \tilde{s}$. In [Appendix A.5](#), we apply the Brouwer's fixed point theorem to show that T admits a fixed point $\hat{s}^* = (\hat{s}_A^*, \hat{s}_B^*)$ such that $T(\hat{s}^*) = \hat{s}^*$. By [Lemma 2](#), the strategies (σ_A, σ_B) (constructed as above) at this fixed point $\hat{s}^* = (\hat{s}_A^*, \hat{s}_B^*)$ are indeed best responses for both colleges. \blacksquare

¹²Colleges' interests in these aspects are often genuine. In that case, our insight suggests that colleges would over-emphasize them in admission decisions. This will be shown formally in the next section.

¹³Note that existence of an (arbitrary) equilibrium follows from the Fan-Glicksberg theorem, since each college's strategy space is compact and convex, and each college's payoff function is concave in its own strategy. A proof is required here only because the special structure of behavior we require with MME.

¹⁴As usual, these formulae are valid only if the associated sets in (3.2) are nonempty. If they are empty, then threshold values are set equal to one for \tilde{s}_A and zero for \tilde{s}_B .

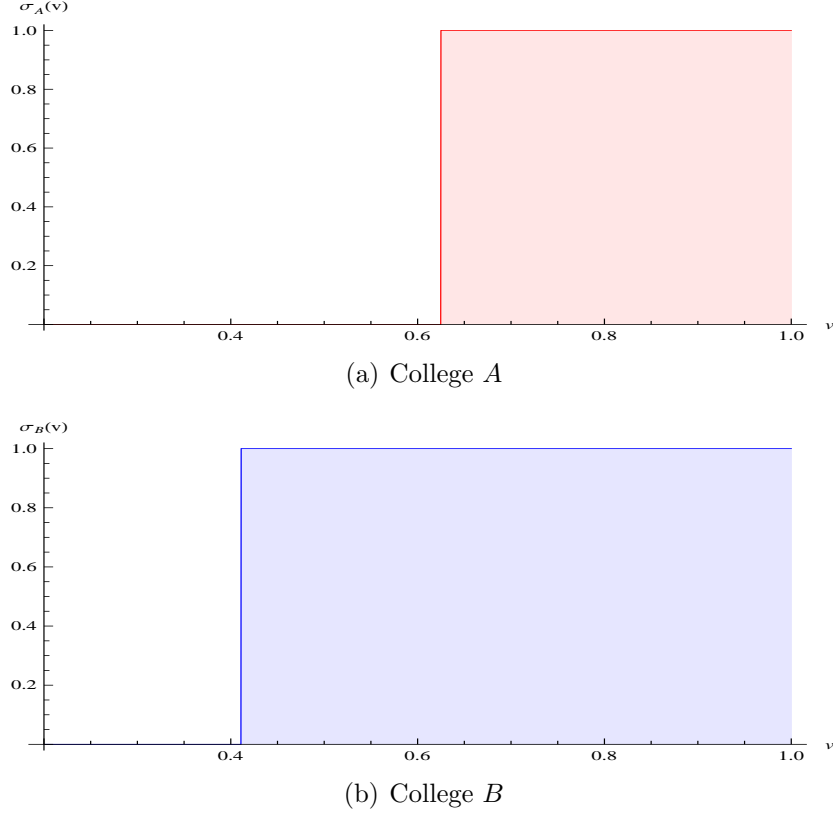


Figure 3.3: Cutoff Equilibrium

It is important to recognize that a randomization by each college arises from its attempt to avoid competition for students within the middle range of scores. In this sense, as long as a competitive equilibrium admits the intermediate region, i.e., if $\check{v} = \max\{\underline{v}_A, \underline{v}_B\} < \hat{v} = \min\{\bar{v}_A, \bar{v}_B\}$, one can say that equilibrium involves strategic targeting. We say a MME exhibits **strategic targeting** if $\check{v} < \hat{v}$.

When do MMEs exhibit strategic targeting? Note that MME does not preclude a competitive equilibrium in which $\hat{v} < \check{v}$. **Example 2** shows such a possibility with $\underline{v}_B < \bar{v}_B < \underline{v}_A < \bar{v}_A$, as depicted in **Figure 3.3**.

Example 2. Let $v \sim U[0, 1]$, $\lambda = 20$, $\kappa = 0.32$ and $\mu(s) = \frac{1}{7}s + \frac{5}{7}$. Then, there is a competitive equilibrium with

\hat{s}_A	\hat{s}_B	\underline{v}_B	\bar{v}_B	\underline{v}_A	\bar{v}_A
0.97	0.02	0.41	0.54	0.57	0.62

As before, college i admits students with $v > \bar{v}_i$ and rejects those with $v < \underline{v}_i$. Observe that college A does not admit any student with $v \in [\underline{v}_A, \bar{v}_A]$, since college B admits them for sure (because $\bar{v}_B < \underline{v}_A$). Despite the fact that colleges have targeting incentives in this

example, the resulting equilibrium is indistinguishable from the cutoff equilibria featured in the existing research.

A natural question is when such an equilibrium can be ruled out. The exact condition for its existence appears difficult to find, but we show that if the two colleges are sufficiently symmetric ex ante, then any MME must exhibit strategic targeting. To be precise, redefine the share μ to depend on parameter $t \in [0, 1]$. That is, for a given t , $\mu(s, t)$ is the mass of students preferring A over B in state s . We assume that $\mu(s, t)$ is continuous in (s, t) and $\mu(s, 0) = 1 - \mu(s, 0)$ for all s , i.e., the colleges are ex ante symmetric when $t = 0$.

Theorem 3. *There exists $\eta > 0$ such that for all $t < \eta$, any competitive MME under $\mu(\cdot, t)$ involves strategic targeting. In other words, any MME exhibits strategic targeting if the colleges are sufficiently symmetric.*¹⁵

Proof. See [Appendix A.6](#). ■

Although the result is not limited to symmetric equilibria, the intuition can be seen more clearly when we focus on symmetric equilibria (in the case that colleges are symmetric). Suppose to the contrary that both colleges adopt cutoff strategies using the same cutoff \hat{v} . Namely, they compete for students of types $[\hat{v}, 1]$. Suppose now a college deviates by shifting its admissions from students of types $[\hat{v}, \hat{v} + \varepsilon)$ to types $[\hat{v} - \varepsilon', \hat{v})$. Notice the former students receive a competing offer, so they entail uncertainty in enrollment; but not the latter. For small enough ε and ε' , chosen to keep the expected yield unchanged, the resulting drop in the quality of admission pool is negligible but the benefit in reducing the uncertainty is of first order importance. Hence, the (symmetric) cutoff equilibrium cannot be sustained.

3.1 Properties of Equilibria

We have seen that the equilibrium outcome involves strategic targeting. We now consider the properties of the equilibria in welfare and fairness.

A few definitions are necessary. For each state s , an **assignment** is a mapping from $\mathcal{V} \times \{A, B\}$ into $[0, 1]$ that specifies the fraction of students of a given type that are enrolled at each college. An **outcome** is a mapping from a state to an assignment, i.e., the realized allocation in state s . Welfare and fairness can be defined based on the colleges' and the students' (ordinal) preferences. We say that a student has **justified envy** at state s if at that state she is not admitted by her favorite college even though it enrolls a student with a lower score. An outcome is **fair** if for almost every state, almost all students have no justified envy.

¹⁵[Theorem 3](#) holds more generally beyond MME. In fact, any competitive equilibrium in which both colleges admit students within $[\check{v}, \hat{v}]$ exhibits strategic targeting.

Since admissions may involve randomness (as with our strategic targeting equilibrium), we can define a student’s ex ante preference over lotteries via first order stochastic dominance and evaluate welfare and fairness based on it. We say that for any two lotteries $x, y \in \{(z_A, z_B, z_\emptyset) \in \mathbb{R}_+^3 \mid z_A + z_B + z_\emptyset = 1\}$, a student **FOSD-prefers** x to y , if the probabilities of being admitted by her favorite college and being admitted by either college are both (weakly) higher at x than at y , and the ranking is strict for at least one category. We say that a student has **ex ante justified envy** prior to random selection if the student FOSD-prefers a lottery obtained by a lower-score student to the lottery she obtains. An outcome is said to be **ex ante fair** if for almost every state, almost all students have no ex ante justified envy.¹⁶ If an outcome fails to be ex ante fair, or equivalently **ex ante unfair**, then a positive measure of students are disadvantaged for having better scores than others in an ex ante strong sense: they FOSD envy lower-score students. Note that when the assignment is deterministic, the ex ante notion of fairness collapses to the (ex post) notion of fairness, so we shall use the latter.

Next, an outcome is **Pareto efficient** if the associated assignment is Pareto undominated for almost every state—namely, there is no other assignment in which both colleges and all students are weakly better off and at least one college or a positive measure of students is strictly better off. It is also useful to study the welfare of one side of the market, taking the other side simply as resources. We say that an outcome is **student efficient** if for almost every state, there is no other assignment in which all students are weakly better off and a positive measure of students is strictly better off. An outcome is said to be **college efficient** if for almost every state, no other assignment can make both colleges weakly better off and at least one college strictly better off relative to the assignment that the outcome selects. Note that Pareto efficiency does not imply student efficiency or college efficiency.

The next theorem states properties of equilibria that arise in decentralized matching.

Theorem 4. (i) *Every competitive equilibrium is student, college and Pareto inefficient.*

(ii) *Every competitive MME is unfair if and only if it exhibits strategic targeting.*

(iii) *Suppose $\underline{v}_A \neq \underline{v}_B$ and $\bar{v}_A \neq \bar{v}_B$ and $\max\{\underline{v}_A, \underline{v}_B\} < \min\{\bar{v}_A, \bar{v}_B\}$ in a MME. Then, the outcome is ex ante unfair.*

(iv) *Every non-competitive equilibrium is unfair, student inefficient, but college efficient.*

(v) *Every non-competitive equilibrium is Pareto inefficient unless almost every student admitted by one college has higher score than those admitted by the other college.*

Proof. See [Appendix A.7](#). ■

¹⁶Note that this is a weak notion of ex ante fairness. One may consider a stronger notion which requires (almost) all students to FOSD-prefer the lotteries they obtain in comparison with those obtained by lower-score students in (almost) all states. We consider the weak notion because equilibria fail even the weak notion. The DA mechanism ensures a stronger notion of fairness, but the assignment is deterministic in this case, so the ex ante notion of fairness reduces to the (ex post) fairness.

Part (i) follows since some seats are unfilled even though it would be Pareto improving to allocate them to students who are unmatched. Part (ii) follows easily from the fact that some students in the targeting competition region are rejected by their favorite college (or by both colleges), even though that college admits lower-score students. Part (iii) follows from the ex ante non-monotonicity observed in [Corollary 1](#). The unfairness as well as student inefficiency in Part (iv) follows from the fact that two colleges simply admit different sets of students. In such an equilibrium, high-score students receive an admission from only one college, and some of these students, rejected by their top choice, will have justified envy with lower-score students who were assigned their top choice. The outcome is still college efficient since the colleges cannot be made both better off. Nevertheless it admits a Pareto improving reallocation, as stated in Part (v).

4 Multidimensional Performance Measures and Evaluation Distortion

In the baseline model, we have assumed that colleges assess students based on a common criterion. In practice, colleges consider multiple dimensions of students' attributes and performances, academic as well as non-academic. Some dimensions are common among colleges; for instance, SAT scores or GPA of students are commonly observed and interpreted virtually the same by colleges. Others such as essays and college entrance exams are specific to individual colleges.¹⁷ Non-academic measures are often less correlated among colleges since they usually focus on different aspects and/or interpret them differently. For instance, students' community service or leadership activities may weigh heavily for some colleges, whereas extracurricular activities such as musical or athletic talents may be important for others. We show that strategic targeting takes a particular form in this environment: *Colleges bias their admissions criteria toward non-common performances.*

To illustrate this point, we extend our model as follows. A student's type is described as a triple $(v, e_A, e_B) \in \mathcal{V} \times E_A \times E_B \equiv [0, 1]^3$, where v is the common measure or score, and e_A and e_B are college specific measures considered respectively by colleges A and B . One interpretation is that v is a student's test score of the nationwide exam, and e_A and e_B are her performances on college-specific essays or tests. Alternatively, v can be an academic performance measure observed by both colleges, and e_A and e_B correspond to different dimensions of extracurricular activities that the two colleges emphasize.

¹⁷College essay topics vary greatly across different colleges in the US. In Japan, there is a nation-wide exam, called National Center Test (NCT), and each university has its own exam. Public universities usually use both NCT and their own exams, and private ones use their own exams only. Similarly, students in Korea take a nationwide exam, and each college often has its own essay tests and/or oral interviews.

As before, v is distributed according to $G(\cdot)$, and e_i , $i = A, B$, is conditionally independent on v and is distributed according to $X_i(\cdot|v)$ which admits a density $x_i(\cdot|v)$. We also assume that $\frac{\partial}{\partial v}X_i(e_i|v) < 0$ for any $e_i \in [0, 1]$. That is, a student with larger v has a higher probability of scoring high e_i . We also assume full support of G and X_i for all i . College i only values (v, e_i) . Specifically, it derives payoff $U_i(v, e_i)$ from matriculating a student with type (v, e_A, e_B) , where U_i is strictly increasing and differentiable in both arguments.

College i 's strategy is now described as a mapping $\sigma_i : \mathcal{V} \times E_i \rightarrow [0, 1]$, whereby it admits a fraction $\sigma_i(v, e_i)$ of students of type (v, e_i) . Enrollment uncertainty facing college i generated by students of type (v, e_i) depends on whether they receive an offer from college j , $j \neq i$. Since e_j is conditionally uncorrelated with e_i , the probability of such event is $\bar{\sigma}_j(v) := \mathbb{E}[\sigma_j(v, e_j)|v]$.

For given $\bar{\sigma}_A(\cdot)$ and $\bar{\sigma}_B(\cdot)$, the mass of students enrolling in college i in state s is

$$m_i(s) = \int_0^1 \int_0^1 \sigma_i(v, e_i) (1 - \bar{\sigma}_j(v) + \mu_i(s) \bar{\sigma}_j(v)) dX_i(e_i|v) dG(v).$$

Let \hat{s}_i be such that $m_i(\hat{s}_i) = \kappa$ as before. Then, college i 's payoff is described as follows:

$$\begin{aligned} \pi_i &= \int_0^1 \int_0^1 U_i(v, e_i) \sigma_i(v, e_i) (1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)] \bar{\sigma}_j(v)) dX_i(e_i|v) dG(v) \\ &\quad - \lambda \mathbb{E}_s[m_A(s) - \kappa | s \in S_i] \text{Prob}(s \in S_i) \\ &= \int_0^1 \int_0^1 \sigma_i(v, e_i) H_i(v, e_i, \bar{\sigma}_j(v)) dX_i(e_i|v) dG(v) + \lambda \kappa \text{Prob}(s \in S_i), \end{aligned}$$

where $S_A := \{s | s \geq \hat{s}_A\}$ and $S_B := \{s | s \leq \hat{s}_B\}$, and

$$\begin{aligned} H_i(v, e_i, \bar{\sigma}_j(v)) &:= U_i(v, e_i) (1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)] \bar{\sigma}_j(v)) \\ &\quad - \lambda \text{Prob}(s \in S_i) (1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s) | s \in S_i] \bar{\sigma}_j(v)). \end{aligned} \quad (4.1)$$

As before, this marginal payoff equals the student's value $U_i(v, e_i)$ to college i multiplied by the probability of her accepting i 's admission minus the capacity cost the student adds to i . It is worth noting that the capacity cost only depends on the common measure v but not on the non-common measure e_i . This is because a student with high v is more likely to have a competing offer, but conditional on v , e_i is independent with e_j . Intuitively, a student scoring high in e_i is less likely to incur enrollment uncertainty than a student scoring high in v .

We focus on a cutoff strategy equilibrium in which college i admits student type (v, e_i) if and only if $e_i \geq \eta_i(v)$ for some η_i nonincreasing in v . [Figure 4.1](#) depicts a typical cutoff strategy. In the figure, the solid line represents the locus of η_i , so the shaded area depicts the

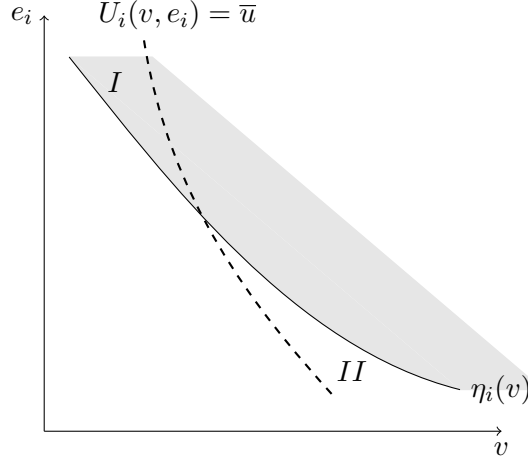


Figure 4.1: College's Cutoff Locus

types of students college i admits under a cutoff strategy, whereas the dotted line is i 's true indifference curve. [Appendix A.8](#) provides a condition under which cutoff equilibrium exists. Such an equilibrium is quite plausible since the use of non-common performance measure lessens head-on competition and enrollment uncertainty.

We shall now show that in such a cutoff equilibrium, the colleges place more weight on the non-common measures relative to their common measures. The reasoning of [Lemma 2](#) implies that a college i must accept student type (v, e_i) if and only if $H_i(v, e_i, \bar{\sigma}_j(v)) \geq 0$. In particular, the cutoff locus $e_i = \eta_i(v)$ must satisfy $H_i(v, \eta_i(v), \bar{\sigma}_j(v)) = 0$ whenever $\eta_i(v) \in (0, 1)$. Its slope $-\eta'_i(v)$ shows the “relative worth” of the student’s common performance v in college i 's evaluation in terms of the units of the student’s non-common performance. The higher this value is, the larger weight the college places on common performance. In particular, we shall say that the college *under-weights a student’s common performance v and over-weights her non-common performance e_i* if for all v ,

$$-\eta'_i(v) \leq \frac{\partial U_i(v, \eta_i(v))/\partial v}{\partial U_i(v, \eta_i(v))/\partial e_i}$$

and the inequality is strict for a positive measure of v .¹⁸ Since students with high v contributes more to enrollment uncertainty than students with high e_i , as seen from (4.1), evaluation distortion arises in a cutoff equilibrium:

Theorem 5. *In a cutoff equilibrium, each college under-weights a student’s common performance and over-weights her non-common performance.*

¹⁸Suppose for instance $U_i(v, e_i) = (1 - \rho)v + \rho e_i$. Then, the condition means $-\eta'_i(v) \leq \frac{1-\rho}{\rho}$, so the college places a weight less than $1 - \rho$ to common performance v and a weight more than ρ to non-common performance e_i .

Proof. See [Appendix A.8](#). ■

This particular form of strategic targeting again entails justified envy in a positive measure of states. Among students who prefer A to B , those who are in region II (at the bottom between dotted and solid lines) in [Figure 4.1](#) have justified envy toward those students in region I (at the top between solid and dotted lines). The outcome is also Pareto inefficient since a college have vacant seats in a positive measure of states, which could have been filled with unmatched students.

Theorem 6. *A cutoff equilibrium is unfair and students, college and Pareto inefficient.*

5 Restriction on Applications: Self-Targeting

So far, we have studied decentralized college admissions in the most stylized format. In the current and following sections, we study two common ways for colleges to manage their enrollment uncertainty. We assume, as in the baseline model, that students' type is single dimensional.

One common method used is to limit the number of applications that students can submit. For instance, students cannot apply to both Cambridge and Oxford in the UK, and applicants in Japan can apply to at most two public universities.¹⁹ In Korea, schools (more precisely, college-department pairs) are partitioned into three groups, and students are allowed to apply to only one in each group. Some US colleges place restrictions on application in their Early Admissions plans.²⁰

Limiting the number of applications a student can make forces her to “self-target” colleges. Since students apply to schools they are likely to accept when admitted, this method improves the colleges' yield rates and reduces their enrollment uncertainty. In our model with two colleges, if the number of applications is restricted to one, colleges face no enrollment uncertainty because a student admitted by a college will never turn down its offer, so their admission behavior is non-strategic; namely, they admit students in the descending order of v until their capacities are filled. As will be seen, however, students' application behavior now becomes strategic. Thus, the overall welfare is unclear a priori.

¹⁹Public colleges in Japan may hold three exams. The first one is called “zenki(former period)-exam” and the last one is called “koki(later period)-exam.” There are very small number of schools that have exam between these two exams. Students can apply to at most one public school at each exam date.

²⁰Early admissions consist of Early Decision which requires students to enroll, if admitted, and Early Action which does not involve such commitment. While pursuing admission under an Early Decision plan, students may apply to other institutions, but may have only one Early Decision application pending at any time ([NACAC, 2012](#)). Some Early Action plans place restrictions on student applications to other early plans. Selective universities such as Harvard, Stanford, Yale and Princeton restricted applicants to a single private university in their 2014 Early Action plans.

We now provide a simple model showing students' application behavior when they are limited to one application. To this end, we introduce students' cardinal preferences for colleges. Each student has a taste $y \in \mathcal{Y} \equiv [0, 1]$, which is independent of score $v \in [0, 1]$. A student with taste y obtains payoff y from attending college A and $1 - y$ from attending college B . Thus, students with $y \in [0, \frac{1}{2}]$ prefer B to A , and those with $y \in [\frac{1}{2}, 1]$ prefer A to B . To facilitate analysis, we assume that colleges observe an applicant's score v but not her preference y , while each student knows her preference y but not her score v .²¹ In reality, even though students submit their records to colleges, they do not know precisely how they are ranked by colleges. See [Avery and Levin \(2010\)](#) for the same treatment.

A student's taste y is drawn according to a distribution that depends on the underlying state. For a given s , let $K(y|s)$ be the distribution of y with density function $k(y|s)$, which is continuous and obeys (strict) monotone likelihood ratio property: For any $y' > y$ and $s' > s$,

$$\frac{k(y'|s')}{k(y|s')} > \frac{k(y'|s)}{k(y|s)}, \quad (5.1)$$

meaning that a student's value of A is likely to be high when s is high. We further assume that there exists $\delta > 0$ such that $\left| \frac{k_y(y|s)}{k(y|s)} \right| < \delta$ for any $y \in [0, 1]$ and $s \in [0, 1]$, which indicates that students' tastes change moderately relative to the state. Each student with taste y forms a posterior belief about the state s , given by the following conditional density:

$$l(s|y) := \frac{k(y|s)}{\int_0^1 k(y|s) ds}.$$

Before proceeding, we make the following observations: First, for each student, applying to a school dominates not applying at all. Second, since a student does not know her score and her preference is independent of the score, the student's application depends solely on the preference. Third, since each student's preference depends on the state, the mass of students applying to each college varies across states. Let $n_i(s)$ be the mass of students who apply to college $i = A, B$ in state s .

We next consider a college's admission strategy. Since a college faces no enrollment uncertainty, it is optimal to admit all students up to a cutoff

$$c_i(s) := \inf \{c \in [0, 1] \mid n_i(s)[1 - G(c)] \leq \kappa\}.$$

If $n_i(s) \geq \kappa$ in state s , then college i will set its cutoff so as to admit students up to its

²¹Note that introducing cardinal preferences does not alter the previous analyses, because even if students have cardinal preferences, it is still a weak dominant strategy for students to apply to both colleges. Nonobservability of scores by students makes it a strict dominant strategy for students to apply to both colleges. Hence, the previous analyses still remain with this assumption.

capacity. Otherwise, it will admit all applicants.

Consider now students' application decisions. Fix any strategy $\sigma : \mathcal{Y} \rightarrow [0, 1]$ which specifies a probability of applying to A for each $y \in \mathcal{Y}$. The mass of students applying to A in each state s is then given by

$$n_A(s|\sigma) := \int_0^1 \sigma(y)k(y|s) dy.$$

Clearly, $n_B(s|\sigma) = 1 - n_A(s|\sigma)$. A student with taste y expects to be admitted by college i with probability

$$P_i(y|\sigma) \equiv \mathbb{E}[1 - G(c_i(s)) | y, \sigma] = \int_0^1 q_i(s|\sigma)l(s|y)ds,$$

where $q_i(s|\sigma) := \min \{ \kappa/n_i(s|\sigma), 1 \}$ for $i = A, B$. This probability depends on the student's preference y since it is correlated with the underlying state. Note that a student with taste y will apply to A if and only if

$$yP_A(y|\sigma) \geq (1 - y)P_B(y|\sigma).$$

We show that students follow a cutoff strategy in any equilibrium, given a moderate value of δ .

Lemma 4. *Suppose $\delta \leq \frac{1}{2}$. In any equilibrium, there exists a cutoff \hat{y} such that students with $y \geq \hat{y}$ apply to A and those with $y < \hat{y}$ apply to B . And such an equilibrium exists.*

Proof. See [Appendix A.9](#). ■

We now show that an equilibrium involves strategic application by students if one school is more popular than the other.

Theorem 7. *Suppose $\mu(s) > \frac{1}{2}$ ($\mu(s) = \frac{1}{2}$) for almost all s . Then, $\hat{y} \in (\frac{1}{2}, 1)$ ($\hat{y} = \frac{1}{2}$), where \hat{y} is the equilibrium cutoff.*

Proof. See [Appendix A.9](#). ■

The intuition behind [Theorem 7](#) is clear. If college A is more popular than B , then A becomes more difficult to get in than B , all else equal. Hence, students who prefer B ($y \leq \frac{1}{2}$) will definitely apply to B . But even students who mildly prefer A (i.e., y is greater than but close to $\frac{1}{2}$) will apply to B instead of A . This behavior leads to a cutoff $\hat{y} > \frac{1}{2}$ as depicted in [Figure 5.1](#).

Example 3. Suppose there are two states a and b , each arising with probability $\frac{1}{2}$. Let $K(y|a) = y^2$, $K(y|b) = y$ and $\kappa = 0.4$. Then, we have

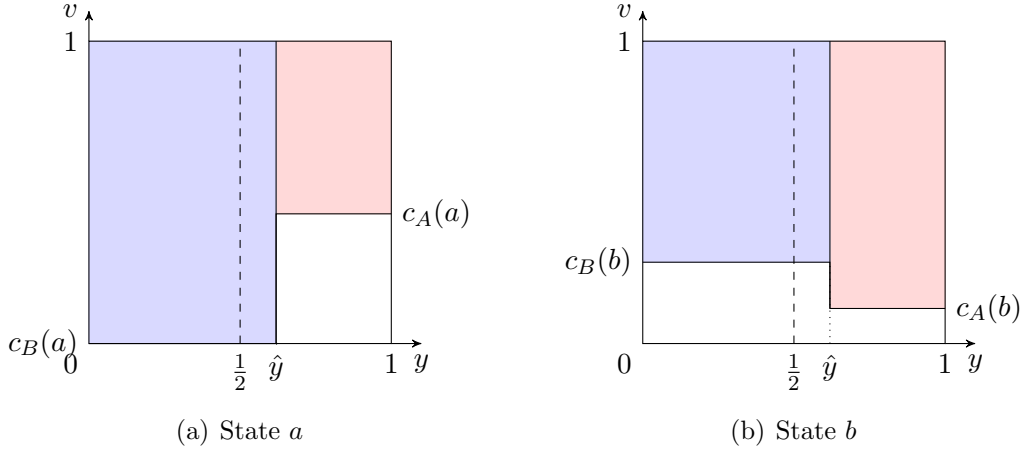


Figure 5.1: Equilibrium Assignment when $\kappa = 0.4$

\hat{y}	$n_A(a)$	$n_B(a)$	$c_A(a)$	$c_B(a)$	$n_A(b)$	$n_B(b)$	$c_A(b)$	$c_B(b)$
0.547	0.701	0.299	0.429	0	0.453	0.547	0.116	0.269

Observe that if $n_i(s) \geq \kappa$ for all s and for all $i = A, B$, then self-targeting eliminates colleges' yield control problem, since each college fills its capacity with the best students among those who applied to it. But, a college may be undersubscribed; for instance, the mass of applicants to college B in state a is smaller than its capacity ($n_B(a) = 0.299 < \kappa = 0.4$).

Let us now consider welfare and fairness properties of the equilibrium outcome. First, the equilibrium is unfair. Justified envy arises in that (i) students who happen to have applied to a more popular college for a given state may be unassigned even though their scores are good enough for the other college (the area on the bottom right below $c_A(a)$ of [Figure 5.1\(a\)](#)); and (ii) students who prefer but avoid an ex ante more popular college get into an ex ante less popular college, but they could have gotten into the former when it becomes ex post less popular (the shaded area between $\frac{1}{2}$ and \hat{y} of [Figure 5.1\(b\)](#)).

Second, a college may be undersubscribed in equilibrium so that its capacity is not filled even though there are unassigned, acceptable students. By assigning those students to unfilled seats of that college, students and the college will be both better off. Thus, the equilibrium outcome is still student, college and Pareto inefficient.

Theorem 8. *The outcome of the restricted applications is unfair. Suppose $K(\hat{y}|s) < \kappa$ for a positive measure of states. Then, college B suffers from under-subscription, and the outcome is student, college and Pareto inefficient.*

Proof. See [Appendix A.9](#). ■

6 Sequential Admissions: Wait-listing

Colleges also manage enrollment uncertainty by offering admissions sequentially. According to this method, a college would admit some applicants and wait-list others in the first round. Later it would admit students from the wait list when some offers are rejected. This process may repeat in several rounds. Wait-listing is adopted by most colleges in France and Korea. In the US, nearly 45% of four-year colleges adopted wait lists in 2011, up from 32% in 2002 (NACAC, 2012). In a typical application, the acceptance decisions cannot be deferred and/or the number of iterations is limited. Hence, even though wait-listing allows for more admission offers and acceptances than the baseline model or restricted applications, it does not fully eliminate congestion. For this reason, strategic targeting remains an issue.

To see this, we consider a simple extension of our baseline model. There are three colleges, A , B and C , each with a mass $\kappa < \frac{1}{3}$ capacity. There is a unit mass of students with score v , where v is distributed from $[0, 1]$ according to $G(\cdot)$ as before. All students prefer A and B to C , but C is significantly better than not attending any school. A college's utility is given by students' scores, but for each student, there is a probability ε that each of colleges A and B finds the student unacceptable. College C admits students simply based on their scores.

There are two states, a and b , each arising with probability $\frac{1}{2}$. In state $i = a, b$, a fraction s_i of students gets utility u from A and $u'(< u)$ from B , and the remaining $1 - s_i$ students have the opposite preference, where $s_a = 1 - s_b > \frac{1}{2}$. In either state, a student gets utility u'' from C , where $(1 - \varepsilon)u < u'' < u$ so that entering C with certainty is better than entering A with probability $1 - \varepsilon$. In state a , the mass of students who prefer A to B is larger than that of those who prefer B to A ($s_a > \frac{1}{2} > 1 - s_a$), and in state b , the opposite is true ($s_b < \frac{1}{2} < 1 - s_b$).

Suppose also the capacity cost is prohibitively high so that whenever a college makes an admission decision, it must make sure that the capacity constraint is not violated. We consider the following model of wait-listing. In each round, each college admits a set of students and wait-lists the remaining. A student who has received an offer must accept or reject it immediately. After the first round, colleges A and B learn the state, so the game effectively ends in two rounds.

We show that there is no symmetric equilibrium in which both colleges A and B use a cutoff strategy (i.e., admit the top κ acceptable students) in the first round.

Theorem 9. *There is no symmetric equilibrium in which both colleges A and B offer admissions to the top κ acceptable students in the first round.*

Proof. See [Appendix A.10](#). ■

The intuition behind this result is as follows. Suppose A and B admit the best candidates up to their capacities while keeping the next best group in mind in case some offers are turned

down. The problem with this strategy is that when some of those admitted turn down their offers, the second-best students the colleges have in mind may not be available. The reason is that those students are uncertain about whether A or B would find them acceptable, hence if they receive an admission offer from C , they would simply accept it. This implies that the students who remain after the first round are likely to be far worse than the second-best group. Hence, a college would deviate profitably by skipping over some of the top κ students and preemptively admitting some of the second-best students.

Theorem 9 implies that strategic targeting must occur in any symmetric equilibrium. Strategic targeting here can be attributed to the uncertainty facing the colleges about the quality of students remaining after each round. This uncertainty rests on the uncertainty students face on whether better offers will emerge should they turn down current offers. Without allowing deferral of decisions, either for colleges in admitting students or for students in accepting offers, the uncertainties will result in strategic targeting.²²

Again, strategic targeting—i.e., a non-cutoff equilibrium—means that the equilibrium outcome involves justified envy and is thus unfair. It is also student inefficient because there are two sets of students with the second-best group such that one set prefers A but is admitted only by B and the other prefers B but is admitted only by A . In sum, the undesirable properties of decentralized matching are not eliminated by wait-listing.

Such a pattern of strategic targeting is observed in practice. We present one evidence from the admissions decisions made by Hanyang University in Korea.²³ **Figure 6.1** depicts the distribution of the nation-wide College Scholastic Ability Test (CSAT) scores earned by the students who were admitted by the Department of Economics and Finance (DEF) at different sequential rounds in years 2011, 2012 and 2013.²⁴ The horizontal axis represents students' CSAT scores, and the vertical axis is the number of students admitted in each round of admission.²⁵

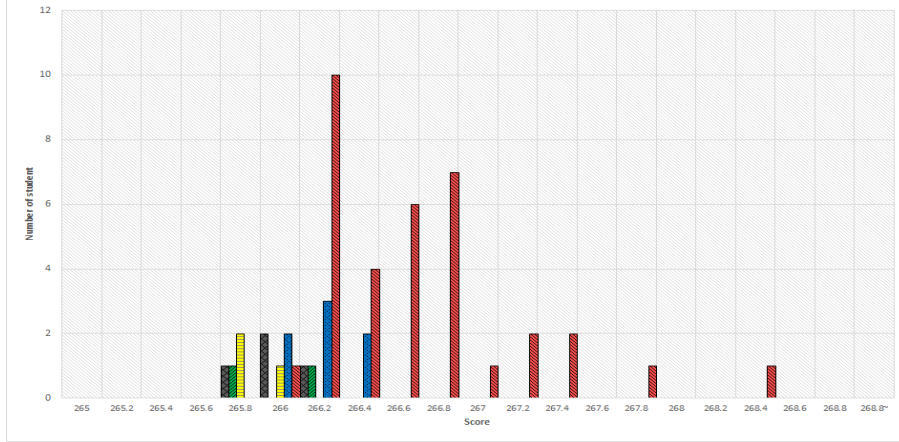
The figure reveals a pattern of strategic targeting. In 2011, 133 students applied to DEF, and DEF admitted 35 students in the first round, 7 students in the second round and additional 9 students in the subsequent rounds. The average score of the top four students admitted in the second round (266.155) is higher than that of the bottom four students in the

²²As will be seen in the next section, the deferral of decisions allowed in the Gale-Shapley's algorithm solves this problem.

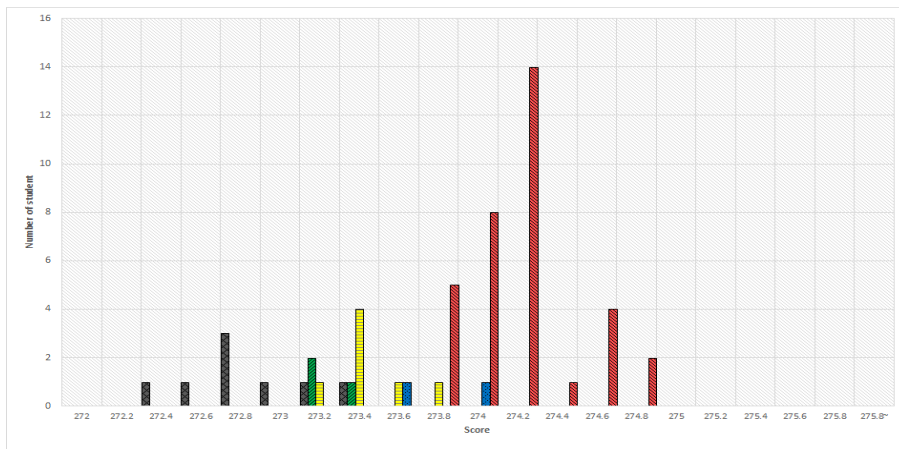
²³We gratefully acknowledge the Hanyang University for providing their admissions data.

²⁴As noted in **Section 5**, college-department pairs in Korea are divided into three groups, called groups Ga , Na and Da , and students can apply to at most one in each group. Thus, an unit in each group competes with other units in the same group but does not face competition with units in other groups. DEF divides its quota into two groups, Ga and Na , and admits students separately for the two groups. We focus on the admissions decision on the group Ga applications, which is the primary target group of DEF. The quota assigned for group Ga is twice as many as the quota for group Na .

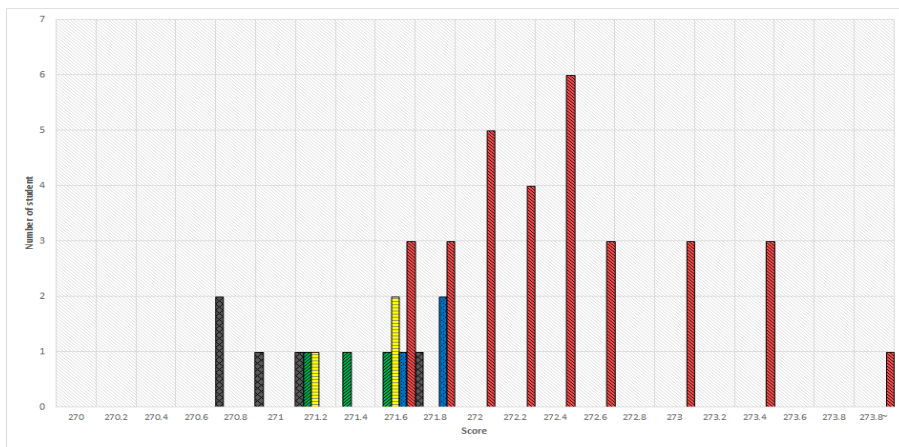
²⁵The total score of CSAT is normalized as 280 by the admission office, while the actual score may depend on the subjects taken by students.



(a) 2011



(b) 2012



(c) 2013

■ 1st ■ 2nd ■ 3rd ■ 4th ■ ≥5th

Note: “*i*-th,” $i = 1, \dots, 4$, means the *i*-th round of admission and “ ≥ 5 th” includes all rounds after 5th round.

Figure 6.1: Admissions on Wait Lists

first round (266.027). In 2012, DEF admitted 54 students out of 103 applicants through 11 rounds. It admitted 34 students in the first round and 2 students in the second round. The highest score in the second round (273.844) is the same as the 27th highest score in the first round. In 2013, 45 students were admitted out of 124 applicants in 8 rounds. DEF admitted 31 and 3 students in the first and the second rounds, respectively. The scores of the top two students in the second round are higher than those of the bottom three students in the first round. The reason for the observed non-monotonicity is that DEF offers a significant number of its admissions based on a measure that “garbles” a student’s CSAT score by another less-informative measure.²⁶

7 Centralized Matching via Deferred Acceptance

The most systemic response to enrollment uncertainty is to organize centralized matching between students and colleges. Centralized admissions are adopted in countries such as Australia, China, Germany, Taiwan, Turkey and the UK.²⁷ In this section, we consider a centralized matching with a Gale and Shapley’s Deferred Acceptance algorithm (henceforth DA). Not only is the DA employed in many centralized markets, such as public school admissions and medical residency assignments, but it has a number of desirable properties compared with the outcomes of decentralized matching, as we shall highlight below.

In the DA algorithm, students and colleges report their ordinal preferences to the clearinghouse, which then uses the information to simulate the following multi-round procedure. In each round, students propose to the best schools that have not yet rejected them. The colleges then tentatively accept the most preferred students up to their capacities and reject the rest. This process is repeated until no further proposals are made, in which case each student is assigned to a college that has tentatively accepted her proposal.²⁸

²⁶A college in Korea typically considers an applicant’s CSAT score and his/her high school GPA. But for the high school GPA, a college is prohibited by law from adjusting it for the quality of the high school the student is attending. Since the quality of high schools differs significantly across regions and between “special-purpose” schools and regular schools, a student’s CSAT score is widely regarded as a more reliable indicator of his/her ability than his/her high school GPA. In keeping with this, DEF, as well as many other departments, at Hanyang University awards a small number of the so-called “priority” admissions in its first round admission based solely on applicants’ CSAT scores. But for the remaining admissions in the first and the subsequent rounds, DEF makes selection based on the sum of an applicant’s CSAT score and his/her GPA (unadjusted for high school quality). Accordingly, the students receiving priority admissions have higher CSAT scores than all other admitted students. But, because of the “garbling” of the CSAT scores by the high school GPA for the remaining admissions, the students admitted in the earlier rounds need not have higher CSAT scores than those admitted in later rounds. [Figure 6.1](#) includes the students receiving priority admissions as part of the first-round admittees.

²⁷See [Chen and Kesten \(2013\)](#) for Shanghai mechanism and [Westkamp \(2013\)](#) for Germany medical school matchings.

²⁸The outcome of college-proposing DA is the same as that of student-proposing DA in our model, since colleges have a uniform rank on students. See also [Abdulkadiroğlu, Che and Yasuda \(forthcoming\)](#) and

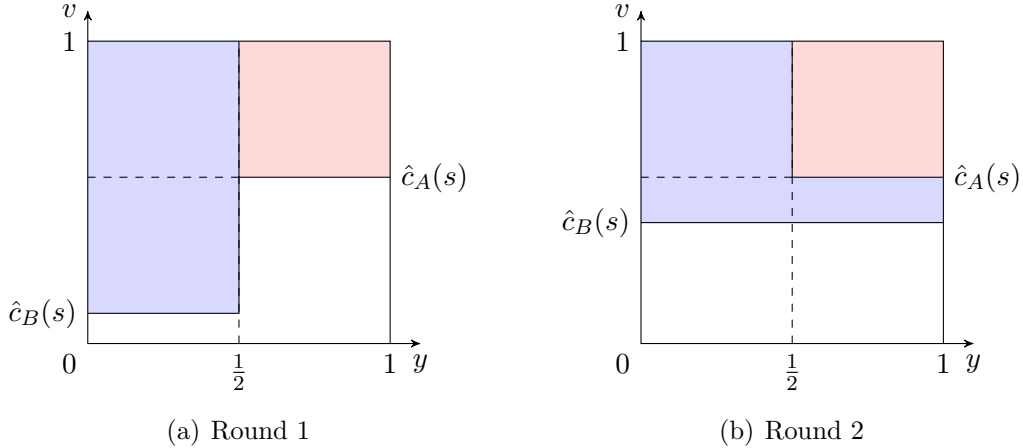


Figure 7.1: Deferred Acceptance Algorithm

Figure 7.1 illustrates the process for the case $\mu(s) \geq \frac{1}{2}$. In the first round, a fraction $\mu(s)$ of students proposes to college A , and the remaining students propose to college B . Each college tentatively admits the top κ students among the applicants. Thus, colleges' cutoffs in this round, denoted by $\hat{c}_i(s)$, $i = A, B$, satisfy $\mu(s)[1 - G(\hat{c}_A(s))] = \kappa$ and $(1 - \mu(s))[1 - G(\hat{c}_B(s))] = \kappa$ (see Figure 7.1(a)). Unassigned students then propose to another college at the second round, and again, colleges reselect the top κ students from those admitted tentatively in the first round together with the new applicants. Thus, colleges' cutoffs in this round satisfy $\mu(s)[1 - G(\hat{c}_A(s))] = \kappa$ and $1 - G(\hat{c}_B(s)) = 2\kappa$ (see Figure 7.1(b)). Since there are no more colleges unassigned students can apply to, the assignment is finalized in the second round in our model.

Consider now the equilibrium properties of the DA outcome. DA is strategy-proof for the students, so they have a dominant strategy of reporting their preferences truthfully (Dubins and Freedman, 1981; Roth, 1982). In addition, colleges in our model also report their rankings and capacities truthfully in an ex post equilibrium; namely, truthful reporting forms a Nash equilibrium for any profile of preferences students may report.

Lemma 5. *Given the common college preferences, it is an ex post equilibrium for colleges to report their rankings and capacities truthfully.*

Proof. See Appendix A.11. ■

The matching in the equilibrium involves no justified envy (Gale and Shapley, 1962; Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003) since the assignment is non-random for each realized state.²⁹ The matching is Pareto efficient (an implication of stability)

Azevedo and Leshno (2012) for a model of DA in which a continuum mass of students is matched to a finite number of schools.

²⁹It also involves no ex ante weak justified envy as well; that is, the outcome is strongly fair.

and is student efficient (because colleges' preferences are acyclic in the sense of [Ergin \(2002\)](#)). It also eliminates colleges' yield control problem completely. Colleges never exceed their capacities (because it is never allowed by the algorithm) and have no seats left unfilled in the presence of acceptable unmatched students (a consequence of stability).

In fact, given the homogeneous preferences of the colleges, there exists a single cutoff such that a student is assigned a college under DA if and only if her score exceeds that cutoff. In other words, only those with the top 2κ scores are assigned to a college. This outcome is jointly optimal for the two colleges—that is, the outcome would be chosen if the colleges were to merge into a single entity. In particular, the outcome is college efficient.

By contrast, a competitive equilibrium in decentralized matching entails unfilled seats for a college in low demand states and overfilled seats in high demand states, so the assignment is far from jointly optimal. This observation suggests that at least one college must be strictly better off from a shift from decentralized matching to a centralized matching via the deferred acceptance algorithm. Despite the overall benefit from switching centralization via DA, it is possible for one college to be worse off. To see this, consider the following example.

Example 4. Let $v \sim U[0, 1]$, $\lambda = 5$, $\kappa = 0.45$ and $\mu(s) = \frac{2}{5}s + \frac{3}{5}$. Then, in a decentralized admission, there is a MME such that $\check{v} = \underline{v}_A < \bar{v}_B = \hat{v}$ and colleges' payoffs in the equilibrium are $\pi_A = 0.283$ and $\pi_B = 0.180$. Suppose now that DA is adopted. Then, their payoffs are $\pi_A^{DA} = 0.321$ and $\pi_B^{DA} = 0.174$. Notice $\pi_A^{DA} + \pi_B^{DA} = 0.495 > \pi_A + \pi_B = 0.463$ (total payoff of the colleges increases), $\pi_A^{DA} > \pi_A$ (college A is strictly better off), but $\pi_B^{DA} < \pi_B$ (college B is worse off).

In this example, college A is more popular than B for all states. Yet, decentralized matching enables college B to attract some good students it cannot attract under DA. This may explain why centralized matching is not as common in college admissions as in other contexts such as public high-school admissions. In the latter situation, schools are largely under the control of central authority which serves the interest of the students. In contrast, colleges are independent strategic players with their own interest to pursue.

Equilibrium properties of the outcome under DA are summarized as follows.

Theorem 10. *Under DA, the equilibrium outcome is fair, and student, college and Pareto efficient. However, some college may be worse off compared to decentralized matching.*

8 Conclusion

The current paper has studied a new model of decentralized college admissions. In this model, colleges face enrollment uncertainty that arises from students' (aggregately) uncertain preferences. We have shown that colleges respond to enrollment uncertainty by strategically

targeting their admissions to students who are likely overlooked by their competitors. When colleges also consider students' performance in college-specific essays or tests, or their non-academic performance or extracurricular activities, strategic targeting takes the form of colleges' placing excessive weights on these measures in their admissions decisions. We have shown that this equilibrium behavior of colleges leads to justified envy and Pareto inefficiency.

We have also studied alternative arrangements in which colleges limit the number of applications students can submit, in which they admit students in multiple rounds via wait-listing, and in which they centralize admissions via DA. Both restricted application and wait-listing alleviate colleges' yield control burden, but targeting and enrollment uncertainty remain. So do inefficiencies and justified envy. Centralized matching via DA completely eliminates enrollment uncertainty and justified envy and achieves efficiency. However, not all colleges may benefit from such a centralized matching. This last observation may explain why college admissions remain decentralized in many countries.

Our analyses have several other implications.

Early Admissions. Early admissions are widely adopted by colleges in the US and Korea. Early admissions programs allow students to apply to sponsoring colleges early, and the colleges in turn process their applications prior to the regular admissions round (with binding or non-binding requirements for students to accept them early). The remaining students and seats are then allocated through regular admissions. This process resembles sequential admissions studied in [Section 6](#). In addition, some early admissions programs restrict applications just as in [Section 5](#). Although the model involving both sequential admissions and restricted application is not tractable (especially with aggregate uncertainty), our analyses imply that these features can help colleges to cope with enrollment uncertainty. We believe this is an important function of early admissions not emphasized by other recent papers ([Avery and Levin, 2010](#); [Lee, 2009](#)). Despite this benefit to colleges, our analyses suggest that the outcomes of these programs are unlikely to be efficient or fair.

Loyalty and Legacy. It is well documented that colleges favor students who show eagerness to attend them. Students who signal their interests through campus visits, essays, letter of intention, and webcam interviews are known to be favored by colleges. According to the 2004 and 2005 NACAC Admission Trends Survey, 59% of the colleges surveyed assigned some level of importance to a student's "demonstrated interest" in their admissions decisions. [Table 1](#) shows the extent to which certain applicant activities would be considered as a "plus factor" in the admission process by institutions with varying yield rates.

Early admissions, as [Avery and Levin \(2010\)](#) argue, also serve as a tool for colleges to identify enthusiastic applicants and favor them in the admission. It is entirely plausible that

Table 1: Percentage of institutions that consider a “plus factor” in the admission process

Yield	Campus visit		Interview (on or off campus)		Frequent contact the admission office		Applying Early Action or Decision		Particular academic or professional focus		Contact with faculty members on campus	
	2004	2005	2004	2005	2004	2005	2004	2005	2004	2005	2004	2005
< 30%	61.9	59.0	58.3	56.6	42.7	41.3	35.8	26.9	39.8	39.6	43.3	38.5
30 ~ 40%	51.3	47.2	54.1	42.5	37.0	34.0	24.4	26.5	38.0	36.4	34.8	34.4
46 ~ 60%	30.9	32.3	36.3	32.3	25.9	26.2	10.3	11.1	32.5	36.9	8.9	31.8
> 60%	32.6	45.0	37.0	52.5	23.9	39.3	17.4	7.3	33.3	48.3	17.8	32.2

Source: [NACAC \(2012\)](#)

these preferences by colleges are intrinsic, as postulated by [Avery and Levin \(2010\)](#). But, our theory suggests that such preference could also arise endogenously from the desire to manage enrollment uncertainty. Like the students without an competing offer in our model, those with demonstrated interest add less to the capacity cost, since their preference is unlikely to vary much with the popularity of the college among regular students.³⁰ Accordingly, even a college with no intrinsic preference for loyal students has a reason to favor them. Consistent with this view, [Table 1](#) shows that colleges suffering lower yield rates tend to favor students with “demonstrated interest” more than the colleges enjoying higher yield rates. A similar explanation applies to the favoring of legacy students (who have a family history with the school).³¹

Yield Manipulation and Strategic Targeting. As noted in [Remark 1](#), colleges in our model exhibit (endogenous) preference for a high yield. While such a preference arises in our model from the desire to manage enrollment uncertainty, in practice, colleges may prefer a high yield rate for an intrinsic reason—favorable perception and brand image. Importantly, “yield and acceptance rates for a college’s entering class account for one-fourth of the ‘student selectivity’ score in the influential US News & World Report annual rankings of colleges and

³⁰To see this, suppose there are two groups of students, loyal (L) and regular (R), for college A . A regular student prefers college A with probability $\mu(s)$ in state $s \in [0, 1]$ as before, but a loyal student prefers A with probability $\mu_L(s) = \mu(s) + a$, for some constant $a > 0$. That is, a higher fraction of loyal students prefer A than regular students at every state. Suppose there are two students, one loyal and one regular, and suppose both receive an offer from B . The loyal student adds to capacity cost λ whenever college A over-enrolls. This latter probability conditional on the loyal student accepting A ’s offer is

$$(1 - s_A) \frac{\mathbb{E}[\mu_L(s)|s > s_A]}{\mathbb{E}[\mu_L(s)]} = (1 - s_A) \frac{\mathbb{E}[\mu(s)|s > s_A] + a}{\mathbb{E}[\mu(s)] + a},$$

which is strictly lower than $(1 - s_A) \frac{\mathbb{E}[\mu(s)|s > s_A]}{\mathbb{E}[\mu(s)]}$, the corresponding probability of A over-enrolling conditional on the regular student accepting A ’s offer.

³¹[Espenshade, Chung and Walling \(2004\)](#) show that legacy applicants have nearly three times the likelihood of being accepted as non-legacies.

universities, enough to raise or lower a school’s position by several spots.”³² Our analysis implies that “yield-motivated” colleges will behave similarly to colleges in our model; namely, they will strategically target students overlooked by other (particularly stronger) colleges and reject students who are unlikely to accept their offers (or “too good for them”). This behavior is consistent with anecdotal evidence. The *Wall Street Journal* article cited in [footnote 32](#) reports one such practice employed by Franklin and Marshall College:

By wait-listing top applicants who didn’t visit the campus or interview with college representatives, the college bumped up its yield for the next school year to 27% from 25%. It also improved its acceptance rate – the ratio of acceptances to total applications – to a more selective 51% from 53%. Such numbers could help Franklin and Marshall rise in the US News ranking of national liberal-arts colleges from its current position of 33rd.

References

- Abdulkadiroğlu, Atila, and Tayfun Sönmez.** 2003. “School Choice: A Mechanism Design Approach.” *American Economic Review*, 93(3): 729–747. [30](#)
- Abdulkadiroğlu, Atila, Yeon-Koo Che, and Yosuke Yasuda.** forthcoming. “Expanding ‘Choice’ in School Choice.” *American Economic Journal: Microeconomics*. [29](#)
- Albrecht, James, Pieter A. Gautier, and Susan Vroman.** 2006. “Equilibrium Directed Search with Multiple Applications.” *Review of Economic Studies*, 73(4): 869–891. [6](#)
- Avery, Christopher, and Jonathan Levin.** 2010. “Early Admissions at Selective College.” *American Economic Review*, 100(5): 2125–2156. [5](#), [23](#), [32](#), [33](#)
- Avery, Christopher, Andrew Fairbanks, and Richard Zeckhauser.** 2003. *The Early Admissions Game: Joining the Elite*. Harvard University Press. [2](#)
- Azevedo, Eduardo M., and Jacob D. Leshno.** 2012. “A Supply and Demand Framework For Two-Sided Matching Market.” Mimeo. [30](#)
- Balinski, Michel, and Tayfun Sönmez.** 1999. “A Tale of Two Mechanisms: Student Placement.” *Journal of Economic Theory*, 84(1): 73–94. [30](#)
- Burdett, Kenneth, Shouyong Shi, and Randall Wright.** 2001. “Pricing and Matching with Frictions.” *Journal of Political Economy*, 109(5): 1060–1085. [6](#)

³²“Glass Floor: Colleges Reject Top Applicants, Accepting Only the Students Likely to Enroll,” by Daniel Golden, *Wall Street Journal*, May 29, 2001.

- Chade, Hector, and Lones Smith.** 2006. “Simultaneous Search.” *Econometrica*, 74(5): 1293–1307. 5
- Chade, Hector, Gregory Lewis, and Lones Smith.** forthcoming. “Student Portfolios and the College Admissions Problem.” *Review of Economic Studies*. 5, 14
- Chen, Yan, and Onur Kesten.** 2013. “From Boston to Shanghai to Deferred Acceptance: Theory and Experiments on A Famila of School Choice Mechanisms.” Mimeo. 29
- Coles, Peter, Alexey Kushnir, and Muriel Neiderle.** 2013. “Preference Signaling in Matching Markets.” *American Economic Journal: Microeconomics*, 5(2): 99–134. 5
- Dubins, L. E., and D. A. Freedman.** 1981. “Machiavelli and the Gale-Shapley Algorithm.” *American Mathematical Monthly*, 88(7): 485–494. 30
- Ergin, Haluk.** 2002. “Efficient Resoure Allocation on the Basis of Priorities.” *Econometrica*, 79(6): 2489–2497. 31
- Espenshade, Thomas J., Chang Y. Chung, and Joan L. Walling.** 2004. “Admissions Preferences for Minority Student, Athletes, and Legacies at Elite Universities.” *Social Science Quarterly*, 85(5): 1422–1446. 33
- Gale, D., and L. Shapley.** 1962. “College Admissions and the Stability of Marriage.” *American Mathematical Monthly*, 69(1): 9–15. 30
- Galenianos, Manolis, and Philipp Kircher.** 2009. “Directed Search with Multiple Job Applications.” *Journal of Economic Theory*, 144(2): 445–471. 6
- Julien, Benoît, John Kennes, and Ian King.** 2000. “Bidding for Labor.” *Review of Economic Dynamcis*, 3(4): 619–649. 6
- Kircher, Philipp.** 2009. “Efficiency of Simultaneous Search.” *Journal of Political Economy*, 117(5): 861–913. 6
- Lee, Sam-Ho.** 2009. “Jumping the Curse: Early Contracting with Private Information in University Admissions.” *International Economic Review*, 50(1): 1–38. 5, 32
- Lizzeri, Alessandro, and Nicola Persico.** 2001. “The Provision of Public Goods Under Alternative Electoral Incentives.” *American Economic Review*, 91(1): 225–239. 5
- Lizzeri, Alessandro, and Nicola Persico.** 2005. “A Drawback of Electoral Competition.” *Journal of the European Economic Association*, 3(6): 1318–1348. 5

- Montgomery, James D.** 1991. “Equilibrium Wage Dispersion and Interindustry Wage Differentials.” *Quarterly Journal of Economics*, 106(1): 163–179. [6](#)
- National Association for College Admission Counseling (NACAC).** 2012. “2012 State of College Admissions.” [2](#), [4](#), [22](#), [26](#), [33](#)
- Neiderle, Muriel, and Leeat Yariv.** 2009. “Decentralized Matching with Aligned Preferences.” *National Bureau of Economic Research*, No. 14840. [5](#)
- Ok, Efe A.** 2007. *Real Analysis with Economic Applications*. Princeton University Press. [69](#)
- Roth, Alvin E.** 1982. “The Economics of Matching: Stability and Incentives.” *Mathematics of Operations Research*, 7(4): 617–628. [30](#)
- Roth, Alvin E., and Xiaolin Xing.** 1997. “Turnaround Time and Bottlenecks in Market Clearing: Decentralized Matching in the Market for Clinical Psychologists.” *Journal of Political Economy*, 105(2): 284–329. [2](#), [5](#)
- Westkamp, Alexander.** 2013. “An Analysis of the German University Admissions System.” *Economic Theory*, 53(3): 561–589. [29](#)

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A Appendix A: Proofs

A.1 Proof of **Lemma 1**

Claim 1. *Suppose \mathcal{V}_{AB} has zero measure. Then, the following results hold.*

(i) $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$.

(ii) *Almost every student with $v \geq G^{-1}(1 - 2\kappa)$ receives an admission.*

Proof. (i) Since \mathcal{V}_{AB} is a measure zero set, $m_i(s)$ is constant across states for all $i = A, B$. If $m_i(s) < \kappa$, then college i can benefit by admitting some students with measure less than $\kappa - m_i(s)$. Similarly, if $m_i(s) > \kappa$, then it can benefit by rejecting some students with measure less than $m_i(s) - \kappa$.

(ii) Observe that $\mathcal{V}_A \cup \mathcal{V}_B$ cannot have a gap, otherwise at least one college can benefit by replacing a positive measure of low score students with the same measure of students in the gap. So, it must be an interval with $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$. Since $m_A(s) = m_B(s) = \kappa$ for all s by Part (i), this means that almost every top 2κ students are admitted. ■

Note that the proofs for Parts (i), (ii) and (iv) of the lemma for non-competitive equilibrium follow from **Claim 1**. We thus consider competitive equilibrium in what follows. We prove in the sequence of Parts (i), (iii) and (ii).

Proof of Part (i). Consider a competitive equilibrium. Suppose $m_A(1) < \kappa$. Let college A admit a mass $\kappa - m_A(1)$ of students. Then, the mass of students attending A , denoted by $\tilde{m}_A(s)$, satisfies that for any $s < 1$,

$$m_A(s) < m_A(s) + \mu(s)[\kappa - m_A(1)] \leq \tilde{m}_A(s) \leq m_A(s) + [\kappa - m_A(1)] < \kappa,$$

where the first and the last inequality follow from the fact that $m_A(s) < m_A(1)$ for $s < 1$ (since $\mu(\cdot)$ is strictly increasing in s). Observe that A benefits from such deviation since it admits more students without having over-enrollment. Hence, we must have $\kappa \leq m_A(1)$ in equilibrium. Similarly, if $m_A(0) > \kappa$, then A can benefit by rejecting a mass $m_A(0) - \kappa$ of students. Therefore, we must have $m_A(0) \leq \kappa \leq m_A(1)$ in any competitive equilibrium. The proof for college B is analogous. ■

Proof of Part (iii). We consider college A here. The proof for college B will be analogous. Since $\mu(\cdot)$ is strictly increasing and continuous in s , so is $m_A(\cdot)$. Thus, there exists $\hat{s}_A \in [0, 1]$ such that $m_A(\hat{s}_A) = \kappa$ by Part (i). We show that $\hat{s}_A \neq 0, 1$ in what follows.

Suppose $\hat{s}_A = 0$. Then, $m_A(s) > m_A(0) = \kappa$ for all $s > 0$. Thus, college A 's payoff is

$$\pi_A = \int_0^1 v\sigma_A(v)(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) - \lambda \int_0^1 [m_A(s) - \kappa]ds,$$

where $\bar{\mu} := \mathbb{E}[\mu(s)]$ and

$$m_A(s) = \int_0^1 \sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v).$$

Let A reject a positive measure of students, say $(c, c + \delta) \subset \mathcal{V}_A$. Then, its payoff is

$$\tilde{\pi}_A = \int_{[0,1] \setminus (c, c+\delta)} v\sigma_A(v)(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 [\tilde{m}_A(s) - \kappa]ds,$$

where

$$\tilde{m}_A(s) = m_A(s) - \int_c^{c+\delta} \sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \quad (\text{A.1.1})$$

and \tilde{s}_A is such that $\tilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A > \hat{s}_A = 0$ since $\tilde{m}_A(s) < m_A(s)$. Now, we can choose δ such that $\tilde{s}_A < \varepsilon$ for sufficiently small $\varepsilon > 0$. Then, A 's net payoff from the deviation is

$$\begin{aligned} & - \int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 [\tilde{m}_A(s) - \kappa]ds + \lambda \int_0^1 [m_A(s) - \kappa]ds \\ = & - \int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) + \lambda \int_{\tilde{s}_A}^1 [m_A(s) - \tilde{m}_A(s)]ds + \lambda \int_0^{\tilde{s}_A} [m_A(s) - \kappa]ds \\ = & - \int_{\tilde{s}_A}^1 \left(\int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ & - \int_0^{\tilde{s}_A} \left(\int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ & + \lambda \int_{\tilde{s}_A}^1 \left(\int_c^{c+\delta} \sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ & + \lambda \int_0^1 [m_A(s) - \kappa]ds \\ = & \int_{\tilde{s}_A}^1 \left(\int_c^{c+\delta} (\lambda - v)\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ & - \int_0^{\tilde{s}_A} \left(\int_c^{c+\delta} v\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ & + \lambda \int_0^{\tilde{s}_A} [m_A(s) - \kappa]ds \\ > & 0, \end{aligned}$$

where the second equality follows from (A.1.1) and the last inequality holds for sufficiently small ε .

Next, suppose $\hat{s}_A = 1$. Then, $m_A(s) < m_A(1) = \kappa$ for all $s < 1$. Let A admit all students in $(c, c + \delta) \not\subseteq \mathcal{V}_A$ for some $c < 1$. Then, the mass of students attending A becomes

$$\tilde{m}_A(s) = m_A(s) + \int_c^{c+\delta} (1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v). \quad (\text{A.1.2})$$

Let \tilde{s}_A be such that $\tilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A < \hat{s}_A = 1$ since $\tilde{m}_A(s) > m_A(s)$. We can choose δ such that $1 - \tilde{s}_A < \varepsilon$ for sufficiently small ε . A 's net payoff from the deviation is

$$\begin{aligned} & \int_c^{c+\delta} v(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 (\tilde{m}_A(s) - \kappa)ds \\ &= \int_c^{c+\delta} v(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) \\ & \quad - \lambda \int_{\tilde{s}_A}^1 \left(m_A(s) + \int_c^{c+\delta} (1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) - \kappa \right) ds \\ &= \int_c^{c+\delta} v(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 \left(\int_c^{c+\delta} (1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ & \quad + \lambda \int_{\tilde{s}_A}^1 [\kappa - m_A(s)]ds \\ &= \int_0^{\tilde{s}_A} \left(\int_c^{c+\delta} v(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ & \quad - \int_{\tilde{s}_A}^1 \left(\int_c^{c+\delta} (\lambda - v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds + \lambda \int_{\tilde{s}_A}^1 [\kappa - m_A(s)]ds \\ &> 0 \end{aligned}$$

where the first equality follows from (A.1.2) and the last inequality holds for sufficiently small ε . ■

Proof of Part (ii). We first show $\sup(\mathcal{V}_A \cup \mathcal{V}_B) = 1$ and then show that $\mathcal{V}_A \cup \mathcal{V}_B$ is an interval and $\inf(\mathcal{V}_A \cup \mathcal{V}_B) > 0$.

Step 1. $\sup(\mathcal{V}_A \cup \mathcal{V}_B) = 1$.

Proof. Suppose to the contrary that $\bar{c} := \sup(\mathcal{V}_A \cup \mathcal{V}_B) < 1$. We show that at least one college can benefit by rejecting some students in favor of those with $(\bar{c}, 1]$.

Suppose $\mathcal{V}_i \setminus \mathcal{V}_{AB}$ contains an open interval with positive measure for some $i = A, B$. Then, it is clear that college i can benefit by rejecting a positive measure of students from the bottom of $\mathcal{V}_i \setminus \mathcal{V}_{AB}$ and admits the same measure of students from 1.

Suppose now it is not the case. Let college A reject students in $(c, c + \delta) \subset \mathcal{V}_{AB}$ and admit those in $(1 - \varepsilon, 1]$ instead, where δ and ε satisfy

$$\int_{1-\varepsilon}^1 v dG(v) = \int_c^{c+\delta} v dG(v) \quad (\text{A.1.3})$$

and

$$\int_{1-\varepsilon}^1 1 dG(v) = \int_c^{c+\delta} \sigma_A(v)[1 - \sigma_B(v) + \mu(\hat{s}_A)\sigma_B(v)]dG(v), \quad (\text{A.1.4})$$

for given \hat{s}_A such that $m_A(\hat{s}_A) = \kappa$. The mass of students attending A from this deviation is

$$\tilde{m}_A(s) = m_A(s) + \int_{1-\varepsilon}^1 1 dG(v) - \int_c^{c+\delta} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v).$$

Note that $\tilde{m}_A(\hat{s}_A) = m_A(\hat{s}_A)$. Denote college A 's payoff from the deviation by $\tilde{\pi}_A$. Then, its net payoff from the deviation is

$$\begin{aligned} \tilde{\pi}_A - \pi_A &= \int_{1-\varepsilon}^1 v dG(v) - \int_c^{c+\delta} v \sigma_A(v)(1 - \sigma_B(v) + \bar{\mu}\sigma_B(v))dG(v) \\ &\quad - \lambda \mathbb{E}[\tilde{m}_A(s) - m_A(s) | s \in S_A] \text{Prob}(s \in S_A) \\ &\geq \int_{1-\varepsilon}^1 v dG(v) - \int_c^{c+\delta} v dG(v) \\ &\quad - \lambda \int_{\hat{s}_A}^1 \left(\int_{1-\varepsilon}^1 1 dG(v) - \int_c^{c+\delta} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds \\ &= -\lambda \int_{\hat{s}_A}^1 \left(\int_{1-\varepsilon}^1 1 dG(v) - \int_c^{c+\delta} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds \\ &> -\lambda \int_{\hat{s}_A}^1 \left(\int_{1-\varepsilon}^1 1 dG(v) - \int_c^{c+\delta} \sigma_A(v)[1 - \sigma_B(v) + \mu(\hat{s}_A)\sigma_B(v)]dG(v) \right) ds \\ &= 0, \end{aligned}$$

where the first inequality holds since $\sigma_A(v), \sigma_B(v), \bar{\mu} = \mathbb{E}[\mu(s)] \leq 1$ for any v , the second equality follows from (A.1.3), the penultimate inequality follows from the fact that $\mu(\cdot)$ is strictly increasing in s , and the last equality follows from (A.1.4). \square

Step 2. $\mathcal{V}_A \cup \mathcal{V}_B$ is an interval.

Proof. Suppose that there is gap in $\mathcal{V}_A \cup \mathcal{V}_B$. The proof is analogous to Step 1, where $(1 - \varepsilon, 1]$ is now replaced by the gap in $\mathcal{V}_A \cup \mathcal{V}_B$. We omit the details. \square

Step 3. $\inf(\mathcal{V}_A \cup \mathcal{V}_B) > 0$

Proof. Suppose to the contrary that $\inf(\mathcal{V}_A \cup \mathcal{V}_B) = 0$. Suppose $\inf(\mathcal{V}_A) = 0$. Let A reject a small fraction of students at the bottom, $[0, \varepsilon)$, where $2\varepsilon < 1 - \hat{s}_A$ and \hat{s}_A is such that $m_A(\hat{s}_A) = \kappa$. Then, the mass of students attending A from the deviation is

$$\tilde{m}_A(s) = \int_{\varepsilon}^1 \sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v).$$

Let \tilde{s}_A be the state such that $\tilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A > \hat{s}_A$ since $\tilde{m}_A(s) < m_A(s)$. Hence, we can choose ε such that $\tilde{s}_A - \hat{s}_A < \varepsilon$. Then, A 's net payoff from the deviation is

$$\tilde{\pi}_A - \pi_A = - \underbrace{\int_0^{\varepsilon} v\sigma_A(v)[1 - \sigma_B(v) + \bar{\mu}\sigma_B(v)]dG(v)}_{(*)} + \lambda \left[\underbrace{\int_{\hat{s}_A}^1 (m_A(s) - \kappa)ds - \int_{\tilde{s}_A}^1 (\tilde{m}_A(s) - \kappa)ds}_{(**)} \right].$$

Note that

$$\begin{aligned} (*) &= \int_0^1 \left(\int_0^{\varepsilon} v\sigma_A(v)(1 - \sigma_B(v) + \mu(s)\sigma_B(v))dG(v) \right) ds \\ &< \varepsilon \int_{\tilde{s}_A}^1 \left(\int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds \\ &\quad + \varepsilon \int_0^{\tilde{s}_A} \left(\int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds \end{aligned}$$

and

$$\begin{aligned} (**) &= \int_{\tilde{s}_A}^1 (m_A(s) - \kappa)ds + \int_{\hat{s}_A}^{\tilde{s}_A} (m_A(s) - \kappa)ds - \int_{\tilde{s}_A}^1 (\tilde{m}_A(s) - \kappa)ds \\ &= \int_{\tilde{s}_A}^1 \left(\int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds + \int_{\hat{s}_A}^{\tilde{s}_A} (m_A(s) - \kappa)ds \\ &> \int_{\tilde{s}_A}^1 \left(\int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds \end{aligned}$$

where the last inequality holds since $m_A(s) > \kappa$ for any $s \in (\hat{s}_A, \tilde{s}_A)$. Thus, we have

$$\begin{aligned} \tilde{\pi}_A - \pi_A &> (\lambda - \varepsilon) \int_{\tilde{s}_A}^1 \left(\int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds \\ &\quad - \varepsilon \int_0^{\tilde{s}_A} \left(\int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(s)\sigma_B(v)]dG(v) \right) ds \\ &> (\lambda - \varepsilon)(1 - \tilde{s}_A) \int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(\tilde{s}_A)\sigma_B(v)]dG(v) \\ &\quad - \varepsilon \tilde{s}_A \int_0^{\varepsilon} \sigma_A(v)[1 - \sigma_B(v) + \mu(\tilde{s}_A)\sigma_B(v)]dG(v) \end{aligned}$$

$$\begin{aligned}
&= (\lambda(1 - \tilde{s}_A) - \varepsilon) \int_0^\varepsilon \sigma_A(v)[1 - \sigma_B(v) + \mu(\tilde{s}_A)\sigma_B(v)]dG(v) \\
&> \varepsilon(\lambda - 1) \int_0^\varepsilon \sigma_A(v)[1 - \sigma_B(v) + \mu(\tilde{s}_A)\sigma_B(v)]dG(v) \\
&\geq 0
\end{aligned}$$

where the penultimate inequality holds since

$$\lambda(1 - \tilde{s}_A) - \varepsilon = \lambda((1 - \hat{s}_A) - (\tilde{s}_A - \hat{s}_A)) - \varepsilon > \lambda\varepsilon - \varepsilon = \varepsilon(\lambda - 1)$$

because $\tilde{s}_A - \hat{s}_A < \varepsilon$ and $2\varepsilon < 1 - \hat{s}_A$. \square ■

A.2 Proof of Lemma 2

“If” part. We show that the strategy profile satisfying the stated conditions forms a best response. First, let $\tilde{\sigma}_i(v) \in [0, 1]$ be an arbitrary strategy and define $\sigma_i(v, t) := t\tilde{\sigma}_i(v) + (1 - t)\sigma_i(v)$ for $t \in [0, 1]$ and for $i = A, B$. Let $\hat{s}_i(t)$ be the cutoff state in equilibrium for given $\sigma_i(v; t)$, and $S_i(t)$ be such that $S_A(t) := \{s \mid s \geq \hat{s}_A(t)\}$ and $S_B(t) := \{s \mid s \leq \hat{s}_B(t)\}$. Next, let

$$\begin{aligned}
W(t, \hat{s}_i(t)) &:= \int_0^1 v\sigma_i(v; t)(1 - \sigma_j(v) + \mathbb{E}[\mu_i(s)]\sigma_j(v))dG(v) \\
&\quad - \lambda \int_{S_i(t)} \left[\int_0^1 \sigma_i(v; t)(1 - \sigma_j(v) + \mu_i(s)\sigma_j(v))dG(v) - \kappa \right] ds,
\end{aligned}$$

and denote it by $V(t) := W(t, \hat{s}_i(t))$. Observe that $\pi_i(\tilde{\sigma}_i) = V(1)$ and $\pi_i(\sigma_i) = V(0)$. Therefore, the proof is completed by showing that $V(1) \leq V(0)$. Because $\tilde{\sigma}_i(\cdot)$ is arbitrary, this proves that $\sigma_i(\cdot)$ is a best response for a given $\sigma_j(\cdot)$, where $j \neq i$. To do this, we establish the following lemmas.

Lemma A1. $V(\cdot)$ is concave in t for any $t \in [0, 1]$.

Proof. Observe that $\sigma_i(\cdot, t)$ is linear in t , clearly. Hence, it suffices to show that π_i is concave in σ_i because if so, we have for any $\eta \in [0, 1]$ and $t, t' \in [0, 1]$,

$$\begin{aligned}
V(\eta t + (1 - \eta)t') &= \pi_i(\sigma_i(v, \eta t + (1 - \eta)t')) \\
&= \pi_i(\eta\sigma_i(v, t) + (1 - \eta)\sigma_i(v, t')) \\
&\geq \eta\pi_i(\sigma_i(v, t)) + (1 - \eta)\pi_i(\sigma_i(v, t')) \\
&= \eta V(t) + (1 - \eta)V(t'),
\end{aligned}$$

where the second equality follows from the linearity of $\sigma_i(\cdot, t)$. For the concavity of π_i , recall that

$$\begin{aligned} \pi_i &= \int_0^1 \left[\int_0^1 v \sigma_i(v) (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) dG(v) \right] ds \\ &\quad - \lambda \int_0^1 \max \left\{ \int_0^1 \sigma_i(v) (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) dG(v) - \kappa, 0 \right\} ds. \end{aligned} \tag{A.2.1}$$

Consider any feasible σ_i and σ'_i . Note that the first part of (A.2.1) is linear in σ_i clearly, and the second part is convex in σ_i , since for $\eta \in [0, 1]$,

$$\begin{aligned} &\max \left\{ \int_0^1 (\eta \sigma_i(v) + (1 - \eta) \sigma'_i(v)) (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) dG(v) - \kappa, 0 \right\} \\ &= \max \left\{ \eta \left[\int_0^1 \sigma_i(v) (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) dG(v) - \kappa \right] \right. \\ &\quad \left. + (1 - \eta) \left[\int_0^1 \sigma'_i(v) (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) dG(v) - \kappa \right], 0 \right\} \\ &\leq \eta \max \left\{ \int_0^1 \sigma_i(v) (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) dG(v) - \kappa, 0 \right\} \\ &\quad + (1 - \eta) \max \left\{ \int_0^1 \sigma'_i(v) (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) dG(v) - \kappa, 0 \right\}. \end{aligned}$$

Therefore, we have $\pi_i(\eta \sigma_i + (1 - \eta) \sigma'_i) \geq \eta \pi_i(\sigma_i) + (1 - \eta) \pi_i(\sigma'_i)$. ■

Lemma A2. $V'(0) \leq 0$.

Proof. Observe that

$$V'(t) = W_1(t, \hat{s}_i(t)) + W_2(t, \hat{s}_i(t)) \hat{s}'_i(t),$$

where

$$W_1(t, \hat{s}_i(t)) = \int_0^1 (\tilde{\sigma}_i(v) - \sigma_i(v)) \left[v (1 - \sigma_j(v) + \mathbb{E}[\mu_i(s)] \sigma_j(v)) - \lambda \int_{S_i(t)} (1 - \sigma_j(v) + \mu_i(s) \sigma_j(v)) ds \right] dG(v)$$

and

$$W_2(t, \hat{s}_i(t)) = \lambda \left[\int_0^1 \sigma_i(v; t) (1 - \sigma_j(v) + \mu_i(\hat{s}_i(t)) \sigma_j(v)) dG(v) - \kappa \right].$$

Notice that $W_2(0, \hat{s}_i(0)) = 0$ by definition of \hat{s}_i . Therefore, we have

$$V'(0) = W_1(0, \hat{s}_i(0)) = \int_0^1 (\tilde{\sigma}_i(v) - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) \leq 0,$$

where the inequality holds since if $H_i(v, \sigma_j(v)) > 0$ for some v , then $\sigma_i(v) = 1$ and $\tilde{\sigma}_i(v) \leq$

1; if $H_i(v, \sigma_j(v)) < 0$ for some v , then $\sigma_i(v) = 0$ and $\tilde{\sigma}_i(v) \geq 0$; and $H_i(v, \sigma_j(v)) = 0$ otherwise. \blacksquare

Now, note that

$$\pi_i(\tilde{\sigma}_i) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_i(\sigma_i),$$

where the first inequality follows from the concavity of $V(\cdot)$ and the second inequality follows from [Lemma A2](#). This completes the proof.

“Only if” part. We now show that in any competitive equilibrium, the strategy profile must satisfy the stated conditions. Let

$$\mathcal{V}_+ := \{v \mid H_i(v, \sigma_j(v)) > 0\} \quad \text{and} \quad \mathcal{V}_- := \{v \mid H_i(v, \sigma_j(v)) < 0\}.$$

Suppose to the contrary that in equilibrium, $\sigma_i(\cdot)$ does not satisfy either (i) or (ii) (or both); that is, either $\sigma_i(v) < 1$ for $v \in \mathcal{V}_+$, or $\sigma_i(v) > 0$ for $v \in \mathcal{V}_-$ (or both). Consider a deviating strategy $\tilde{\sigma}_i(\cdot)$ such that $\tilde{\sigma}_i(v) = 1$ for every $v \in \mathcal{V}_+$, $\tilde{\sigma}_i(v) = 0$ for every $v \in \mathcal{V}_-$, and $\tilde{\sigma}_i(v) = \sigma_i(v)$ for all other v 's. Now, define $\sigma_i(v, t) := t\tilde{\sigma}_i(v) + (1-t)\sigma_i(v)$ for $t \in [0, 1]$ and $V(t)$ similar as above. In what follows, we show that $V'(0) > 0$ so there exists $\sigma_i(\cdot, t)$ for small t that will be profitable. To see this, observe that

$$\begin{aligned} V'(0) &= W_1(0, \hat{s}_A(0)) \\ &= \int_{\mathcal{V}_+} (\tilde{\sigma}_i(v) - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) + \int_{\mathcal{V}_-} (\tilde{\sigma}_i(v) - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) \\ &\quad + \int_{\mathcal{V} \setminus \{\mathcal{V}_+ \cup \mathcal{V}_-\}} (\tilde{\sigma}_i(v) - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) \\ &= \int_{\mathcal{V}_+} (1 - \sigma_i(v)) H_i(v, \sigma_j(v)) dG(v) - \int_{\mathcal{V}_-} \sigma_i(v) H_i(v, \sigma_j(v)) dG(v) \\ &> 0, \end{aligned}$$

where the last equality follows from the construction of $\tilde{\sigma}_i(\cdot)$, and the inequality holds since $\tilde{\sigma}_i(v) = 1 > \sigma_i(v)$, $H_i(v, \sigma_j(v)) > 0$ for $v \in \mathcal{V}_+$, and $\tilde{\sigma}_i(v) = 0 < \sigma_i(v)$, $H_i(v, \sigma_j(v)) < 0$ for $v \in \mathcal{V}_-$.

A.3 Non-Competitive Equilibrium

In this section, we show that when $\kappa < \frac{1}{2}$ is not too small or $\lambda > 1$ is not too large, there does not exist a non-competitive equilibrium.

Lemma A3. *Suppose that \mathcal{V}_{AB} has zero measure. Then, we have the followings:*

- (i) *There is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, one college has an incentive to deviate.*
- (ii) *There is $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, one college has an incentive to deviate.*

Proof. Since \mathcal{V}_{AB} has zero measure, $m_i(s) = \kappa$ for all s and

$$\pi_i = \int_{\mathcal{V}_i} v dG(v), \quad i = A, B.$$

Now, let $\underline{c}_i := \inf(\mathcal{V}_i)$ and $\bar{c}_i := \sup(\mathcal{V}_i)$.

Proof of Part (i). Let $\underline{c}_A = \inf(\mathcal{V}_A \cup \mathcal{V}_B)$, without loss of generality. Then, $\underline{c}_A = G^{-1}(1 - 2\kappa)$ by [Lemma 1](#). We show that college A has an incentive to deviate. Suppose A rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but accepts those in $[\bar{c}_B - \varepsilon, \bar{c}_B]$, where ε and δ are such that

$$G(\bar{c}_B) - G(\bar{c}_B - \varepsilon) = G(\underline{c}_A + \delta) - G(\underline{c}_A). \quad (\text{A.3.1})$$

Note that the mass of students attending A under this deviation is

$$\begin{aligned} \tilde{m}_A(s) &= \int_{\bar{c}_B - \varepsilon}^{\bar{c}_B} \mu(s) dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} 1 dG(v) \\ &= \mu(s)(G(\bar{c}_B) - G(\bar{c}_B - \varepsilon)) + \kappa - (G(\underline{c}_A + \delta) - G(\underline{c}_A)) \\ &\leq \kappa, \end{aligned}$$

where the second equality holds since $m_A(s) = \kappa$ for all s , and the last inequality follows from [\(A.3.1\)](#) and the fact that $\mu(s) \leq 1$ for all s .

Since A is never over-demanded, its payoff from the deviation is

$$\tilde{\pi}_A = \bar{\mu} \int_{\bar{c}_B - \varepsilon}^{\bar{c}_B} v dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} v dG(v) = \bar{\mu} \int_{\bar{c}_B - \varepsilon}^{\bar{c}_B} v dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v dG(v),$$

where $\bar{\mu} = \mathbb{E}[\mu(s)]$. Therefore,

$$\begin{aligned} \tilde{\pi}_A - \pi_A &= \bar{\mu} \int_{\bar{c}_B - \varepsilon}^{\bar{c}_B} v dG(v) - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v dG(v) \\ &= \bar{\mu} \left[\bar{c}_B G(\bar{c}_B) - (\bar{c}_B - \varepsilon) G(\bar{c}_B - \varepsilon) - \int_{\bar{c}_B - \varepsilon}^{\bar{c}_B} G(v) dv \right] \\ &\quad - \left[(\underline{c}_A + \delta) G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \int_{\underline{c}_A}^{\underline{c}_A + \delta} G(v) dv \right] \\ &> \bar{\mu} \left[\bar{c}_B G(\bar{c}_B) - (\bar{c}_B - \varepsilon) G(\bar{c}_B - \varepsilon) - \varepsilon G(\bar{c}_B) \right] \\ &\quad - \left[(\underline{c}_A + \delta) G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \delta G(\underline{c}_A) \right] \end{aligned}$$

$$= (G(\bar{c}_B) - G(\bar{c}_B - \varepsilon)) (\bar{\mu} \bar{c}_B - \underline{c}_A - \bar{\mu} \varepsilon - \delta), \quad (\text{A.3.2})$$

where the second equality follows from the integration by parts, and the last equality follows from (A.3.1). Observe that if $\bar{\mu} > \frac{\underline{c}_A}{\bar{c}_B}$, then (A.3.2) is strictly positive for sufficiently small ε and δ , hence $\tilde{\pi}_A > \pi_A$. Note that since $\underline{c}_A = G^{-1}(1 - 2\kappa)$ and $m_i(s) = \kappa$ for all s and $i = A, B$, we have that $G(\bar{c}_B) \geq 1 - \kappa$; that is, $\bar{c}_B \geq G^{-1}(1 - \kappa)$. (Otherwise, college A must be admitting more than measure κ of students.) Therefore,

$$\frac{\underline{c}_A}{\bar{c}_B} \leq \frac{G^{-1}(1 - 2\kappa)}{G^{-1}(1 - \kappa)}. \quad (\text{A.3.3})$$

Since the RHS of (A.3.3) is continuous in κ and converges to zero as κ approaches to $\frac{1}{2}$, there is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, $\bar{\mu} > \frac{\underline{c}_A}{\bar{c}_B}$ for any given $\bar{\mu}$. \square

Proof of Part (ii). Let $\bar{c}_B = \sup(\mathcal{V}_A \cup \mathcal{V}_B)$, without loss of generality. Then, $\bar{c}_B = 1$ by Lemma 1. We show that college A has an incentive to deviate. Suppose A rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but admits students in $[1 - \varepsilon, 1]$, where ε and δ satisfy

$$\mu(1 - \underline{c}_A)(1 - G(1 - \varepsilon)) = G(\underline{c}_A + \delta) - G(\underline{c}_A). \quad (\text{A.3.4})$$

The mass of students attending A in state s under the deviation is

$$\tilde{m}_A(s) = \int_{1-\varepsilon}^1 \mu(s) dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} 1 dG(v) = \mu(s)(1 - G(1 - \varepsilon)) + \kappa - (G(\underline{c}_A + \delta) - G(\underline{c}_A)).$$

Let \hat{s}_A be such that $\tilde{m}_A(\hat{s}_A) = \kappa$, i.e., $\mu(\hat{s}_A)(1 - G(1 - \varepsilon)) = (G(\underline{c}_A + \delta) - G(\underline{c}_A))$. Since $\mu(\cdot)$ is strictly increasing in s , $\hat{s}_A = 1 - \underline{c}_A$ by (A.3.4).

Thus, A 's payoff from the deviation is

$$\begin{aligned} \tilde{\pi}_A &= \bar{\mu} \int_{1-\varepsilon}^1 v dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A + \delta]} v dG(v) - \lambda \int_{\hat{s}_A}^1 (\tilde{m}(s) - \kappa) ds \\ &= \bar{\mu} \int_{1-\varepsilon}^1 v dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v dG(v) \\ &\quad - \lambda \left[(1 - G(1 - \varepsilon)) \int_{\hat{s}_A}^1 \mu(s) ds - (G(\underline{c}_A + \delta) - G(\underline{c}_A))(1 - \hat{s}_A) \right]. \end{aligned}$$

and the net payoff from the deviation is

$$\tilde{\pi}_A - \pi_A = \bar{\mu} \int_{1-\varepsilon}^1 v dG(v) - \int_{\underline{c}_A}^{\underline{c}_A + \delta} v dG(v)$$

$$\begin{aligned}
& -\lambda \left[(1 - G(1 - \varepsilon)) \int_{\hat{s}_A}^1 \mu(s) ds - (G(\underline{c}_A + \delta) - G(\underline{c}_A))(1 - \hat{s}_A) \right] \\
& > \bar{\mu}(1 - \varepsilon)[1 - G(1 - \varepsilon)] - (\underline{c}_A + \delta)(G(\underline{c}_A + \delta) - G(\underline{c}_A)) \\
& -\lambda \left[(1 - G(1 - \varepsilon)) \int_{\hat{s}_A}^1 \mu(s) ds - (G(\underline{c}_A + \delta) - G(\underline{c}_A))(1 - \hat{s}_A) \right] \\
& = [1 - G(1 - \varepsilon)] \left(\bar{\mu} - \eta \underline{c}_A - \bar{\mu} \varepsilon - \eta \delta - \lambda \left[\int_{\hat{s}_A}^1 \mu(s) ds - \eta(1 - \hat{s}_A) \right] \right), \tag{A.3.5}
\end{aligned}$$

where $\eta = \mu(1 - \underline{c}_A)$ and the last equality follows from (A.3.4).

Observe that if $\bar{\mu} - \eta \underline{c}_A - \lambda \left[\int_{\hat{s}_A}^1 \mu(s) ds - \eta(1 - \hat{s}_A) \right] > 0$, then (A.3.5) is strictly positive for sufficiently small ε and δ . Note that

$$\bar{\mu} - \eta \underline{c}_A - \lambda \left[\int_{\hat{s}_A}^1 \mu(s) ds - \eta(1 - \hat{s}_A) \right] = \bar{\mu} - \lambda \int_{\hat{s}_A}^1 \mu(s) ds + (\lambda - 1) \eta \underline{c}_A,$$

Since $\bar{\mu} = \int_0^1 \mu(s) ds > \int_{\hat{s}_A}^1 \mu(s) ds$ (which follows from the fact that $\hat{s}_A < 1$), there exists $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, $\tilde{\pi}_A > \pi_A$. \square ■

A.4 Proof of Lemma 3

Observe that $H_i(\cdot, x)$ is strictly increasing in v , since for $v' > v$,

$$H_i(v', x) - H_i(v, x) = (1 - x + \mathbb{E}[\mu_i(s)]x)(v' - v) > 0,$$

where the inequality holds since $\mathbb{E}[\mu_i(s)] > 0$ because $\mu(\cdot)$ is strictly increasing in s .

Next, $H_i(v, x)$ satisfies the strict single crossing property with respect to x ; that is, if $H_i(v, x) \leq 0$ for some $x \in (0, 1)$, then $H_i(v, x') < 0$ for any $x' > x$. Suppose for any $x \in (0, 1)$,

$$H_i(v, x) = (1 - x)(v - \lambda \text{Prob}(s \in S_i)) + x \mathbb{E}[\mu_i(s)] \left(v - \lambda \text{Prob}(s \in S_i) \frac{\mathbb{E}[\mu_i(s)|s \in S_i]}{\mathbb{E}[\mu_i(s)]} \right) \leq 0. \tag{A.4.1}$$

Consider any $x' > x$. If $v < \lambda \text{Prob}(s \in S_i)$, then

$$H_i(v, x') = (1 - x')(v - \lambda \text{Prob}(s \in S_i)) + x' \mathbb{E}[\mu_i(s)] \left(v - \lambda \text{Prob}(s \in S_i) \frac{\mathbb{E}[\mu_i(s)|s \in S_i]}{\mathbb{E}[\mu_i(s)]} \right) < 0,$$

where the inequality follows from (A.4.1) and the facts that $x' > x$ and $\mathbb{E}[\mu_i(s)|s \in S_i] > \mathbb{E}[\mu_i(s)]$. If $v \geq \lambda \text{Prob}(s \in S_i)$, then

$$H_i(v, x) - H_i(v, x') = (x' - x) \left[v(1 - \mathbb{E}[\mu_i(s)]) - \lambda \text{Prob}(s \in S_i)(1 - \mathbb{E}[\mu_i(s)|s \in S_i]) \right]$$

$$\begin{aligned}
&> (x' - x)(v - \lambda \text{Prob}(s \in S_i))(1 - \mathbb{E}[\mu_i(s)|s \in S_i]) \\
&\geq 0,
\end{aligned}$$

where the first inequality holds since $x' > x$ and $\mathbb{E}[\mu_i(s)|s \in S_i] > \mathbb{E}[\mu_i(s)]$, and the second inequality holds since $v \geq \lambda \text{Prob}(s \in S_i)$. Since $H_i(v, x) \leq 0$, we have $H_i(v, x') < 0$.

A.5 Proof of Theorem 2

Step 1. Existence of a profile of admission strategies (σ_A, σ_B) that forms local best responses.

We first establish existence of a profile of admission strategies (σ_A, σ_B) such that for each $v \in [0, 1]$, $\sigma_i(v)$ is given by (3.1) for $i = A, B$. Now, fix any $\hat{s} = (\hat{s}_A, \hat{s}_B) \in S \equiv [0, 1]^2$ and consider the resulting profile $(\sigma_A(\cdot; \hat{s}), \sigma_B(\cdot; \hat{s}))$. This strategy profile in turn induces the mass of students enrolling in each college i :

$$m_i(s; \hat{s}) = \int_0^1 \sigma_i(v; \hat{s}) [1 - \sigma_j(v; \hat{s}) + \mu_i(s) \sigma_j(v; \hat{s})] dG(v).$$

Observe that $m_A(\cdot; \hat{s})$ and $m_B(\cdot; \hat{s})$ in turn yield a new profile of cutoff states:

$$\tilde{s}_A = \inf \{s \in [0, 1] | m_A(s; \hat{s}) - \kappa > 0\}, \quad (\text{A.5.1})$$

if the set in the RHS is nonempty, or else $\tilde{s}_A \equiv 1$, and

$$\tilde{s}_B = \sup \{s \in [0, 1] | m_B(s; \hat{s}) - \kappa > 0\}, \quad (\text{A.5.2})$$

if the set in the RHS is nonempty, or else $\tilde{s}_B \equiv 0$.

Next, define a mapping T such that $T(\hat{s}) = \tilde{s}$, where $\tilde{s} = (\tilde{s}_A, \tilde{s}_B)$ is given by (A.5.1) and (A.5.2). The next lemma shows that T is continuous. Therefore, it has a fixed point by the Brouwer's fixed point theorem. From the construction of T , it is immediate that given the fixed point, say $\hat{s}^* = (\hat{s}_A^*, \hat{s}_B^*)$, the profile $(\sigma_A(\cdot; \hat{s}^*), \sigma_B(\cdot; \hat{s}^*))$ satisfies the local incentives.

Lemma A4. $T(\cdot)$ is continuous in s for $s \in S$.

Proof. Note that \bar{v}_A and \underline{v}_A are continuous in \hat{s}_A , and \bar{v}_B and \underline{v}_B are continuous in \hat{s}_B . Let

$$\underline{v} := \min \{\underline{v}_A, \underline{v}_B\}, \quad \tilde{v} := \max \{\underline{v}_A, \underline{v}_B\}, \quad \hat{v} := \min \{\bar{v}_A, \bar{v}_B\}, \quad \bar{v} := \max \{\bar{v}_A, \bar{v}_B\}.$$

For any given \hat{s} , $T(\hat{s}) = \tilde{s}$ is given by (A.5.1) and (A.5.2). Consider any $\hat{s}' = (\hat{s}'_A, \hat{s}'_B) \in S$,

where $\hat{s}' \neq \hat{s}$. Then, $\sigma'_A \equiv \sigma_A(\cdot; \hat{s}')$ and $\sigma'_B \equiv \sigma_B(\cdot; \hat{s}')$ are defined by (3.1). Let

$$\underline{v}' := \min \{\underline{v}'_A, \underline{v}'_B\}, \quad \check{v}' := \max \{\underline{v}'_A, \underline{v}'_B\}, \quad \hat{v}' := \min \{\bar{v}'_A, \bar{v}'_B\}, \quad \bar{v}' := \max \{\bar{v}'_A, \bar{v}'_B\}.$$

Again, $\tilde{s}' = (\tilde{s}'_A, \tilde{s}'_B) \in S$ is defined by T through (A.5.1) and (A.5.2). Next, let

$$\begin{aligned} v_1 &:= \min \{v, v'\}, & v_2 &:= \max \{v, v'\}, & v_3 &:= \min \{\check{v}, \check{v}'\}, & v_4 &:= \max \{\check{v}, \check{v}'\}, \\ v_5 &:= \min \{\hat{v}, \hat{v}'\}, & v_6 &:= \max \{\hat{v}, \hat{v}'\}, & v_7 &:= \min \{v, \bar{v}'\}, & v_8 &:= \max \{\bar{v}, \bar{v}'\}, \end{aligned}$$

and consider a partition of $[0, 1]$ such that

$$\mathcal{V}_1 = (\cup_{i=2,4,6,8} [v_{i-1}, v_i]) \cap [0, 1], \quad \mathcal{V}_2 = [v_4, v_5] \cap [0, 1], \quad \mathcal{V}_3 = [0, 1] \setminus (\mathcal{V}_1 \cup \mathcal{V}_2).$$

Consider σ_i and σ'_i for $i = A, B$. For any $v \in [0, 1]$, we have

$$\int_0^1 |\sigma'_i(v) - \sigma_i(v)| dG(v) = \sum_{k=1}^3 \int_0^1 |\sigma'_i(v) - \sigma_i(v)| \mathbf{1}_{\mathcal{V}_k}(v) dG(v),$$

where $\mathbf{1}_{\mathcal{V}_k}(v)$ is 1 if $v \in \mathcal{V}_k$ or 0 otherwise. Observe, first, that by the continuity of \underline{v}_i and \bar{v}_i , there is $\delta_1 > 0$ such that for any $\varepsilon > 0$, if $\|\hat{s}' - \hat{s}\| < \delta_1$, then

$$\int_0^1 \mathbf{1}_{\mathcal{V}_1}(v) dG(v) < \frac{\varepsilon}{6}. \tag{A.5.3}$$

Second, for any $v \in \mathcal{V}_2$, the continuity of $\sigma_i^o(\cdot)$, given by

$$\sigma_i^o(v) := \frac{v - \lambda \text{Prob}(s \in S_j)}{v(1 - \mathbb{E}[\mu_j(s)]) - \lambda \text{Prob}(s \in S_j) \left(1 - \mathbb{E}[\mu_j(s) | s \in S_j]\right)},$$

implies that there is δ_2 such that $\|\hat{s}' - \hat{s}\| < \delta_2$ implies

$$|\sigma'_i(v) - \sigma_i(v)| = \left| \sigma_i^o(v) - \sigma_i^o(v) \right| < \frac{\varepsilon}{6}, \tag{A.5.4}$$

Lastly, for any $v \in \mathcal{V}_3$, $\sigma'_i(v)$ and $\sigma_i(v)$ are either 0 or 1 at the same time, hence we have that

$$|\sigma'_i(v) - \sigma_i(v)| = 0. \tag{A.5.5}$$

Now, let $\delta := \min \{\delta_1, \delta_2\}$ and suppose $\|\hat{s}' - \hat{s}\| < \delta$. Then, we have

$$\int_0^1 |\sigma'_i(v) - \sigma_i(v)| dG(v) = \int_0^1 |\sigma'_i(v) - \sigma_i(v)| \mathbf{1}_{\mathcal{V}_1}(v) dG(v) + \int_0^1 \left| \sigma_i^o(v) - \sigma_i^o(v) \right| \mathbf{1}_{\mathcal{V}_2}(v) dG(v)$$

$$< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}, \quad (\text{A.5.6})$$

where the equality follows from (A.5.5) and the inequality follows from (A.5.3) and (A.5.4).

Observe that

$$\begin{aligned} & \left| m_i(s; \hat{s}') - m_i(s; \hat{s}) \right| \\ &= \left| \int_0^1 \sigma'_i(v)[1 - \sigma'_j(v) + \mu_i(s)\sigma'_j(v)] dG(v) - \int_0^1 \sigma_i(v)[1 - \sigma_j(v) + \mu_i(s)\sigma_j(v)] dG(v) \right| \\ &= \left| \int_0^1 \left[[\sigma'_i(v) - \sigma_i(v)] - (1 - \mu_i(s))[\sigma'_i(v)\sigma'_j(v) - \sigma_i(v)\sigma_j(v)] \right] dG(v) \right| \\ &\leq \int_0^1 |\sigma'_i(v) - \sigma_i(v)| dG(v) + (1 - \mu_i(s)) \int_0^1 |\sigma'_i(v)\sigma'_j(v) - \sigma_i(v)\sigma_j(v)| dG(v) \quad (\text{A.5.7}) \end{aligned}$$

The first part of (A.5.7) is smaller than $\varepsilon/3$ by (A.5.6). The second part of (A.5.7) is

$$\begin{aligned} & \int_0^1 |\sigma'_i(v)\sigma'_j(v) - \sigma_i(v)\sigma_j(v)| dG(v) \\ &= \int_0^1 |\sigma'_i(v)\sigma'_j(v) - \sigma'_i(v)\sigma_j(v) + \sigma'_i(v)\sigma_j(v) - \sigma_i(v)\sigma_j(v)| dG(v) \\ &\leq \int_0^1 |\sigma'_j(v) - \sigma_j(v)| dG(v) + \int_0^1 |\sigma'_i(v) - \sigma_i(v)| dG(v) \\ &< \frac{2}{3}\varepsilon, \end{aligned}$$

where the first inequality holds since $\sigma'_i(v), \sigma_j(v) \leq 1$, and the last inequality follows from (A.5.6). Therefore, if $\|\hat{s}' - \hat{s}\| < \delta$, then

$$\left| \int_0^1 \sigma'_i(v)[1 - \sigma'_j(v) + \mu_i(s)\sigma'_j(v)]dG(v) - \int_0^1 \sigma_i(v)[1 - \sigma_j(v) + \mu_i(s)\sigma_j(v)]dG(v) \right| < \varepsilon. \quad (\text{A.5.8})$$

Hence, we conclude that there is $\delta > 0$ such that for any $\varepsilon > 0$, if $\|\hat{s}' - \hat{s}\| < \delta$, then $\|\tilde{s}' - \tilde{s}\| < \varepsilon$. Since \hat{s} is chosen arbitrary, T is continuous on S . \blacksquare

Step 2. \mathcal{V}_{AB} has a positive measure in the strategy profile identified in Step 1.

Suppose to the contrary that \mathcal{V}_{AB} has measure zero. Then, $\hat{s}_B^* = 0$ and $\hat{s}_A^* = 1$. But in that case, $H_A(v, 1) > 0$ and $H_B(v, 1) > 0$ for all v . Hence, $\bar{v}_A = \bar{v}_B = 0$. Therefore, we cannot have a non-competitive equilibrium.

A.6 Proof of Theorem 3

Let $\mu(s, t)$ be the mass of students preferring A over B in state s for a given $t \in [0, 1]$. We assume that (i) $\mu(\cdot, \cdot)$ is continuous in (s, t) ; (ii) $\mu(\cdot, t)$ is strictly increasing in s for a given t , as before; and (iii) $\mu(s, 0)$ is symmetric, i.e., $\mu(s, 0) = 1 - \mu(s, 0)$ for all s .

Now, for a given t , let $T(\cdot, t)$ be a map from state space to itself as in [Appendix A.5](#). Let $S(t)$ be the set of fixed points in competitive equilibrium, and $\underline{v}_i(\hat{s}(t))$ and $\bar{v}_i(\hat{s}(t))$, $i = A, B$, be the associated cutoffs for $\hat{s}(t) \in S(t)$, where $\hat{s}(t) = (\hat{s}_A(t), \hat{s}_B(t))$.

The proof consists of two steps. We first show that when $t = 0$ (i.e., $\mu(\cdot, 0)$ is symmetric), any competitive equilibrium exhibits strategic targeting. We next show that there is $\eta > 0$ such that for any $t < \eta$, every competitive equilibrium involves strategic targeting.

Step 1. *At $t = 0$, any competitive equilibrium exhibits strategic targeting.*

Proof. Let $t = 0$. Suppose to the contrary that $\hat{v} \leq \check{v}$ in a competitive equilibrium. Suppose further that³³

$$\underline{v}_B < \bar{v}_B \leq \underline{v}_A < \bar{v}_A, \quad (\text{A.6.1})$$

without loss of generality, where the first and the last strict inequalities follows from [Theorem 1](#). Note that we must have $\bar{v}_A \in (0, 1)$ in equilibrium, since if $\bar{v}_A = 1$, then $m_A(s) = 0$ for all s , and if $\bar{v}_A = 0$, then $\underline{v}_B = \bar{v}_B = \underline{v}_A = \bar{v}_A = 0$, so $\inf(\mathcal{V}_A \cup \mathcal{V}_B) = 0$, which contradicts [Lemma 1](#).

In equilibrium, we have

$$m_A(\hat{s}_A) = \mu(\hat{s}_A, 0)[1 - G(\bar{v}_A)] = \kappa, \quad (\text{A.6.2})$$

and

$$m_B(\hat{s}_B) = (1 - \mu(\hat{s}_B, 0))[1 - G(\bar{v}_A)] + G(\bar{v}_A) - G(\underline{v}_B) = \kappa. \quad (\text{A.6.3})$$

From [\(A.6.2\)](#), $1 - G(\bar{v}_A) = \frac{\kappa}{\mu(\hat{s}_A)}$. Substituting this into [\(A.6.3\)](#), we have

$$G(\bar{v}_A) - G(\underline{v}_B) = \kappa \left(\frac{\mu(\hat{s}_A, 0) + \mu(\hat{s}_B, 0) - 1}{\mu(\hat{s}_A, 0)} \right).$$

Since $\underline{v}_B < \bar{v}_A$, this implies

$$\mu(\hat{s}_A, 0) + \mu(\hat{s}_B, 0) > 1 \Leftrightarrow \mu(\hat{s}_B, 0) > 1 - \mu(\hat{s}_A, 0) = \mu(1 - \hat{s}_A, 0),$$

where the last equality follows from the symmetry of $\mu(\cdot, 0)$. Since $\mu(\cdot, 0)$ is strictly increasing, we have $\hat{s}_B > 1 - \hat{s}_A$, and so $\underline{v}_B = \lambda \hat{s}_B > \lambda(1 - \hat{s}_A) = \underline{v}_A$ which contradicts [\(A.6.1\)](#). \blacksquare

³³At $t = 0$, equilibrium cutoffs depend on $\hat{s}_i(0)$; that is, $\underline{v}_i(\hat{s}(0))$ and $\bar{v}_i(\hat{s}(0))$. For notational simplicity, we suppress the dependence and denote them by \underline{v}_i and \bar{v}_i , respectively.

Step 2. *There exists $\eta > 0$ such that for any $t < \eta$, every competitive equilibrium under $\mu(\cdot, t)$ involves strategic targeting.*

Proof. Let V_0 be the set of limit cutoff points at $t = 0$, i.e., for $i = A, B$,

$$V_0 := \{(\underline{v}_i, \bar{v}_i) \mid \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \min\{|\underline{v}_i(\hat{s}(t)) - \underline{v}_i|, |\bar{v}_i(\hat{s}(t)) - \bar{v}_i|\} < \varepsilon, \forall t < \delta, \forall \hat{s}(t) \in S(t)\}.$$

Then, for any $v = (\underline{v}_i, \bar{v}_i)_{i=A,B} \in V_0$, there is a sequence of t_n going to zero such that $\hat{s}(t_n)$ converges, and $v(\hat{s}(t_n)) = (\underline{v}_i(\hat{s}(t_n)), \bar{v}_i(\hat{s}(t_n)))_{i=A,B}$ converges to v .³⁴ Let \hat{s}^* be the limit of $\{\hat{s}(t_n)\}$. Then,

$$\hat{s}^* := \lim_{n \rightarrow \infty} \hat{s}(t_n) = \lim_{n \rightarrow \infty} T(\hat{s}(t_n), t_n) = T(\hat{s}^*, 0),$$

where the second equality follows from the fact that $\hat{s}(t_n)$ is a fixed point of T , and the last one follows from the continuity of T . Therefore, \hat{s}^* is an equilibrium at $t = 0$. This shows that every $v = (\underline{v}_i(\hat{s}^*), \bar{v}_i(\hat{s}^*))_{i=A,B} \in V_0$ is supported as an equilibrium when $t = 0$, so it must involve strategic targeting by Step 1 above.

We now show that $S(0)$, the set of fixed points at $t = 0$, is closed. To this end, consider any sequence $\hat{s}_n \in S(0)$ for each n such that $|\hat{s}_n - \hat{s}| \rightarrow 0$ as $n \rightarrow \infty$ for some \hat{s} . Then,

$$|T(\hat{s}, 0) - \hat{s}| \leq |T(\hat{s}, 0) - T(\hat{s}_n, 0)| + |T(\hat{s}_n, 0) - \hat{s}| = |T(\hat{s}, 0) - T(\hat{s}_n, 0)| + |\hat{s}_n - \hat{s}|,$$

where the last equality holds since $T(\hat{s}_n, 0) = \hat{s}_n$. From the continuity of T , we have that $|T(\hat{s}, 0) - T(\hat{s}_n, 0)| \rightarrow 0$ as $n \rightarrow \infty$, implying $T(\hat{s}, 0) = \hat{s}$, and so $\hat{s} \in S(0)$.

Next, observe that

$$\bar{V}_0 := \{v \mid v = (\underline{v}_i(\hat{s}), \bar{v}_i(\hat{s}))_{i=A,B} \text{ for } \hat{s} \in S(0)\}$$

is closed since v is continuous and $S(0)$ is closed, implying that \bar{V}_0 is compact. Hence,

$$\xi := \min_{v \in \bar{V}_0} [\min\{\bar{v}_A, \bar{v}_B\} - \max\{\underline{v}_A, \underline{v}_B\}]$$

is well defined. Furthermore, $\xi > 0$ due to Step 1. Lastly, from the definitions of V_0 and \bar{V}_0 , the above arguments show that $V_0 \subseteq \bar{V}_0$. Hence, for each $v \in V_0$, we have

$$\min\{\bar{v}_A, \bar{v}_B\} - \max\{\underline{v}_A, \underline{v}_B\} \geq \xi > 0,$$

implying that there exists $\eta > 0$ such that for any $t < \eta$, every competitive equilibrium under $\mu(\cdot, t)$ involves strategic targeting. ■

³⁴Since $\hat{s}(t_n) \in [0, 1]$ lies in a compact set, a sequence of $\hat{s}(t_n)$ admits a convergent subsequence. For any such converging (sub)sequence, $v(\hat{s}(t_n))$ is convergent since v is continuous as shown in [Lemma A4](#).

A.7 Proof of **Theorem 4**

Proof of Part (i). Recall that there are cutoff states (\hat{s}_A, \hat{s}_B) such that colleges have a mass of unfilled seats in a positive measure of states, $[0, \hat{s}_A)$ for A and $(\hat{s}_B, 1]$ for B , despite the fact that there are unmatched and acceptable students ($\inf(\mathcal{V}_A \cup \mathcal{V}_B) > 0$ in **Lemma 1**). By assigning those unmatched students to a college with excess capacity, both the students and college are better off. Thus, it is student, college and Pareto inefficient. ■

Proof of Part (ii). Suppose a MME exhibits strategic targeting; i.e., $\check{v} = \max\{v_A, v_B\} < \hat{v} = \min\{\bar{v}_A, \bar{v}_B\}$. Fix a state s such that $\mu(s) \neq 0, 1$. For students in $[\check{v}, \hat{v}]$, there is a positive measure of students who are assigned to a college, say B , but prefer A , and their scores are higher than those of a positive measure of students who are assigned to A , even though both colleges prefer the high-score students. Moreover, students in $[\check{v}, \hat{v}]$ get no admission from either college with positive probabilities even if their scores are high. Thus, it entails justified envy for a positive measure of students for almost every state.

Suppose now a MME does not exhibit strategic targeting; i.e., $\hat{v} < \check{v}$. Let $\bar{v}_B < v_A$, as depicted in **Figure 3.3**, without loss of generality, so students in $[v_B, \bar{v}_A]$ admitted only by college B and those in $(\bar{v}_A, 1]$ are admitted by both colleges. Observe that only the students who are not admitted by either college or admitted only by B may have envies. However, the students whom they envy have higher scores. So, no justified envy arises in any state s , making the outcome fair. ■

Proof of Part (iii). This part follows directly from **Corollary 1**. Let $k := \arg \max\{v_A, v_B\}$, $\ell \neq k$, $i \in \arg \max\{\bar{v}_A, \bar{v}_B\}$, and $j \neq i$, where $k, \ell, i, j = A, B$. For almost every state s , a positive measure of students with $v \in (\hat{v}, \bar{v}_i)$ prefer college i to college j , but are not admitted by i . These students have ex ante justified envy with students with $\hat{v} = \bar{v}_j$, who, as noted in **Corollary 1**, are admitted by college i with probability one.

Also, for almost every state s , a positive measure of students with $v \in [\check{v}, \check{v} + \epsilon)$, for some $\epsilon > 0$, prefer college ℓ to college k but are not admitted by ℓ with positive probability. These students have ex ante justified envy with students with (v_ℓ, \check{v}) , who, as noted in **Corollary 1**, are admitted by college i with probability one. ■

Proof of Part (iv). Consider any non-competitive equilibrium. For each state s except $\mu(s) = 0$ or 1 , the equilibrium must admit a positive measure of students who prefer A but are assigned to B and a positive measure of students who are assigned to A but have scores lower than those of the first group of students; that is, justified envy exists. Since justified envy arises for a positive measure of students for almost every state, the outcome is unfair. Also, for almost every state, there must be a positive measure of students assigned to A but prefer B and a positive measure of students assigned to B but prefer A . Thus, the outcome is student inefficient.

Next, the equilibrium is college efficient. To see this, recall that in any non-competitive equilibrium, almost all top 2κ students are assigned to either college. Suppose to the contrary that for a given state, there is another assignment that makes both colleges weakly better off and at least one college strictly better off. Then, it must also admit almost all top 2κ students, or else at least one college is strictly worse off. Therefore, it is a reallocation of the initial assignment, hence if one college is strictly better off, then the other college must be strictly worse off. Thus, we reach a contraction. ■

Proof of Part (v). Suppose that almost all top κ students are assigned to one college, and the next top κ students are assigned to the other college. Then, any change of assignments by positive measure of students will leave the former college strictly worse off, hence it is Pareto efficient.

Suppose this is not the case in a non-competitive equilibrium. Note that for a fixed s , there are some $\mathcal{V}'_i, \mathcal{V}''_i \subset \mathcal{V}_i$ and $\mathcal{V}'_j \subset \mathcal{V}_j$, $i \neq j$, all with positive measures, such that $v' < \hat{v} < v''$ whenever $v' \in \mathcal{V}'_i$, $v'' \in \mathcal{V}''_i$ and $\hat{v} \in \mathcal{V}'_j$. Let $i = A$ and $j = B$ without loss of generality. We can choose $\mathcal{V}'_A, \mathcal{V}''_A$ and \mathcal{V}'_B that satisfy

$$\frac{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v dG(v)}{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} 1 dG(v)} = \frac{\int_{\mathcal{V}'_B} v dG(v)}{\int_{\mathcal{V}'_B} 1 dG(v)} \quad (\text{A.7.1})$$

and

$$(1 - \mu(s)) \int_{\mathcal{V}'_A \cup \mathcal{V}''_A} 1 dG(v) = \mu(s) \int_{\mathcal{V}'_B} 1 dG(v). \quad (\text{A.7.2})$$

(If either (A.7.1) or (A.7.2) is violated, we can adjust $\mathcal{V}'_A, \mathcal{V}''_A$ and/or \mathcal{V}'_B by adding or subtracting a positive mass of students.) Note that the LHS (resp. RHS) of (A.7.2) is the measure of students who prefer B (resp. A) in $\mathcal{V}'_A \cup \mathcal{V}''_A$ (resp. \mathcal{V}'_B). From (A.7.1), we have

$$\begin{aligned} & \frac{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}'_A \cup \mathcal{V}''_A} 1 dG(v)} = \frac{\int_{\mathcal{V}'_B} v dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}'_B} 1 dG(v)} \\ \Leftrightarrow & \frac{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v dG(v)}{\mu(s) \int_{\mathcal{V}'_B} 1 dG(v)} = \frac{\int_{\mathcal{V}'_B} v dG(v)}{(1 - \mu(s)) \int_{\mathcal{V}'_B} 1 dG(v)} \\ \Leftrightarrow & (1 - \mu(s)) \int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v dG(v) = \mu(s) \int_{\mathcal{V}'_B} v dG(v), \end{aligned}$$

where the first equivalence follows from (A.7.2). The last equivalence shows that the average value of students who prefer B in $\mathcal{V}'_A \cup \mathcal{V}''_A$ is the same as that of students who prefer A in \mathcal{V}'_B . Thus, in state s , a fraction $1 - \mu(s)$ of students in $\mathcal{V}'_A \cup \mathcal{V}''_A$ who prefer B to A can be swapped with a fraction of $\mu(s)$ of students in \mathcal{V}'_B who prefer A to B . This reassignment leaves both colleges the same in welfare and makes all students weakly better off and some

positive measure of students strictly better off. Since this argument holds for all s except $\mu(s) = 0$ or 1 , the outcome is Pareto inefficient. \blacksquare

A.8 Proof of **Theorem 5** and Existence of a Cutoff Equilibrium

A.8.1 Proof of **Theorem 5**

Suppose there is a cutoff equilibrium with strategy profiles (σ_A, σ_B) where $\sigma_i(v, e_i) = \mathbb{1}_{\{e_i \geq \eta(v)\}}$, $i = A, B$, for some $\eta_i(\cdot)$ which is nonincreasing. Since $\partial U_i / \partial e_i > 0$, by the Implicit Function Theorem, $H_i(v, e_i, \bar{\sigma}_j(v)) = 0$, $j \neq i$, implicitly defines $\eta_i(v)$. Since $\mathbb{E}[\mu_i(s) | s \in S_i] > \mathbb{E}[\mu_i(s)]$, we must have

$$1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s) | s \in S_i] \bar{\sigma}_j(v) > 1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)] \bar{\sigma}_j(v).$$

Then, $H_i(v, \eta_i(v), \bar{\sigma}_j(v)) = 0$ implies that by (4.1),

$$U_i(v, \eta_i(v)) > \lambda \text{Prob}(s \in S_i). \quad (\text{A.8.1})$$

Next, totally differentiate H_i to obtain:

$$\begin{aligned} & \frac{\partial U_i(v, \eta_i(v))}{\partial v} + \frac{\partial U_i(v, \eta_i(v))}{\partial e_i} \eta_i'(v) \\ &= \frac{[U_i(v, \eta_i(v))(1 - \mathbb{E}[\mu_i(s)]) - \lambda \text{Prob}(s \in S_i)(1 - \mathbb{E}[\mu_i(s) | s \in S_i])] \bar{\sigma}_j'(v)}{1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)] \bar{\sigma}_j(v)}. \end{aligned} \quad (\text{A.8.2})$$

Since college j adopts a cutoff strategy, $\bar{\sigma}_j(v) = 1 - X_j(\eta_j(v) | v)$, we have that

$$\bar{\sigma}_j'(v) = -x_j(\eta_j(v) | v) \eta_j'(v) - \frac{\partial X_j(\eta_j(v) | v)}{\partial v} > 0, \quad (\text{A.8.3})$$

where the inequality holds since $\eta_j'(v) \leq 0$ and $\frac{\partial}{\partial v} X_j(e | v) < 0$.³⁵ Further, $\mathbb{E}[\mu_i(s)] < \mathbb{E}[\mu_i(s) | s \in S_i] \leq 1$, so it follows from (A.8.1) that the RHS of (A.8.2) is strictly positive for any v such that $\eta_i(v) \in (0, 1)$. Hence, for all v ,

$$-\eta_i'(v) \leq \frac{\partial U_i(v, \eta_i(v)) / \partial v}{\partial U_i(v, \eta_i(v)) / \partial e_i},$$

and the inequality is strict for a positive measure of v .

³⁵When v and e_i are independent, $\bar{\sigma}_j'(v) = -x_j(\eta_j(v)) \eta_j'(v) \geq 0$. This implies that each college under-weights a students' common performance and over-weights her non-common performance at least weakly and one college does so strictly. Further, together with college j 's condition (total differentiation of H_j), one can show that $\bar{\sigma}_j'(v) > 0$ for a positive measure of v , generically.

A.8.2 Existence of a Cutoff Equilibrium

Step 1. Existence of a profile of cutoff strategies for A and B .

Define

$$\delta := \max_{v, e_A, e_B} \left\{ x_A(e_A|v) \left(\frac{\partial U_A(v, e_A)}{\partial v} \right) - \frac{\partial X_A(e_A|v)}{\partial v} \vee x_B(e_B|v) \left(\frac{\partial U_B(v, e_B)}{\partial v} \right) - \frac{\partial X_B(e_B|v)}{\partial v} \right\}.$$

Let \mathcal{M} be the set of Lipschitz-continuous function from $[0,1]$ to $[0,1]$ with Lipschitz bound given by δ . We define an operator $T : [0, 1]^2 \times \mathcal{M}^2 \rightarrow [0, 1]^2 \times \mathcal{M}^2$ as follows.

For any $(\hat{s}_A, \hat{s}_B, \bar{\sigma}_A, \bar{\sigma}_B) \in [0, 1]^2 \times \mathcal{M}^2$, the third component of $T(\hat{s}_A, \hat{s}_B, \bar{\sigma}_A, \bar{\sigma}_B)$ is a function $\bar{\alpha}$ defined as follows. First, $\eta_A(v)$ is implicitly defined via $H_A(v, \eta_A(v), \bar{\sigma}_B(v)) = 0$ according to the Implicit Function Theorem (since $\partial U_A / \partial e_A > 0$). For v such that $\eta_A(v) \in (0, 1)$, the same argument as in the proof of [Theorem 5](#) implies that

$$-\eta'_A(v) \leq \frac{\partial U_A(v, \eta_A(v)) / \partial v}{\partial U_A(v, \eta_A(v)) / \partial e_A}. \quad (\text{A.8.4})$$

We now define $\sigma_A(v, e) := \mathbf{1}_{\{e_A \geq \eta_A(v)\}}$. Let $\bar{\alpha}(v) := \mathbb{E}[\sigma_A(v, e_A)|v]$. Then,

$$\bar{\alpha}(v) = 1 - X_A(\eta_A(v)|v)$$

and

$$\bar{\alpha}'(v) = -x_A(\eta_A(v)|v)\eta'_A(v) - \frac{\partial X_A(\eta_A(v)|v)}{\partial v} \leq x_A(\eta_A(v)|v) \frac{\frac{\partial U_A(v, \eta_A(v))}{\partial v}}{\frac{\partial U_A(v, \eta_A(v))}{\partial e_A}} - \frac{\partial X_A(\eta_A(v)|v)}{\partial v} \leq \delta,$$

where the first inequality follows from [\(A.8.4\)](#). It thus follows that $\bar{\alpha} \in \mathcal{M}$.

The fourth component of $T(\hat{s}_A, \hat{s}_B, \bar{\sigma}_A, \bar{\sigma}_B)$, labeled $\bar{\beta}$, is analogously constructed via $e_B = \eta_B(v)$ determined implicitly by $H_B(v, \eta_B(v), \bar{\sigma}_A) = 0$, analogously, and belongs to \mathcal{M} .

The first two components (\hat{s}'_A, \hat{s}'_B) are determined by the $m_A(\hat{s}'_A) = m_B(\hat{s}'_B) = \kappa$, much as in the earlier proofs, using σ_A and σ_B , along with (\hat{s}_A, \hat{s}_B) as input.

In sum, the operator T maps from $(\hat{s}_A, \hat{s}_B, \bar{\sigma}_A, \bar{\sigma}_B) \in [0, 1]^2 \times \mathcal{M}^2$ to $(\hat{s}'_A, \hat{s}'_B, \bar{\alpha}, \bar{\beta}) \in [0, 1]^2 \times \mathcal{M}^2$. By Arzela-Ascoli theorem, the set \mathcal{M} endowed with sup norm topology is compact, bounded and convex. Hence, the same holds for the Cartesian product $[0, 1]^2 \times \mathcal{M}^2$. Following the techniques used in [Appendix B](#), the mapping T is continuous (with respect to sup norm). Hence, by the Schauder's theorem, T has a fixed point. The fixed point then identifies a profile of cutoff strategies (σ_A, σ_B) via $\sigma_A(v, e_A) = \mathbf{1}_{\{e_A \geq \eta_A(v)\}}$ and $\sigma_B(v, e_B) = \mathbf{1}_{\{e_B \geq \eta_B(v)\}}$. See [Appendix B](#) for technical details.

Step 2. The cutoff strategies identified in Step 1 form an equilibrium under a

condition.

Consider the following conditions:

$$\left(\frac{\partial U_i(v, e_i)/\partial v}{U_i(v, e_i)} + \frac{\partial X_j(\eta_j(v)|v)}{\partial v} \Psi_i(s) \right) \frac{\partial U_j(v, \tilde{e})/\partial e_j}{\partial U_j(v, \tilde{e})/\partial v} \geq x_j(\tilde{e}|v) \Psi_i(s)$$

for all v, e_i, \tilde{e}, s , where

$$\Psi_i(s) := \frac{\mathbb{E}[\mu_i(s)|s \in S_i] - \mathbb{E}[\mu_i(s)]}{\mathbb{E}[\mu_i(s)|s \in S_i] \mathbb{E}[\mu_i(s)]}. \quad (\text{A.8.5})$$

Since the RHS of each inequality is bounded by some constant, the conditions can be interpreted as requiring that each college values the non-common performance sufficiently highly. For instance, if $U_i(v, e_i) = (1 - \rho)v + \rho e_i$ for all $i = A, B$, then the LHS of each inequality will be no less than $\rho - \gamma$, where $\gamma := \max_{v, e_A, e_B} \left\{ \left| \frac{\partial X_A(e_A|v)}{\partial v} \right|, \left| \frac{\partial X_B(e_B|v)}{\partial v} \right| \right\}$, whenever $\mathbb{E}[\mu_i(s)] \geq \rho$. So the condition will hold if the RHS of each inequality is less than $\rho - \gamma$.

We now show the cutoff strategies identified by Step 1 form an equilibrium, given this condition. For the proof, it suffices to show that

$$\frac{\partial H_i(v, e_i, \bar{\sigma}_j(v))}{\partial v} \geq 0 \text{ whenever } H_i(v, e_i, \bar{\sigma}_j(v)) = 0.$$

This result holds since

$$\begin{aligned} & \text{sgn} \left(\frac{\partial H_i(v, e_i, \bar{\sigma}_j(v))}{\partial v} \right) \\ &= \frac{\partial U_i(v, e_i)}{\partial v} - \frac{[U_i(v, e_i)(1 - \mathbb{E}[\mu_i(s)]) - \lambda \text{Prob}(s \in S_i)(1 - \mathbb{E}[\mu_i(s)|s \in S_i])]}{1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\bar{\sigma}_j(v)} \bar{\sigma}'_j(v) \\ &= \frac{\partial U_i(v, e_i)}{\partial v} - \frac{U_i(v, e_i)}{1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\bar{\sigma}_j(v)} \times \\ & \quad \left((1 - \mathbb{E}[\mu_i(s)]) - \frac{1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\bar{\sigma}_j(v)}{1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)|s \in S_i]\bar{\sigma}_j(v)} (1 - \mathbb{E}[\mu_i(s)|s \in S_i]) \right) \bar{\sigma}'_j(v) \\ &= \frac{\partial U_i(v, e_i)}{\partial v} - U_i(v, e_i) \frac{\mathbb{E}[\mu_i(s)|s \in S_i] - \mathbb{E}[\mu_i(s)]}{(1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)]\bar{\sigma}_j(v))(1 - \bar{\sigma}_j(v) + \mathbb{E}[\mu_i(s)|s \in S_i]\bar{\sigma}_j(v))} \bar{\sigma}'_j(v) \\ &\geq \frac{\partial U_i(v, e_i)}{\partial v} - U_i(v, e_i) \Psi_i(s) \bar{\sigma}'_j(v) \\ &= \frac{\partial U_i(v, e_i)}{\partial v} + U_i(v, e_i) \Psi_i(s) \left(x_j(\eta_j(v)|v) \eta'_j(v) + \frac{\partial X_j(\eta_j(v)|v)}{\partial v} \right) \\ &\geq \frac{\partial U_i(v, e_i)}{\partial v} - U_i(v, e_i) \Psi_i(s) \left(x_j(\eta_j(v)|v) \frac{\partial U_j(v, \eta_j(v))/\partial v}{\partial U_j(v, \eta_j(v))/\partial e_j} - \frac{\partial X_j(\eta_j(v)|v)}{\partial v} \right) \\ &\geq 0, \end{aligned}$$

where $\Psi_i(s)$ is given by (A.8.5), the second equality is obtained by substituting $H_i(v, e_i, \bar{\sigma}_j(v)) =$

0, the first inequality follows since $\mathbb{E}[\mu_i(s)], \mathbb{E}[\mu_i(s)|s \in S_i] \leq 1$, the penultimate equality follows from the fact that $\bar{\sigma}_j(v) = 1 - X_j(\eta_j(v)|v)$, the second inequality follows from (A.8.4), and the last inequality follows from the above conditions.

We last show that the identified strategies are nonincreasing in v . Note that

$$\frac{d\eta_i(v)}{dv} = -\frac{\partial H_i(v, e_i, \bar{\sigma}_j(v))/\partial v}{\partial H_i(v, e_i, \bar{\sigma}_j(v))/\partial e_i} \leq 0,$$

where the equality follows from the Implicit Function Theorem, and the inequality holds since $\partial H_i(v, e_i, \bar{\sigma}_j(v))/\partial v \geq 0$ and

$$\frac{\partial H_i(v, e_i, \bar{\sigma}_j(v))}{\partial e_i} = \frac{\partial U_i(v, e_i)}{\partial e_i} (1 - \bar{\sigma}_j + \mathbb{E}[\mu_i(s)]\bar{\sigma}_j) > 0.$$

A.9 Proofs of Lemma 4, Theorem 7 and Theorem 8

It is convenient to define $T(y|\sigma) := y P_A(y|\sigma) - (1 - y) P_B(y|\sigma)$ for the proofs.

A.9.1 Proof of Lemma 4

Fix any σ . To prove the optimality of the cutoff strategy, we show that $T'(y|\sigma) > 0$ for any y . Note that

$$\begin{aligned} T'(y|\sigma) &= P_A(y|\sigma) + P_B(y|\sigma) + y P'_A(y|\sigma) - (1 - y) P'_B(y|\sigma) \\ &\geq y [P_A(y|\sigma) + P'_A(y|\sigma)] + (1 - y) [P_B(y|\sigma) - P'_B(y|\sigma)] \\ &= y \int_0^1 q_A(s|\sigma) [l(s|y) + l_y(s|y)] ds + (1 - y) \int_0^1 q_B(s|\sigma) [l(s|y) - l_y(s|y)] ds. \end{aligned}$$

Observe that

$$l(s|y) + l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s) ds} \left[1 + \frac{k_y(y|s)}{k(y|s)} - \frac{\int_0^1 k_y(y|s) ds}{\int_0^1 k(y|s) ds} \right] > \frac{k(y|s)}{\int_0^1 k(y|s) ds} (1 - 2\delta),$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} > -\delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s) ds}{\int_0^1 k(y|s) ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)} k(y|s) ds}{\int_0^1 k(y|s) ds} < \delta$$

because $\left| \frac{k_y(y|s)}{k(y|s)} \right| < \delta$. Similarly,

$$l(s|y) - l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s) ds} \left[1 - \frac{k_y(y|s)}{k(y|s)} + \frac{\int_0^1 k_y(y|s) ds}{\int_0^1 k(y|s) ds} \right] > \frac{k(y|s)}{\int_0^1 k(y|s) ds} (1 - 2\delta),$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} < \delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s) ds}{\int_0^1 k(y|s) ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)} k(y|s) ds}{\int_0^1 k(y|s) ds} > -\delta.$$

Therefore, we have that $T'(y|\sigma) > 0$ since $\delta \leq \frac{1}{2}$.

It remains to show that there exists an equilibrium in cutoff strategy. Let \hat{y} be a cutoff. Then, we have $n_A(s|\hat{y}) = \int_{\hat{y}}^1 k(y|s) dy = 1 - K(\hat{y}|s)$. Hence,

$$P_A(y|\hat{y}) = \int_0^1 \min \left\{ \frac{\kappa}{1 - K(\hat{y}|s)}, 1 \right\} l(s|y) ds \quad \text{and} \quad P_B(y|\hat{y}) = \int_0^1 \min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} l(s|y) ds,$$

Now, let

$$T(y|\hat{y}) := yP_A(y|\hat{y}) - (1 - y)P_B(y|\hat{y}).$$

Note that

$$T(0|\hat{y}) = -P_B(0|\hat{y}) = - \int_0^1 \min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} l(s|0) ds < 0,$$

where the inequality holds since $\min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} > 0$ and $l(s|0) \geq 0$ for all s , and $l(s|0) > 0$ for a positive measure of states. Similarly, $T(1|\hat{y}) > 0$. By the continuity of $T(\cdot|\hat{y})$, there is a \tilde{y} such that $T(\tilde{y}|\hat{y}) = 0$. Moreover, such a \tilde{y} is unique since $T'(y|\hat{y})|_{y=\tilde{y}} > 0$.

Next, let $\tau : [0, 1] \rightarrow [0, 1]$ be the map from \hat{y} to \tilde{y} , which is implicitly defined by $T(\tau(\hat{y})|\hat{y}) = 0$ according to the Implicit Function Theorem (since $T'(y|\hat{y})|_{y=\tilde{y}} > 0$). Since $P_A(y|\cdot)$ is nondecreasing and $P_B(y|\cdot)$ is nonincreasing \hat{y} , $\tau(\cdot)$ is decreasing. Therefore, there is a fixed point such that $\tau(\hat{y}) = \hat{y}$, and hence there is \hat{y} such that $T(\hat{y}|\hat{y}) = 0$.

A.9.2 Proof of **Theorem 7**

We first show $\hat{y} < 1$. Suppose $\hat{y} = 1$. Then, $n_A(s|1) = 1 - K(1|s) = 0$, so $P_A(y|\hat{y}) = 1$ for any y . Hence, $T(1|1) = P_A(1|1) = 1$, which contradicts the fact that $T(\hat{y}|\hat{y}) = 0$.

We now show that $\hat{y} > \frac{1}{2}$ whenever $\mu(s) > \frac{1}{2}$. Suppose to the contrary $\hat{y} \leq \frac{1}{2}$. We then

have $\frac{1}{2} < \mu(s) = 1 - K(\frac{1}{2}|s) \leq 1 - K(\hat{y}|s)$, so $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. Therefore,

$$P_A(y|\hat{y}) - P_B(y|\hat{y}) = \int_0^1 \min \left\{ \frac{\kappa}{1 - K(\hat{y}|s)}, 1 \right\} l(s|y) ds - \int_0^1 \min \left\{ \frac{\kappa}{K(\hat{y}|s)}, 1 \right\} l(s|y) ds \leq 0. \quad (\text{A.9.1})$$

Hence, if $\hat{y} < \frac{1}{2}$, then

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1 - \hat{y})P_B(\hat{y}|\hat{y}) < \frac{1}{2} [P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})] \leq 0, \quad (\text{A.9.2})$$

where the first inequality holds since $\hat{y} < \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) < 0$, a contradiction. Suppose now $\hat{y} = \frac{1}{2}$. Notice that since $K(\hat{y}|s) < 1 - K(\hat{y}|s)$, we have $K(\frac{1}{2}|s) < \frac{1}{2} < 1 - \kappa$, where the the second inequality holds since $\kappa < \frac{1}{2}$. So, $\kappa/(1 - K(\frac{1}{2}|s)) < 1$. Therefore, the last inequality of (A.9.1) becomes strict, and hence

$$T(\hat{y}|\hat{y}) = \frac{1}{2} [P_A(\frac{1}{2}|\frac{1}{2}) - P_B(\frac{1}{2}|\frac{1}{2})] < 0,$$

a contradiction again.

Lastly, let $\mu(s) = \frac{1}{2}$. If $\hat{y} < \frac{1}{2}$, then $\frac{1}{2} = \mu(s) = 1 - K(\frac{1}{2}|s) < 1 - K(\hat{y}|s)$, so we have $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. By (A.9.1) and (A.9.2), we reach a contradiction. If $\hat{y} > \frac{1}{2}$, then $\frac{1}{2} = \mu(s) = 1 - K(\frac{1}{2}|s) > 1 - K(\hat{y}|s)$ and so $K(\hat{y}|s) > 1 - K(\hat{y}|s)$. We then have $P_A(y|\hat{y}) - P_B(y|\hat{y}) \geq 0$ and

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1 - \hat{y})P_B(\hat{y}|\hat{y}) > \frac{1}{2} [P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})] \geq 0,$$

where the first inequality holds since $\hat{y} > \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) > 0$, a contradiction again.

A.9.3 Proof of Theorem 8

For the first part of the theorem, observe that for a given s , justified envy arises whenever $c_A(s) \neq c_B(s)$ as depicted in Figure 5.1. We thus show that there is a positive measure of states in which $c_A(s) \neq c_B(s)$. Suppose to the contrary $c_A(s) = c_B(s)$ for almost all s . Recall that equilibrium admission cutoff of each college satisfies

$$G(c_A(s)) = \max \left\{ 1 - \frac{\kappa}{1 - K(\hat{y}|s)}, 0 \right\} \quad \text{and} \quad G(c_B(s)) = \max \left\{ 1 - \frac{\kappa}{K(\hat{y}|s)}, 0 \right\}.$$

Since $G(\cdot)$ is strictly increasing, if $c_A(s) = c_B(s)$, then we must have either $n_i(s) < \kappa$ for all $i = A, B$ (so that $c_A(s) = c_B(s) = 0$) or $n_A(s) = n_B(s) \geq \kappa$.

First, we cannot have $n_i(s) < \kappa$ for all i in equilibrium, since this means that all applicants are admitted by either college, which contradicts the fact that $2\kappa < 1$. Second, suppose

$n_A(s) = n_B(s) \geq \kappa$. This implies that $K(\hat{y}|s) = \frac{1}{2}$ for all s (recall that $n_A(s) = 1 - K(\hat{y}|s)$ and $n_B(s) = K(\hat{y}|s)$). However, by (5.1), we have $K(\hat{y}|s') < K(\hat{y}|s)$ for all $s' > s$. Therefore, we reach a contradiction again.

To see the second part of the theorem, recall that for given \hat{y} in equilibrium, the mass of students applying to B is $K(\hat{y}|s)$. Thus, if there is a positive measure of states in which $K(\hat{y}|s) < \kappa$, college B faces under-subscription in such states. Therefore, the equilibrium outcome is inefficient.

A.10 Proof of Theorem 9

Suppose there is a symmetric equilibrium as described in the theorem. Then, colleges A and B will admit all acceptable students with $v > \tilde{v}$, where \tilde{v} is such that each of A and B fills its capacity in the popular state, i.e., $s_a(1 - \varepsilon)[1 - G(\tilde{v})] = \kappa$ and $(1 - s_b)(1 - \varepsilon)[1 - G(\tilde{v})] = \kappa$, and wait-lists the remaining students. College C will offer admissions to all of these students (i.e., those whose scores are above \tilde{v}), knowing that exactly measure ε^2 of them will accept its offer. It will also offer $\kappa - \varepsilon^2$ admissions to all students with $v \in [\hat{v}, \tilde{v}]$, where \hat{v} is such that $G(\tilde{v}) - G(\hat{v}) = \kappa - \varepsilon^2$.

The students in $[\hat{v}, \tilde{v}]$ now have a choice to make. If a student accepts C , then she will get u'' for sure, but if she turns down C 's offer, then with probability $1 - \varepsilon$ the less popular one between A and B will offer an admission to her (assuming all other students admitted by C have accepted that offer), and the student will earn the payoff u if she happens to like the college, or u' otherwise. Since $u'' > (1 - \varepsilon)u$, she will accept C .

Given this, consider now the incentive for deviation of college A . If it does not deviate, there will be seats left in the less popular state, equal to $\kappa - s_b(1 - \varepsilon)[1 - G(\hat{v})]$. Thus, A will fill those vacant seats with students whose scores are below \hat{v} . Its payoff in this case is

$$\begin{aligned} \pi_A &= \frac{1}{2}s_a(1 - \varepsilon) \int_{\tilde{v}}^1 v dG(v) + \frac{1}{2} \left[s_b(1 - \varepsilon) \int_{\tilde{v}}^1 v dG(v) + (1 - \varepsilon) \int_{\tilde{v}}^{\hat{v}} v dG(v) \right] \\ &= \frac{1}{2}(1 - \varepsilon) \left[\int_{\tilde{v}}^1 v dG(v) + \int_{\tilde{v}}^{\hat{v}} v dG(v) \right], \end{aligned}$$

where \check{v} is such that

$$(1 - \varepsilon)[G(\hat{v}) - G(\check{v})] = \kappa - s_b(1 - \varepsilon)[1 - G(\tilde{v})]. \quad (\text{A.10.1})$$

and the second equality follows from $s_a = 1 - s_b$.

Suppose now A admits a small fraction, say δ' , of (acceptable) students just below \tilde{v} instead of admitting those who are acceptable and slightly above \tilde{v} , say $[\tilde{v}, \tilde{v} + \delta]$, where δ

and δ' are such that

$$s_a[G(\tilde{v} + \delta) - G(\tilde{v})] = G(\tilde{v}) - G(\tilde{v} - \delta'). \quad (\text{A.10.2})$$

Notice that students in $[\tilde{v} - \delta', \tilde{v}]$ accept A 's admission offer, since they prefer it over C . Hence, A fills its capacity in the popular state, i.e.,

$$s_a(1 - \varepsilon)[1 - G(\tilde{v} + \delta)] + (1 - \varepsilon)[G(\tilde{v}) - G(\tilde{v} - \delta')] = s_a(1 - \varepsilon)[1 - G(\tilde{v})] = \kappa.$$

So, A 's payoff under the deviation is

$$\begin{aligned} \pi_A^d &= (1 - \varepsilon) \int_{\tilde{v} - \delta'}^{\tilde{v}} v dG(v) + \frac{1}{2} s_a(1 - \varepsilon) \int_{\tilde{v} + \delta}^1 v dG(v) \\ &\quad + \frac{1}{2} \left[s_b(1 - \varepsilon) \int_{\tilde{v} + \delta}^1 v dG(v) + (1 - \varepsilon) \int_{\bar{v}}^{\tilde{v}} v dG(v) \right] \\ &= (1 - \varepsilon) \int_{\tilde{v} - \delta'}^{\tilde{v}} v dG(v) + \frac{1}{2} (1 - \varepsilon) \left[\int_{\tilde{v} + \delta}^1 v dG(v) + \int_{\bar{v}}^{\tilde{v}} v dG(v) \right], \end{aligned}$$

where \bar{v} satisfies

$$(1 - \varepsilon)[G(\hat{v}) - G(\bar{v})] = \kappa - (1 - \varepsilon)[G(\tilde{v}) - G(\tilde{v} - \delta')] - s_b(1 - \varepsilon)[1 - G(\tilde{v} + \delta)], \quad (\text{A.10.3})$$

that is, \bar{v} is set to meet the capacity in the less popular state. Observe that $\bar{v} > \tilde{v}$, since from (A.10.1) and (A.10.3),

$$\begin{aligned} (1 - \varepsilon)[G(\bar{v}) - G(\hat{v})] &= -s_b(1 - \varepsilon)[G(\tilde{v} + \delta) - G(\tilde{v})] + (1 - \varepsilon)[G(\tilde{v}) - G(\tilde{v} - \delta')] \\ &= (s_a - s_b)(1 - \varepsilon)[G(\tilde{v} + \delta) - G(\tilde{v})] \\ &> 0 \end{aligned} \quad (\text{A.10.4})$$

where the second equality follows from (A.10.2) and the inequality holds since $s_a > s_b$. We thus have

$$\begin{aligned} \frac{2(\pi_A^d - \pi_A)}{1 - \varepsilon} &= 2 \int_{\tilde{v} - \delta'}^{\tilde{v}} v dG(v) - \left[\int_{\tilde{v}}^{\tilde{v} + \delta} v dG(v) + \int_{\tilde{v}}^{\bar{v}} v dG(v) \right] \\ &= 2 \left[\tilde{v} G(\tilde{v}) - (\tilde{v} - \delta') G(\tilde{v} - \delta') - \int_{\tilde{v} - \delta'}^{\tilde{v}} G(v) dv \right] \\ &\quad - \left[(\tilde{v} + \delta) G(\tilde{v} + \delta) - \tilde{v} G(\tilde{v}) - \int_{\tilde{v}}^{\tilde{v} + \delta} G(v) dv \right] \\ &\quad - \left[\bar{v} G(\bar{v}) - \tilde{v} G(\tilde{v}) - \int_{\tilde{v}}^{\bar{v}} G(v) dv \right] \\ &= 2\tilde{v}[G(\tilde{v}) - G(\tilde{v} - \delta')] + 2 \left[\delta' G(\tilde{v} - \delta') - \int_{\tilde{v} - \delta'}^{\tilde{v}} G(v) dv \right] \end{aligned}$$

$$\begin{aligned}
& -\tilde{v}[G(\tilde{v} + \delta) - G(\tilde{v})] - \left[\delta G(\tilde{v} + \delta) - \int_{\tilde{v}}^{\tilde{v} + \delta} G(v) dv \right] \\
& - \left[\bar{v} G(\bar{v}) - \check{v} G(\check{v}) - \int_{\check{v}}^{\bar{v}} G(v) dv \right] \\
> & 2\tilde{v}[G(\tilde{v}) - G(\tilde{v} - \delta')] - 2\delta'[G(\tilde{v}) - G(\tilde{v} - \delta')] \\
& - \tilde{v}[G(\tilde{v} + \delta) - G(\tilde{v})] - \delta[G(\tilde{v} + \delta) - G(\tilde{v})] - \bar{v}[G(\bar{v}) - G(\check{v})] \\
= & (2s_a(\tilde{v} - \delta') - \tilde{v} - \delta - (s_a - s_b)\bar{v})[G(\tilde{v} + \delta) - G(\tilde{v})] \\
= & ((s_a - s_b)(\tilde{v} - \bar{v}) - (2s_a\delta' + \delta))[G(\tilde{v} + \delta) - G(\tilde{v})]
\end{aligned}$$

where the inequality holds since $\delta'G(\tilde{v}) > \int_{\tilde{v}-\delta'}^{\tilde{v}} G(v)dv$, $\int_{\tilde{v}}^{\tilde{v}+\delta} G(v)dv > \delta G(\tilde{v})$ and $\int_{\check{v}}^{\bar{v}} G(v)dv > (\bar{v} - \check{v})G(\check{v})$, the penultimate equality follows from (A.10.2) and (A.10.4), and the last inequality holds since $s_a + s_b = 1$. Therefore, for sufficiently small δ , we have $\pi_A^d > \pi_A$, which shows that the admission strategies described in the theorem are not equilibrium.

A.11 Proof of Lemma 5

Recall that when both colleges A and B report truthfully up to the capacity, they achieve jointly optimal matching for the two colleges. Now suppose college A unilaterally deviates by either reporting untruthfully about its preferences or its capacity and is strictly better off for some state s . Then, college B must be strictly worse off. Thus, there must exist a positive measure set of students whom A must obtain from the deviation which it prefers to some students it had before the deviation. At the same time, it must be the case that either college B gets a positive measure set of students who are worse than the former set of students or it has some unfilled seats left after A 's deviation. Note that students in the former set (who are assigned to A in the new matching) must prefer B , or else the original matching would not be stable. But then since B prefer each of those students to some students it has in the new matching, this means that the new matching is not stable (given the stated preferences).

B Appendix B: More than Two Colleges

Our main model in [Section 2](#) considers the case with two colleges. In this section, we show that our analysis extends to the case with more than two colleges. While the extension works for any arbitrary number of colleges, we provide the result for the three-college case for expositional simplicity. It will become clear that the method also extends to larger numbers.

Suppose there are three colleges, each with mass $\kappa < \frac{1}{3}$ capacity. Let $\sigma_i : \mathcal{V} \rightarrow [0, 1]$ be college i 's admission strategy, where $i = 1, 2, 3$. In each state $s \in [0, 1]$, let $\mu_{ijk}(s)$, where $i, j, k = 1, 2, 3$, denote the mass of students whose preference ordering is $i \succ j \succ k$.

Define the following notations: $\mu_{i>j}(s) := \mu_{ijk}(s) + \mu_{ikj}(s) + \mu_{kij}(s)$ (the mass of students who prefer i over j in state s), $\mu_{i>j,k}(s) := \mu_{ijk}(s) + \mu_{ikj}(s)$ (the mass of students who prefer i over j and k in state s) and

$$\bar{\mu}_{i>j} := \int_0^1 \mu_{i>j}(s) ds, \quad \bar{\mu}_{i>j,k} := \int_0^1 \mu_{i>j,k}(s) ds.$$

We assume that for any distinct i, j, k and s, s' ,

$$\mu_{i>j}(s) > \mu_{i>j}(s') \Leftrightarrow \mu_{i>k}(s) > \mu_{i>k}(s') \Leftrightarrow \mu_{i>j,k}(s) > \mu_{i>j,k}(s') \quad (\text{B.0.1})$$

and

$$\frac{|\mu_{i>j}(s) - \mu_{i>j}(s')|}{\mathbb{E}[\mu_{i>j}(s)]} < \frac{|\mu_{i>j,k}(s) - \mu_{i>j,k}(s')|}{\mathbb{E}[\mu_{i>j,k}(s)]}. \quad (\text{B.0.2})$$

The first assumption, [\(B.0.1\)](#), simply says that when college i is popular relative to college j , it will be more popular relative to college k . The second assumption, [\(B.0.2\)](#), means that $\mu_{i>j,k}(\cdot)$ is more sensitive to the students' preference toward i than does $\mu_{i>j}(\cdot)$.

For given $\sigma_i(\cdot)$, $i = 1, 2, 3$, let $n_i(v|s)$ be the probability that a student with score v attends college i in state s when she is admitted by i . That is,

$$\begin{aligned} n_i(v|s) := & (1 - \sigma_j(v))(1 - \sigma_k(v)) + \mu_{i>j}(s)\sigma_j(v)(1 - \sigma_k(v)) \\ & + \mu_{i>k}(s)\sigma_k(v)(1 - \sigma_j(v)) + \mu_{i>j,k}(s)\sigma_j(v)\sigma_k(v). \end{aligned} \quad (\text{B.0.3})$$

The student will attend college i if she is admitted only by i , which happens with probability $(1 - \sigma_j(v))(1 - \sigma_k(v))$; or is admitted by college i and one of the less preferred colleges, which happens with probability $\mu_{i>j}(s)\sigma_j(v)(1 - \sigma_k(v)) + \mu_{i>k}(s)\sigma_k(v)(1 - \sigma_j(v))$ in state s ; or is admitted by both of the other colleges but prefers i the most, which happens with probability $\mu_{i>j,k}(s)\sigma_j(v)\sigma_k(v)$ in state s .

Thus, for a given profile of admission strategies, $\sigma = (\sigma_i)_{i=1,2,3}$, the mass of students who

attend college i in state s is

$$m_i(s) := \int_0^1 \sigma_i(v) n_i(v|s) dG(v), \quad (\text{B.0.4})$$

and college i 's payoff is

$$\pi_i = \int_0^1 v \sigma_i(v) \bar{n}_i(v) dG(v) - \lambda \int_0^1 \max \{m_i(s) - \kappa, 0\} ds,$$

where

$$\bar{n}_i(v) := \int_0^1 n_i(v|s) ds.$$

Recall that in the two-school case, the monotonicity of $\mu(\cdot)$ yields cutoff states (\hat{s}_A, \hat{s}_B) that trigger over-enrollment for each college, and the set of over-enrolled states for each of them is an interval, $(\hat{s}_A, 1]$ for college A and $[0, \hat{s}_B)$ for college B . Using this, we project the admission strategies to state space to use the Brouwer's fixed point theorem. When there are more than two colleges, however, we do not know the structure of the set of over-enrolled states in general, so we cannot directly define a map on the cutoff states. Nonetheless, the main idea of the proof can be carried over, although we use a fixed point theorem (Schauder) in a functional space.

Define a **subdistribution** $F_i : [0, 1] \rightarrow [0, 1]$, $i = 1, 2, 3$, such that $F_i(0) = 0$ and

$$F_i(s) := \text{Prob}(m_i(t) > \kappa \text{ for } t < s). \quad (\text{B.0.5})$$

The subdistribution of college i places a positive mass only on the states in which college i over-enrolls. Observe that $F_i(\cdot)$ is nondecreasing and

$$0 \leq F_i(s') - F_i(s) \leq s' - s, \quad \forall s' \geq s. \quad {}^{36}$$

Let \mathcal{F}_i be the set of all such subdistributions and $\mathcal{F} := \times_{i=1}^3 \mathcal{F}_i$. (It will become clear that these subdistributions will play a similar role to the cutoff states in the two-school case.)

³⁶The second inequality holds because

$$\begin{aligned} F_i(s') - F_i(s) &= \text{Prob}(m_i(t) > \kappa \text{ for } t < s') - \text{Prob}(m_i(t) > \kappa \text{ for } t < s) \\ &= \text{Prob}(m_i(t) > \kappa \text{ for } s < t < s') \\ &\leq \text{Prob}(s < t < s') \\ &= s' - s. \end{aligned}$$

Using the subdistributions, each college's payoff is now given by³⁷

$$\begin{aligned}\pi_i &= \int_0^1 v \sigma_i(v) \bar{n}_i(v) dG(v) - \lambda \int_0^1 (m_i(s) - \kappa) dF_i(s) \\ &= \int_0^1 \sigma_i(v) H_i(v, \sigma_j(v), \sigma_k(v)) dG(v) + \lambda \int_0^1 \kappa dF_i(s),\end{aligned}$$

where

$$H_i(v, \sigma_j(v), \sigma_k(v)) := v \bar{n}_i(v) - \lambda \int_0^1 n_i(v|s) dF_i(s)$$

is college i 's marginal payoff from admitting a student with score v . Observe that H_i can be decomposed as follow:

$$\begin{aligned}H_i(v, \sigma_j(v), \sigma_k(v)) &= (1 - \sigma_j(v))(1 - \sigma_k(v))H_i(v, 0, 0) + \sigma_j(v)(1 - \sigma_k(v))H_i(v, 1, 0) \\ &\quad + (1 - \sigma_j(v))\sigma_k(v)H_i(v, 0, 1) + \sigma_j(v)\sigma_k(v)H_i(v, 1, 1),\end{aligned}$$

where $H_i(v, 0, 0)$ is college i 's marginal payoff from admitting a student with score v if she is refused by both of the other colleges, $H_i(v, 1, 0)$ and $H_i(v, 0, 1)$ are the marginal payoffs if the student is admitted by college j (k) but rejected by k (j , respectively), and $H_i(v, 1, 1)$ is the marginal payoff if the student is admitted by both of the other colleges.

Let us now define v_i^{11} , v_i^{10} , v_i^{01} and v_i^{00} such that

$$H_i(v_i^{11}, 1, 1) = 0, \quad H_i(v_i^{10}, 1, 0) = 0, \quad H_i(v_i^{01}, 0, 1) = 0, \quad H_i(v_i^{00}, 0, 0) = 0.$$

Lemma B1. $v_i^{00} < \min\{v_i^{10}, v_i^{01}\} \leq \max\{v_i^{10}, v_i^{01}\} < v_i^{11}$.

Proof. See [Appendix B.1](#). ■

Lemma B2. (i) If $v < v_i^{00}$, then $H_i(v, x, y) < 0$, and if $v > v_i^{11}$, then $H_i(v, x, y) > 0$, for all x and y .

(ii) If $v < v_i^{10}$, then $H_i(v, x, y) \geq 0$ if $x \leq \hat{x}(y)$ for some $\hat{x}(y) \in [0, 1]$ for each y .

(iii) If $v < v_i^{01}$, then $H_i(v, x, y) \geq 0$ if $y \leq \hat{y}(x)$ for some $\hat{y}(x) \in [0, 1]$ for each x .

(iv) For any $v \notin [\min\{v_i^{10}, v_i^{01}\}, \max\{v_i^{10}, v_i^{01}\}]$, if $H_i(v, x, y) \leq 0$, then $H_i(v, x', y') < 0$ for $(x', y') > (x, y)$.

Proof. Write

$$H_i(v, x, y) = (1 - x)(1 - y)(v - v_i^{00}) + x(1 - y) \bar{\mu}_{i>j}(v - v_i^{10})$$

³⁷Note that since F_i is Lipschitz continuous, it is absolute continuous. Thus, the integration is well defined.

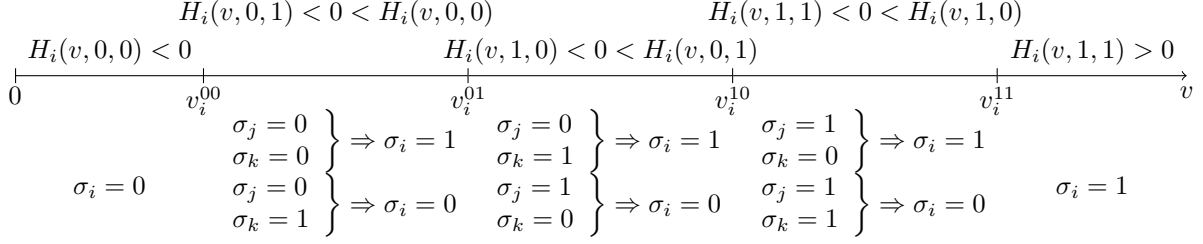


Figure B.1: College i 's Admission Decision

$$+ (1 - x)y \bar{\mu}_{i>k}(v - v_i^{01}) + xy \bar{\mu}_{i>j,k}(v - v_i^{11}).$$

Part (i) follows easily from [Lemma B1](#). To see part (ii), suppose $v < v_i^{10}$. Then,

$$H_i(v, 1, y) = (1 - y) \bar{\mu}_{i>j}(v - v_i^{10}) + y \bar{\mu}_{i>j,k}(v - v_i^{11}) < 0,$$

by [Lemma B1](#). Since H_i is linear in x , part (ii) follows. Part (iii) follows analogously.

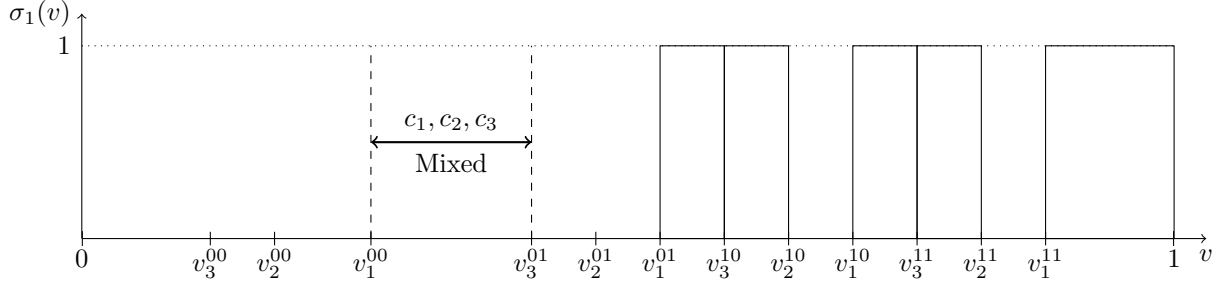
We now prove part (iv). If $v < \min\{v_i^{10}, v_i^{01}\}$, then the result follows from parts (ii) and (iii), since both $H_i(v, 1, y) < 0$ and $H_i(v, x, 1) < 0$ for any x, y . Next suppose $v > \max\{v_i^{10}, v_i^{01}\}$, then both $H_i(v, 0, y) > 0$ and $H_i(v, x, 0) > 0$ for any x, y . If $H_i(v, x, y) < 0$ for some (x, y) , it must be that $\partial H_i / \partial x < 0$ and $\partial H_i / \partial y < 0$. Linearity of H_i in x and y implies the result. \blacksquare

As discussed in [Remark 1](#), [Lemma B1](#) and [Lemma B2](#) reveal that a college's marginal payoff from students with fewer offers is higher than those with more competing offers, and so the college prefers the former students than the latter. The lemmas also imply that $H_i(v, \sigma_j, \sigma_k)$ partitions the students' type space, as depicted in [Figure B.1](#) for the case that $v_i^{01} \leq v_i^{10}$. College i admits type- v students for sure if $H_i(v, 1, 1) > 0$ and rejects them if $H_i(v, 0, 0) < 0$. In the case $H_i(v, 1, 1) < 0 < H_i(v, 0, 0)$, college i admits type- v students only when $H_i(v, 1, 0) > 0$ or $H_i(v, 0, 1) > 0$; that is, those students are worthy only in the case that they are admitted by one of the other colleges. This shows that colleges engage in *strategic targeting* for those intermediate range of scores.

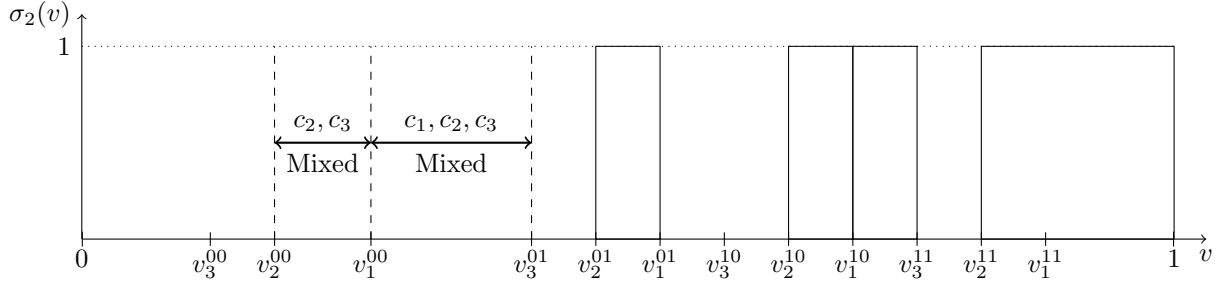
As in the two-school case, there are many ways that colleges could coordinate for the students in the middle. To be consistent, we consider the maximally mixed equilibrium (MME) as before. [Figure B.2](#) depicts a competitive MME in the case that for instance,

$$v_3^{00} < v_2^{00} < v_1^{00} < v_3^{01} < v_2^{01} < v_1^{01} < v_3^{10} < v_2^{10} < v_1^{10} < v_3^{11} < v_2^{11} < v_1^{11}.$$

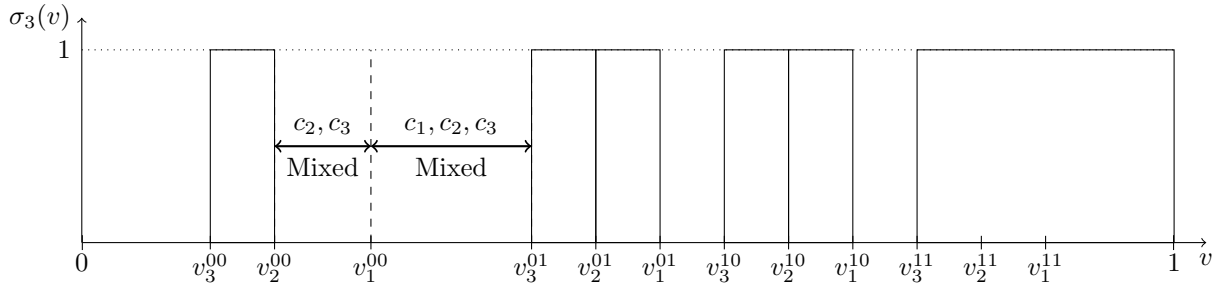
We now provide the existence of MME. For a given profile of subdistributions $(F_i)_{i=1}^3$, let $\sigma := (\sigma_i)_{i=1}^3$ be the profile of admission strategies that satisfy the local conditions described above. Then, such σ in turn determines a new profile of subdistributions, $(F_i)_{i=1}^3$ via [\(B.0.5\)](#).



(a) College 1 (c_1)



(b) College 2 (c_2)



(c) College 3 (c_3)

Figure B.2: Admission Strategies

Next, we define $T : \mathcal{F} \rightarrow \mathcal{F}$, a self-map from the set of subdistributions to itself, where $\mathcal{F} = \times_{i=1}^3 \mathcal{F}_i$. The existence of equilibrium is achieved when T has a fixed point (on the functional space of \mathcal{F}). As mentioned earlier, the idea of proving the existence of equilibrium is similar to that of [Theorem 2](#), projecting the strategy profile into a simpler space. The difference is that in the two-school case, the strategy profiles are projected into the state space, but in the general case, they are projected into the set of subdistributions \mathcal{F} .

Theorem 11. *There exists a maximally mixed equilibrium.*

We show that \mathcal{F} is a compact and convex subset of a normed linear space, and $T : \mathcal{F} \rightarrow \mathcal{F}$ is continuous. Then, T has a fixed point by Schauder's fixed point theorem.³⁸ We provide a

³⁸Schauder's fixed point theorem is a generalization of Brouwer's theorem on a normed linear space. It

formal proof in [Appendix B.2](#).

B.1 Proof of [Lemma B1](#)

We first establish a useful lemma.

Lemma B3. *For any distinct i, j, k , $\mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa] > \bar{\mu}_{i>j}$ and*

$$\frac{\mathbb{E}[\mu_{i>j,k}(s)|m_i(s) > \kappa]}{\bar{\mu}_{i>j,k}} > \frac{\mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa]}{\bar{\mu}_{i>j}}.$$

Proof. Observe first that if $m_i(s) > m_i(s')$ for some s and s' , then by [\(B.0.4\)](#) it holds either $\mu_{i>j}(s) > \mu_{i>j}(s')$ or $\mu_{i>k}(s) > \mu_{i>k}(s')$ or $\mu_{i>j,k}(s) > \mu_{i>j,k}(s')$. But, if one of this holds, then [\(B.0.1\)](#) implies that the other two must also hold.

Now, let $F_+(x) := \text{Prob}(s \leq x \text{ and } m_i(s) > \kappa)$ and $F_-(x) := \text{Prob}(s \leq x \text{ and } m_i(s) \leq \kappa)$, and define $\phi_+(y) := \inf\{z | F_+(z) \geq y\}$ and $\phi_-(y) := \inf\{z | F_-(z) \geq y\}$. Note that for each $s \in \phi_+([0, 1])$, $m_i(s) > \kappa$ and for each $s \in \phi_-([0, 1])$, $m_i(s) \leq \kappa$.

To see the first part, note that

$$\begin{aligned} \mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa] - \bar{\mu}_{i>j} &= \text{Prob}(m_i(s) \leq \kappa) (\mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa] - \mathbb{E}[\mu_{i>j}(s)|m_i(s) \leq \kappa]) \\ &= \text{Prob}(m_i(s) \leq \kappa) \left(\int_0^1 \mu_{i>j}(s) dF_+(s) - \int_0^1 \mu_{i>j}(s) dF_-(s) \right) \\ &= \text{Prob}(m_i(s) \leq \kappa) \left(\int_0^1 \mu_{i>j}(\phi_+(y)) dy - \int_0^1 \mu_{i>j}(\phi_-(y)) dy \right) \\ &= \text{Prob}(m_i(s) \leq \kappa) \int_0^1 [\mu_{i>j}(\phi_+(y)) - \mu_{i>j}(\phi_-(y))] dy \\ &> 0, \end{aligned}$$

where the inequality follows from [\(B.0.1\)](#), since $m_i(\phi_+(y)) > \kappa \geq m_i(\phi_-(y))$ implies that $\mu_{i>j}(\phi_+(y)) > \mu_{i>j}(\phi_-(y))$. Similarly, for the second inequality,

$$\begin{aligned} &\frac{\mathbb{E}[\mu_{i>j,k}(s)|m_i(s) > \kappa]}{\bar{\mu}_{i>j,k}} - \frac{\mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa]}{\bar{\mu}_{i>j}} \\ &= \frac{\mathbb{E}[\mu_{i>j,k}(s)|m_i(s) > \kappa] - \bar{\mu}_{i>j,k}}{\bar{\mu}_{i>j,k}} - \frac{\mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa] - \bar{\mu}_{i>j}}{\bar{\mu}_{i>j}} \\ &= \text{Prob}(m_i(s) \leq \kappa) \left(\frac{\mathbb{E}[\mu_{i>j,k}(s)|m_i(s) > \kappa] - \mathbb{E}[\mu_{i>j,k}(s)|m_i(s) \leq \kappa]}{\bar{\mu}_{i>j,k}} \right) \end{aligned}$$

guarantees that every continuous self-map on a nonempty, compact, convex subset of a normed linear space has a fixed point (see [Ok, 2007](#)).

$$\begin{aligned}
& - \text{Prob}(m_i(s) \leq \kappa) \left(\frac{\mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa] - \mathbb{E}[\mu_{i>j}(s)|m_i(s) \leq \kappa]}{\bar{\mu}_{i>j,k}} \right) \\
& = \text{Prob}(m_i(s) \leq \kappa) \int_0^1 \left(\frac{\mu_{i>j,k}(\phi_+(y)) - \mu_{i>j,k}(\phi_-(y))}{\bar{\mu}_{i>j,k}} - \frac{\mu_{i>j}(\phi_+(y)) - \mu_{i>j}(\phi_-(y))}{\bar{\mu}_{i>j}} \right) dy \\
& > 0,
\end{aligned}$$

where the inequality follows from (B.0.2), since $m_i(\phi_+(y)) > \kappa \geq m_i(\phi_-(y))$ implies that $\mu_{i>j,k}(\phi_+(y)) - \mu_{i>j,k}(\phi_-(y)) > 0$ and $\mu_{i>j}(\phi_+(y)) - \mu_{i>j}(\phi_-(y)) > 0$. \blacksquare

We now prove **Lemma B1**. Let $v_i^{01} = \min\{v_i^{10}, v_i^{01}\}$ and $v_i^{10} = \max\{v_i^{10}, v_i^{01}\}$ without loss of generality. Then

$$v_i^{01} - v_i^{00} = \lambda \text{Prob}(m_i(s) > \kappa) \left(\frac{\mathbb{E}[\mu_{i>k}(s)|m_i(s) > \kappa]}{\mathbb{E}[\mu_{i>k}(s)]} - 1 \right) > 0,$$

where the equality follows from the construction of v_i^{01} , v_i^{00} and F_i , and the inequality follows from the first part of **Lemma B3**. Similarly,

$$v_i^{11} - v_i^{10} = \lambda \text{Prob}(m_i(s) > \kappa) \left(\frac{\mathbb{E}[\mu_{i>j,k}(s)|m_i(s) > \kappa]}{\mathbb{E}[\mu_{i>j,k}(s)]} - \frac{\mathbb{E}[\mu_{i>j}(s)|m_i(s) > \kappa]}{\mathbb{E}[\mu_{i>j}(s)]} \right) > 0,$$

where the inequality follows from the second part of **Lemma B3**.

B.2 Proof of **Theorem 11**

For given $(F_i)_{i=1,2,3}$, consider colleges' strategy profile $(\sigma_i)_{i=1,2,3}$ which satisfies the following local conditions:

- $\sigma_i(v) = 1$ if $H_1(v, 1, 1) > 0$, $i = 1, 2, 3$.
- $\sigma_1(v) = 0$ if $H_1(v, 1, 1) < 0$, $H_2(v, 1, 1) > 0$, $H_3(v, 1, 1) > 0$.
 $\sigma_2(v) = 0$ if $H_1(v, 1, 1) > 0$, $H_2(v, 1, 1) < 0$, $H_3(v, 1, 1) > 0$.
 $\sigma_3(v) = 0$ if $H_1(v, 1, 1) > 0$, $H_2(v, 1, 1) > 0$, $H_3(v, 1, 1) < 0$.

- $\sigma_1(v) = 1$, $\sigma_2(v) = 0$, $\sigma_3(v) = 1$ if $\begin{cases} H_1(v, 1, 1) < 0, H_1(v, 0, 1) > 0 \\ H_2(v, 1, 1) < 0 \\ H_3(v, 1, 1) > 0 \end{cases}$

- $\sigma_1(v) = 0, \sigma_2(v) = 1, \sigma_3(v) = 1$ if $\begin{cases} H_1(v, 1, 1) < 0 \\ H_2(v, 1, 1) < 0, H_2(v, 0, 1) > 0 \\ H_3(v, 1, 1) < 0, H_3(v, 0, 1) > 0 \end{cases}$
- $\sigma_1(v) = 1, \sigma_2(v) = 0, \sigma_3(v) = 1$ if $\begin{cases} H_1(v, 1, 1) < 0, H_1(v, 0, 1) > 0 \\ H_2(v, 1, 1) < 0 \\ H_3(v, 1, 1) < 0, H_3(v, 1, 0) > 0 \end{cases}$
- $\sigma_1(v) = 1, \sigma_2(v) = 1, \sigma_3(v) = 0$ if $\begin{cases} H_1(v, 1, 1) < 0, H_1(v, 1, 0) > 0 \\ H_2(v, 1, 1) < 0, H_2(v, 1, 0) > 0 \\ H_3(v, 1, 1) < 0 \end{cases}$
- $\sigma_1(v) = 1, \sigma_2(v) = 0, \sigma_3(v) = 0$ if $\begin{cases} H_1(v, 1, 1) < 0, H_1(v, 0, 0) > 0 \\ H_2(v, 1, 1) < 0, H_2(v, 1, 0) < 0 \\ H_3(v, 1, 1) < 0, H_3(v, 1, 0) < 0 \end{cases}$
- $\sigma_1(v) = 0, \sigma_2(v) = 1, \sigma_3(v) = 0$ if $\begin{cases} H_1(v, 1, 1) < 0, H_1(v, 1, 0) < 0 \\ H_2(v, 1, 1) < 0, H_2(v, 0, 0) > 0 \\ H_3(v, 1, 1) < 0, H_3(v, 0, 1) < 0 \end{cases}$
- $\sigma_1(v) = 0, \sigma_2(v) = 0, \sigma_3(v) = 1$ if $\begin{cases} H_1(v, 1, 1) < 0, H_1(v, 0, 1) < 0 \\ H_2(v, 1, 1) < 0, H_2(v, 0, 1) < 0 \\ H_3(v, 1, 1) < 0, H_3(v, 0, 0) > 0 \end{cases}$
- $\sigma_i(v) = 0$ if $H_1(v, 0, 0) < 0, i = 1, 2, 3$.
- $\sigma_i(v)$'s satisfy $H_1(v, \sigma_2(v), \sigma_3(v)) = H_2(v, \sigma_1(v), \sigma_3(v)) = H_3(v, \sigma_1(v), \sigma_2(v)) = 0$, if

$$\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 < \min_{i=1,2,3} \{H_i(v, 0, 0)\}$$

- $\sigma_i(v)$ and $\sigma_j(v)$ satisfy $H_i(v, \sigma_j, 0) = 0$ and $H_j(v, \sigma_i, 0) = 0$ if $H_k(v, 0, 0) < 0$ and

$$\max \{H_i(v, 1, 0), H_j(v, 1, 0)\} < 0 < \min \{H_i(v, 0, 0), H_j(v, 0, 0)\}$$

Now, let $\mathbf{CB}([0, 1])$ be the space of continuous and bounded real maps on $[0, 1]$. Then, $\mathbf{CB}([0, 1])$ is a normed linear space, with a sup norm: for any $F, F' \in \mathbf{CB}([0, 1])$,

$$\|F - F'\| = \sup_{s \in [0, 1]} |F(s) - F'(s)|.$$

Lemma B4. \mathcal{F} is compact and convex.

Proof. We first show that \mathcal{F}_i , $i = 1, 2, 3$, is closed. To this end, consider any sequence $\{F_i^n\}$, where $F_i^n \in \mathcal{F}_i$ for each n , such that $\|F_i^n - F_i\| \rightarrow 0$ as $n \rightarrow \infty$. We prove that $F_i \in \mathcal{F}_i$.

Observe first that F_i is nondecreasing. Suppose to the contrary that $F_i(s') - F_i(s) < 0$ for some $s' > s$. But then,

$$\begin{aligned} \|F_i^n - F_i\| &\geq \max \{|F_i^n(s') - F_i(s')|, |F_i^n(s) - F_i(s)|\} \\ &\geq \frac{1}{2} (|F_i^n(s') - F_i(s')| + |F_i^n(s) - F_i(s)|) \\ &\geq \frac{1}{2} |F_i^n(s') - F_i(s') + F_i(s) - F_i^n(s)| \\ &\geq \frac{1}{2} |F_i(s) - F_i(s')| \\ &> 0 \end{aligned}$$

which is a contradiction. Similarly, for $s' > s$, we must have that $F_i(s') - F_i(s) \leq s' - s$. If $F_i(s') - F_i(s) > s' - s$, then

$$\begin{aligned} \|F_i^n - F_i\| &\geq \max \{|F_i^n(s') - F_i(s')|, |F_i^n(s) - F_i(s)|\} \\ &\geq \frac{1}{2} (|F_i^n(s') - F_i(s')| + |F_i^n(s) - F_i(s)|) \\ &\geq \frac{1}{2} |F_i(s') - F_i(s) + F_i^n(s) - F_i^n(s')| \\ &\geq \frac{1}{2} |F_i(s') - F_i(s) - (s' - s)| \\ &> 0, \end{aligned}$$

which is a contradiction again. Combining these, we have $F_i \in \mathcal{F}_i$, proving that \mathcal{F}_i is closed.

Next, we show that \mathcal{F}_i is compact. Note that for any $F_i \in \mathcal{F}_i$ and $s, s' \in [0, 1]$,

$$|F_i(s') - F_i(s)| \leq |s' - s|,$$

Hence, \mathcal{F}_i is Lipschitz continuous and so is equicontinuous and bounded. By the Arzèla-Ascoli theorem,³⁹ \mathcal{F}_i is compact.

We now show that \mathcal{F} is convex. Observe that for any $F_i, F'_i \in \mathcal{F}$ and $s, s' \in [0, 1]$, for

³⁹Arzèla-Ascoli theorem gives conditions for a set of $\mathbf{C}(T)$ to be compact, where $\mathbf{C}(T)$ is the space of continuous maps on T and T is a compact metric space. A subset of $\mathbf{C}(T)$ is compact if and only if it is closed, bounded, and equicontinuous.

and $\eta \in (0, 1)$,

$$\begin{aligned} (\eta F_i + (1 - \eta)F'_i)(s') - (\eta F_i + (1 - \eta)F'_i)(s) &= \eta(F_i(s') - F_i(s)) + (1 - \eta)(F'_i(s') - F'_i(s)) \\ &\leq \eta(s' - s) + (1 - \eta)(s' - s) \\ &= s' - s, \end{aligned}$$

which proves that \mathcal{F}_i is convex.

Since \mathcal{F}_i is compact and closed, so is its Cartesian product $\mathcal{F} = \times_{i=1}^3 \mathcal{F}_i$ (with respect to the product topology). \blacksquare

Lemma B5. *T is continuous.*

Proof. The proof involves several steps. For the first step, with slight abuse of notation, let v_i^{jk} be such that $j = 1$ if $\sigma_j(v) = 1$ or $j = 0$ if $\sigma_j(v) = 0$; and $k = 1$ if $\sigma_k(v) = 1$ or $k = 0$ if $\sigma_k(v) = 0$.

Step 1. v_i^{jk} are continuous on F_1, F_2, F_3 .

Proof. Fix any $F_i \in \mathcal{F}_i$ and $\varepsilon > 0$. Let $v_i^{jk} = v_i^{11}$. Take $\delta = \frac{\bar{\mu}_{i>j,k}}{2\lambda} \varepsilon$. Then, for any $F_i, F'_i \in \mathcal{F}_i$ such that $\|F_i - F'_i\| < \delta$, we have that

$$\begin{aligned} \left| v_i^{jk} - v_i^{jk'} \right| &= \left| \frac{\lambda}{\bar{\mu}_{i>j,k}} \int_0^1 \mu_{i>j,k}(s) [dF_i(s) - dF'_i(s)] \right| \\ &= \frac{\lambda}{\bar{\mu}_{i>j,k}} \left| \mu_{i>j,k}(1) [F_i(1) - F'_i(1)] - \int_0^1 \mu'_{i>j,k}(s) [F_i(s) - F'_i(s)] ds \right| \\ &\leq \frac{2\lambda}{\bar{\mu}_{i>j,k}} \|F_i(s) - F'_i(s)\| \\ &< \varepsilon, \end{aligned}$$

where the second equality follows from the integration by parts and $F_i(0) = F'_i(0) = 0$, and the penultimate inequality holds since $\int_0^1 \mu'_{i>j,k}(s) ds = \mu_{i>j,k}(1) - \mu_{i>j,k}(0) \leq 1$. The proofs for other cases $(v_i^{10}, v_i^{01}, v_i^{00})$ are similar, so we omit them. \square

Step 2. σ_i 's in mixed-strategies are continuous.

Proof. Consider, at first, students with score v such that

$$H_k(v, 0, 0) < 0, \tag{B.2.1}$$

$$H_i(v, 1, 0) < 0 < H_i(v, 0, 0), \tag{B.2.2}$$

$$H_j(v, 1, 0) < 0 < H_j(v, 0, 0). \tag{B.2.3}$$

That is, college k puts zero probability for those students (by (B.2.1)), and colleges i and j use mixed-strategies σ_i and σ_j which satisfy $H_i(v, \sigma_j, 0) = 0$ and $H_j(v, \sigma_i, 0) = 0$.

Now, let $J_i : \mathcal{F}_i \times \mathcal{F}_j \times [0, 1]^2 \rightarrow [0, 1]$ such that

$$J_i(F_i, F_j, \sigma_i, \sigma_j) \equiv H_i(v, \sigma_j, 0) = v[(1 - \sigma_j) + \bar{\mu}_{i>j}\sigma_j(v)] - \lambda \int_0^1 [(1 - \sigma_j) + \mu_{i>j}(s)\sigma_j(v)] dF_i(s),$$

$$J_j(F_i, F_j, \sigma_i, \sigma_j) \equiv H_j(v, \sigma_i, 0) = v[(1 - \sigma_i) + \bar{\mu}_{j>i}\sigma_i(v)] - \lambda \int_0^1 [(1 - \sigma_i) + \mu_{j>i}(s)\sigma_i(v)] dF_j(s).$$

Then, σ_i and σ_j are the solutions to $J_i = 0$ and $J_j = 0$ in terms of F_i and F_j . Observe that

$$J_i = (1 - \sigma_j)H_i(v, 0, 0) + \sigma_jH_i(v, 1, 0).$$

Hence,

$$\frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + H_i(v, 1, 0) < 0,$$

where inequality follows from (B.2.2). Similarly, we also have by (B.2.3)

$$\frac{\partial J_j}{\partial \sigma_i} = -H_j(v, 0, 0) + H_j(v, 1, 0) < 0.$$

Therefore,

$$\Delta_{ij} := \begin{vmatrix} \frac{\partial J_i}{\partial \sigma_i} & \frac{\partial J_i}{\partial \sigma_j} \\ \frac{\partial J_j}{\partial \sigma_i} & \frac{\partial J_j}{\partial \sigma_j} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial J_i}{\partial \sigma_j} \\ \frac{\partial J_j}{\partial \sigma_i} & 0 \end{vmatrix} = -\frac{\partial J_i}{\partial \sigma_j} \frac{\partial J_j}{\partial \sigma_i} < 0.$$

Since $\Delta_{ji} \neq 0$, the Implicit function theorem implies that there are unique σ_i and σ_j such that

$$J_i(F_i, F_j, \sigma_i, \sigma_j) = 0 \quad \text{and} \quad J_j(F_i, F_j, \sigma_i, \sigma_j) = 0.$$

Furthermore, such σ_i and σ_j are continuous.

Consider now the case that $H_1(v, \sigma_2, \sigma_3) = H_2(v, \sigma_1, \sigma_3) = H_3(v, \sigma_1, \sigma_2) = 0$ when

$$\max_{i=1,2,3} \{H_i(v, 1, 0), H_i(v, 0, 1)\} < 0 < \min_{i=1,2,3} \{H_i(v, 0, 0)\}. \quad (\text{B.2.4})$$

Similar as before, let

$$J_1(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_1(v, \sigma_2, \sigma_3) = 0,$$

$$J_2(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_2(v, \sigma_1, \sigma_3) = 0,$$

$$J_3(F_1, F_2, F_3, \sigma_1, \sigma_2, \sigma_3) \equiv H_3(v, \sigma_1, \sigma_2) = 0.$$

Observe that

$$\begin{aligned} J_i &= (1 - \sigma_j)(1 - \sigma_k)H_i(v, 0, 0) + \sigma_j(1 - \sigma_k)H_i(v, 1, 0) + (1 - \sigma_j)\sigma_k H_i(v, 0, 1) + \sigma_j\sigma_k H_i(v, 1, 1) \\ &= (1 - \sigma_j)H_i(v, 0, 0) + \sigma_j(1 - \sigma_k)H_i(v, 1, 0) - (1 - \sigma_j)\sigma_k H_i(v, 1, 0) + \sigma_j\sigma_k H_i(v, 1, 1). \end{aligned}$$

where the second equality holds after some rearrangement using the fact that $1 - \mu_{i>k}(s) = \mu_{k>i}(s)$. Therefore,

$$\frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + (1 - \sigma_k)H_i(v, 1, 0) + \sigma_k H_i(v, 1, 0) + \sigma_k H_i(v, 1, 1) < 0,$$

where the inequality holds since $H_i(v, 0, 0) > 0$, $H_i(v, 1, 0) < 0$, $H_k(v, 1, 0) < 0$ and $H_i(v, 1, 1) < 0$ by (B.2.4). This implies that

$$\Delta := \begin{vmatrix} \frac{\partial J_1}{\partial \sigma_1} & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\ \frac{\partial J_2}{\partial \sigma_1} & \frac{\partial J_2}{\partial \sigma_2} & \frac{\partial J_2}{\partial \sigma_3} \\ \frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & \frac{\partial J_3}{\partial \sigma_3} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\ \frac{\partial J_2}{\partial \sigma_1} & 0 & \frac{\partial J_2}{\partial \sigma_3} \\ \frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & 0 \end{vmatrix} = \frac{\partial J_1}{\partial \sigma_2} \frac{\partial J_2}{\partial \sigma_3} \frac{\partial J_3}{\partial \sigma_1} + \frac{\partial J_1}{\partial \sigma_3} \frac{\partial J_2}{\partial \sigma_1} \frac{\partial J_3}{\partial \sigma_2} < 0$$

Using the Implicit function theorem again, we conclude that such $\sigma_1, \sigma_2, \sigma_3$ exist and they are continuous. \square

Observe that from Step 1 and Step 2, $H_i(v, \sigma_j, \sigma_k)$, $i = 1, 2, 3$, is continuous in $(F_i)_{i=1,2,3}$ for a given s and fixed v .

Step 3. $m_i(s)$ is continuous.

Proof. Consider any $F_i, F'_i \in \mathcal{F}_i$ such that $\|F_i - F'_i\| < \delta$ for all $i = 1, 2, 3$. Let σ_i and σ'_i are admission strategies of college i which correspond to F_i and F'_i , respectively. Then, for a given s and v , $n_i(v|s)$ is defined by (B.0.3) and $n'_i(v|s)$ is defined similarly using σ'_i .

Let $X := \{v \in [0, 1] \mid |\sigma_i(v) - \sigma'_i(v)| \geq \varepsilon/2\}$. Clearly,

$$|\sigma_i(v) - \sigma'_i(v)| = |\sigma_i(v) - \sigma'_i(v)| \mathbf{1}_X(v) + |\sigma_i(v) - \sigma'_i(v)| \mathbf{1}_{X^c}(v),$$

where $\mathbf{1}_X(v)$ is the indicator function which is 1 if $v \in X$ or 0 otherwise, and X^c is the complementary set of X . Since v_i^{jk} are continuous by Step 1, we have

$$\int_0^1 \mathbf{1}_X(v) dG(v) < \frac{\varepsilon}{2}. \quad (\text{B.2.5})$$

For $v \in X^c$, it must be the case that either $\sigma_i = \sigma'_i$, or σ_i and σ'_i are the mixed-strategies. Thus, we have for $v \in X^c$,

$$|\sigma_i(v) - \sigma'_i(v)| < \frac{\varepsilon}{2}. \quad (\text{B.2.6})$$

Observe that

$$\begin{aligned}
\int_0^1 |\sigma_i(v) - \sigma'_i(v)| dG(v) &= \int_0^1 |\sigma_i(v) - \sigma'_i(v)| \mathbf{1}_X(v) dG(v) + \int_0^1 |\sigma_i(v) - \sigma'_i(v)| \mathbf{1}_{X^c}(v) dG(v) \\
&< \int_0^1 \mathbf{1}_X(v) dG(v) + \int_0^1 |\sigma_i(v) - \sigma'_i(v)| \mathbf{1}_{X^c}(v) dG(v) \\
&< \varepsilon,
\end{aligned}$$

where the first inequality holds since $\sigma_i, \sigma'_i \leq 1$, and the last inequality follows from (B.2.5) and (B.2.6). Thus, there exists δ_1 such that $\|F_i - F'_i\| < \delta_1$, for all $i, i' = 1, 2, 3$, implies

$$\begin{aligned}
&\int_0^1 |\sigma_i(1 - \sigma_j)(1 - \sigma_k) - \sigma'_i(1 - \sigma'_j)(1 - \sigma'_k)| dG(v) \\
&\leq \int_0^1 \left[|\sigma_i - \sigma'_i| (1 - \sigma_j)(1 - \sigma_k) + |\sigma_j - \sigma'_j| \sigma'_i(1 - \sigma_k) + |\sigma_k - \sigma'_k| \sigma'_i(1 - \sigma'_j) \right] dG(v) \\
&< \frac{\varepsilon}{4}
\end{aligned}$$

Similarly, there are δ_t , $t = 2, 3, 4$, such that $\|F_i - F'_i\| < \delta_t$ respectively imply that

$$|\sigma_i \sigma_j (1 - \sigma_k) - \sigma'_i \sigma'_j (1 - \sigma'_k)| < \frac{\varepsilon}{4}, \quad |\sigma_i \sigma_k (1 - \sigma_j) - \sigma'_i \sigma'_k (1 - \sigma'_j)| < \frac{\varepsilon}{4}, \quad |\sigma_i \sigma_j \sigma_k - \sigma'_i \sigma'_j \sigma'_k| < \frac{\varepsilon}{4}.$$

Now, let $\delta = \min_{t=1,2,3,4} \{\delta_t\}$. We have that $\|F_i - F'_i\| < \delta$ implies

$$|m_i(s) - m'_i(s)| \equiv \left| \int_0^1 \sigma_i(v) n_i(v|s) dG(v) - \int_0^1 \sigma'_i(v) n'_i(v|s) dG(v) \right| < \varepsilon.$$

That is, $m_i(s)$ is continuous on $(F_i)_{i=1,2,3}$. \square ■

Lemma B5 proves the existence admission strategies that satisfy the local conditions. The proof that those strategies are mutual (global) best responses is analogous to that of the two college case.