Gains from Trade

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Abstract

In a social choice context, we ask whether there exists a system of transfers and regulations whereby gains from trade can always be realized under trade liberalization. We consider a resource allocation problem in which the set of commodities to be traded is variable. We propose an axiom stating that enlarging the set of commodities and reapplying the rule hurts nobody. We obtain two results. Suppose that we extend the allocation rule in two steps, first from autarky to a class of smaller sets of commodities and second to the entire set of commodities. Our first result is that if we apply the Walrasian solution in the first step, it is impossible to extend the rule in the second step in order to satisfy the above axiom, even when compensation or regulation is allowed. Our second result is that if the rule satisfies an allocative efficiency axiom and an informational efficiency axiom stating that only preferences over tradable commodities should matter, together with the above axiom, gains from trade can be given to only one individual in the first step.

1 Introduction

A classical rationale for markets is that they allow gains from trade to be realized; at the very least, no agent can be made worse off than her initial holding. However, this basic

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comparative static only holds generally when starting from autarky. If a group of agents trade some goods on the market, but others are untraded, opening markets in the untraded goods can potentially hurt some of the agents. The intuition for this is simple: opening trade in new goods can alter the equilibrium price of already traded goods to accommodate the potential tradeoffs for newly traded goods.

In the international trade literature, this is known as a negative terms-of-trade effect (see Krugman, Obstfeld and Melitz [5] for example). A related phenomenon occurs in the context of financially incomplete markets. Hart [4] offers an example establishing that opening a market in new securities result in a Pareto loss. Elul [2] and Cass and Citanna [1] have shown that such worsening is generic.

A very basic question remains. While unregulated markets do not in general produce gains from trade except in the special case of autarky, there may be room for transfers or subsidies or regulations which allow such a result to be restored more generally. To this end, our question does not take the competitive market solution in the Walrasian sense as given. We ask: Is it possible to allocate resources, allowing redistribution of income or resources and any other compensation or any price regulation, so that that opening trade in new goods never makes anybody worse off?

Somewhat surprisingly, we show that the answer is generally negative. To qualify this statement, we first ask what our social choice function (SCF) should or has to meet. First, we ask that our SCF always respect weak Pareto efficiency. Secondly, we ask that our SCF be sufficiently *decentralized*, in that it only take into account preferences and endowments of traded commodities. We call this constraint (rather than a normative postulate) Independence of Untraded Commodities. Any SCF going beyond this constraint would require extreme bureaucratic involvement on the part of a social planner, requiring sophisticated knowledge of preferences over untraded commodities. More to the point, the revealed preference paradigm dictates that if commodities are not tradeable, it is by definition impossible to infer preferences over these commodities from choice behavior. Hence, if we interpret preference as *revealed preference*, the condition is a *necessary* requirement for any mechanism in the environment we study. Removing the condition would result in a framework involving elements which cannot be identified economically.

Now, the Walrasian solution, for example, satisfies these two properties. As our third and final condition, we also ask that nobody be made worse off when opening markets to trade in new goods. We call this No Loss from Trade.

We obtain two results. Imagine that we extend the SCF in two steps, first from autarky to a class of smaller sets of commodities and second to the entire set of commodities, where the preference domain satisfies certain minimal richness conditions. Our first result is that as long as we accept the Walrasian solution in the first step, it is impossible to extend the SCF in the second step in a manner which does not hurt anybody, even when arbitrary compensation or regulation is permitted.

The second result does not require acceptance of the Walrasian solution in the first step. However, we establish that gains from trade requires there to be a dominant individual who reaps all of the gains; all other agents remain at the welfare level of their endowment.

Related literature

Our result is related to several results in the literature in social choice in exchange economies, for example Moulin and Thomson [6]. A major theme of this literature relates to whether everybody can benefit systematically when the set of available objects increases somehow. The aforementioned result establishes that, in an exchange economy environment without endowments, it is very hard for each agent to benefit when more of each commodity is introduced. Our result follows this theme by considering the introduction of new commodities, rather than introducing more of existing commodities.

Our independence axiom may resemble an independence axiom proposed by Fleurbaey and Tadenuma [3], stating that in the setting of variable sets of *physically present* commodities only preferences over physically present commodities should matter. The difference here is that we fix the set of physically present commodities and vary the sets of tradable commodities, where individuals consume their initial endowment of untradable commodities. Also, we view our independence axiom as a natural constraint rather than as a normative postulate.

2 Model and axioms

2.1 Model

Let I be the set of individuals. Let X be a finite set of commodities which are *physically present* in the world. Fix a list of initial endowments $\omega = (\omega_1, \cdots, \omega_{|I|}) \in \mathbb{R}^{I\times X}_{++}$ Let $\mathcal{R}$

be the set of convex and strongly monotone preferences over $\mathbb{R}^X_+$, and for each $i \in I$ let $\mathcal{D}_i \subset \mathcal{R}$ be the domain of i's preferences. Let $\mathcal{D} = \prod_{i\in I} \mathcal{D}_i$ be the domain of preference profiles. We will discuss the properties $\mathcal{D}$ satisfies later.

Let $\mathcal{T} \subset 2^X$ be the family of admissible sets of tradable commodities.

An *economy* is a pair $(\succsim, T)$, which consists of a list of preference relations $\succsim = (\succsim_1, \cdots, \succsim_{|I|}) \in \mathcal{D}$ and a set of *tradable* commodities $T \in \mathcal{T}$. We take $\mathcal{D} \times \mathcal{T}$ to be the domain of economies.

Further, let $F(T) \subset \mathbb{R}^{I\times X}_+$ denote the set of feasible allocations when T is tradable. Formally,

$$F(T) = \left\{ x \in \mathbb{R}^{I\times X}_+ : \begin{array}{c} \sum_{i\in I} x_{ik} \leq \sum_{i\in I} \omega_{ik},\ \forall k \in T \\ x_{ik} = \omega_{ik},\ \forall i \in I,\ \forall k \in T^c \end{array} \right\}.$$

A *social choice function (SCF)* is a mapping φ carrying each economy $(\succsim, T) \in \mathcal{D} \times \mathcal{T}$ into an element of $F(T)$. An SCF specifies how trade in any given economy should be undertaken. The use of an abstract rule allows us to study the properties we wish our allocations to satisfy.

2.2 Axioms

We list here our properties for SCFs. The first states that, for any given economy, it should be impossible to reallocate tradable resources in a fashion that makes everybody strictly better off.

Axiom 1 (**Weak Pareto**): For all $(\succsim, T) \in \mathcal{D} \times \mathcal{T}$, there is no $x \in F(T)$ such that $x_i \succ_i \varphi_i(\succsim, T)$ for $i \in I$.

The second condition is our motivating criterion: when opening up trade in new commodities, nobody should be hurt.

Axiom 2 (**No Loss from Trade**): For all $\succsim \in \mathcal{D}$ and $T, T' \in \mathcal{T}$ with $T \subset T'$, we have

$$\varphi_i(\succsim, T') \succsim_i \varphi_i(\succsim, T)$$

for all $i \in I$.

Note that no loss from trade implies the following individual rationality axiom:

Axiom 3 (**Individual Rationality**): For all $(\succsim, T) \in \mathcal{D} \times \mathcal{T}$,

$$\varphi_i(\succsim, T) \succsim_i \omega_i$$

for all $i \in I$.

Finally, we specify our decentralization condition. Formally this is an independence condition, specifying that only the preferences over tradable commodities should be taken into account. Any SCF not satisfying this property will necessarily be extremely complicated. By itself, this does not necessarily preclude rules violating the axiom. However, there is a more serious concern. We do not view this condition as a normative requirement, or even necessarily as an "informational simplicity" assumption. Rather, it is a positive restriction placed on any mechanism in the environment we study. The reasoning here follows the revealed-preference paradigm. Recall that preferences are simply a summary of choice behavior: it is only conceptually possible to infer preferences over objects amongst which individuals may choose. Even ignoring the strategic issues with many agents, if it is impossible to trade some commodity, it is by definition *impossible* to infer an individual's preferences over that commodity. Hence, the need for such a constraint.

This type of independence condition can also be viewed as a formal notion capturing the idea of partial equilibrium analysis. Trade liberalization typically can take place only in a gradual manner, in which we do not see the final goal. In such circumstances a "partial equilibrium" approach isolates the issue under consideration from the rest of the economy, ignoring preferences over non-marketed goods.

Our point is that it is indeed a constraint, and the independence axiom formalizes this constraint. Thus we take this as an informational constraint which any "market-like," or economically meaningful, allocation rule has to obey, rather than a normatively desirable postulate.

Given $\succsim_i \in \mathcal{D}_i$ and $T \in \mathcal{T}$, let $\succsim_i |_T$ denote the ordering over $\mathbb{R}^T_+$ defined by

$$x \succsim_i |_T y \iff (x, \omega_{iT^c}) \succsim_i (y, \omega_{iT^c})$$

for all $x, y \in \mathbb{R}^T_+$. Given $\succsim \in \mathcal{D}$ and $T \in \mathcal{T}$, let $\succsim |_T = (\succsim_i |_T)_{i \in I}$.

Axiom 4 (**Independence of Untraded Commodities**): For all $\succsim, \succsim' \in \mathcal{D}$ and $T \in \mathcal{T}$, if $\succsim |_T = \succsim^* |_T$ then

$$\varphi(\succsim, T) = \varphi(\succsim', T).$$

Related to the revealed preference justification of the condition, independence of untraded commodities is related to a condition of immunity to manipulation. To see this, suppose that when the SCF is applied to $T \in \mathcal{T}$ and the social planner let each individual reports her preference. The preferences over consumptions of commodities in T are known to the social planner and cannot be misreported, as such piece of information is presumably revealed during the trading process via demand choices. However, the individuals may misreport other aspects of preferences involving untraded commodities T^c, since they cannot be revealed via demand choices. For example, they may misreport (i) rankings over consumption of T^c, and (ii) the marginal rate of substitution *between* T and T^c even when the rankings over consumptions of T^c are known. The latter kind of misrepresentation may be used to claim that one receives a relatively low level of "utility" from the consumption of goods in T^c, and hence he should receive more "utility" from T. Under the minimal domain richness conditions we introduce later, only the latter kind of information will be relevant as the source of manipulation.

The condition below states that the SCF should be immune to such manipulation.

Axiom 5 (**Strategy-Proofness with respect to Untraded Commodities**): For all $\succsim \in \mathcal{D}$ and $T \in \mathcal{T}$, for all $i \in I$ and $\succsim'_i \in \mathcal{R}$ with $(\succsim'_i, \succsim_{-i}) \in \mathcal{D}$, and $\succsim_i |_T = \succsim'_i |_T$ it holds

$$\varphi(\succsim, T) \succsim_i \varphi(\succsim', T).$$

The following lemma is immediate.

Lemma 1 Independence of Untraded Commodities implies Strategy-Proofness with respect to Untraded Commodities.

Examples of SCFs satisfying all but one of the properties follow.

Example 1 **No-trade solution** which gives $\varphi(\succsim, T) = \omega$ for all $(\succsim, T) \in \mathcal{D} \times \mathcal{T}$ satisfies No Loss from Trade, Independence of Untraded Commodities but violates Weak Pareto.

Example 2 **Monotone path solution** is defined as follows. For all $\succsim \in \mathcal{D}$, fix a profile of utility representations $u = (u_i)_{i \in I}$.

For all $(\succsim, T) \in \mathcal{D} \times \mathcal{T}$, define

$$\varphi(\succsim, T) \in \arg\max_{x \in F(T)} \min_{i \in I} u_i(x_i),$$

in which the way of selection when multiplicity occurs is arbitrary.

This satisfies Weak Pareto, No Loss from Trade but violates Independence of Untraded Commodities. The reason this violation occurs is because the profile of utility functions depends on the preference profile under consideration. In order to guarantee independence, two preference profiles which induce the same preferences on $T \in \mathcal{T}$ should therefore map to the same induced utility functions on $T \in \mathcal{T}$. In general, there is no way to construct a system of utility functions, depending on preference profiles, which has this property.

Example 3 Consider any selection of the **Walrasian solution**, in which the selection in the case of multiplicity depends only on the induced preference over the tradable commodities.

This satisfies Weak Pareto, Independence of Untraded Commodities but violates No Loss from Trade.

To illustrate the result, let us describe the following natural procedure, which all economists will find familiar and obviously results in gains from trade at each stage. Start from autarky. Open trade in a collection of commodities and find a Walrasian equilibrium. Now, open trade in a collection of new commodities, taking the old Walrasian equilibrium as the endowment. Find a new Walrasian equilibrium. It is clear that all individuals will be made better off at each stage here, so it is instructive to ask how our primitives preclude this rule.

The issue is the following. Suppose we have two disjoint sets of commodities, K and K^c. Depending on which set of commodities we open trade to first, the resulting final allocation will be different. There is a path dependence of the previous procedure on the order in which trade is opened. This means that, in general, the procedure we demonstrated cannot be compatible with a *rule* which works independent of path.

And in fact, this is roughly the main source of difficulty leading to our result. The agents that benefit from trading in the set of commodities K can be quite different from those that benefit from trade in K^c. However, the allocation resulting from trade in all commodities in X should give all agents a consumption bundle that is weakly preferred to both the bundle obtained under K and the bundle obtained under K^c. However, the preferences on X may exhibit arbitrarily strong complementarities between the commodities of group K and those of group K^c. Hence, the utility of an average of the allocations assigned under K and K^c can then be arbitrarily close to the minimum of the utilities under K and K^c;

and it is not possible to satisfy the no loss from trade axiom.[1]

3 The minimal domain richness conditions

Before proceeding, it is worth attempting to convey some of the intuition of our construction. Unfortunately, we have not been able to establish any general results on the domain of strictly convex, strictly monotone, and continuous preferences. The reasoning will hopefully become clear. Imagine two disjoint sets of commodities, K and K^c. Imagine that a social planner has information as to the preferences over these commodities, passing through the endowment point. In general, one would suspect there are many degrees of freedom in "completing" these preferences. That is, we would think that there are many preferences over X which induce the given preference profiles over K and K^c.

We have one obvious restriction on the preference over X. The indifference curve over X which passes through the endowment must intersect the indifference curves of the original preference profiles over K and K^c which pass through the endowment. This is a matter of definition. Now, take the convex and upper comprehensive hull of the indifference curves passing through the endowment over K and K^c. This looks almost like an upper contour set for a preference over X, except in general it is not strictly monotone. In order to be compatible with our domain restrictions, we need to be able to ensure that there is an indifference curve over X passing through the indifference curves over K and K^c which is strictly monotone. It eases our argument to make sure the upper contour set of this indifference curve is as "small" as possible; that is, it is as close to the convex hull described as possible.

Now, consider a bundle of goods x_K in commodity space K, and another bundle x_{K^c} in K^c, such that each bundle is strictly preferred the the endowment in each commodity space. There is nothing tying the indifference curves passing through these two points in the two dimensions together—they are totally disjoint. It seems reasonable that we may therefore complete the preference so that this pair is ranked arbitrarily. That is—we can effectively choose the marginal rate of substitution between the goods K and K^c to be whatever we want. In fact, more seems to be true. For any continuous utility representations of the preferences over K and K^c which agree at the endowment, we can complete the preference

[1] We are grateful to an anonymous referee from the *American Economic Review* for suggesting this intuition.

relation to agree with these utility functions. The construction would consist of taking, for any utility level, the convex and upper comprehensive hull of the preferences giving at least that utility on the two commodity subspaces and then "extending" the preference elsewhere to retain strict monotonicity and continuity.

These arguments seem intuitive, and we believe them to be true on the domain of convex, continuous, strictly monotone preferences. Unfortunately, we were unable to come up with a rigorous proof. The difficulty stems from finding a general procedure for guaranteeing strict monotonicity and continuity of a preference relation outside of the convex hulls we discussed in the previous paragraph. [3] use a similar extension construction, except they do not have to worry about what happens outside of these convex hulls. The reason is that their induced preferences always appear on the boundary of commodity space; the convex hull construction covers the entire space.

In light of these difficulties, we have isolated the conditions necessary for the completion of our argument. These are what we term the "minimal richness condition." As we stated, we believe this condition is satisfied by the domain of convex, strictly monotone, and continuous preferences. At the same time, to ensure that the conditions are not vacuous, we have demonstrated the existence of a domain satisfying them.

Our main result is then a characterization theorem, establishing that a mechanism satisfying the properties must feature one individual maximizing their preference subject to the other individual's rationality constraints. And, even on the domain of all continuous, strictly monotone, and convex preferences, this result will hold when restricted to the domain we consider. This, at the very minimum, tells us that on the full preference domain there is no mechanism which is Pareto efficient, satisfies our independence condition, and results in strict gains from trade to all individuals whenever possible.

Our argument will require that the domain of admissible preferences be sufficiently "rich." To ensure the result is as powerful as possible, we postulate a pair of relatively weak richness conditions.

First, we restrict attention to a simple family of admissible sets of tradable commodities.

Minimal Richness Condition (MRC) 1: Assume $|X| \geq 4$. Let $K \subset X$ with $2 \leq |K| \leq |X| - 2$, and assume

$$\mathcal{T} = \{\emptyset, K, K^c, X\}.$$

To illustrate, imagine a two-step procedure in which the society starts with autarky and

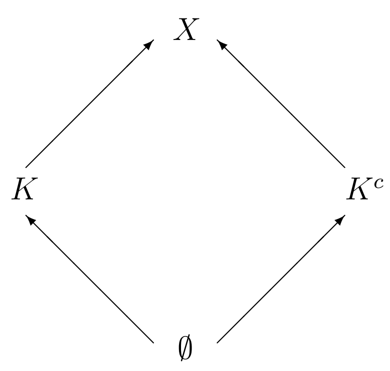

Figure 1: Admissible sets of tradable commodities

the first step is to choose between applying the SCF to K or to K^c, and the second step is to extend the SCF to the entire set of commodities X (see Figure 1).

The second condition is about minimal richness of the preference domain.

Minimal Richness Condition (MRC) 2: For all $i \in I$:

(i) For all $\succsim_i, \succsim_i' \in \mathcal{D}_i$, it holds $\succsim_i |_K = \succsim_i' |_K$ and $\succsim_i |_{K^c} = \succsim_i' |_{K^c}$;

(ii) For all $\succsim_i \in \mathcal{D}_i$ and any fixed $x_i \in \mathbb{R}_+^X$, there exists $\succsim_i^* \in \mathcal{D}_i$ such that $R(x_i, \succsim_i^*)$ is arbitrarily close to

$$co\,(R_K(x_i, \succsim_i) \times \{\omega_{iK^c}\} \cup \{\omega_{iK}\} \times R_{K^c}(x_i, \succsim_i))$$

(iii) For all $\succsim_i \in \mathcal{D}_i$ and any $x_i \in \mathbb{R}_+^X$ such that $(x_{iK}, \omega_{iK^c}) \succ_i \omega_i$ and $(\omega_{iK}, x_{iK^c}) \succ_i \omega_i$, there exist $\succsim_i', \succsim_i'' \in \mathcal{D}_i$ such that

$$(x_{iK}, \omega_{iK^c}) \succ_i' (\omega_{iK}, x_{iK^c})$$

and

$$(x_{iK}, \omega_{iK^c}) \prec_i'' (\omega_{iK}, x_{iK^c})$$

hold respectively.

MRC2-(i) says that we are dealing with only a **fixed profile** of rankings when it comes to consumptions of K (resp. K^c) alone given that K^c (resp. K) is untradable. This leads us to consider only manipulations of information about marginal rates of substitution **between** K and K^c. In this sense we are dealing with a quite "small" preference domain.

MRC2-(ii) says that given a preference and a consumption bundle we can find a related preference which is arbitrarily "more demanding." A "more demanding" preference is one in which preference is more difficult to improve upon by allowing trades of all goods in X. Hence the "most demanding" one is such that the corresponding upper contour set is the convex hull, which is taken in the entire consumption space, of the two upper contour sets respectively in the affine subspace where K is the set of traded commodities trade and in the one where K^c is the set of traded commodities. This most demanding preference is not strongly monotone, so our condition says we should be able to take a strongly monotone one which is arbitrarily close to it.

MRC2-(iii) says that the ranking between consumption of goods in K when goods in K^c are untradable and consumption of goods in K^c when good in K are untradable is indeterminate, provided these bundles are strictly individually rational.

The following observation is easy to see.

Lemma 2 Under Minimal Richness Condition 1 and 2, Independence of Untraded Commodities implies that for all $\succsim, \succsim' \in \mathcal{D}$ it holds

$$\varphi(\succsim, K) = \varphi(\succsim', K)$$

and

$$\varphi(\succsim, K^c) = \varphi(\succsim', K^c).$$

Here is a more specific construction of preference domain which satisfies MRC1 and MRC2.

Example 4 Construct $\mathcal{D}_i$ for each $i \in I$ as follows.

1. Fix concave and strictly increasing functions $v_{iK} : \mathbb{R}^K_+ \to \mathbb{R}$ and $v_{iK^c} : \mathbb{R}^{K^c}_+ \to \mathbb{R}$.

2. For any $\gamma_i \in (0, 1)$, let

$$u_i(x_i|\gamma_i) \quad = \quad \gamma_i \left(v_{iK}(x_{iK}) - v_{iK}(\omega_{iK})\right) + (1 - \gamma_i) \left(v_{iK^c}(x_{iK^c}) - v_{iK^c}(\omega_{iK^c})\right)$$

3. For any $\gamma_i \in (0, 1)$ and $\varepsilon_i \in (0, 1]$, let

$$u_i(x_i|\gamma_i, \varepsilon_i) = \max_{(1-\varepsilon_i)V(u)+\varepsilon_i W(u)\ni x} u,$$

   where

$$V(u) = co\left(\{(z_{iK}, \omega_{iK^c}) : u_i(z_{iK}, \omega_{iK^c}|\gamma_i) \geqq u\} \cup \{(\omega_{iK}, z_{iK^c}) : u_i(\omega_{iK}, z_{iK^c}|\gamma_i) \geqq u\}\right)$$

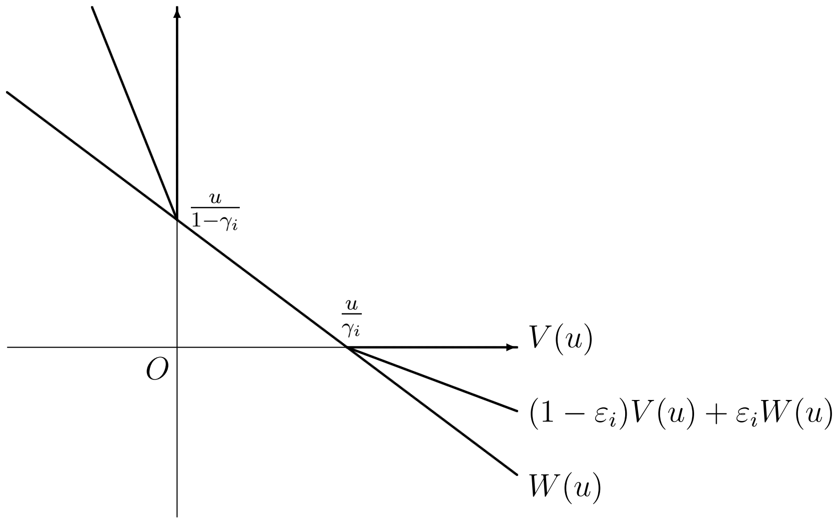

Figure 2: Construction of $\mathcal{D}_v$

and

$$W(u) = \{z : u_i(z|\gamma_i) \geqq u\}.$$

4. Given a pair of functions $v_i = (v_{iK}, v_{iK^c})$, let $\mathcal{D}_{v_i}$ let be the set of preference relations which are represented by $u_i(\cdot|\gamma_i, \varepsilon_i)$ for some $\gamma_i \in (0,1)$ and $\varepsilon_i \in (0,1]$.

5. Let

$$\mathcal{D}_i = \mathcal{D}_{v_i}$$

Let us pretend that the above convex-hull operation is done in the utility space. See Figure 2, in which $v_{iK}(x_{iK}) - v_{iK}(\omega_{iK})$ is taken on the horizontal axis and $v_{iK^c}(x_{iK^c}) - v_{iK^c}(\omega_{iK^c})$ is taken on the vertical axis. Here $W(u)$ is the half space which is upper-right to the straight line passing through $\left(\frac{u}{\gamma_i}, 0\right)$ and $\left(0, \frac{u}{1-\gamma_i}\right)$, whereas $V(u)$ is the intersection of $W(u)$ and the non-negative orthant. Since the preference giving $V(u)$ is not strongly monotone, we take a convex combination $(1-\varepsilon_i)V(u) + \varepsilon_i W(u)$ where ε_i is sufficiently small.

Here MRC2-(i) and (ii) are met by construction. To see that MRC2-(iii) is met, pick any $i \in I$, $\gamma_i \in (0,1)$ and $\varepsilon_i \in (0,1]$. Suppose

$$\begin{aligned} u_i(x_{iK}, \omega_{iK^c}|\gamma_i, \varepsilon_i) &= \gamma_i \left(v_{iK}(x_{iK}) - v_{iK}(\omega_{iK})\right) \\ &> 0 = u_i(\omega_i|\gamma_i, \varepsilon_i) \end{aligned}$$

and

$$
\begin{aligned}
u_i(\omega_{iK}, x_{iK^c}|\gamma_i, \varepsilon_i) &= (1-\gamma_i)\left(v_{iK^c}(x_{iK^c}) - v_{iK^c}(\omega_{iK^c})\right) \\
&> 0 = u_i(\omega_i|\gamma_i, \varepsilon_i)
\end{aligned}
$$

Then one can take $\gamma_i' \in (0,1)$ such that

$$
\begin{aligned}
u_i(x_{iK}, \omega_{iK^c}|\gamma_i', \varepsilon_i) &= \gamma_i'\left(v_{iK}(x_{iK}) - v_{iK}(\omega_{iK})\right) \\
&> (1-\gamma_i')\left(v_{iK^c}(x_{iK^c}) - v_{iK^c}(\omega_{iK^c})\right) \\
&= u_i(\omega_{iK}, x_{iK^c}|\gamma_i', \varepsilon_i),
\end{aligned}
$$

and similarly for the opposite direction.

4 Impossibility of opening markets without hurting anybody

Our first result is that there is generally no system of transfers, taxes, subsidies or price regulation which worsens nobody in the second step, once we accept the Walrasian solution for the first step.

Let W denote the Walrasian correspondence.

Definition 1 Say that φ is *Walrasian in the first step* if

$$\varphi(\succsim, K) \in W(\succsim, K) \text{ and } \varphi(\succsim, K^c) \in W(\succsim, K^c)$$

for all $\succsim \in \mathcal{D}$.

Theorem 1 Assume MRC1 and MRC2. Also assume that for some $\succsim \in \mathcal{D}$ the Walrasian solutions $W(\succsim, K)$ and $W(\succsim, K^c)$ are single-valued and strictly individually rational. Then there is no allocation rule which is Walrasian in the first step and satisfies No Losses from Trade.

For the proof we establish the following lemma.

Lemma 3 Suppose φ satisfies Efficiency and No Loss from Trade. For any $\succsim \in \mathcal{D}$, suppose that $\varphi(\succsim, K) \in W(\succsim, K)$, $\varphi(\succsim, K^c) \in W(\succsim, K^c)$ and $|W(\succsim, K)| = |W(\succsim, K^c)| = 1$. Then $\varphi(\succsim, K)$ and $\varphi(\succsim, K^c)$ are Pareto-ranked.

Proof.

Graphical intuition: To illustrate, suppose that $I = \{a, b\}$, $|X| = 4$ and $|K| = |K^c| = 2$. Consider a four-dimensional Edgeworth box (which obviously has no trivial graphical representation). So imagine its two affine subspaces, $\mathbb{R}^K \times \{\omega_{K^c}\}$ and $\{\omega_K\} \times \mathbb{R}^{K^c}$, as in Figure 3. Let $(x_K, \omega_{K^c}) = \varphi(\succsim, K)$ and $(\omega_K, x_{K^c}) = \varphi(\succsim, K^c)$ here. Suppose now $(x_{aK}, \omega_{aK^c}) \succ_a (\omega_{aK}, x_{aK^c})$ and $(x_{bK}, \omega_{bK^c}) \prec_b (\omega_{bK}, x_{bK^c})$ for the sake of contradiction.

Then, when a's indifference surface giving the indifference curve I_a passing through x_K in the left affine subspace $\mathbb{R}^K \times \{\omega_{K^c}\}$ intersects the right affine subspace $\{\omega_K\} \times \mathbb{R}^{K^c}$ it gives the dotted indifference curve I_a passing strictly beyond x_{K^c} from a's viewpoint. Also, when b's indifference surface giving the indifference curve I_b passing through x_{K^c} in the right affine subspace $\{\omega_K\} \times \mathbb{R}^{K^c}$ intersects the left affine subspace $\mathbb{R}^K \times \{\omega_{K^c}\}$ it gives the dotted indifference curve I_b passing strictly beyond x_K from b's viewpoint.

Then, in the left affine subspace a's upper contour setting corresponding to I_a and b's upper contour set corresponding to the dotted I_b are separated by the budget line passing through ω_K. Likewise, in the right affine subspace a's upper contour setting corresponding to the dotted I_a and b's upper contour set corresponding to I_b are separated by the budget line passing through ω_{K^c}.

Now from (ii) in Minimal Richness Condition 2 we can take $\succsim^* \in \mathcal{D}$ such that a's upper contour set above (x_{aK}, ω_{aK^c}) is arbitrarily close to the convex hull of the solid I_a in the left and the dotted I_a in the right in the entire space $\mathbb{R}^X$, and b's upper contour set above (ω_{bK}, x_{bK^c}) is arbitrarily close to the convex hull of the dotted I_b in the left and the solid I_b in the right in the entire space $\mathbb{R}^X$.

Then, a's upper contour set above (x_{aK}, ω_{aK^c}) and b's upper contour set above (ω_{bK}, x_{bK^c}) are separated by the hyperplane spanned by the budget line in the left and the budget line in the right, implying that they are disjoint.

No Loss from Trade requires now that $\varphi(\succsim^*, X)$ be in both a's upper contour set and b's upper contour set, which is impossible.

Proof of the Lemma: Let

$$\begin{aligned} P(x_i, \succsim_i) &= \{z_i \in \mathbb{R}^X_+ : z_i \succ_i x_i\} \\ R(x_i, \succsim_i) &= \{z_i \in \mathbb{R}^X_+ : z_i \succsim_i x_i\} \\ P_K(x_i, \succsim_i) &= \{z_{iK} \in \mathbb{R}^K_+ : (z_{iK}, \omega_{iK^c}) \succ_i x_i\} \\ R_K(x_i, \succsim_i) &= \{z_{iK} \in \mathbb{R}^K_+ : (z_{iK}, \omega_{iK^c}) \succsim_i x_i\} \end{aligned}$$

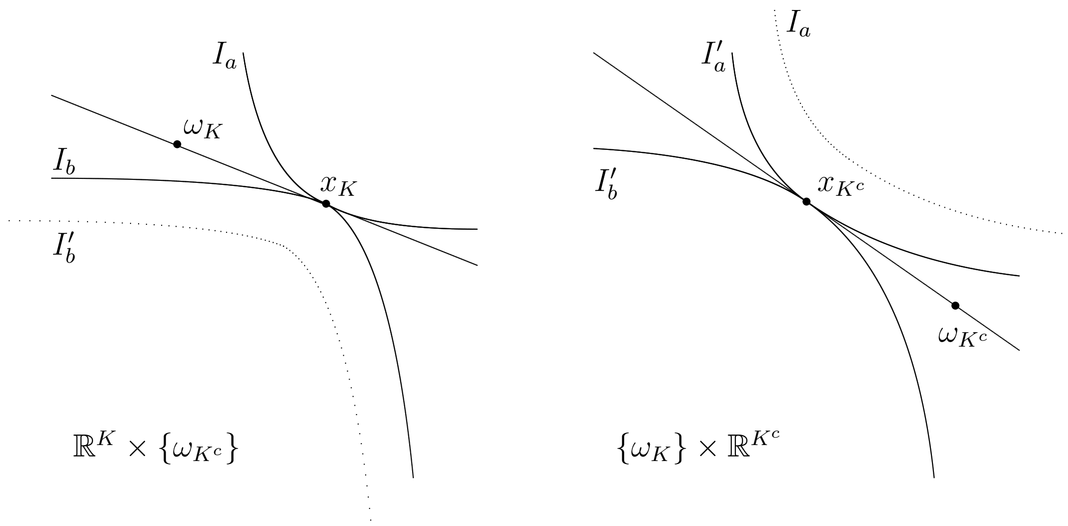

Figure 3: Walrasian allocations

for each i, and define the last two similarly for K^c.

Let

$$\begin{aligned} I_1 &= \{i \in I : \varphi_i(\succsim, K) \succ_i (e)\varphi_i(\succsim, K^c)\} \\ I_2 &= \{i \in I : \varphi_i(\succsim, K) \prec_i (e)\varphi_i(\succsim, K^c)\} \\ I_3 &= \{i \in I : \varphi_i(\succsim, K) \sim_i (e)\varphi_i(\succsim, K^c)\} \end{aligned}$$

and suppose $I_1, I_2 \neq \emptyset$.

Let p_K be the price vector corresponding to $W(\succsim, K)$. Then we have

$$p_K \omega_{iK} \leq \inf_{z_{iK} \in R_K(\varphi_i(\succsim, K), \succsim_i)} p_K z_{iK}$$

for all $i \in I$.

For each $i \in I_2$, by assumption that $\varphi_i(\succsim, K) \prec_i \varphi_i(\succsim, K^c)$, it follows $R_K(\varphi_i(\succsim, K^c), \succsim_i$ $) \subsetneqq P_K(\varphi_i(\succsim, K), \succsim_i)$. Hence we have

$$p_K \omega_{iK} < \inf_{z_{iK} \in R_K(\varphi_i(\succsim, K^c), \succsim_i)} p_K z_{iK}$$

for all $i \in I_2$.

Thus we have

$$\begin{aligned} p_K \sum_{i \in I} \omega_{iK} \leq & \sum_{i \in I_1 \cup I_3} \inf_{z_{iK} \in R_K(\varphi_i(\succsim, K), \succsim_i)} p_K z_{iK} \\ & + \sum_{i \in I_2} \inf_{z_{iK} \in R_K(\varphi_i(\succsim, K^c), \succsim_i)} p_K z_{iK} \end{aligned}$$

Let p_{K^c} be the price vector corresponding to $W(\succsim, K^c)$. Then we have

$$p_{K^c}\omega_{iK^c} \leq \inf_{z_{iK^c}\in R_{K^c}(\varphi_i(\succsim,K^c),\succsim_i)} p_{K^c}z_{iK^c}$$

for all $i \in I$.

For each $i \in I_1$, by assumption that $\varphi_i(\succsim, K) \succ_i \varphi_i(\succsim, K^c)$, it follows $R_K(\varphi_i(\succsim, K), \succsim_i) \subsetneq P_K(\varphi_i(\succsim, K^c), \succsim_i)$. Hence we have

$$p_{K^c}\omega_{iK^c} < \inf_{z_{iK^c}\in R_{K^c}(\varphi_i(\succsim,K),\succsim_i)} p_{K^c}z_{iK^c}$$

for all $i \in I_1$.

Thus we have

$$\begin{aligned} p_{K^c}\sum_{i\in I}\omega_{iK^c} \quad &\leq \quad \sum_{i\in I_1\cup I_3} \inf_{z_{iK^c}\in R_{K^c}(\varphi_i(\succsim,K),\succsim_i)} p_{K^c}z_{iK^c} \\ &\quad +\sum_{i\in I_2} \inf_{z_{iK^c}\in R_{K^c}(\varphi_i(\succsim,K^c),\succsim_i)} p_{K^c}z_{iK^c} \end{aligned}$$

Now take $\succsim^*\in \mathcal{D}_v$ such that $R(\varphi_i(\succsim, K), \succsim_i^*)$ is arbitrarily close to

$$co\,(R_K(\varphi_i(\succsim, K), \succsim_i) \times \{\omega_{iK^c}\} \cup \{\omega_{iK}\} \times R_{K^c}(\varphi_i(\succsim, K), \succsim_i))$$

for all $i \in I_1 \cup I_3$ and $R(\varphi_i(\succsim, K^c), \succsim_i^*)$ is arbitrarily close to

$$co\,(R_K(\varphi_i(\succsim, K^c), \succsim_i) \times \{\omega_{iK^c}\} \cup \{\omega_{iK}\} \times R_{K^c}(\varphi_i(\succsim, K^c), \succsim_i))$$

for all $i \in I_2$

Then we have

$$p_K\omega_{iK} + p_{K^c}\omega_{iK^c} < \inf_{z_i\in R(\varphi_i(\succsim,K),\succsim_i^*)} p_K z_{iK} + p_{K^c}z_{iK^c}$$

for all $i \in I_1 \cup I_3$ and

$$p_K\omega_{iK} + p_{K^c}\omega_{iK^c} < \inf_{z_i\in R(\varphi_i(\succsim,K^c),\succsim_i^*)} p_K z_{iK} + p_{K^c}z_{iK^c}$$

for all $i \in I_2$.

Since $\varphi(\succsim^*, K) = \varphi(\succsim, K)$ and $\varphi(\succsim^*, K) = \varphi(\succsim, K^c)$, No Loss from Trade requires

$$\varphi(\succsim^*, X) \in R(\varphi_i(\succsim, K), \succsim_i^*)$$

for all $i \in I_1 \cup I_3$ and

$$\varphi(\succsim^*, X) \in R(\varphi_i(\succsim, K^c), \succsim_i^*)$$

for all $i \in I_2$.

Hence we have

$$p_K \omega_{iK} + p_{K^c} \omega_{iK^c} < p_K \varphi_{iK}(\succsim^*, X) + p_{K^c} \varphi_{iK^c}(\succsim^*, X)$$

for all $i \in I$, meaning

$$p_K \sum_{i \in I} \omega_{iK} + p_{K^c} \sum_{i \in I} \omega_{iK^c} < p_K \sum_{i \in I} \varphi_{iK}(\succsim^*, X) + p_{K^c} \sum_{i \in I} \varphi_{iK^c}(\succsim^*, X)$$

However, this contradicts to $\sum_{i \in I} \varphi_i(\succsim^*, X) = \sum_{i \in I} \omega_i$. ■

Proof of Theorem 1. The conclusion of the previous lemma is impossible, since from MRC2-(i) and MRC2-(iii) we can always take $\succsim' \in \mathcal{D}$ such that $W_a(\succsim', K) \succ_a W_a(\succsim', K^c)$ and $W_b(\succsim', K) \prec_b W_b(\succsim', K^c)$.

Graphically speaking, in Figure 3 we can always "glue" the indifference curve I_a in the left with an arbitrary indifference curve above x_{K^c} like the dotted I_a, and we can always "glue" the indifference curve I_b in the right with an arbitrary indifference curve above x_{K^c} (from b's viewpoint) like the dotted I_b. ■

5 A no-mutual-gains result

Our second result is that whenever the SCF satisfies Weak Pareto, No Loss from Trade, and Independence of Untraded Commodities, gains from trade can be given to only one individual in the first step.

We will have the Pareto-ranking property between the solutions in K and K^c like in the previous section, but the separation argument there does not work here. See Figure 4. Here the line weakly separating the two upper contour sets in the left affine subspace must pass above the endowment, while the separating line in the right affine subspace must pass below the endowment. Hence the two lines cannot span a hyperplane in the entire space.

Thus we impose another richness condition.

Minimal Richness Condition 3: For any $\succsim \in \mathcal{D}$ and any fixed $x \in \mathbb{R}_+^{I\times X}$ such that

$$\begin{aligned}\sum_{i\in I}\omega_{iK} &\notin \sum_{i\in I}R_K(x_i,\succsim_i)\\ \sum_{i\in I}\omega_{iK^c} &\notin \sum_{i\in I}R_{K^c}(x_i,\succsim_i)\end{aligned}$$

and

$$\begin{aligned}\omega_{iK} &\notin R_K(x_i,\succsim_i)\\ \omega_{iK^c} &\notin R_{K^c}(x_i,\succsim_i)\end{aligned}$$

for all $i \in I$, there exists $\succsim^* \in \mathcal{D}$ such that

$$\sum_{i\in I}\omega_i \notin \sum_{i\in I}R(x_i,\succsim_i^*)$$

The intuition of MRC3 is similar to that of MRC2-(ii). It says that given a preference and a consumption bundle such that its welfare levels are not attainable by trading K only or by trading K^c, we can make the preference "demanding" so that its welfare levels are not attainable even by allowing trades of all goods in X.

One might wonder that MRC3 is implied by MRC2-(ii) through the argument made in the proof of Lemma 3. Figure 4 shows that it does not work as it says. In the left affine subspace the only (weakly) separating line must pass above ω_K, while in the left affine subspace the only (weakly) separating line must pass below ω_{K^c}. Hence the separating lines cannot span a hyperplane in the full space. To our knowledge of mathematics, we have to take MRC3 to be a direct assumption.

Here is an example showing that there exists a domain satisfying the above minimal richness condition.

Example 5 Let $I = \{a, b\}$. Let $X = \{1, 2, 3, 4\}$ and $K = \{1, 2\}$. Consider the domain $\mathcal{D}_{v_a} \times \mathcal{D}_{v_b}$, where v_a, v_b are given as follows.

Fix $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (0, 1)$ and $\omega \in \mathbb{R}_{++}^X$ such that

$$\omega_{a1} + \omega_{b1} = \omega_{a2} + \omega_{b2}, \quad \omega_{a3} + \omega_{b3} = \omega_{a4} + \omega_{b4}.$$

For each $i \in \{a, b\}$, let

$$\begin{aligned}v_{iK}(x_{iK}) &= \min\{\alpha_1 x_{i1} + (1-\alpha_1)x_{i2}, \alpha_2 x_{i1} + (1-\alpha_2)x_{i2}\}\\ v_{iK^c}(x_{iK^c}) &= \min\{\alpha_3 x_{i3} + (1-\alpha_3)x_{i4}, \alpha_4 x_{i3} + (1-\alpha_4)x_{i4}\}\end{aligned}$$

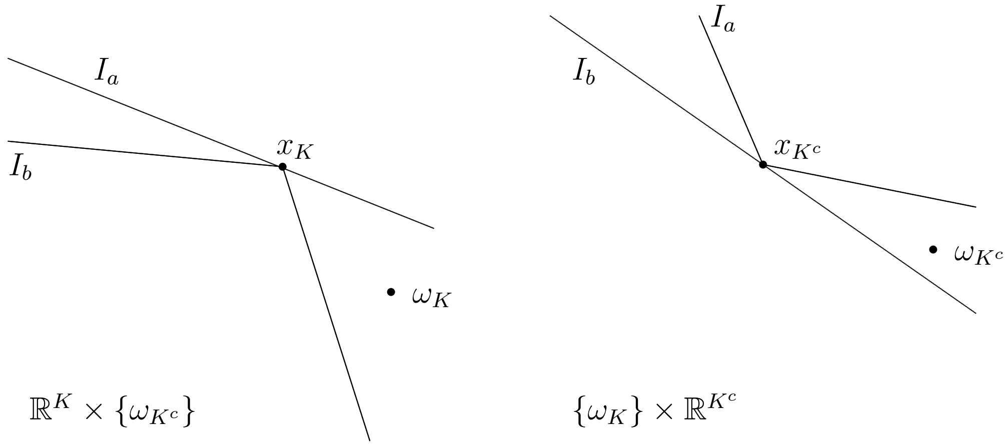

Figure 4: Failure of separation

As in Figure 5, whenever two upper-contour sets in the Edgeworth box in the left for K are disjoint they are separated by a line passing through the endowment point ω_K, and similarly for K^c. Therefore, one can take $\varepsilon'_a, \varepsilon'_b \in (0,1)$ to be sufficiently small so that the two upper-contour sets in the Edgeworth box for X given by $u_a(\cdot|\gamma_a, \varepsilon'_a)$ and $u_b(\cdot|\gamma_b, \varepsilon'_b)$ are separated by the hyperplane spanned by the two separating lines.

Now we state our main result.

Theorem 2 Assume MRC1,2,3. Assume also that for some $\succsim \in \mathcal{D}$, ω is not Pareto efficient in $(\succsim, K)$ or $(\succsim, K^c)$. Suppose that φ satisfies Weak Pareto, No Loss from Trade, and Independence of Untraded Commodities. Then there is $i \in I$ such that for all $\succsim \in \mathcal{D}$ it holds $\varphi_j(\succsim, K) \sim_j \omega_j$ and $\varphi_j(\succsim, K^c) \sim_j \omega_j$ for all $j \neq i$.

By abandoning the Walrasian mechanism, we allow the possibility that individuals may strictly gain in the second step. However, the flexibility afforded by richness allows us to establish that in the first step, only one individual can strictly gain.

This result allows that more than one individuals may gain from trade in the second step. However, because we cannot worsen the dominant individual either it puts a bound on the gains the other individuals may get in the second step. In other words, those other individuals can gain from trade only as a residual which "trickles-down" after the dominant individual takes her gains.

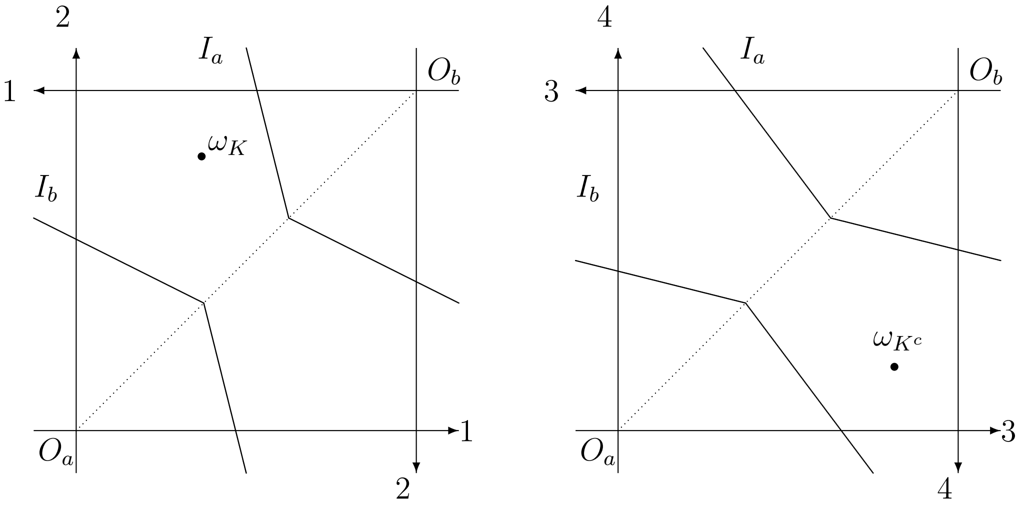

Figure 5: Example meeting MRC3

First we prove a lemma saying that the outcomes of the rule applied to two economies with mutually disjoint sets of tradable commodities must be Pareto ranked.

Lemma 4 Assume MRC1,2,3. Suppose that φ satisfies Weak Pareto, No Loss from Trade, and Independence of Untraded Commodities. Then for every $\succsim \in \mathcal{D}$, $\varphi(\succsim, K)$ and $\varphi(\succsim, K^c)$ are Pareto-ranked.

Proof.

Graphical intuition: To illustrate, consider again that $I = \{a, b\}$, $|X| = 4$ and $|K| = |K^c| = 2$. Consider again a four-dimensional Edgeworth box and imagine its two affine subspaces, $\mathbb{R}^K \times \{\omega_{K^c}\}$ and $\{\omega_K\} \times \mathbb{R}^{K^c}$ like in Figure 6. Let $(x_K, \omega_{K^c}) = \varphi(\succsim, K)$ and $(\omega_K, x_{K^c}) = \varphi(\succsim, K^c)$ here. Suppose now $(x_{aK}, \omega_{aK^c}) \succ_a (\omega_{aK}, x_{aK^c})$ and $(x_{bK}, \omega_{bK^c}) \prec_b (\omega_{bK}, x_{bK^c})$ for the sake of contradiction.

Then, when A's indifference surface giving the indifference curve I_a passing through x_K in the left affine subspace $\mathbb{R}^K \times \{\omega_{K^c}\}$ intersects the right affine subspace $\{\omega_K\} \times \mathbb{R}^{K^c}$ it gives the dotted indifference curve I_a passing strictly beyond x_{K^c} from A's viewpoint. Also, when B's indifference surface giving the indifference curve I_b passing through x_{K^c} in the right affine subspace $\{\omega_K\} \times \mathbb{R}^{K^c}$ intersects the left affine subspace $\mathbb{R}^K \times \{\omega_{K^c}\}$ it gives the dotted indifference curve I_b passing strictly beyond x_K from B's viewpoint.

Then, in the left affine subspace A's upper contour setting corresponding to I_a and B's upper contour set corresponding to the dotted I_b are separated by the budget line passing

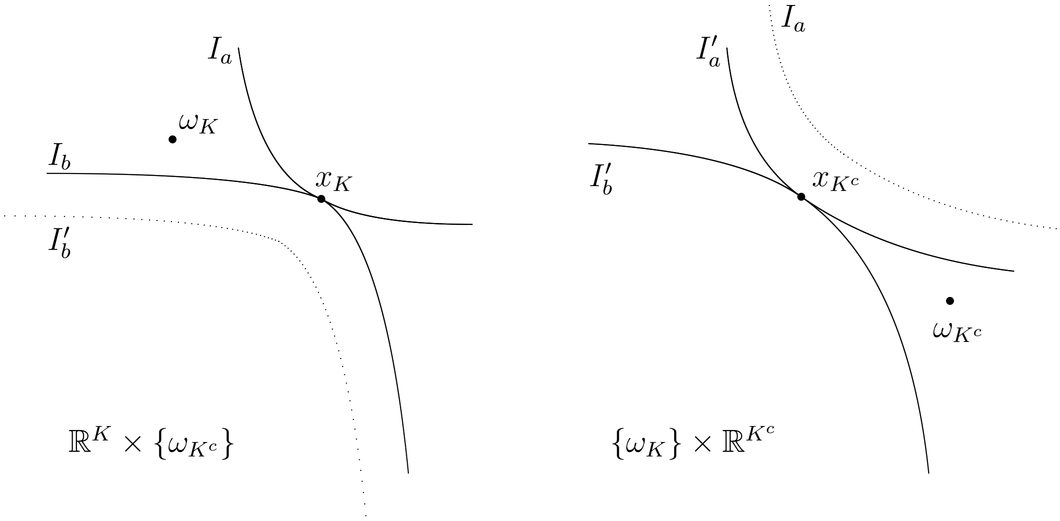

Figure 6: Pareto-ranking property

through ω_K. Likewise, in the right affine subspace A's upper contour setting corresponding to the dotted I_a and B's upper contour set corresponding to I_b are separated by the budget line passing through ω_{K^c}.

From MRC3 we can take $\succsim^* \in \mathcal{D}$ such that A's upper contour set above (x_{aK}, ω_{aK^c}) and B's upper contour set above (ω_{bK}, x_{bK^c}) are disjoint and neither contains the endowment point.

No Loss from Trade requires now that $\varphi(\succsim^*, X)$ be in both A's upper contour set and B's upper contour set, which is impossible.

Proof of the Lemma: Let

$$\begin{aligned} I_1 &= \{i \in I : \varphi_i(\succsim, K) \succ_i \varphi_i(\succsim, K^c)\} \\ I_2 &= \{i \in I : \varphi_i(\succsim, K) \prec_i \varphi_i(\succsim, K^c)\} \\ I_3 &= \{i \in I : \varphi_i(\succsim, K) \sim_i \varphi_i(\succsim, K^c)\} \end{aligned}$$

and suppose $I_1, I_2 \neq \emptyset$.

By Weak Pareto we have

$$\sum_{i \in I} \omega_{iK} \notin \sum_{i \in I} P_K(\varphi_i(\succsim, K), \succsim_i)$$

For each $i \in I_2$, by assumption that $\varphi_i(\succsim, K) \prec_i \varphi_i(\succsim, K^c)$, it follows $R_K(\varphi_i(\succsim, K^c), \succsim_i$

$) \subsetneqq P_K(\varphi_i(\succsim, K), \succsim_i)$. Therefore we have

$$\sum_{i\in I} \omega_{iK} \notin \sum_{i\in I_1\cup I_3} R_K(\varphi_i(\succsim, K), \succsim_i) + \sum_{i\in I_2} R_K(\varphi_i(\succsim, K^c), \succsim_i)$$

By Weak Pareto we have

$$\sum_{i\in I} \omega_{iK^c} \notin \sum_{i\in I} P_{K^c}(\varphi_i(\succsim, K^c), \succsim_i)$$

For each $i \in I_1$, by assumption that $\varphi_i(\succsim, K) \succ_i \varphi_i(\succsim, K^c)$, it follows $R_{K^c}(\varphi_i(\succsim, K), \succsim_i$ $) \subsetneqq P_{K^c}(\varphi_i(\succsim, K^c), \succsim_i)$. Therefore we have

$$\sum_{i\in I} \omega_{iK^c} \notin \sum_{i\in I_1\cup I_3} R_{K^c}(\varphi_i(\succsim, K), \succsim_i) + \sum_{i\in I_2} R_{K^c}(\varphi_i(\succsim, K^c), \succsim_i)$$

Then we can take $\succsim^* \in \mathcal{D}_{\tilde{v}}$ such that $\succsim^* |_K = \succsim_K$, $\succsim^* |_{K^c} = \succsim_{K^c}$ and

$$\sum_{i\in I} \omega_i \quad \notin \quad \sum_{i\in I_1\cup I_3} R(\varphi_i(\succsim, K), \succsim_i^*) + \sum_{i\in I_2} R(\varphi_i(\succsim, K^c), \succsim_i^*)$$

Since $\varphi(\succsim^*, K) = \varphi(\succsim, K)$ and $\varphi(\succsim^*, K^c) = \varphi(\succsim^*, K^c)$ follow from Independence of Untraded Commodities, No Loss from Trade requires

$$\begin{aligned} \sum_{i\in I} \varphi_i(\succsim^*, X) \quad &\in \quad \sum_{i\in I_1\cup I_3} R(\varphi_i(\succsim^*, K), \succsim_i^*) + \sum_{i\in I_2} R(\varphi_i(\succsim^*, K^c), \succsim_i^*) \\ &= \quad \sum_{i\in I_1\cup I_3} R(\varphi_i(\succsim, K), \succsim_i^*) + \sum_{i\in I_2} R(\varphi_i(\succsim, K^c), \succsim_i^*), \end{aligned}$$

which is a contradiction to $\sum_{i\in I} \varphi_i(\succsim^*, X) = \sum_{i\in I} \omega_i$. ■

Proof.

Graphical intuition: Consider again that $I = \{a, b\}$, $|X| = 4$ and $|K| = |K^c| = 2$. Suppose $\varphi_a(\succsim, K) \succ_a \omega_a$ and $\varphi_b(\succsim, K) \succ_b \omega_b$.

Without loss of generality, assume that $\varphi_a(\succsim, K^c) \succ_a \omega_a$ and $\varphi_b(\succsim, K^c) \succ_b \omega_b$ hold as well, whereas the boundary cases are treated in the formal proof.

Then we have a situation as depicted in Figure 7, where $(x_K, \omega_{K^c}) = \varphi(\succsim, K)$ and $(\omega_K, x_{K^c}) = \varphi(\succsim, K^c)$.

Then we can "glue" the indifference curve I_a in the left with an arbitrary indifference curve above x_{K^c} like the dotted I_a, and we can "glue" the indifference curve I_b in the left

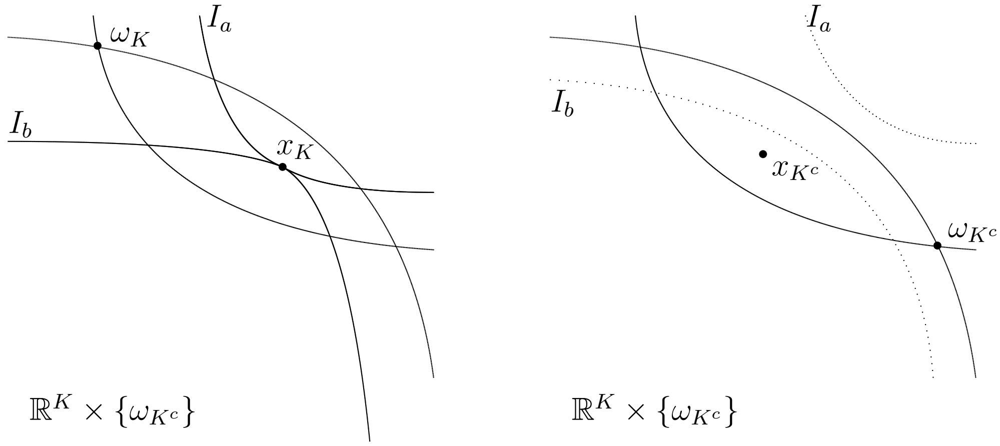

Figure 7: No mutual gains

with an arbitrary indifference curve below x_{K^c} (from b's viewpoint) like the dotted I_b. Then we obtain a contradiction to the conclusion of the previous lemma.

Proof of the Theorem: Suppose there exists $i, j \in I$ such that $\varphi_i(\succsim, K) \succ_i \omega_i$ and $\varphi_j(\succsim, K) \succ_j \omega_j$.

By the individual rationality condition we have $\varphi_i(\succsim, K^c) \succsim_i \omega_i$ and $\varphi_j(\succsim, K^c) \succsim_j \omega_j$.

Case 1: Suppose $\varphi_i(\succsim, K^c) \succ_i \omega_i$ and $\varphi_j(\succsim, K^c) \succ_j \omega_j$.

By MRC2-(i) and MRC2-(iii) we can take $\succsim^* \in \mathcal{D}_{\widetilde{v}}$ which satisfies $\succsim^* |_K = \succsim_K$ and $\succsim^* |_{K^c} = \succsim |_{K^c}$, and

$$\varphi_i(\succsim, K) \succ_i^* \varphi_i(\succsim, K^c)$$

$$\varphi_j(\succsim, K) \prec_j^* \varphi_j(\succsim, K^c)$$

By Independence of Untraded Commodities, this is equivalent to

$$\varphi_i(\succsim^*, K) \succ_i^* \varphi_i(\succsim^*, K^c)$$

$$\varphi_j(\succsim^*, K) \prec_j^* \varphi_j(\succsim^*, K^c)$$

However, this is a contradiction to the previous lemma.

Case 2: Suppose $\varphi_i(\succsim, K^c) \succ_i \omega_i$ and $\varphi_j(\succsim, K^c) \sim_j \omega_j$.

By MRC2-(i) and MRC2-(iii) we can take $\succsim^* \in \mathcal{D}_{\tilde{v}}$ which satisfies $\succsim^* |_K = \succsim_K$ and $\succsim^* |_{K^c} = \succsim |_{K^c}$, and

$$\varphi_i(\succsim, K) \prec_i^* \varphi_i(\succsim, K^c)$$

On the other hand, from the assumption we have

$$\varphi_j(\succsim, K) \succ_j^* \varphi_j(\succsim, K^c)$$

By Independence of Untraded Commodities, this is equivalent to

$$\varphi_i(\succsim^*, K) \prec_i^* \varphi_i(\succsim^*, K^c)$$

$$\varphi_j(\succsim^*, K) \succ_j^* \varphi_j(\succsim^*, K^c)$$

However, this is a contradiction to the previous lemma.

Case 3: Suppose $\varphi_i(\succsim, K^c) \sim_i \omega_i$ and $\varphi_j(\succsim, K^c) \succ_j \omega_j$. Then we can follow the argument similar to Case 2.

Case 4: Suppose $\varphi_i(\succsim, K^c) \sim_i \omega_i$ and $\varphi_j(\succsim, K^c) \sim_j \omega_j$.

Then by Weak Pareto and the assumption that ω is not Pareto-efficient in $(\succsim, K^c)$ there exists $k \neq i, j$ such that $\varphi_k(\succsim, K^c) \succ_k \omega_k$. By the individual rationality condition it holds $\varphi_k(\succsim, K) \succsim_k \omega_k(\succsim, K)$. Then we can follow the argument similar to one of the above cases.

Likewise, there is $\widehat{i} \in I$ such that $\varphi_j(\succsim, K^c) \sim_j \omega_j$ for all $j \neq \widehat{i}$.

If $i \neq \widehat{i}$ we have

$$\varphi_i(\succsim, K) \succ_i \varphi_i(\succsim, K^c)$$

$$\varphi_{\widehat{i}}(\succsim, K) \prec_{\widehat{i}} \varphi_{\widehat{i}}(\succsim, K^c),$$

which is a contradiction to the previous lemma. Therefore $\widehat{i} = i$.

Since $\succsim |_K = \succsim' |_K$ and $\succsim |_{K^c} = \succsim' |_{K^c}$ for all $\succsim, \succsim' \in \mathcal{D}_{\tilde{v}}$, from Independence of Untraded Commodities such i who takes all the gains from trade in the first-step is common across all preference profiles. ■

6 Conclusion

This paper initiates a formal study of trade liberalization in a social choice context. We have asked a very basic question: whether, in fact, the invisible hand could be modified to guide agents in Pareto improvements when opening markets to new trade. We have shown that the answer is negative.

As we have noted, our results do not preclude the possibility that *given* a preference profile one can find a *particular path or order* of trade liberalization along with everybody gets strictly better off. This can be easily accomplished by using the Walrasian mechanism at each stage, taking as endowment the consumption chosen at the previous stage. However, we have shown that it cannot happen as a property of a *rule* which is applied across different preference profiles and different sets of tradable commodities. This demonstrates that any "fair" method of allocation will necessarily be path-dependent on the order in which commodities become available for trade. Any kind of path dependence obviously opens the door for manipulation via bureaucrats or social planners.

One possibility remains for escaping the impossibility. From the outset, we have assumed that our SCF is single-valued. Dropping this constraint would require modifying the no loss from trade condition, but at the very least, some types of generalizations seem to pass the test. For example, consider a multi-valued rule which selects the entire Pareto correspondence. For any efficient allocation, when opening trade in new commodities, there is an efficient allocation for the larger set which weakly Pareto dominates the original allocation. The issue, as we see it, remains that a multi-valued rule is open to manipulation via bureaucrats and social planners.

References

[1] Cass, David, and Alessandro Citanna. "Pareto improving financial innovation in incomplete markets." *Economic Theory* 11.3 (1998): 467-494.

[2] Elul, Ronel. "Welfare effects of financial innovation in incomplete markets economies with several consumption goods." *Journal of Economic Theory* 65.1 (1995): 43-78.

[3] Fleurbaey, Marc, and Koichi Tadenuma. "Do irrelevant commodities matter?." *Econometrica* 75.4 (2007): 1143-1174.

[4] Hart, Oliver D. "On the optimality of equilibrium when the market structure is incomplete." *Journal of Economic Theory* 11.3 (1975): 418-443.

[5] Krugman, Paul R., Obstfeld, Maurice, and Marc Melitz. *International economics: theory and policy*. Pearson, 2012.

[6] Moulin, Hervé, and William Thomson. "Can everyone benefit from growth?: Two difficulties." *Journal of Mathematical Economics* 17.4 (1988): 339-345.