Search with Adverse Selection*

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Abstract

This paper introduces a sequential search model with adverse selection. We study information aggregation by the price—how close the equilibrium prices are to the full information prices—when the search frictions are small. We identify circumstances under which prices fail to aggregate information well even when the search frictions are small. We trace this to a strong form of the winner's curse that is present in the sequential search model. The failure of information aggregation may result in inefficient allocations.

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1 Introduction

This paper analyzes a sequential search model with asymmetric information of the common value variety. The main objective is to understand how the combination of search activity and information asymmetry affects prices and welfare. We specifically study the extent of information aggregation by the price—how close the equilibrium prices are to the full information prices—when the search frictions are small. Roughly speaking, we conclude that information is aggregated less well in the sequential search model than it is in a standard common value auction. In fact, when the search frictions are small, the equilibrium prices may be entirely independent of the common value of the transaction, even when there are exceedingly informative signals. We trace this failure of information aggregation to a stronger form of the winner's curse that is present with sequential search. The stronger winner's curse is a central insight of this paper. We also look at the efficiency perspective and relate the extent of potential inefficiencies to the informativeness of the signal technology available to the uninformed. In the analysis leading to these results, we develop a simple measure for the relative informativeness of signals that is finer than what has been used in the related literature.

The searching agent—the buyer—samples sequentially trading partners—sellers—for a transaction that involves information asymmetry. For each sampled seller, the buyer incurs a sampling cost. The buyer has private information about a parameter (type) $w \in \{1, 2, \cdots, m\}$ that determines the sellers' cost, c_w, where $c_1 < \cdots < c_m$. Upon being sampled, a seller observes a noisy signal of the cost but not the buyer's search history. Then the buyer and that seller bargain over the price. The outcome of the bargaining is affected by the seller's belief that results from updating the common prior over the buyer's type with the information contained in being sampled and in the observed signal. This induces the buyer to search for sellers who would receive a favorable signal in order to trade at a lower price. The strength of this incentive and hence the resulting search intensity generally vary across the different types of the buyer and this feeds back into the sellers' beliefs. The equilibrium concept is perfect Bayesian equilibrium. Each agent's behavior is optimal given the behavior of others, and sellers' beliefs are Bayesian updates of the prior given the signals and the understanding of the buyer's equilibrium behavior. The effect of the buyer's equilibrium search behavior on sellers' beliefs and, hence, on the price, is the main element that distinguishes the nature of price formation and information aggregation in this search environment from that in a related auction environment.

Although our model is not formulated with a specific application in mind, it provides a general model for a number of important scenarios in economics. One concrete scenario is that of procurement by an individual (the buyer) who needs a repair service and searches among potential providers (the sellers). Another scenario is that of a loan market: The buyer is a potential borrower who seeks funding for an investment of uncertain quality and the sellers are potential lenders. Finally, our model also captures the standard examples of "lemons markets," that is, sales of objects of uncertain quality by an informed seller. For this interpretation, we have to only reverse the roles of what we call buyer and sellers.

While this model depicts a single searcher looking for a single transaction, it is isomorphic to a two-sided search and matching model where instead of one buyer there is a population of buyers each of whom behaves like the buyer in our model. Upon completing their transactions, buyers exit the market, and the market is kept in steady state by constant flows of new buyers whose types are distributed independently according to the prior distribution.[1] This matching model would not require a different analysis, simply a straightforward translation of the notation and prose to the language of two-sided matching models.

Information aggregation by prices is a central problem of the economic theory of markets.[2] In the context of the present model, the price aggregates the information contained in the sellers' signals perfectly if the price equals the true cost; the price does not aggregate any information if it is independent of the true cost. We show that, when the sampling cost is small, the extent of information aggregation by the equilibrium price depends on a simple measure of the informativeness of the signal technology. Let F_w denote the distribution of the signal x on a common support $[\underline{x}, \overline{x}]$ when the buyer is of type $w \in \{1, 2, \cdots, m\}$. Low realizations of the signal indicate a higher likelihood of the low cost type—the likelihood ratio f_w/f_{w+1} is decreasing. We present a simple pairwise measure of relative informativeness $\lambda_{w,w'}$, $w < w'$, which is related to the rate of change of the likelihood ratio $\frac{f_w(x)}{f_{w'}(x)}$ near $\underline{x}$. We show that equilibrium prices aggregate the information perfectly when the sampling cost becomes negligible if and only if $\lambda_{w,w+1} = \infty$ for all w. This is a stronger condition than the existence of unboundedly informative signals, $\lim_{x \to \underline{x}} f_w(x)/f_{w+1}(x) = \infty$ for all w. If signals are only boundedly informative, that is, $\lim_{x \to \underline{x}} f_w(x)/f_{w+1}(x) < \infty$ for all w, the result is complete pooling—the buyer trades at the same price independent of her type and no information is aggregated at all. This is also the case when there are unboundedly informative signals, that is, $\lim_{x \to \underline{x}} f_w(x)/f_{w+1}(x) = \infty$ for all w, but $\lambda_{1,m} = 0$. A range of intermediate situations lies between these two extremes and exhibit imperfect information aggregation.

Generically, in a sense that is made precise later, the limit equilibrium price schedule in the case of unboundedly informative signals has the form depicted in Figure 1. That is, the set of types is partitioned into "pools" of adjacent types. If type w is in a pool to itself, it means that, when the buyer is of type w, the information is aggregated perfectly (in the limit) and the expected equilibrium price $\overline{p}_w$ is equal to c_w. If a pool contains several types, information is not aggregated perfectly when the buyer is of one of these types and $\overline{p}_w = E[\text{cost} \mid \text{type is in pool with w}]$. The exact configuration in terms of the number of pools and their sizes is determined by the λ measures mentioned above and any such shape between perfect aggregation and complete pooling is possible. The failure of prices to aggregate the information perfectly is caused by the incentives for higher cost buyer types to mimic the search behavior of lower cost types, which diminishes the informative value of signals.

[1] The large market analog of our model is one of Akerlof-style idiosyncratic uncertainty about individual buyers' types.

[2] The term "information aggregation" is used here to describe the collection of information that is dispersed among the sellers and the reflection of this information in prices. This is of course related to the question of whether the equilibrium prices are "pooling" or "separating," and occasionally we use these terms as well. However, the term "aggregation" emphasizes the coalescence of dispersed information, which is the focus of our paper.

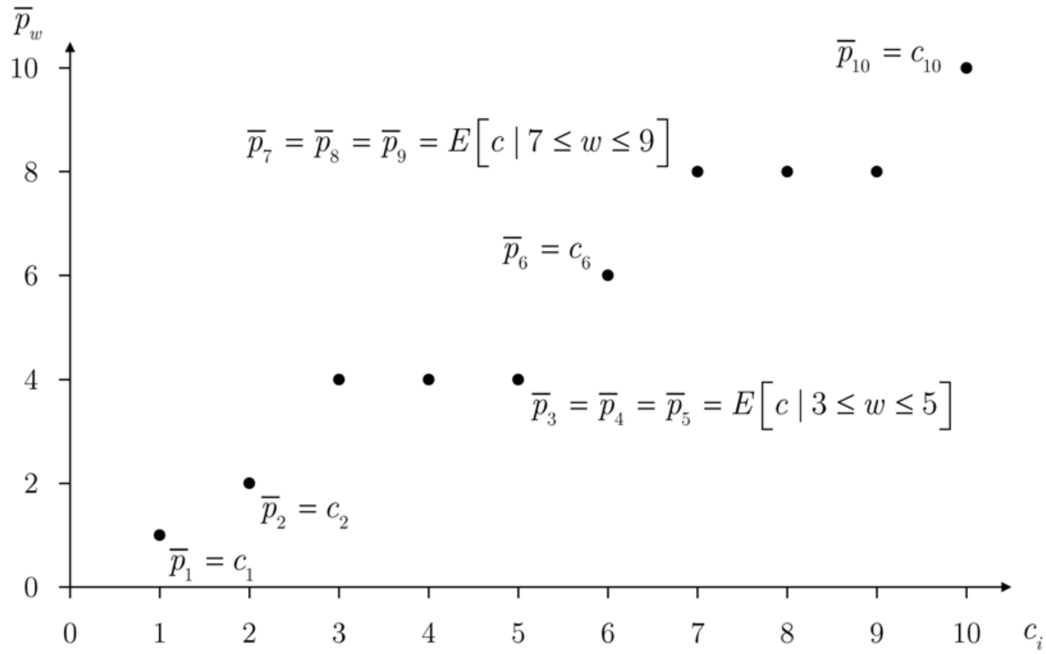

Figure 1: Example with 10 types. Here, types $\{3,4,5\}$ and types $\{7,8,9\}$ are pooled, while the others are separated.

Let us note in passing that the fact that much of what goes on can be traced back to the simple and interpretable measure λ is an interesting insight in its own right.

The basic features of the environment of the present search model resemble those of a standard common value (procurement) auction. Although there are a number of differences between the models, the crucial one is between the endogenous sampling of sellers in the search model and the exogenously fixed set of bidders in the auction model.[3] We compare our results to their counterparts in a setting in which a buyer conducts a procurement auction (with the same signal structure) and we look at the case in which the number of bidders is large. Milgrom (1979) and Wilson (1977) show that the equilibrium winning bid of a common value auction aggregates the information perfectly (in the limit as the number of bidders increases indefinitely) if and only if there are unboundedly informative signals. In contrast, as mentioned above, perfect information aggregation in the search model requires a stronger condition concerning the rate at which the informativeness of signals increases as $x \to \underline{x}$. Indeed, the equilibrium of the search model may exhibit complete pooling in the limit even when there are unboundedly informative signals. In contrast, the auction equilibrium never involves complete pooling in the limit, even when signals are only boundedly informative. In other words, in the counterpart of the above diagram for the auction, the schedule is always strictly increasing. In this sense, the search model aggregates information more poorly than the corresponding auction model. The reason for this difference is that, in the standard auction model, the number of

[3]This assertion will be discussed later.

sellers/bidders is fixed and independent of the true cost, whereas in the search model, this number is endogenous and dependent on the true cost. This exacerbates the winner's curse in the search model relative to its counterpart in the corresponding auction model and impedes the aggregation of information by prices.[4]

We also consider the welfare implications. In our base model, trade is always beneficial and always takes place. Hence, welfare coincides with the negative of the accumulated search costs because the price is just a transfer. We evaluate welfare in the limit as the sampling cost goes to zero. Generically, the limit equilibrium is efficient, regardless of the exact form of the pooling regions in the above figure. This is not an immediate consequence of the negligible sampling cost. In fact, this is not the case in some non-generic cases that we discuss in the paper. In a variation on the basic model introduced in Section 7, the volume of trade is also determined endogenously. Here, imperfect aggregation of information might be translated to an inefficient volume of trade, so even though the (limit) search cost is zero, the allocation might be inefficient.

A key feature of the model and the environments it represents is the sellers' inability to observe the buyer's history. This is a natural assumption in many important settings. For example, while lenders are aware that a borrower may have applied for a loan elsewhere, they do not generally observe how many other lenders a borrower has contacted before nor what happened in those prior meetings. Obviously, this leaves out interesting environments in which significant parts of the searcher's history are observable.[5]

This paper is related to three bodies of work. One deals with the question of information aggregation in the interaction of a large group of players. We have already mentioned Milgrom (1979) and Wilson (1977) who address this question in the context of a single unit auction. Feddersen and Pesendorfer (1997) and Duggan and Martinelli (2001) consider information aggregation in the context of a voting model, Smith and Sorenson (2000) consider it in the context of social learning, Pesendorfer and Swinkels (1997) consider it in the context of a multi-unit auction.

Another related body of work deals with search with adverse selection, for example, Inderst (2005); Moreno and Wooders (2010); and Guerrieri, Shimer and Wright (2010). While these papers are not directly related to ours in terms of the model and questions, they do have in common with our paper the idea that, in a search model, the distribution of types is determined endogenously. For example, in Inderst (2005) the distribution of types adjusts to sustain the Rothschild-Stiglitz best separating outcome as an equilibrium, which would not necessarily be the case for an arbitrary exogenous distribution of types.

There are a few papers in the intersection of these literatures. Wolinsky (1990) and Blouin and Serrano (2001) show that, in a two sided search model with binary signals and actions, information is not aggregated perfectly even as the frictions are made negligible. Duffie and Manso (2007) and Duffie, Malamud, and Manso (2009) characterize information percolation in markets where agents truthfully exchange their information with each other whenever they are matched. However, in our model the private information is idiosyncratic to each buyer, while those papers

[4] We discuss an auction model in which the number of bidders is endogenous in Section 8.

[5] The observability of the history is discussed further in Section 12.

study private information about a market-wide state of nature.

A third body of the related literature concerns dynamic trade with adverse selection. One strand studies the separation of quality types through differences in preferences over the timing and probability of trade. Owing to a single crossing condition, high quality types trade more slowly. It includes Evans (1989), Vincent (1989), Deneckere and Liang (2006), and Hörner and Vieille (2009). Another strand studies the effect of the gradual resolution of uncertainty through public signals, especially Bar-Isaac (2003), Kremer and Skrzypacz (2007), and Daley and Green (2012), expanding on Swinkels (1999). In contrast, in our model, types' preferences over the timing of trade are identical and we document the limited separability of types through private signals. Zhu (2012) introduces a finite variation of our model with recall and studies the resulting trading dynamics. Zhu applies the model to opaque financial over-the-counter markets. Over-the-counter asset markets are natural applications of our model, because these markets are typically characterized by price determination through private bilateral negotiations.

2 The Model and Preliminary Analysis

2.1 The Setup

A single buyer samples sequentially among a large number of sellers in search for a single transaction. The gross value for the buyer of transacting with one of the sellers is u and she incurs a sampling cost $s > 0$ for each seller sampled. The set of sellers is the interval $[0, 1]$, and the buyer's draws from this set are independent and uniformly distributed.

All sellers have identical cost to provide the service. Their common cost depends on the buyer's type $w \in W = \{1, 2, \cdots, m\}$ and is denoted by c_w, with $c_1 < c_2 \cdots < c_m$. The prior probability of type w is ρ_w. The ex-ante expected cost is $E[c] = \sum_{i \in W} \rho_i c_i$. The buyer knows her type w but not the sellers. It is assumed that $u > c_m$, so that trade is efficient for all types of the buyer. We relax this assumption later and study $c_1 < u < c_m$ in Section 7.

Upon meeting the buyer, the seller obtains a signal $x \in X = [\underline{x}, \overline{x}]$ from a c.d.f. F_w that depends on the buyer's type w and has a continuously differentiable density f_w strictly positive on $(\underline{x}, \overline{x})$. A lower signal value indicates a strictly higher likelihood of lower types—the likelihood ratio $\frac{f_w(x)}{f_v(x)}$ is strictly decreasing in x on $(\underline{x}, \overline{x})$ for all $w < v$. The distribution of the likelihood ratios is well behaved in the following sense. First, the (maximal) expected likelihood ratio, $E[\frac{f_1}{f_m}|w = 1] = \int_{\underline{x}}^{\bar{x}} \frac{f_1(x)}{f_m(x)} dF_1(x)$, is finite. Second, for all $i < j$,

$$\lim_{x \to \underline{x}} \frac{-\frac{d}{dx}\left(\frac{f_i(x)}{f_j(x)}\right)}{\frac{f_i(x)}{F_i(x)}}$$

exists, allowing it to be ∞ as well. We do not view these two conditions as restrictive. If they rule out anything, it would be a somewhat pathological case.

After a seller is sampled and observed the realization of the signal, bargaining unfolds: Nature draws a price $p \in [0, u]$ from a c.d.f. G that has a continuous density

g strictly positive on $[0, u]$. Given the price, the seller decides first whether to accept the price. If the seller accepts, then the buyer decides whether to accept as well. Acceptance by both parties ends the game. Rejection by either party terminates this match and the buyer continues sampling.

The "random proposals" bargaining model has been used in the related literature by Wilson (2001) and Compte and Jehiel (2010). It provides a robust model of bargaining with asymmetric information that avoids modeling complications, such as off-path beliefs, tangential to our focus. We discuss this modeling choice and alternative bargaining models in Section 10.

Before making the acceptance decision, the sampled seller observes both the signal and the price, but does not observe anything else about the history of the game. In particular, the seller does not observe how many other sellers the buyer has sampled before. The buyer observes the price, and she knows her private history. We assume that the buyer does not observe the seller's signal. However, nothing depends on this assumption.

A history of the process up to a certain point records the sequence of all signal realizations, prices, as well as the acceptance decisions by the buyer and all the sellers she has encountered up to that point. A terminal history is a history that ends with a trade or an infinite history with no trade.[6]

A finite terminal history determines a terminal outcome $\left(n^t, p^t, x^t, j^t\right)$, where p^t, x^t, j^t are the price, signal, and the identity of the seller in the terminal trade, and n^t is the number of sellers sampled throughout.

Buyer type w's payoff after a finite terminal history is

$$u_b\left(n^t, p^t, x^t, j^t, w\right) = u - p^t - n^t s;$$

the payoff after an infinite history is $-\infty$.

Seller j^t's payoff from transacting with buyer type w is

$$p^t - c_w.$$

Payoffs are zero for all other sellers.

A pure strategy for seller j is an acceptance set of prices $A_j(x) \subset [0, u]$ for each signal value since the current signal is all the seller observes. A pure strategy for a buyer with type w is an acceptance set for any history φ, denoted $B_w(\varphi) \subset [0, u]$.

The profile $(B, \mathcal{A}) = \left((B_w)_{w\in W}, (A_j)_{j\in[0,1]}\right)$ together with the prior over the set of types W induce a distribution on the set of terminal histories and, hence, over terminal outcomes. The expected payoff of the buyer is

$$V_w(B, \mathcal{A}) = E\left[u_b\left(n^t, p^t, x^t, j^t, w\right)|w; B, \mathcal{A}\right] = u - E\left[p^t|w; B, \mathcal{A}\right] - sE\left[n^t|w; B, \mathcal{A}\right],$$

where the expectation is with respect to the said distribution.

We assume that the sampling costs are small enough such that

$$u \geq \int_{c_m}^{u} p\frac{dG(p)}{1 - G(c_m)} + \frac{s}{1 - G(c_m)}. \tag{1}$$

[6] Formally, a history is a sequence $[w, \left(j_i, x_i, p_i, \alpha_i^S, \alpha_i^B\right)_{i=1}^t]$, where j_i is the identity of the i'th seller sampled, x_i and p_i are the signal and the price realizations in the encounter with j_i, and α_i^S and α_i^B denote the acceptance decisions of the seller and the buyer at that encounter. The equality $\alpha_i^S = \alpha_i^B =$"accept" may hold only if $i = t$ and in such a case it is a terminal history.

Thus, the buyer's payoff is positive even if the sellers accept only prices that are above c_m.

Let $\Pi(w|j, x; B, \mathcal{A})$ denote seller j's belief that the buyer's type is w, conditional on the event that seller j is sampled and observes signal x, when the strategy profile is $(B, \mathcal{A})$.

Equilibrium. The solution concept is perfect Bayesian equilibrium consisting of a strategy profile, $B = (B_w)_{w \in W}$ and $\mathcal{A} = (A_j)_{j \in [0,1]}$ and probability $\Pi(w|j, x; B, \mathcal{A})$ such that: (i) after any history φ, B_w maximizes the expected payoff of the buyer of type w given $\mathcal{A}$. (ii) for any signal realization x, $A_j(x)$ maximizes seller j's expected profit given B and $\Pi(w|j, x; B, \mathcal{A})$. (iii) $\Pi(w|j, x; B, \mathcal{A})$ is consistent with Bayesian updating in an environment with atomless random variables.

2.2 Equilibrium Strategies and Beliefs

Buyer's equilibrium strategy. Recall that, given a strategy profile $(B, \mathcal{A})$, the buyer's interim expected payoff before any sampling takes place is $V_w(B, \mathcal{A})$. If the buyer's acceptance strategy is optimal at all histories, then the expected continuation payoff is stationary. This is because the distribution of price offers and signals in the future and the acceptance decisions of sellers depend on neither the history of the game nor on the number of periods. It follows that the buyer's optimal decision is to accept a price following any history if and only if $p \leq u - V_w(B, \mathcal{A})$.[7] Thus, $B_w(\varphi) = [0, u - V_w(B, \mathcal{A})]$. That is, the buyer's equilibrium strategy is stationary and described by a cutoff, $u - V_w(B, \mathcal{A})$. Therefore, we omit the argument φ from now on and write $B_w(\varphi) = B_w$.

Sellers' equilibrium strategy. In equilibrium, a seller has a real decision to make only for prices that are acceptable to some type of the buyer, $p \in \cup B_w$. Let $W(p) = \{w|\ p \in B_w\}$ be the set of types that accept price p and let $E[c|j, x, W(p); B, \mathcal{A}]$ denote the expected cost of seller j conditional on: (i) being sampled, (ii) observing signal x, and (iii) $w \in W(p) \subseteq W$, given the strategy profile $(B, \mathcal{A})$. If $W(p) \neq \emptyset$,

$$E[c|j, x, W(p); B, \mathcal{A}] = \frac{\sum_{w \in W(p))} \Pi(w|j, x; B, \mathcal{A}) c_w}{\sum_{w \in W(p)} \Pi(w|j, x; B, \mathcal{A})}. \tag{2}$$

Optimality of $A_j(x)$ for $p \in \cup B_w$ requires that $p \in A_j(x)$ if and only if $p \geq E[c|j, x, W(p); B, \mathcal{A}]$. In particular, if $p > c_m$, then $p \in A_j(x)$ for all x and all j.

Since $\Pi(w|j, x; B, \mathcal{A})$ and, hence, $E[c|j, x, W(p); B, \mathcal{A}]$ are shown below to be independent of j, it follows that for $p \in \cup B_w$ the optimal acceptance strategy is independent of j, and we denote it by $A(x)$.[8] If $p \notin \cup B_w$, then any acceptance decision is optimal. To simplify the exposition, we assume for $p \notin \cup B_w$ that, if $p < c_1$, then $p \notin A(x)$ for all x and, if $p > c_m$, then $p \in A(x)$ for all x. This assumption has no consequences for the equilibrium outcome. Since all sellers use the

[7] Here and in the following, we ignore the multiplicity of optimal acceptance rules at prices at which either the buyer or the seller is indifferent. Behavior at such prices is inessential because such prices occur with zero probability.

[8] A_j might differ across sellers for $p \notin \cup B_w$, but sellers' acceptance decisions for such prices are irrelevant.

same strategy in equilibrium, we identify the strategy profile $\mathcal{A}$ with the individual strategy A and will use A to denote the profile as well.

Equilibrium beliefs. Owing to the nature of the equilibrium strategies, the interim probability $\Pi(w|j, x, B, A)$ takes a particularly simple form. Let

$$n_w(B, A) = E\left[n^t|w; B, A\right],$$

the expected number of sellers that are sampled by type w. Then,

$$\Pi(w|j, x; B, A) = \frac{\rho_w f_w(x)\, n_w(B, A)}{\sum_{i=1}^{m} \rho_i f_i(x)\, n_i(B, A)}. \tag{3}$$

We show that (3) is the natural extension of Bayes' formula to the present environment in which any realization of x and any particular seller being drawn are both zero probability events. Of course, conditioning on the realization of a signal from an atomless distribution with a continuous density is entirely standard. Therefore, if it were just the signal x, we could have simply stated the posterior using the densities f_w as in (3) without further comment.

Updating conditional on a particular seller being sampled is less standard. It is analogous to the inference problem of players in games with population uncertainty, which have been studied by Myerson (1998) and Milchtaich (2004).[9] Their approaches differ from each other but they give an identical answer in our model. Myerson (1998) derives the posterior of the players about the state of the world conditional on being called upon to play the game, by considering the limit of sampling from a large but finite number of potential players. Milchtaich (2004) directly studies the conditional probabilities derived from a suitably defined stochastic process on the continuum, using techniques developed in the theory of point processes. We present next a simple argument in the spirit of Myerson's approach, and we verify this derivation using Milchtaich's approach in the appendix.

Suppose there is a finite number N of sellers but that behavior is described by stationary and symmetric acceptance strategies, B and A.[10] Given these strategies, let $\psi_w = \Pr\left[\{(x, p) : p \notin B_w \cap A(x)\}\,|w\right]$ be the probability that an encounter between buyer type w and a seller ends with disagreement. Note that $n_w(B, A) = \frac{1}{1-\psi_w}$. If the buyer samples uniformly without replacement from N sellers with success probability $1 - \psi_w$, then seller j is sampled with probability

$$\begin{aligned}\Pr[j \text{ sampled}|w;\ N] &= \frac{1}{N} + \frac{N-1}{N}\frac{\psi_w}{N-1} + \ldots + \frac{N-1}{N}\frac{N-2}{N-1}\cdots\frac{1}{2}{\psi_w}^{N-1} \\ &= \frac{n_w(B, A)}{N}\left(1 - {\psi_w}^N\right).\end{aligned}$$

[9] In these games, the number of players is uncertain and the uncertainty may depend on a state of the world. Players infer about the state of the world from being called to play. The probability that any particular player is drawn is zero because the potential number of players in these games is unbounded.

[10] In the appendix, we also allow the sellers' strategies to be not symmetric.

Therefore, the posterior probability conditional on seller j being sampled when there are N sellers is

$$\Pr\left[w|j \text{ sampled}, x;\ N\right] = \frac{\rho_w f_w(x) \frac{n_w(B,A)}{N}\left(1-(\psi_w)^N\right)}{\sum_{i=1}^{m} \rho_i f_i(x) \frac{n_i(B,A)}{N}\left(1-(\psi_i)^N\right)} \underset{N\to\infty}{\longrightarrow} \frac{\rho_w f_w(x) n_w(B,A)}{\sum_{i=1}^{m} \rho_i f_i(x) n_i(B,A)}.$$

Thus, (3) approximates the Bayesian Posterior in a large market.

Since the sellers' posterior is independent of j, we will write from here on $\Pi(w|x;B,A)$ suppressing j. Thus, when conditioning on a signal realization, we implicitly also condition on the event that a seller was sampled.

The compound likelihood ratio. Expression (3) can be written as

$$\Pi(w|x;B,A) = \frac{\frac{\rho_w}{\rho_m}\frac{f_w(x)}{f_m(x)}\frac{n_w(B,A)}{n_m(B,A)}}{\sum_{i=1}^{m-1}\frac{\rho_i}{\rho_m}\frac{f_i(x)}{f_m(x)}\frac{n_i(B,A)}{n_m(B,A)}+1}.$$

The compound likelihood ratio $\frac{\rho_i}{\rho_j}\frac{f_i(x)}{f_j(x)}\frac{n_i(B,A)}{n_j(B,A)}$ between $w=i$ and $w=j$ is a product of the prior likelihood ratio $\frac{\rho_i}{\rho_j}$, the signal likelihood ratio $\frac{f_i(x)}{f_j(x)}$, and the sampling likelihood ratio $\frac{n_i(B,A)}{n_j(B,A)}$. Since the compound likelihood ratio will appear repeatedly in the derivations that will follow, it will be convenient to dedicate to it a special symbol,

$$\eta_{ij}(x;B,A) = \frac{\rho_i}{\rho_j}\frac{f_i(x)}{f_j(x)}\frac{n_i(B,A)}{n_j(B,A)}. \tag{4}$$

Thus, $\Pi(w|x;B,A) = \frac{\eta_{wm}(x;B,A)}{1+\sum_{i=1}^{m-1}\eta_{im}(x;B,A)}$ and the expected interim cost defined in (2) for $W(p) = W$ is

$$E\left[c|x,W;B,A\right] = \frac{c_m + \sum_{i=1}^{m-1}\eta_{im}(x;B,A)\,c_i}{1+\sum_{i=1}^{m-1}\eta_{im}(x;B,A)}. \tag{5}$$

Notice that $\eta_{ij}(x;B,A)$ and, hence, $\Pi(w|x;B,A)$ and $E\left[c|x,w\geq 1;B,A\right]$ depend on the strategies only through the ratios $\frac{n_i(B,A)}{n_j(B,A)}$.

2.3 Equilibrium Behavior, Payoffs, and Equilibrium Existence

Given a strategy profile (B,A), denote the set of signal-price pairs that result in trade given type w by

$$\Omega_w(B,A) = \left\{(x,p) : p \in B_w \cap A(x)\right\}.$$

To streamline the notation from now on we omit the arguments (B,A) and write Ω_w, n_w, η_{ij}, V_w, $\Pi(w|x)$ and $E\left[c|x,W(p)\right]$ with the understanding that these magnitudes depend on the strategy profile.

Given a set Q of signal-price pairs, $\Gamma_w(Q) = (F_w \times G)(Q)$ denotes the probability that an individual meeting between a seller and buyer type w yields a realization $(x,p) \in Q$. Thus, the probability that an individual meeting between a seller and buyer type w ends in trade is

$$\Gamma_w(\Omega_w) = \iint_{(x,p)\in\Omega_w} g(p)\, f_w(x)\, dp\; dx. \tag{6}$$

The expected number of sampled sellers is

$$n_w = E\left[n^t|w\right] = \frac{1}{\Gamma_w(\Omega_w)}. \tag{7}$$

Therefore, type w's expected search cost is

$$S_w = n_w s = \frac{s}{\Gamma_w(\Omega_w)}. \tag{8}$$

The price conditional on trading is $p_w = E\left[p^t|w\right]$, with

$$p_w = E_{(x,p)}\left[p|\,(x,p) \in \Omega_w, w\right] = \iint_{(x,p)\in\Omega_w} pg(p)\, \frac{f_w(x)}{\Gamma_w(\Omega_w)} dp\; dx. \tag{9}$$

The interim expected payoff of the buyer is

$$V_w = u - p_w - S_w. \tag{10}$$

Lemma 1 *In every equilibrium:*

1. *V_w is strictly decreasing in w.*
2. *Up to irrelevant differences,*[11] *acceptance strategies are given by*

$$\begin{aligned} B_w &= [0, u - V_w], \\ A(x) &= \bigcup_{i=1}^{m} \left[E[c|x, w \geq i], u - V_i\right]. \end{aligned}$$

A lower buyer's type generates better signals and qualifies for lower prices, hence the monotonicity of V_w. Given the monotonicity of V_w, the characterization of the acceptance strategies follows from our previous discussion. The lemma is proved in the appendix. From here on out, the proofs are in the appendix unless otherwise specified.

In general, the sellers' acceptance strategies are *not* cutoff strategies. For example, if $u - V_i < E[c|x, w \geq i+1]$ for some type i, then there is a gap in the set of equilibrium prices—no trade takes place at prices that are between $u - V_i$ and $E[c|x, w \geq i+1]$. Type $i+1$ and those above may be willing to trade at such prices, but sellers reject since transacting with those types at those prices involves a loss. Figure 2 illustrates the acceptance strategies and the regions of mutually acceptable prices.

[11] The "irrelevant difference" concern zero probability events and the description of sellers' acceptance decisions for prices that all types of the buyer will reject.

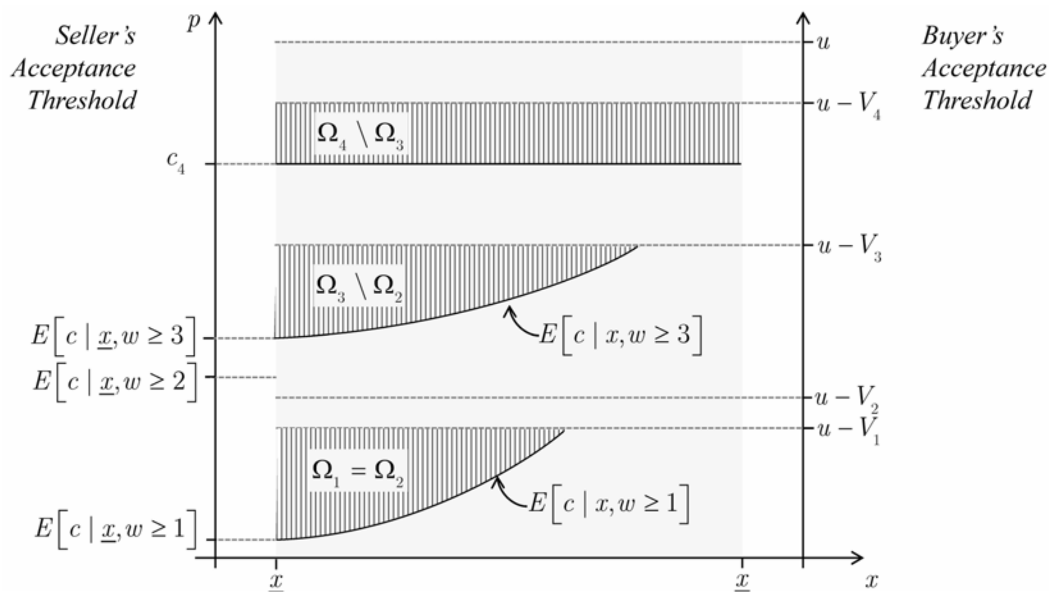

Figure 2: Acceptance strategies and the sets $(\Omega_i)_{i=1}^{4}$.

In the figure, $\Omega_1 = \Omega_2$, although buyer type 2 is willing to also accept prices in the interval $(u - V_1, u - V_2]$. However, these prices are unacceptable to sellers since they are below $E[c|x, w \geq 2]$. The set Ω_3 is the union of the two lower shaded areas while the set Ω_4 is the union of all three shaded areas.

The recursive structure of the sets Ω_i that is evident from the figure is a general implication of Lemma 1, that is,

$$\Omega_1 = \{(x,p) : p \in [E[c|x, w \geq 1], u - V_1]\} \tag{11}$$

$$\Omega_{i+1} = \Omega_i \cup \{(x,p) : p \in [E[c|x, w \geq i+1], u - V_{i+1}]\}. \tag{12}$$

In particular, $\Omega_1 = \Omega_2 = \cdots = \Omega_m$ if $V_m \geq u - E[c|x, w \geq 2]$.

The system (3), (7), (8), (9), (10), (11), and (12) fully determines the equilibrium. The sets Ω_w determine the V_w's and n_w's. The n_w's determine $E[c|x, w \geq i]$ which together with the V_w's determine the Ω_w's. A standard fixed-point argument proves that this system has a solution and, hence, existence of an equilibrium.

Proposition 1 *An equilibrium exists.*

The proof in the appendix utilizes the notation introduced in the following subsection.

2.4 The Cutoff Signals ξ_i

There are cutoff signals $\underline{x} < \xi_1 \leq \xi_2 \leq \cdots \leq \xi_m \leq \bar{x}$ such that a buyer having type i trades only if $x \leq \xi_i$. This subsection rewrites the description of the equilibrium behavior provided above in terms of these cutoffs. This notation is not required for the statement of the main results but is useful for the proofs of Proposition 1 and subsequent results. The cutoffs are defined by

$$\xi_i = \begin{cases} \bar{x} & \text{if} \quad u - V_i > E\left[c|\bar{x}, w \geq i\right], \\ \xi_{i-1} & \text{if} \quad u - V_i < E\left[c|\xi_{i-1}, w \geq i\right], \end{cases}$$

and, otherwise, if $E\left[c|\xi_{i-1}, w \geq i\right] \leq u - V_i \leq E\left[c|\bar{x}, w \geq i\right]$, the cutoff ξ_i solves

$$u - V_i = E\left[c|\xi_i, w \geq i\right].$$

These cutoffs determine the set of mutually acceptable prices. If type i realizes a signal $x \in (\xi_{j-1}, \xi_j]$, $j \leq i$, trade takes place if the realized price p satisfies $E\left[c|x, w \geq j\right] \leq p \leq u - V_j$ or $E\left[c|x, w \geq j+1\right] \leq p \leq u - V_{j+1}$ or ... or $E\left[c|x, w \geq i\right] \leq p \leq u - V_i$. Thus,

$$\Omega_i = \Omega_{i-1} \cup \left\{(x, p) : x \leq \xi_i \text{ and } p \in \left[E\left[c|x, w \geq i\right], u - V_i\right]\right\}, \tag{13}$$

with $\Omega_0 = \varnothing$. The trading probability of type i when following type ℓ's acceptance strategy is

$$\Gamma_i\left(\Omega_\ell\right) = \sum_{j=1}^{\ell} \int_{\underline{x}}^{\xi_j} \left(G\left(u - V_j\right) - G\left(\max\left\{E\left[c|x, w \geq j\right], u - V_{j-1}\right\}\right)\right) f_i\left(x\right) dx, \tag{14}$$

with $V_0 = u$. Using the ξ_j's and (14) to rewrite $S_w = \frac{s}{\Gamma_w(\Omega_w)}$, the expected payoffs (10) can be rearranged and written as

$$s = \sum_{j=1}^{i} \int_{\underline{x}}^{\xi_j} \left(\int_{\max\{E[c|x,w\geq j], u-V_{j-1}\}}^{u-V_j} \left(u - V_i - p\right) g\left(p\right) dp\right) f_i\left(x\right) dx. \tag{15}$$

3 Information Aggregation: Summary of Results

We study the extent to which information is aggregated into the equilibrium prices when sampling costs are small. Aggregation is maximal if the price that each buyer type pays is equal to its cost. Aggregation is minimal when all buyer types pay the same price. Formally, consider a sequence of sampling costs $\left\{s^k\right\}_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} s^k = 0,$$

and a sequence of equilibria $\left\{\left(B^k, A^k\right)\right\}_{k=1}^{\infty}$ associated with it. The superscript k indicates magnitudes arising in equilibrium $\left(B^k, A^k\right)$. So, we use V_w^k, S_w^k, n_w^k, E^k, Ω_w^k, etc. Recall that S_w^k is the expected total search cost paid by type w and p_w^k is the expected price. We denote their limits as

$$\overline{p}_w = \lim_{k \to \infty} p_w^k \qquad \text{and} \qquad \overline{S}_w = \lim_{k \to \infty} S_w^k.$$

Here and elsewhere, when we speak about limits we mean over any subsequence for which they exist. The following analysis investigates $\overline{p}_w$ and $\overline{S}_w$.

3.1 Boundedly Informative Signals

In the case of boundedly informative signals,

$$\lim_{x \to \underline{x}} \frac{f_i(x)}{f_{i+1}(x)} < \infty \qquad \forall i \in W, \tag{16}$$

even the most favorable signal carries only limited information. In this case, when the sampling cost is small, the outcome is complete pooling. In the limit equilibrium, for every type of the buyer, the expected total search cost is zero, and the expected price is equal to the ex-ante expected cost, $E[c]$.

Proposition 2 *Suppose that* $\lim_{x \to \underline{x}} \frac{f_i(x)}{f_{i+1}(x)} < \infty$ *for all* i*; then:*

1. $\overline{S}_w = 0 \qquad \forall w \in W,$
2. $\overline{p}_w = E[c] \qquad \forall w \in W.$

The proof is in Section 4 below. The intuition is that the bounded likelihood ratio implies that it is at most $\frac{f_1(\underline{x})}{f_m(\underline{x})}$ times more costly for the highest cost type $w = m$ to wait for the best signals as it is for the lowest cost type $w = 1$. As we show, the search cost of the lowest type becomes negligible as s^k becomes small. Therefore, the expected search cost of a buyer having type m who mimics the acceptance strategy of a type 1 is also negligible. Thus, these types must end up paying the same price in the limit.

3.2 Unboundedly Informative Signals: Characterization

In the case of unboundedly informative signals,

$$\lim_{x \to \underline{x}} \frac{f_i(x)}{f_{i+1}(x)} = \infty \qquad \forall i \in W, \tag{17}$$

signals close to $\underline{x}$ separate any two types. The main result is that generically, the limit equilibrium has the general form shown in Figure 1 in the introduction. That is, the set of the buyer types is partitioned into "pooling regions" comprised of adjacent types. All types in each pooling region pay the same expected price equal to the expected cost of the types in the pool. In particular, if type w is in a region by itself its expected price is c_w. In addition, all types incur zero expected search costs. To state this result formally, we introduce the following two ingredients.

First, for any pair of types (i, j), $i < j$, let

$$\lambda_{ij} = \lim_{x \to \underline{x}} \frac{-\frac{d}{dx}\left(\frac{f_i(x)}{f_j(x)}\right)}{\frac{f_i(x)}{F_i(x)}}; \tag{18}$$

which may be any value in $[0, \infty]$. Recall that, by assumption, this limit exists. Observe that λ_{ij} is a measure of informativeness—it captures the rate at which the signal's ability to distinguish type i from j normalized by the hazard rate $\frac{f_i}{F_i}$ improves as $x \to \underline{x}$. A larger λ_{ij} means that this ability increases more sharply as $x \to \underline{x}$.

For boundedly informative signals, that is, $\lim_{x\to\underline{x}}\frac{f_i(x)}{f_j(x)}<\infty$, $\lambda_{ij}=0$; a $\lambda_{ij}>0$ requires $\lim_{x\to\underline{x}}\frac{f_i(x)}{f_j(x)}=\infty$. However, $\lim_{x\to\underline{x}}\frac{f_i(x)}{f_j(x)}=\infty$ is not sufficient for $\lambda_{ij}>0$. The measure λ plays an important role in the characterization of equilibrium and is one of the main innovations of this paper. It will be discussed in more detail in Section 9. It is shown there that for every signal distribution that is in a specific sense "generic,"

$$\lambda_{ij}\in\{0,\infty\} \qquad \text{if } i<j. \tag{19}$$

Second, let a partitional configuration be a partition $(I(r))_{r=1}^R$ of W into adjacent sets of consecutive types ("pools"), $I(r)=\left\{\underline{I}(r),\underline{I}(r)+1,\cdots,\bar{I}(r)\right\}$, which start at $\underline{I}(1)=1$ and $\underline{I}(r)=\bar{I}(r-1)+1$. The expected cost conditional on a type from a pool $I\subset W$ is $E[c|w\in I]=\frac{\sum_{i\in I}\rho_i c_i}{\sum_{i\in I}\rho_i}$.

Proposition 3 *Suppose that* $\lim_{x\to\underline{x}}\frac{f_i(x)}{f_{i+1}(x)}=\infty$ *for all* i *and* $\lambda_{ij}\in\{0,\infty\}$ *for all* $i<j$. *Let* $(I(r))_{r=1}^R$ *be the partitional configuration defined by*

$$\bar{I}(r)=\begin{cases}\underline{I}(r) & \text{if } \lambda_{\underline{I}(r),\underline{I}(r)+1}=\infty,\\ \max\left\{i\in W \mid \lambda_{\underline{I}(r),i}=0\right\} & \text{if } \lambda_{\underline{I}(r),\underline{I}(r)+1}=0;\end{cases}$$

then:

1. $\overline{S}_w=0 \qquad \forall w\in W,$
2. $\bar{p}_w=E[c|w\in I(r)] \qquad \forall w\in I(r),\ r\in\{1,...,R\}.$

That is, when $\lambda_{ij}\in\{0,\infty\}$, then the last element $\bar{I}(r)$ of a non-singleton "pool" is the largest type such that $\lambda_{\underline{I}(r),\underline{I}(r)+1}=0$. The limit prices aggregate the information perfectly in the sense that $\bar{p}_w=c_w$ for all w if and only if all the elements of the partition are singletons. Thus, information is perfectly aggregated if and only if $\lambda_{w,w+1}=\infty$ for all w.

If $\lambda_{1,m}=0$, then the partition has a single element, the entire type set W. Hence, in the limit, there is complete pooling and the limit price is equal to the ex-ante expected costs, $\bar{p}_w=E[c]$ for all w.

Proposition 3 is proved in two stages over the following two sections. First, we state and prove Proposition 4 that establishes the partitional structure of the equilibrium without reference to the measure λ**.** Then in Section 5, we relate the structure of the equilibrium partition to the informativeness measure λ and complete the proof.

Proposition 4 *If* $\lim_{x\to\underline{x}}\frac{f_i(x)}{f_{i+1}(x)}=\infty$ *for all* i, *there is a partitional configuration* $(I(r))_{r=1}^R$ *such that*

$$\begin{aligned}\bar{p}_{\underline{I}(r)} &= \cdots=\bar{p}_{\bar{I}(r)-1}=\frac{\sum_{i=\underline{I}(r)}^{\bar{I}(r)-1}\rho_i c_i+\alpha(r)\rho_{\bar{I}(r)}c_{\bar{I}(r)}}{\sum_{i=\underline{I}(r)}^{\bar{I}(r)-1}\rho_i+\alpha(r)\rho_{\bar{I}(r)}},\\ \bar{p}_{\bar{I}(r)} &= \alpha(r)\bar{p}_{\underline{I}(r)}+(1-\alpha(r))c_{\bar{I}(r)},\end{aligned}$$

with $\alpha(r) = \lim_{k\to\infty} \frac{\Gamma_{\bar{I}(r)}(\Omega^k_{\underline{I}(r)})}{\Gamma_{\bar{I}(r)}(\Omega^k_{\bar{I}(r)})} > 0$ *and*

$$\begin{array}{rcl} \overline{S}_{\underline{I}(r)} & = & \cdots = \overline{S}_{\bar{I}(r)-1} = 0, \\ \overline{S}_{\bar{I}(r)} & \left\{ \begin{array}{l} = \\ \le \end{array} \right. & \begin{array}{lll} \alpha(r)\left[c_{\bar{I}(r)} - \bar{p}_{\underline{I}(r)}\right] & \textit{if} & \alpha(r) < 1, \\ c_{\bar{I}(r)} - \bar{p}_{\underline{I}(r)} & \textit{if} & \alpha(r) = 1. \end{array} \end{array}$$

Roughly speaking, if $\alpha(r) = 1$, all types $w \in I(r)$ trade at the same set of signal-price combinations $\Omega^k_{\underline{I}(r)}$ at a price equal to the expected cost conditional on $w \in I(r)$. If $0 < \alpha(r) < 1$, then type $\bar{I}(r)$ "mixes" between searching for $(x,p) \in \Omega^k_{\underline{I}(r)}$ and trading at a price close to its true cost. In the latter case, "mixing" requires that the search cost for $\bar{I}(r)$ to find $(x,p) \in \Omega^k_{\underline{I}(r)}$ are equal to the expected price improvement, $\bar{p}_{\underline{I}(r)} - c_{\bar{I}(r)}$.[12] The proof of the proposition is in Section 4 below.

4 Proofs of Propositions 2 and 4

This section provides a general characterization of the limit equilibrium outcome, which is then used to prove Proposition 2, for boundedly informative signals, and Proposition 4, for the unbounded case. Section 5 uses the λ measure to tighten this characterization and complete the proof of Proposition 3.

Two observations are critical for the characterization. The first lemma says that the set of signals $[\underline{x}, \xi^k_i]$ after which type $i < m$ trades shrinks to the bottom of the support.

Lemma 2 $\lim_{k\to\infty} \xi^k_i \to \underline{x}$ *for all* $i < m$.

This result is intuitive, since, when the search cost is lower, the buyer has a stronger incentive to search for a seller who receives a lower signal. The proof is not immediate, however, because the interim expected cost, and, hence, the search incentive is determined endogenously.

The second lemma says that all types $i \geq j$ that end up with a trade from $\Omega^k_j \backslash \Omega^k_{j-1}$ pay in expectation the same price in the limit and this price is equal to the expected cost of a seller who observes the cutoff signal ξ^k_j.

Lemma 3 *If* $\Omega^k_j \backslash \Omega^k_{j-1} \neq \emptyset$ *for all* k, *then for all* $i \geq j$,

$$\lim_{k\to\infty} E\left[p | (p,x) \in \left(\Omega^k_j \backslash \Omega^k_{j-1}\right), w = i\right] = \lim_{k\to\infty} E^k\left[c | \xi^k_j, w \geq j\right].$$

If the informativeness of the signals is bounded, this Lemma is immediate from the previous observation. This is because the set of signals such that $(x,p) \in \Omega^k_j \backslash \Omega^k_{j-1}$ shrinks to $\underline{x}$ as $k \to \infty$, so that the interim expected cost is essentially

[12] We say "mixing" since $\bar{I}(r)$ is using a pure strategy, so the "mixing" is purified by having an appropriate set of signal and price realizations after which $\bar{I}(r)$ trades at prices close to $c_{\bar{I}(r)}$.

constant on $[\underline{x}, \xi_j^k]$. The proof is not immediate if the informativeness of the signals is unbounded.

Lemmas 2 and 3 imply a number of observations that are used in the subsequent characterization results. The following corollary collects these observations.

Corollary 1 *i.* $\overline{p}_j = \sum_{i=1}^{j} \lim \frac{\Gamma_j\left(\Omega_i^k \backslash \Omega_{i-1}^k\right)}{\Gamma_j\left(\Omega_j^k\right)} \lim E^k\left[c|\xi_i^k, w \geq i\right]$;

ii. *If* $\lim \frac{\Gamma_j\left(\Omega_{j-1}^k\right)}{\Gamma_j\left(\Omega_j^k\right)} = 0$, *then* $\overline{S}_j = 0$. *In particular,* $\bar{S}_1 = 0$;

iii. *If* $\overline{S}_j > 0$ *and* $\lim_{x \to \underline{x}} \frac{f_j(x)}{f_{j+1}(x)} = \infty$, *then* $\lim \frac{\Gamma_{j+1}\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} = 0$ *and* $\overline{S}_{j+1} = 0$;

iv. *If* $\overline{S}_{j+1} = 0$ *and* $\lim \frac{\Gamma_{j+1}\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} > 0$, *then* $\lim \overline{p}_{j+1} = \lim \overline{p}_j$, $\lim \frac{\Gamma_{j+1}\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} = 1$ *and* $\overline{S}_j = 0$;

v. *If* $\lim \frac{\Gamma_{j+1}\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} > 0$ *and* $\lim \overline{p}_{j+1} > \lim \overline{p}_j$, *then* $\overline{S}_{j+1} > 0$;

vi. *If* $\overline{S}_j = 0$ *and* $\lim_{x \to \underline{x}} \frac{f_j(x)}{f_{j+1}(x)} < \infty$, *then* $\lim \frac{\Gamma_{j+1}\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} = 1$ *and* $\overline{S}_{j+1} = 0$.

Proof of Corollary 1: (i) follows from

$$E\left[p|\left(p,x\right) \in \Omega_j^k, w\right] = \sum_{i=1}^{j} \frac{\Gamma_j\left(\Omega_i^k \backslash \Omega_{i-1}^k\right)}{\Gamma_j\left(\Omega_j^k\right)} E\left[p|\left(p,x\right) \in \left(\Omega_i^k \backslash \Omega_{i-1}^k\right), w = j\right],$$

$\overline{p}_j = E\left[p|\left(p,x\right) \in \Omega_j^k, w = j\right]$ and Lemma 3.

(ii) $\lim \frac{\Gamma_j\left(\Omega_{j-1}^k\right)}{\Gamma_j\left(\Omega_j^k\right)} = 0$ and (i) imply together $\overline{p}_j = \lim E^k\left[c|\xi_j^k, w \geq j\right]$. The definition of ξ_j and Lemma 2 imply $\lim V_j^k \geq u - \lim E^k\left[c|\xi_j^k, w \geq j\right]$. Since also $\lim V_j^k = u - \overline{p}_j - \overline{S}_j$, it follows that $\overline{S}_j = 0$.

(iii)

$$\begin{aligned} \overline{S}_{j+1} &= \lim \frac{s^k}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} = \lim \frac{\Gamma_j\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} \frac{s^k}{\Gamma_j\left(\Omega_j^k\right)} \\ &\geq \lim \frac{f_j\left(\xi_j\right)}{f_{j+1}\left(\xi_j\right)} \lim \frac{\Gamma_{j+1}\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} \overline{S}_j \end{aligned} \tag{20}$$

where the inequality follows from $\Gamma_j\left(\Omega_j^k\right) \geq \frac{f_j\left(\xi_j\right)}{f_{j+1}\left(\xi_j\right)} \Gamma_{j+1}\left(\Omega_j^k\right)$ which is implied by (14). Since $\overline{S}_j > 0$, $\lim \frac{\Gamma_{j+1}\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_{j+1}^k\right)} > 0$ would imply $\overline{S}_{j+1} = \infty$ in contradiction to the optimality of type $(j+1)$'s behavior. Therefore, $\lim \frac{\Gamma_{j+1}(\Omega_j^k)}{\Gamma_{j+1}(\Omega_{j+1}^k)} = 0$ and by (ii) $\overline{S}_{j+1} = 0$.

(iv) Using (20), $\overline{S}_{j+1} = 0$ and $\lim \frac{\Gamma_{j+1}(\Omega_j^k)}{\Gamma_{j+1}(\Omega_{j+1}^k)} > 0$ imply $\overline{S}_j = 0$. Now,

$$V_{j+1}^k \geq E_{(p,x)}[u-p|\,(p,x) \in \Omega_j^k, w = j+1] - \frac{s^k}{\Gamma_{j+1}\left(\Omega_j^k\right)} \geq V_j^k - \frac{s^k}{\Gamma_{j+1}\left(\Omega_j^k\right)} \quad (21)$$

where the first inequality follows from the optimality of type $j+1$'s strategy and the second inequality follows from $(u-p) \geq V_j^k$ for all p s.t. $(p,x) \in \Omega_j^k$. Now, (21) and $\lim \frac{s^k}{\Gamma_{j+1}(\Omega_j^k)} = \overline{S}_j / \lim \frac{\Gamma_{j+1}(\Omega_j^k)}{\Gamma_{j+1}(\Omega_{j+1}^k)} = 0$ together imply $\lim_{k\to\infty} V_{j+1}^k \geq \lim_{k\to\infty} V_j^k$. Since Lemma 1 implies $\lim_{k\to\infty} V_{j+1}^k \leq \lim_{k\to\infty} V_j^k$, we have $u - \overline{p}_{j+1} = \lim_{k\to\infty} V_{j+1}^k = \lim_{k\to\infty} V_j^k = u - \overline{p}_j$. Therefore, $\lim \overline{p}_j = \lim \overline{p}_{j+1}$and $\lim \frac{\Gamma_{j+1}(\Omega_j^k)}{\Gamma_{j+1}(\Omega_{j+1}^k)} = 1$.

(v) It may not be that $\overline{S}_{j+1} = 0$, since it would contradict (iv).

(vi) From (21),

$$V_{j+1}^k \geq V_j^k - \frac{\Gamma_j\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_j^k\right)} \frac{s^k}{\Gamma_j\left(\Omega_j^k\right)}. \quad (22)$$

Using (14), we have

$$\begin{aligned} & \frac{\Gamma_j\left(\Omega_j^k\right)}{\Gamma_{j+1}\left(\Omega_j^k\right)} && (23) \\ = & \frac{\sum_{i=1}^{j} \int_{\underline{x}}^{\xi_i} \left(G\left(u - V_i\right) - G\left(\max\left\{E\left[c|x, w \geq i\right], u - V_{i-1}\right\}\right)\right) f_j\left(x\right) dx}{\sum_{i=1}^{j} \int_{\underline{x}}^{\xi_i} \left(G\left(u - V_i\right) - G\left(\max\left\{E\left[c|x, w \geq i\right], u - V_{i-1}\right\}\right)\right) f_{j+1}\left(x\right) dx} \\ \leq & \lim_{x\to\underline{x}} \frac{f_j(x)}{f_{j+1}(x)} < \infty, \end{aligned}$$

where the second inequality follows from the MLRP and $\lim_{x\to\underline{x}} \frac{f_j(x)}{f_{j+1}(x)} < \infty$. Now, (22), (23) and $\lim_{k\to\infty} \frac{s^k}{\Gamma_j(\Omega_j^k)} = \overline{S}_j = 0$ together imply $\lim_{k\to\infty} V_{j+1}^k \geq \lim_{k\to\infty} V_j^k$. Since Lemma 1 implies $\lim_{k\to\infty} V_{j+1}^k \leq \lim_{k\to\infty} V_j^k$, we have $\lim_{k\to\infty} V_{j+1}^k = \lim_{k\to\infty} V_j^k$. This means $\lim \overline{p}_j = \lim \overline{p}_{j+1} + \overline{S}_{j+1} \geq \lim \overline{p}_j$, from $\overline{S}_j = 0$ and $\bar{p}_{j+1} \geq \bar{p}_j$ from MLRP. Therefore, $\lim \overline{p}_j = \lim \overline{p}_{j+1}$, $\overline{S}_{j+1} = 0$ and $\lim \frac{\Gamma_{j+1}(\Omega_j^k)}{\Gamma_{j+1}(\Omega_{j+1}^k)} = 1$. □

Proof of Proposition 2: From Parts (i), (ii), and (vi) of Corollary 1, $\overline{p}_j = \lim E^k\left[c|\xi_1^k, w \geq 1\right]$ and $\overline{S}_j = 0$ for all j. From MLRP and from $\lim \frac{\Gamma_j(\Omega_1^k)}{\Gamma_j(\Omega_j^k)} = 1$ for all j (from Part (vi)),

$$\lim E^k\left[c|\xi_1^k, w \geq 1\right] \geq \lim E^k\left[c|(x,p) \in \Omega_1^k\right] = \sum_{i=1}^{m} \rho_i c_i.$$

The inequality may not be strict since this would imply $\lim E^k\left[c|\xi_1^k, w \geq 1\right] > \lim E\left[p|\,(p,x) \in \Omega_1^k, w = 1\right]$, contrary to Lemma 3. Therefore, $\overline{p}_j = \lim E^k\left[c|\xi_1^k, w \geq 1\right] = \sum_{i=1}^m \rho_i c_i$, as claimed. $\square$

Proof of Proposition 4: The proof is an immediate application of Corollary 1. Let $\underline{I}\,(1) = 1$ and $\bar{I}\,(1)$ be the smallest $r \geq 1$ such that $\lim \frac{\Gamma_{r+1}\left(\Omega_r^k\right)}{\Gamma_{r+1}\left(\Omega_{r+1}^k\right)} = 0$. This implies that, for $r \leq \bar{I}\,(1)$, $\lim E^k\left[c|\xi_r^k, w \geq r\right] = \lim E^k\left[c|\xi_1^k, r \leq w \leq \bar{I}\,(1)\right]$. By Corollary 1-(ii),(iii)&(iv) $\overline{S}_r = 0$, for all $r < \bar{I}\,(1)$, $\frac{\Gamma_{r+1}\left(\Omega_r^k\right)}{\Gamma_{r+1}\left(\Omega_{r+1}^k\right)} = 1$ and $\bar{p}_{r+1} = \bar{p}_r$, and for all $r < \bar{I}\,(1) - 1$. Using these observations and Corollary 1-(i)

$$\overline{p}_r = \begin{cases} \lim E^k\left[c|\xi_1^k, 1 \leq w \leq \bar{I}\,(1)\right] & \text{for} \quad r < \bar{I}\,(1) \\ \lim \frac{\Gamma_{\bar{I}(1)}\left(\Omega_1^k\right)}{\Gamma_{\bar{I}(1)}\left(\Omega_{\bar{I}(1)}^k\right)} \lim E^k\left[c|\xi_1^k, 1 \leq w \leq \bar{I}\,(1)\right] & \\ \quad + \lim \frac{\Gamma_{\bar{I}(1)}\left(\Omega_1^k\right)}{\Gamma_{\bar{I}(1)}\left(\Omega_{\bar{I}(1)}^k\right)} \lim E^k\left[c|\xi_{\bar{I}(1)}^k, w = \bar{I}\,(1)\right] & \text{for} \quad r = \bar{I}\,(1) \end{cases}$$

Let $\alpha(1) = \lim \frac{\Gamma_{\bar{I}(1)}\left(\Omega_1^k\right)}{\Gamma_{\bar{I}(1)}\left(\Omega_{\bar{I}(1)}^k\right)}$. Obviously,

$$\lim E^k\left[c|\xi_1^k, 1 \leq w \leq \bar{I}\,(1)\right] \geq \lim E^k\left[c|(x,p) \in \Omega_1^k\right] = \frac{\sum_{i=\underline{I}(1)}^{\bar{I}(1)-1} \rho_i c_i + \alpha(1)\rho_{\bar{I}(1)} c_{\bar{I}(1)}}{\sum_{i=\underline{I}(1)}^{\bar{I}(1)-1} \rho_i + \alpha(1)\rho_{\bar{I}(1)}}$$

The inequality may not be strict since this would imply $\lim E\left[p|\,(p,x) \in \Omega_1^k, w = 1\right] < \lim E^k\left[c|\xi_1^k, 1 \leq w \leq \bar{I}\,(1)\right]$, contrary to Lemma 3. Therefore,

$$\overline{p}_r = \frac{\sum_{i=\underline{I}(1)}^{\bar{I}(1)-1} \rho_i c_i + \alpha(1)\rho_{\bar{I}(1)} c_{\bar{I}(1)}}{\sum_{i=\underline{I}(1)}^{\bar{I}(1)-1} \rho_i + \alpha(1)\rho_{\bar{I}(1)}} \quad \text{for} \quad r < \bar{I}\,(1)$$

Next observe that $\lim E^k\left[c|\xi_{\bar{I}(1)}^k, w = \bar{I}\,(1)\right] = c_{\bar{I}(1)}$. Therefore, $\bar{p}_{\bar{I}(1)} = \alpha(1)\bar{p}_{\underline{I}(1)} + (1-\alpha(1))c_{\bar{I}(1)}$.

Finally, $u - c_{\bar{I}(1)} \leq \lim V_{\bar{I}(1)} = u - \overline{p}_{\bar{I}(1)} - \overline{S}_{\bar{I}(1)}$ with equality if $\alpha(1) < 1$. Substituting for $\bar{p}_{\bar{I}(1)}$ from above and rearranging, it follows that $\overline{S}_{\bar{I}(1)} \leq \alpha(1)\left[c_{\bar{I}(1)} - \bar{p}_{\underline{I}(1)}\right]$, with equality $\alpha(1) < 1$.

If $\bar{I}\,(1) < m$, let $\underline{I}\,(2) = \bar{I}\,(1)+1$. By Corollary 1-(ii), $\overline{S}_{\underline{I}(2)} = 0$. The construction of the set $\{\underline{I}\,(2), \cdots, \bar{I}\,(2)\}$ is identical to the construction above and so on for any $r > 1$. $\square$

5 Using λ to Characterize Pools

For the subsequent analysis, observe that, for $i < j < \ell$,

$$\lambda_{i\ell} = \lim_{x \to \underline{x}} \frac{-\frac{d}{dx}\left(\frac{f_i(x)}{f_j(x)}\frac{f_j(x)}{f_\ell(x)}\right)}{\frac{f_i(x)}{F_i(x)}} = \lim_{x \to \underline{x}} \frac{-\frac{d}{dx}\left(\frac{f_i(x)}{f_j(x)}\right)}{\frac{f_i(x)}{F_i(x)}} \frac{f_j\,(x)}{f_\ell\,(x)} + \lim_{x \to \underline{x}} \frac{-\frac{d}{dx}\left(\frac{f_j(x)}{f_\ell(x)}\right)}{\frac{f_j(x)}{F_j(x)}} \frac{F_i\,(x)}{F_j\,(x)}. \tag{24}$$

Therefore, if either $\lambda_{ij} > 0$ or $\lambda_{i\ell} > 0$ then $\lambda_{i\ell} = \infty$, since both $\frac{f_j(x)}{f_\ell(x)}$ and $\frac{F_i(x)}{F_j(x)}$ go to ∞. For $\lambda_{i\ell} < \infty$, it is necessary that $\lambda_{ij} = \lambda_{j\ell} = 0$.

The following proposition shows how λ_{ij} shapes the partitional configuration of Propositions 4. It applies to any value of the λ_{ij}'s without the genericity qualification.

Proposition 5 *Let $\{\underline{I}, \cdots, \bar{I}\}$ be any element of the partition described in Proposition 4; then:*

i. *If $\underline{I} < i < \bar{I}$, then $\lambda_{\underline{I}i} = 0$.*

ii. *$\lambda_{\underline{I}\bar{I}} < \infty$.*

iii. *If $\bar{I} < m$, then $\lambda_{\underline{I},\bar{I}+1} > 0$.*

iv. *If $\underline{I} = \bar{I}$, then $\lambda_{\underline{I},\bar{I}+1} = \infty$.*

Recall that $\eta_{ji}^k(x) = \frac{\rho_j}{\rho_i}\frac{f_j(x)}{f_i(x)}\frac{n_j^k}{n_i^k}$. The following Lemma provides a key magnitude: the (limit) search cost for a type $q > \underline{I}$ who mimics the acceptance strategy of type $\underline{I}$.

Lemma 4 *(i) If $\underline{I} < q$, then*

$$\lim_{k\to\infty} \frac{s^k}{\Gamma_q(\Omega_{\underline{I}}^k)} \leq \lambda_{\underline{I}q} \sum_{j=\underline{I}+1}^{\bar{I}} (c_j - c_{\underline{I}}) \lim_{k\to\infty} \frac{\eta_{j\underline{I}}^k(\xi_{\underline{I}}^k)}{\left(\sum_{j=\underline{I}+1}^{\bar{I}} \eta_{j\underline{I}}^k(\xi_{\underline{I}}^k) + 1\right)^2}$$

(ii) If $\underline{I} < q \leq \bar{I}$, then

$$\lim_{k\to\infty} \frac{s^k}{\Gamma_q(\Omega_{\underline{I}}^k)} \geq \lambda_{\underline{I}q}(c_q - c_{\underline{I}}) \lim_{k\to\infty} \frac{\eta_{q\underline{I}}^k(\xi_{\underline{I}}^k)}{\left(\sum_{j=\underline{I}+1}^{\bar{I}} \eta_{j\underline{I}}^k(\xi_{\underline{I}}^k) + 1\right)^2}.$$

The proof of Lemma 4 is relegated to the appendix.

Proof of Proposition 5. Consider a pool $\{\underline{I}, \cdots, \bar{I}\}$. By the definition of a pool, $\lim_{k\to\infty} \frac{\Gamma_{\underline{I}}\left(\Omega_{\underline{I}-1}^k\right)}{\Gamma_{\underline{I}}\left(\Omega_{\underline{I}}^k\right)} = 0$. In addition, for any $q > \bar{I}$, $\lim_{k\to\infty} \frac{\Gamma_q\left(\Omega_{\bar{I}}^k\right)}{\Gamma_q\left(\Omega_q^k\right)} = 0$ and, for all $j \in \{\underline{I}, \cdots, \bar{I}\}$, $\lim_{k\to\infty} \frac{\Gamma_j\left(\Omega_{\bar{I}}^k\right)}{\Gamma_j\left(\Omega_j^k\right)} > 0$. It also follows from the definition of a pool that

$$\lim_{k\to\infty} \eta_{j\underline{I}}^k(\xi_{\underline{I}}^k) \begin{cases} \in (0,\infty) & \text{if} \quad \underline{I} < j \leq \bar{I}, \\ = 0 & \text{if} \quad j > \bar{I}. \end{cases}$$

Thus, for any $q \in \{\underline{I}+1, \cdots, \bar{I}\}$, the right side fractions from Lemma 4 are strictly positive,

$$0 < \lim_{k\to\infty} \frac{\eta_{q\underline{I}}^k(\xi_{\underline{I}}^k)(c_q - c_{\underline{I}})}{\left(\sum_{j=\underline{I}+1}^{\bar{I}} \eta_{j\underline{I}}^k(\xi_{\underline{I}}^k) + 1\right)^2} < \infty.$$

Parts (i) and (ii). If $\lambda_{\underline{I}q} > 0$, for some $\underline{I} < q < \bar{I}$, then (31) from the proof of Lemma 5 implies $\lambda_{\underline{I}\bar{I}} = \infty$. If $\lambda_{\underline{I}\bar{I}} = \infty$, then Lemma 4-(ii) implies $\lim_{k\to\infty} \frac{s^k}{\Gamma_{\bar{I}}\left(\Omega^k_{\underline{I}}\right)} = \infty$. This together with $\lim_{k\to\infty} \frac{\Gamma_{\bar{I}}\left(\Omega^k_{\underline{I}}\right)}{\Gamma_{\bar{I}}\left(\Omega^k_{\bar{I}}\right)} > 0$ implies $\overline{S}_{\bar{I}} = \infty$ hence $\lim_{k\to\infty} V^k_{\bar{I}} = -\infty$ – contradiction. Therefore, $\lambda_{\underline{I}\bar{I}} < \infty$, proving Part (ii). Therefore, (31) implies that $\lambda_{\underline{I}q} = 0$ for any $\underline{I} < q < \bar{I}$, proving Part (i).
Part (iii). Suppose to the contrary that $q > \bar{I}$ and $\lambda_{\underline{I}q} = 0$. Hence, by Lemma 4-(i), $\lim_{k\to\infty} \frac{s^k}{\Gamma_q\left(\Omega^k_{\underline{I}}\right)} = 0$, contradicting $\lim_{k\to\infty} \frac{\Gamma_q\left(\Omega^k_{\underline{I}}\right)}{\Gamma_q\left(\Omega^k_q\right)} = 0$.
Part (iv). From Lemma 4-(i)

$$\lim_{k\to\infty} \frac{s^k}{\Gamma_{\underline{I}+1}\left(\Omega^k_{\underline{I}}\right)} \le \lambda_{\underline{I},\underline{I}+1} \lim_{k\to\infty} \frac{\eta^k_{\underline{I}+1,\underline{I}}(\xi^k_{\underline{I}})(c_{\underline{I}+1} - c_{\underline{I}})}{\left(\eta^k_{\underline{I}+1,\underline{I}}(\xi^k_{\underline{I}}) + 1\right)^2}.$$

If $\lambda_{\underline{I},\underline{I}+1} < \infty$, then together with $\lim_{k\to\infty} \eta^k_{\underline{I}+1,\underline{I}}(\xi^k_{\underline{I}}) = 0$ (which follows from $\underline{I} = \bar{I}$) we have $\lim_{k\to\infty} \frac{s^k}{\Gamma_{\underline{I}+1}\left(\Omega^k_{\underline{I}}\right)} = 0$. However, this contradicts $\lim_{k\to\infty} \frac{\Gamma_{\bar{I}(j)+1}\left(\Omega^k_{\bar{I}(j)}\right)}{\Gamma_{\bar{I}(j)+1}\left(\Omega^k_{\bar{I}(j)+1}\right)} = 0$.
Therefore, $\lambda_{\underline{I},\underline{I}+1} = \infty$. □

Proof of Proposition 3. By hypothesis, for all pairs $\lambda_{i,j} \in \{0, \infty\}$. Given any $\underline{I}(r)$, Proposition 5 implies that $\bar{I}(r)$ is the largest type such that $\lambda_{\underline{I}(r),\bar{I}(r)} = 0$, if such a type exists. From Part (ii) of the Proposition, $\bar{I}(r)$ cannot be larger and from Part (iii) of the Proposition it cannot be smaller. Thus, the partition is as claimed. Now, Lemma 4-(i) implies that $\bar{S}_{\bar{I}(r)} = 0$. Finally, the characterization of $\bar{S}_{\bar{I}(r)}$ from Proposition 4 implies that $\alpha(j) = 1$ whenever $\bar{S}_{\bar{I}(r)} = 0$. Given $\alpha(j) = 1$, Proposition 4 implies that the limit prices are as claimed. □

For later references, we also state the following corollary for the case where $\lambda_{1,m} = 0$. In this case, for sufficiently small sampling costs, the behavior of the buyer is independent of her type, in the sense that Ω^k_w is the same for all w. In contrast to our other results, this is not a limit result but describes equilibrium behavior for strictly positive sampling costs. Note that while the behavior for given (p, x) is independent of the buyer's type, the distribution of the price-signal pairs is not. Thus, the buyer's expected price and search costs are lower when her type is lower. We use the corollary to argue that the failure of information aggregation does not result from the buyer's private information.

Corollary 2 *If $\lambda_{1,m} = 0$, then $\Omega^k_1 = \Omega^k_2 = \cdots = \Omega^k_m$ for k sufficiently large.*

6 The Intermediate Case with $0 < \lambda_{ij} < \infty$

This section considers cases in which λ_{ij} takes values in $(0, \infty)$. We start with the case in which $0 < \lambda_{i,i+1} < \infty$ for all i. In this case, all pools contain exactly two types. Contrary to the generic case of Proposition 3, here the expected search cost may be positive.

Proposition 6 *Suppose that $\lambda_{i,i+1} \in (0, \infty)$ for all $i < m$. For any i that is odd:*

1. ***Search Cost.*** $\overline{S}_i = 0$ *and*
   - *If* $\lambda_{i,i+1} \leq \frac{\rho_{i+1}+\rho_i}{\rho_i}$, *then* $\overline{S}_{i+1} = \lambda_{i,i+1} \frac{\rho_i \rho_{i+1}}{(\rho_i+\rho_{i+1})^2} (c_{i+1} - c_i)$.
   - *If* $\lambda_{i,i+1} > \frac{\rho_{i+1}+\rho_i}{\rho_i}$, *then* $\overline{S}_{i+1} = \frac{1}{\lambda_{i,i+1}} \frac{\rho_i}{\rho_{i+1}} (c_{i+1} - c_i)$.
2. ***Price.*** *If* $i = m$, $\bar{p}_m = c_m$. *Otherwise:*
   - *If* $\lambda_{i,i+1} \leq \frac{\rho_{i+1}+\rho_i}{\rho_i}$, *then* $\bar{p}_i = \frac{\rho_i c_i + \rho_{i+1} c_{i+1}}{\rho_i + \rho_{i+1}} = \bar{p}_{i+1}$.
   - *If* $\lambda_{i,i+1} > \frac{\rho_{i+1}+\rho_i}{\rho_i}$, *then* $\bar{p}_i = c_i + \frac{1}{\lambda_{i,i+1}}(c_{i+1} - c_i) < c_{i+1} - \frac{1}{\lambda_{i,i+1}} \frac{\rho_i}{\rho_{i+1}}(c_{i+1} - c_i) = \bar{p}_{i+1}$.

Proof of Proposition 6: First, Proposition 5 implies that the pools are exactly of size two $\{\underline{I}(r), \bar{I}(r)\}$, so $\underline{I}(r)$ is odd and $\bar{I}(r)$ is even. If m is odd, then the last pool is a singleton. To see why, note that by Part (i), pools cannot be larger and by Part (iv) pools cannot be smaller.

By Proposition 4, $\overline{S}_{\underline{I}(r)} = 0$ for all r. Let

$$\bar{\eta} = \lim_{k \to \infty} \eta^k_{\bar{I}\underline{I}}(\xi^k_{\underline{I}}),$$

considering a convergent subsequence if needed.

By Lemma 4, the search cost of a type $\bar{I} = \bar{I}(r)$ who mimics $\underline{I} = \underline{I}(r)$ is

$$\frac{\bar{\eta}(c_{\bar{I}} - c_{\underline{I}})}{(\bar{\eta} + 1)^2} \lambda_{\underline{I},\bar{I}}. \tag{25}$$

Therefore, recalling that $\alpha = \alpha(r)$ is used to denote the probability with which $\bar{I}$ mimics $\underline{I}$,

$$\overline{S}_{\bar{I}} = \alpha \frac{\bar{\eta}(c_{\bar{I}} - c_{\underline{I}})}{(\bar{\eta} + 1)^2} \lambda_{\underline{I},\bar{I}}. \tag{26}$$

Note that,

$$\bar{p}_{\underline{I}} = \frac{\rho_{\underline{I}} c_{\underline{I}} + \alpha \rho_{\bar{I}} c_{\bar{I}}}{\rho_{\underline{I}} + \alpha \rho_{\bar{I}}} = \frac{c_{\underline{I}} + \bar{\eta} c_{\bar{I}}}{1 + \bar{\eta}}, \tag{27}$$

where the first equality follows from Proposition 4 and the second equality follows from Lemma 3. Therefore,

$$\bar{\eta} = \alpha \frac{\rho_{\bar{I}}}{\rho_{\underline{I}}}. \tag{28}$$

From Proposition 4, $\overline{S}_{\bar{I}} \leq c_{\bar{I}} - \bar{p}_{\underline{I}}$. Upon substituting from (25) and (27),

$$\frac{\bar{\eta}(c_{\bar{I}} - c_{\underline{I}})}{(\bar{\eta} + 1)^2} \lambda_{\underline{I},\bar{I}} \leq \frac{c_{\bar{I}} - c_{\underline{I}}}{1 + \bar{\eta}},$$

with equality if $a < 1$. Rearranging and substituting from (28),

$$\lambda_{\underline{I},\bar{I}} \leq \frac{\alpha \rho_{\bar{I}} + \rho_{\underline{I}}}{\alpha \rho_{\bar{I}}}.$$

Thus, if $\lambda_{\underline{I},\bar{I}} \leq \frac{\rho_{\underline{I}}+\rho_{\bar{I}}}{\rho_{\bar{I}}}$, then $\alpha = 1$ and, if $\lambda_{\underline{I},\bar{I}} > \frac{\rho_{\underline{I}}+\rho_{\bar{I}}}{\rho_{\bar{I}}}$, then $\alpha = \frac{\rho_{\underline{I}}}{(\lambda_{\underline{I},\bar{I}}-1)\rho_{\bar{I}}}$. Substituting from (27) into (26) and then for α, we get the desired expressions. $\square$

The Proposition implies that the ex-ante expected search cost $\sum_{i=1}^{m} \rho_i \overline{S}_i$ is small when all $\lambda_{i,i+1}$ are either small or large, but not for intermediate values of $\lambda_{i,i+1}$. It follows that welfare is not monotone in the informativeness of the signal technology as measured by $\lambda_{i,i+1}$: Less informativeness leads to more efficiency.

The general case of $\lambda_{ij} \in [0,\infty]$. We know from Proposition 4 that equilibria have a partitional structure. The boundaries between the pools are determined as follows. Start with $\underline{I}(1) = 1$. If $\lambda_{1m} = 0$, then Proposition 5 implies that $\bar{I}(1) = m$ and the characterization is complete. Otherwise, let $\hat{\imath}$ be the smallest type such that $\lambda_{1\hat{\imath}} > 0$. If $\lambda_{1\hat{\imath}} = \infty$, then Proposition 5 implies that $\bar{I}(1) = \hat{\imath} - 1$. If $0 < \lambda_{1\hat{\imath}} < \infty$, then either $\bar{I}(1) = \hat{\imath}$ or $\bar{I}(1) = \hat{\imath} - 1$. This is determined by comparing the cost for $\hat{\imath}$ of searching for $\Omega^k_{\underline{I}}$, $\lim_{k\to\infty} \frac{s^k}{\Gamma_i\left(\Omega^k_{\underline{I}}\right)}$, to the price gain, $\lim_{k\to\infty} c_{\hat{\imath}} - E^k\left[c|\xi^k_1, w \geq 1\right]$. If $\hat{\imath} = 2$, then Lemma 4 completely pins down $\lim_{k\to\infty} \frac{s^k}{\Gamma_i\left(\Omega^k_{\underline{I}}\right)}$ and it implies that $\bar{I}(1) = 2$.[13] If $\hat{\imath} \geq 3$, the bounds from Lemma 4 are not tight and therefore do not say whether it is $\bar{I}(1) = \hat{\imath}$ or $\bar{I}(1) = \hat{\imath} - 1$. However, in either case, given $\bar{I}(1)$ the next pool starts at $\underline{I}(2) = \bar{I}(1) + 1$, one looks again for the smallest type $\hat{\imath} \geq \underline{I}(2)$ such that $\lambda_{\underline{I}(2),\hat{\imath}} > 0$ and the construction continues exactly as above. The endpoint $\bar{I}(2)$ is either equal to $\hat{\imath}_2$ or $\hat{\imath}_2 - 1$. This construction is terminated when it reaches a pool that includes type m.

As noted above, this characterization is incomplete because if $0 < \lambda_{\underline{I}(j),\hat{\imath}} < \infty$ and $\hat{\imath} \geq \underline{I}(j) + 2$, then we have not identified the range of $\lambda_{\underline{I}(j),\hat{\imath}}$ for which $\bar{I}(j) = \hat{\imath}$ as opposed to $\hat{\imath} + 1$. It seems intuitively plausible that if $\lambda_{\underline{I}(j),\hat{\imath}}$ is large then $\bar{I}(j) = \hat{\imath} - 1$ rather than $\hat{\imath}$, but we have not proved this conjecture. However, this gap in our characterization seems small because it concerns special subcases that are not generic.

7 Informativeness and the Efficiency of Trade

Because trade is always beneficial and always takes place in this model, the expected surplus is fully determined by the expected search cost incurred by the buyer. We know from Proposition 3 that generically $\overline{S}_i = 0$ for all i. It follows that the limit equilibrium is efficient. We now modify the basic model in a minimal way to introduce efficiency considerations regarding the volume of trade, that is, the allocation.

For the points made by this subsection, it is sufficient to consider the case of two types, $m = 2$. Suppose that the model is as before, except that the buyer's value of the transaction u now satisfies $c_1 < u < c_2$. Therefore, efficiency requires that 1 trades, but 2 does not. To accommodate the possibility of no trade, there is an entry stage. The buyer decides once and for all in the beginning whether to start

[13] Lemma 4 completely pins down the search costs when $\lambda_{12} < \infty$. This makes the characterization in Proposition 6 possible.

the search. If the search is initiated, the process continues until a transaction takes place.

The analysis of this scenario is similar to the previous analysis, except for the buyer's entry decision. Let $e = (e_1, e_2)$ denote the probabilities with which types 1 and 2 enter. A strategy profile (B, A, e) consists of the acceptance decision of the buyer and the sellers, and the entry probabilities. A strategy profile is an equilibrium if the standard optimality conditions hold—the ones posed above and a new one regarding the entry decision. We consider non-trivial equilibria in which either $e_1 > 0$ or $e_2 > 0$ (or both). Trivial equilibria with no entry always exist.

If entry is profitable for type 2, entry must be strictly profitable for type 1. So, in any non-trivial equilibrium, it must be that $e_1 = 1$ when s is sufficiently small. Thus, the probability of type w after the decision to start the search but before any signals are observed is

$$\widehat{\rho}_1(e_2) = \frac{\rho_1}{\rho_1 + e_2\rho_2} \text{ and } \widehat{\rho}_2(e_2) = \frac{e_2\rho_2}{\rho_1 + e_2\rho_2}. \tag{29}$$

Again, we consider a sequence $s^k \to 0$ and a corresponding sequence of non-trivial equilibria, $\left(B^k, A^k, e^k\right)$. The outcome becomes efficient if type 1 enters for sure, it's expected search cost vanishes to zero, and the entry probability of type 2 becomes zero.

In equilibrium, trade takes place only at prices that do not exceed u, and, hence, strictly below c_2. Therefore, there can be no revealing transaction involving type 2. Consequently, if 2 enters in equilibrium, it must be pooling with 1, i.e., $\Omega_2^k = \Omega_1^k$ for all s^k. Thus, in the limit as $s^k \to 0$, the equilibrium price must be equal to the expected cost evaluated at probabilities $\lim \widehat{\rho}_w\left(e_2^k\right)$. That is, given these probabilities, the analysis is identical to Proposition 6, with $\widehat{\rho}_w$ taking the place of the original prior ρ_w. In particular, if $\lambda_{12} = 0$, the outcome is complete pooling on the expected costs conditional on entry, $\widehat{\rho}_1 c_1 + \widehat{\rho}_2 c_2$, and e_2^k stays strictly positive in the limit. If $\lambda_{12} = \infty$, then there is complete separation. Thus, if type 2 were to enter with strictly positive probability in equilibrium, its price would be close to c_2. Therefore, if $\lambda_{12} = \infty$, $e_2^k \to 0$. The proposition follows from the previous arguments. The proof of Proposition 4 is therefore omitted.

Proposition 7 *Suppose that $u \in (c_1, c_2)$. Consider a sequence $s^k \to 0$ and a corresponding sequence of non-trivial equilibria, $\left\{B^k, A^k, e^k\right\}$. The limit equilibrium outcome is efficient if and only if $\lambda_{12} = \infty$.*

In the efficient outcome of this environment only type 1 trades. This outcome is only achieved when the types are fully separated in the limit. In other cases, inefficient trade involving type 2 takes place in equilibrium.

If $\lambda_{12} = 0$, the only inefficiency is excessive trade. If, however, $\lambda_{12} \in (0, \infty)$ then the equilibrium also involves another inefficiency: if type 2 enters, it incurs search cost, as described in Proposition 6.

8 Comparison to Auctions: Sampling Curse vs. Winner's Curse

The literature on auctions addresses a closely related question concerning the extent to which the equilibrium price in a common value auction reflects the information contained in the bidders' signals when the number of bidders is made arbitrarily large (Wilson (1977) and Milgrom (1979)). In the auction version of our model, the buyer/auctioneer faces n sellers in a procurement auction, where n is exogenous. Each of the sellers gets a signal from the same distribution as above. They submit bids simultaneously and the lowest bidder is selected for the transaction. Milgrom's result translated to an auction version of our model is that the equilibrium price of the auction approaches the true cost when $n \to \infty$, if and only if $\lim_{x \to \underline{x}} \frac{f_i(x)}{f_{i+1}(x)} = \infty$. That is, when there are signals that are exceedingly more likely when the true state is i than when it is $i + 1$.

The search model of the present paper differs from the auction environment in a number of ways: the number of "bidders" is determined endogenously through the sampling, the buyer/auctioneer has private information, and the bids are generated randomly. However, the latter two differences are not significant. First, the exact specification of the price determination component of our model is not crucial and it can be changed to let sellers submit bids without essentially affecting the results (see Section 10 below). Second, since the number of bidders in the auction is not determined by the auctioneer, behavior is unaffected by the auctioneer's private information, and we may just as well assume that the auctioneer's information is the same in both models. Conversely, we could assume that the buyer does not know its own type in our model without changing the results qualitatively, see the discussion in the conclusion. The important difference between these two environments is that the number of sellers/bidders is exogenous in the auction environment, whereas in the search environment it is determined endogenously by the informed buyer/auctioneer.

To compare information aggregation across these two environments, we think of the reduction of the sampling cost s in the search model as the counterpart of increasing the number of bidders n in the auction. Under this analogy, the search and auction environments differ with respect to information aggregation. When $\lim_{x \to \underline{x}} \frac{f_i(x)}{f_{i+1}(x)} = \infty$ and $s \to 0$, the search equilibrium prices are not always near the true cost, but only when further conditions on the informativeness of the signals—involving the parameters λ_{ij}—are met.

When $\lambda_{1m} = 0$ (which includes the case of boundedly informative signals, $\lim_{x \to \underline{x}} \frac{f_i(x)}{f_{i+1}(x)} < \infty$ as a special case), the search equilibrium involves complete pooling. Complete pooling never arises in the equilibrium of the auction. Even when $\lim_{x \to \underline{x}} \frac{f_i(x)}{f_{i+1}(x)} < \infty$, the equilibrium price of the auction when $n \to \infty$ is partially revealing: the expected winning bids of the different types are different but do not coincide with the respective costs (see Lauermann and Wolinsky (2012) for an analysis of the two-type case).

Thus, information aggregation is more difficult in the search environment than in the standard auction environment. It is instructive to explain this difference from the perspective of the winner's curse. Suppose that there are just two buyer types,

$m = 2$. In the auction environment, let $\Pr[1|\text{winning}, x]$ be the probability that a bidder assigns to type 1, conditional on signal x and being the winner in a symmetric equilibrium in which bids are strictly increasing in signals,

$$\begin{aligned}\Pr[1|\text{winning}, x] &= \frac{\rho_1 f_1(x)[1-F_1(x)]^{n-1}}{\rho_2 f_2(x)[1-F_2(x)]^{n-1} + \rho_1 f_1(x)[1-F_1(x)]^{n-1}} \\ &= \frac{\frac{\rho_1}{\rho_2}\frac{f_1(x)}{f_2(x)}\frac{[1-F_1(x)]^{n-1}}{[1-F_2(x)]^{n-1}}}{1+\frac{\rho_1}{\rho_2}\frac{f_1(x)}{f_2(x)}\frac{[1-F_1(x)]^{n-1}}{[1-F_2(x)]^{n-1}}},\end{aligned}$$

which reflects the prior, $\frac{\rho_1}{\rho_2}$, the "signal effect", $\frac{f_1(x)}{f_2(x)}$, and the "winner's curse effect", $\frac{[1-F_1(x)]^{n-1}}{[1-F_2(x)]^{n-1}}$. The smaller is $\frac{[1-F_1(x)]^{n-1}}{[1-F_2(x)]^{n-1}}$, the more significantly the winner's course effect reduces the probability of 1. If $\lim_{x\to\underline{x}}\frac{f_i(x)}{f_{i+1}(x)} = \infty$, the signal effect goes to ∞ as $x \to \underline{x}$; the magnitude of the winner's curse effect depends on the rate at which $x \to \underline{x}$ as $n \to \infty$. If we focus on a sequence of signals $\{x_n\}$ at which the probability of winning in a monotone equilibrium, $[1 - F_1(x_n)]^{n-1}$, exceeds some $\varepsilon > 0$, then the winner's curse effect is bounded away from 0 and the signal effect overwhelms it, $\lim_{n\to\infty}\frac{f_1(x_n)}{f_2(x_n)}\frac{[1-F_1(x_n)]^{n-1}}{[1-F_2(x_n)]^{n-1}} = \infty$. It follows that, for large n, $\Pr[1|\text{winning auction}, x_n] \approx 1$. That is, even after allowing for the winner's curse, the winner is almost certain that the buyer is of type 1.

In the search environment, the mere fact of being sampled already implies a form of a winner's curse. The posterior probability that a seller assigns to type 1 conditional on being sampled and observing signal x is

$$\Pr[1|x] = \frac{\frac{\rho_1}{\rho_2}\frac{f_1(x)}{f_2(x)}\frac{n_1}{n_2}}{1+\frac{\rho_1}{\rho_2}\frac{f_1(x)}{f_2(x)}\frac{n_1}{n_2}}.$$

which reflects a "signal effect" $\frac{f_1(x)}{f_2(x)}$ and a "sampling effect" $\frac{n_1}{n_2}$. The latter is in some sense the counterpart of the winner's curse effect in the auction environment. However, while in the auction environment the signal effect prevails over the winner's curse effect, here the sampling effect $\frac{n_1}{n_2}$ might offset the signal effect $\frac{f_1(x)}{f_2(x)}$ even when $\lim_{x\to\underline{x}}\frac{f_1(x)}{f_2(x)} = \infty$. For example, if both 1 and 2 search until they generate a signal-price pair $(x, p) \in \Omega_1^k$, then $\frac{n_1}{n_2} = \frac{\Gamma_2(\Omega_1)}{\Gamma_1(\Omega_1)}$. Consider now a sequence $s^k \to 0$ and the corresponding sequence of equilibria. An implication of our analysis[14] is that, in this case, $\lim_{\xi_1^k\to\underline{x}}\frac{f_1(\xi_1^k)}{f_2(\xi_1^k)}\frac{n_1}{n_2} = \lim_{\xi_1^k\to\underline{x}}\frac{f_1(\xi_1^k)}{f_2(\xi_1^k)}\frac{\Gamma_2(\Omega_1)}{\Gamma_1(\Omega_1)} = \lim_{\xi_1^k\to\underline{x}}\frac{f_1(\xi_1^k)}{f_2(\xi_1^k)}\frac{f_2(\xi_1^k)}{f_1(\xi_1^k)} = 1$. Thus, $\Pr^k[1|\xi_1^k] \to \rho_1$, implying that the expected cost remains near the ex-ante expected cost (hence, away from the actual cost) even when the signal technology is very informative.

In both the search and auction models, revelation of information requires signals that make 1 exceedingly more likely to counteract the negative effect of the sampling or the winner's curse. However, the more substantial negative effect in the search model requires that there are signals that separate 1 from 2 even in a more pronounced way than in the large auction model.

[14]This follows from Lemma 3 together with Proposition 3.

As noted above, whereas in the search model the number of "bidders" depends endogenously on the buyer's type, in the auction model the number of bidders is exogenous and, hence, independent of the buyer's type. One may think of an alternative auction environment in which the buyer decides on the number of bidders, incurring a cost per bidder. If the bidders do not observe how many bidders they are competing against, such an auction model is closer to the search model in the sense that the number of bidders depends on the buyer's type, and bidders learn some information from the mere fact of being selected. In fact, this can be thought of as a simultaneous search model. In a companion paper, Lauermann and Wolinsky (2012), we consider this scenario. In the case of boundedly informative signals, when the sampling cost is negligible, that model has a partially revealing equilibrium like the auction with a large commonly known n considered above. In addition, under some conditions it also has complete pooling equilibrium like the search model of the present paper.

9 More About λ

This section discusses further the measure λ_{ij} and the claim that generically $\lambda_{ij} \in \{0, \infty\}$. An example illustrates the genericity notion and the result of Proposition 3. This measure plays a central role in the characterization. It is striking that this simple magnitude has such a decisive role in determining the equilibrium.

First, we provide an alternative interpretation of λ_{ij}. Using integration-by-parts, it can be shown that λ_{ij} can be also expressed as

$$\lambda_{ij} = \lim_{x_* \to \underline{x}} \int_{\underline{x}}^{x_*} \left(\frac{f_i(x)}{f_j(x)} - \frac{f_i(x_*)}{f_j(x_*)} \right) \frac{f_i(x)}{F_i(x_*)} dx.$$

That is, λ_{ij} measures the expected improvement of the likelihood ratio $\frac{f_i(x)}{f_j(x)}$ relative to $\frac{f_i(x_*)}{f_j(x_*)}$ when sampling for a signal $x \leq x_*$, as x_* approaches $\underline{x}$.

Next, let us turn to the genericity claim**.** Consider a continuum of signal distributions, $(F_z)_{z \in [0,1]}$, together with an atomless distribution on its parameter $z \in [0, 1]$. We assume the signal distributions are such that for all $z < z'$, $\lim_{x \to \underline{x}} \frac{f_z(x)}{f_{z'}(x)} = \infty$, and $\lambda_{zz'}$ exists. A finite sample of m independent draws of $z \in [0, 1]$, with realizations $\{z_1, \cdots, z_m\}$, defines a distribution over finite families of signal distributions, with realizations $\{F_{z_1}, \cdots, F_{z_m}\}$. Without loss of generality, $z_1 < \cdots < z_m$. We say that a property is generic if the set of samples for which it holds has probability one. In other words, our claim about the genericity of $\lambda_{ij} \in \{0, \infty\}$ means that, for any choice of $(F_z)_{z \in [0,1]}$ and for any atomless distribution on this set that is as described above, the probability that a randomly selected finite sample of signal distributions contains a pair of signal distributions $\left(F_{z_i}, F_{z_j}\right)$ for which $0 < \lambda_{ij} < \infty$ is zero.

Lemma 5 *For a generic family of signal distributions,*

$$\lambda_{ij} \in \{0, \infty\} \qquad \forall i < j. \tag{30}$$

Proof: From (24) and the subsequent discussion, for $i < j < \ell$,

$$\lambda_{i\ell} < \infty \Rightarrow \lambda_{i,j} = \lambda_{j,\ell} = 0. \tag{31}$$

It follows immediately from (31) that, if $\lambda_{zz'} \in (0, \infty)$, then $\lambda_{zz''} = 0$, for all $z'' \in (z, z')$ and $\lambda_{zz''} = \infty$, for all $z'' > z'$. Therefore, for any parameter $z \in [0, 1]$, there is at most one parameter $z' > z$ such that $\lambda_{zz'} \in (0, \infty)$ and at most one parameter $z'' < z$ such $\lambda_{z''z} \in (0, \infty)$. This implies the result. $\square$

Example of a Signal Distribution. An example illustrates the genericity result and Proposition 3. Consider a family of signal distributions, indexed by $z \in [0, 1]$. The common support of the distributions is $(-\infty, 0]$, and the distributions are defined by their densities

$$f_z(x) = (-x)^{-2z} \frac{e^x}{\mu_z},$$

with $\mu_z = \int_{-\infty}^{0} (-t)^{-2z} e^t dt$. The likelihood ratios are

$$\frac{f_z}{f_{z'}} = (-x)^{2(z'-z)} \frac{\mu_{z'}}{\mu_z}.$$

The signal is unboundedly informative: For $z' > z$, the likelihood ratio satisfies $\lim_{x \to -\infty} \frac{f_z(x)}{f_{z'}(x)} = \infty$. Moreover,

$$\lambda_{z,z'} = \lim_{x \to -\infty} \frac{-(-x)^{2(z'-z)-1} 2(z'-z)}{\frac{(-x)^{-2z} e^x}{\int_{-\infty}^{x} (-t)^{-2z} e^t dt}}.$$

Evaluating this limit shows that

$$\lambda_{z,z'} = \begin{cases} \infty & \text{if} \quad z' - z > 0.5, \\ 1 & \text{if} \quad z' - z = 0.5, \\ 0 & \text{if} \quad z' - z < 0.5. \end{cases}$$

Let $\{z_1, \cdots, z_m\}$ be a generic sample from $[0, 1]^m$, with $z_1 < z_2 < \cdots < z_m$. Suppose that for type $w \in \{1, 2, \cdots, m\}$ the distribution of signals is given by f_{z_w}. Genericity rules out that $z_{w'} - z_w = 0.5$. Therefore, $\lambda_{w,w'} \in \{0, \infty\}$ for all pairs $w < w'$. Moreover, let $\hat{w}$ be the lowest type such that $z_{\hat{w}} - z_1 > 0.5$. Then, $\lambda_{1,w} = 0$ for all $w < \hat{w}$, $\lambda_{1,\hat{w}} = \infty$, and $\lambda_{\hat{w},w} = 0$ for all $w > \hat{w}$.

By Proposition 3, when sampling costs are small, types $w < \hat{w}$ are pooled at a common price equal to $E[c|w < \hat{w}]$ and types $w > \hat{w}$ are pooled at a common price equal to $E[c|w \geq \hat{w}]$, respectively.

10 Modeling Bargaining

The "random proposals" bargaining protocol employed in this paper has been used in the related literature, for example, Wilson (2001), Compte and Jehiel (2010), and Albrecht et. al. (2009).

This model can be criticized for the artificial character of the exogenously generated random proposals and for restricting the players to acceptance decisions. In particular, this implies that players sometimes fail to strike a mutually beneficial deal, even when there are commonly known gains from trade. Nevertheless, in its defense it should be noted that bargaining undoubtedly involves random elements, which this model captures.

The main advantage of this bargaining protocol over others, however, is that it is simple and that it avoids tangential modeling complications such as off-path beliefs. Two alternative approaches in the literature are (i) to let the uninformed market side make offers, or (ii) to let the informed market side make offers and employ refinements to deal with the multiplicity problems; see Ausubel, Cramton, and Deneckere (2002) for an overview of bargaining with asymmetric information. Our results hold for these approaches as well, demonstrating the robustness of our qualitative findings.

First, in an earlier version of this paper, we modeled the bargaining component as a take-it-or-leave-it offer by the buyer. Since the buyer has private information, this gives rise to multiplicity through the freedom of selecting off-path beliefs in perfect Bayesian equilibria.[15] In that version, equilibria that survive a certain refinement that seems most convincing for the modeled situation[16] give rise to same results we derive in the present version of the paper. We also considered a variation on this model, wherein the buyer offers a direct mechanism instead of a price, which the seller accepts or rejects. This variation also gives rise to the same results, upon using the same refinement. The adoption of the random proposals bargaining eliminates the need to grapple with off-path beliefs and their refinement.

In another variation of the model that we have studied the sellers make the offers. Here, to circumvent Diamond's paradox,[17] in each period the buyer samples two or more sellers who observe the same signal and simultaneously offer prices. The buyer then either trades with the seller who offered the lowest price or continues to search. Apart from the modified bargaining component, all other details of the model remain the same. Since the uninformed sellers make the offers, out-of-equilibrium beliefs play no role and do not generate the multiplicity noted above. Because multiple sellers sampled in each round observe the same signal, they play a one-shot Bertrand game, which drives the prices to the expected costs conditional on the buyer's acceptance decision. The equilibrium outcomes of this version are the same as the refined outcomes of the version in which the buyer makes all the offers and, hence, give rise to the same results we derive with the random proposals model of this paper. While the Bertrand version of the model shares with the random proposals version the advantage of not needing a refinement, the assumption concerning the sampling of multiple sellers in each round is maybe less attractive.

11 Discussion—Variations on the Model

This section collects remarks about possible variations and extensions of the model.

[15] For example, there is a trivial pooling equilibrium in which the buyer offers c_m independently of its type. The seller rejects any price offer below c_m, because following such an offer a seller believes that the buyer must be type m.

[16] We use the undefeated equilibrium refinement by Mailath, Okuno-Fujiwara, and Postlewaite (1993).

[17] Diamond (1970)'s result that the combination of sequential search and positive sampling cost may preclude equilibrium dispersion and search. In our model, if the buyer has no bargaining power and if there is no competition between sellers, then the equilibrium price is monopolistic (equal to u), regardless of the search cost.

Two sided search and matching model. For simplicity of the exposition we chose to present the model in the language of a single searcher. However, as already mentioned in the introduction, it is straightforward to embed this model in a two-sided search and matching model. In that version, instead of one buyer there is a population of buyers each of whom behaves like the buyer in our model. Upon completing their transactions, buyers exit the market. The market is kept in steady state by constant flows of new buyers whose types are distributed independently according to the prior distribution. The interim probabilities in our model $\Pi(w|x)$ would coincide with the actual population distribution in the steady state equilibrium of the matching model. The analogy is complete. The matching model would not require a different analysis but just translation of the description to the language of those models.

The assumption that the types of the entering buyers are distributed independently is important. It would make this model a two-sided search version of Akerlof (1970)'s market in which the uncertainty pertains to each individual transaction as opposed to uncertainty concerning a market-wide state as in the rational expectations literature.

Observable history. The sellers' inability to observe the buyer's history plays a role in generating the search induced winner's curse. Our central insight concerning complete pooling when signals are not informative enough would not survive in its sharp form if parts of the history of the buyer's search history were observable to sellers. Consider a scenario in which the limit equilibrium of our model involves complete pooling. Suppose that sellers can observe the buyer's time on the market but not past signals or prices. This is a natural assumption for some situations like labor markets where the duration of unemployment might be observable. It is straightforward to see that a complete pooling outcome cannot be an equilibrium in this case since it would necessarily involve longer search for higher cost buyer types. This would imply that prices increase with time on the market, and hence, higher cost types would end up paying higher expected price—even when sampling costs vanish. Since a pooling outcome is ruled out and since lower cost types always do better for the same reason they do in our model, it must be that in any equilibrium lower cost types pay lower expected prices. The actual characterization of the equilibria of this model requires more work and is not carried out in this paper. Nevertheless, the above argument establishes that, independent of the exact form of the equilibrium, it necessarily involves some separation of types even with boundedly informative signals. The fact that greater observability results in more separation is not surprising. Intuitively, one expects that more information mitigates the adverse selection.

Obviously there are interesting environments in which the searcher's history is largely unobservable, as we assume, and there are other interesting environments in some of the searcher's history is observable. The present model does not cover the latter situations, whose analysis may be a worthwhile direction for continued research.

Buyer does not know own type. The assumption that the buyer is informed about the cost is not critical for our results. The critical assumptions are that the costs are correlated across sellers and that sellers observe different signals. This can be formally illustrated most easily for the case where $\lambda_{1,m} = 0$, with the help of Corollary 2. In this case, for sufficiently small s^k, any equilibrium in which the buyer knows its type is also an equilibrium if the buyer does not know its type. By Corollary 2, the acceptance strategy of the buyer is essentially independent of its type and, hence, remains an equilibrium strategy if the buyer does not know its type. "Essentially" here refers to the fact that although a higher type may be willing to accept higher prices than a lower type, sellers reject these prices anyway, making this difference of the acceptance strategies irrelevant.

Heterogenous valuations. This model assumes that the buyer's valuation u is known and independent of the cost type. The introduction of heterogeneity of valuations would be an interesting extension that may facilitate further insights into the welfare implications of the failure of information aggregation. In particular, the failure of information aggregation may lead to unraveling of trade, since some types of buyers with low costs who should trade may not find acceptable prices. As explained in Section 7, efficiency considerations with respect to the volume of trade can, however, already be generated in the present model by considering the case of u falling inside the range of possible costs. We therefore did not include heterogeneity of the buyer's value in addition to the buyer's private information about the sellers' cost.

12 Conclusion

This paper introduced a model of trade with adverse selection in a sequential search environment. A buyer has private information about the cost of a transaction and sellers observe noisy signals. We studied the extent to which equilibrium prices aggregate the information contained in the sellers' signals when sampling costs are small. First, if sellers' signals are boundedly informative, prices are independent of costs. Second, if sellers' signals are unboundedly informative, the relationship between prices and costs depends on the rate at which the informativeness of signals increases in the tail of the signal distribution. This rate is summarized by a simple measure λ. If informativeness increases too slowly—the pairwise λ are zero—prices are independent of costs even with unboundedly informative signals. Prices equal costs if and only if the informativeness increases very quickly—the pairwise λ are infinite.

In the corresponding auction environment with a large number of bidders, prices are never independent of costs, even if signals are boundedly informative. If signals are unboundedly informative, prices equal costs without additional conditions. The source of the difference is a stronger winner's curse in the search model that owes to the longer search duration of the bad types. When the search history is unobservable, being sampled is bad news, potentially overwhelming the informative content of signals entirely.

The potential failure of information aggregation has welfare consequences. It might generate an inefficiently high volume of trade, and it gives rise to wasteful search activities, which may be significant even when the one time sampling cost is negligible. A natural extension of the model in which to further study welfare consequences would be to allow for a two dimensional buyer's type that captures heterogeneity with respect to the valuation as well. In such an extension, the failure of information aggregation may lead to unraveling of trade: Some types of buyers with low costs who should trade may not find acceptable prices.

A Appendix 1: Derivation of Interim Beliefs

In the body of the paper the interim beliefs

$$\Pi(w|x) = \frac{\rho_w f_w(x) n_w}{\sum_{i \in W} \rho_i f_i(x) n_i}.$$

are as the limit of the posteriors from a sequence of finite models as the number of potential sellers grows large. This appendix employs Milchtaich (2004)'s approach for an alternative derivation that treats the uncertainty about the number of contacted sellers as a point process on the continuum set of sellers. Besides its advantage of operating directly within the model, this approach also facilitates derivation of the conditional probabilities for the case of non-symmetric sellers' strategies. For an introduction to point processes and the definition of conditional probabilities that is used here, see, for example, Daley and Vere-Jones (2008, especially Sections 9.1 and 13.1) and Milchtaich (2004).

Let Φ be the set of terminal histories in our model. Every terminal history $\varphi \in \Phi$ is a sequence of the form $((w, \left(j_i, x_i, p_i, \alpha_i^S, \alpha_i^B\right)_{i=1}^n)$; see Footnote 2.1. Φ_w is the set of terminal histories with state w. $\mathcal{F}$ is the Borel σ-algebra of Φ.

The set $\hat{J} = [0,1]$ is the set of sellers and $\mathcal{X} = \hat{J} \cup \{1, \cdots, m\}$, with Borel σ-algebra $\mathcal{B}_\mathcal{X}$. Each realization φ induces a simple counting measure ζ on $\mathcal{B}_\mathcal{X}$ as follows. Let $1_j(\varphi) = 1$ if seller j is sampled in history φ and let $1_w(\varphi) = 1$ if the state is w. The induced counting measure of a set $A \in \mathcal{B}_\mathcal{X}$ is

$$\zeta(A) = \sum_{j \in A} 1_j(\varphi) + \sum_{w \in A} 1_w(\varphi).$$

Let $\mathcal{N}$ be the set of all simple counting measures on $\mathcal{B}_\mathcal{X}$ and $\mathcal{B}_\mathcal{N}$ the Borel σ-algebra on $\mathcal{N}$.

Fix any strategy profile $(B, \mathcal{A})$. This strategy profile and the chance moves (buyer's type, signal realizations, and price proposals) induce a probability measure μ on $\Phi, \mathcal{F}$. The probability space $(\Phi, \mathcal{F}, \mu)$, in turn, defines a probability measure—a *point process*—$\mathcal{P}$ on $\mathcal{B}_\mathcal{N}$. Let $U_w = \{\zeta \in \mathcal{N} | \zeta(w) = 1\} \in \mathcal{B}_\mathcal{N}$, the event that the state is w. We are interested in

$$\Pi_j(w) = \Pr[U_w | \zeta(j) = 1],$$

the probability of w conditional on j being sampled. In the theory of point processes, $\Pi_{(\cdot)}(w)$ is referred to as a *Palm probability,* see Daley and Vere-Jones (2008, Chapter 13). The conditional probability $\Pi_{(\cdot)}(w)$ is almost everywhere uniquely defined by the requirement

$$\int_\Phi \left(\sum_{j \in J | 1_j(\varphi)=1} \Pi_j(w) \right) \mu(d\varphi) = \int_\Phi \left(\sum_{j \in J | 1_j(\varphi)=1} 1_w(\varphi) \right) \mu(d\varphi) \qquad \forall J \in \mathcal{B}_\mathcal{X}; \tag{32}$$

see Proposition 13.1.IV, Daley and Vere-Jones.

Lemma 6 *If $\Pi_{(\cdot)}(w)$ satisfies (32), then*

$$\Pi_j(w) = \frac{\rho_w n_w}{\sum_{i\in W}\rho_i n_i} \qquad a.e.\ j \in [0,1].$$

Proof: First, suppose the seller's acceptance strategies are symmetric to simplify exposition. The (Lebesgue-)measure of a set $J \in \mathcal{B}_{\mathcal{X}}$ is $\lambda(J) = \int_J dj$. Rewriting the right side of (32),

$$\begin{aligned}
\int_\Phi \Big(\sum_{j\in J|1_j(\varphi)=1} 1_w(\varphi)\Big)\mu(d\varphi) &= \int_{\Phi_w} |\{j \in J : 1_j(\varphi) = 1\}|\,\mu(d\varphi) \\
&= \rho_w E[n(\varphi)\,|\varphi \in \Phi_w]\,\lambda(J) \\
&= \rho_w n_w \lambda(J).
\end{aligned}$$

Thus, the right side of (32) depends on J only through its measure $\lambda(J)$. Therefore, $\Pi_j(w)$ must be constant for almost all j. Denote this constant by $\Pi(w)$. Suppose $\lambda(J) > 0$ and rewrite (32) further:

$$\begin{aligned}
\Pi(w)\int_\Phi |\{j \in J : 1_j(\varphi) = 1\}|\,\mu(d\varphi) &= \rho_w n_w \lambda(J) \Leftrightarrow \\
\Pi(w)\sum_{i\in W}\rho_i n_i \lambda(J) &= \rho_w n_w \lambda(J) \Leftrightarrow \\
\Pi(w) &= \frac{\rho_w n_w}{\sum_{i\in\{1,\cdots,m\}}\rho_i n_i},
\end{aligned}$$

as claimed.

Asymmetric acceptance strategies. For any non-zero set of sellers, we define

$$q_w(J) = \frac{1}{\lambda(J)} \iiint\limits_{\{(j,x,p):j\in J, p\in A_j(x), p\in B_w\}} g(p)\, f_w(x)\, dp dx dj,$$

the probability to trade conditional on sampling a seller $j \in J$ and state w. For $\varepsilon > 0$, let J_ε be any set with Lebesgue measure $\lambda(J_\varepsilon) = \varepsilon$. Abbreviate $q_w^\varepsilon = q_w(J_\varepsilon)$, $q_w^{\neg\varepsilon} = q_w([0,1]\setminus J_\varepsilon)$ and define $q_w = \lim_{\varepsilon\to 0} q_w^\varepsilon$, $q_w^\neg = \lim_{\varepsilon\to 0} q_w^{\neg\varepsilon}$, considering convergent subsequences if necessary.

Condition (32) holds for all J_ε. We multiply both sides by $1/\lambda(J_\varepsilon)$ and take the limit as $\varepsilon \to 0$. The limit of the right side of $\frac{(32)}{\lambda(J_\varepsilon)}$ is

$$\lim_{\varepsilon\to 0} \frac{1}{\lambda(J_\varepsilon)} \int_\Phi \left(\sum_{j\in J_\varepsilon | 1_j(\varphi)=1} 1_w(\varphi) \right) \mu(d\varphi) =$$

$$\lim_{\varepsilon\to 0} \left(\rho_w q_w^\varepsilon + \frac{\rho_w}{\lambda(J_\varepsilon)} \sum_{n=2}^{\infty} \sum_{k=1}^{n} k \binom{n-1}{k-1} \lambda(J_\varepsilon) q_w^\varepsilon \left[\lambda(J_\varepsilon)(1-q_w^\varepsilon)\right]^{k-1} \left[(1-\lambda(J_\varepsilon))(1-q_w^{\neg\varepsilon})\right]^{n-k} \right.$$

$$\left. + \frac{\rho_w}{\lambda(J_\varepsilon)} \sum_{n=2}^{\infty} \sum_{k=1}^{n} k \binom{n-1}{k} q_w^{\neg\varepsilon} (1-\lambda(J_\varepsilon)) \left[\lambda(J_\varepsilon)(1-q_w^\varepsilon)\right]^{k} \left[(1-\lambda(J_\varepsilon))(1-q_w^{\neg\varepsilon})\right]^{n-k-1} \right)$$

$$= \rho_w (\sum_{n=0}^{\infty} q_w (1-q_w^{\neg})^n - \sum_{n=1}^{\infty} q_w q_w^{\neg} (1-q_w^{\neg})^{n-1} n + n_w)$$

$$= \rho_w (\frac{q_w}{q_w^{\neg}} - \frac{q_w}{q_w^{\neg}} + n_w) = \rho_w n_w.$$

Following the first equality, n counts the total number of contacted sellers and k counts those from J_ε. The first two terms correspond to histories that end with the buyer trading with a seller from J_ε and the third term corresponds to those that do not. The second equality follows from the fact that all terms involving $k \geq 2$ converge to zero.

Take any number Π such that there is a sequence $\{J_{\varepsilon^k}\}_{k=1}^{\infty}$ such that $\lambda(J_{\varepsilon^k}) = \varepsilon^k > 0$, $\lim_{k\to\infty} \varepsilon^k = 0$, and $|\Pi_j(w) - \Pi| \leq \frac{1}{k}$ for all $j \in J_{\varepsilon^k}$ (there is a positive measure of sellers with posteriors close to Π). Using the previous evaluation of the limit of the right side of (32), $\lim_{\varepsilon\to 0} \frac{(32)}{\lambda(J_\varepsilon)}$ becomes

$$\Pi \lim_{k\to\infty} \frac{1}{\lambda(J_{\varepsilon^k})} \int_\Phi |\{j \in J_{\varepsilon^k} | 1_j(\varphi) = 1\}| \mu(d\varphi) = \rho_w n_w.$$

Rewriting the left side of (32) analogous to the previous rewriting of its right side shows that

$$\lim_{k\to\infty} \Pi \frac{1}{\lambda(J_{\varepsilon^k})} \int_\Phi |\{j \in J_{\varepsilon^k} | 1_j(\varphi) = 1\}| \mu(d\varphi) = \Pi \sum_{i\in W} \rho_i n_i.$$

Together,

$$\Pi = \frac{\rho_w n_w}{\sum_{i\in W} \rho_i n_i},$$

which implies that $\Pi_j(w)$ is equal to the same number for almost all j, as claimed. □

A similar argument establishes

$$\Pi_j(w|x) = \frac{\rho_w f_w(x) n_w}{\sum_{i\in W} \rho_i f_i(x) n_i},$$

for almost all j and x. The derivation of the posterior conditional on a signal realization $x \in [\underline{x}, \bar{x}]$ is entirely conventional, however, and so we omit its detailed derivation.

B Appendix 2: Proofs

B.1 Proof of Lemma 1 (preliminary characterization)

We have already established that the equilibrium strategies are for the buyer $B_w = [0, u - V_w]$ and for the seller $A(x) = \{p | p \geq E[c|x, W(p)]$, where $W(p) = \{w | p \in B_w\}$. In other words,

$$A(x) = \bigcup_{i=1}^{m} [E[c|x, \{w|V_w \leq V_i\}], u - V_i] \tag{33}$$

To verify Part (i), suppose that V_w is decreasing for $w \leq j-1$ and that $V_{j-1} > V_w$ for all $w > j-1$, but $j \neq i = \arg\max_{w>j-1} V_w$. (This includes the case of $j = 1$ by letting $V_0 = u$). It follows from (33) and of $B_w = [0, u - V_w]$ that

$$\Omega_i = \left\{ (x,p) | p \in \bigcup_{\ell=1}^{j-1} [E[c|x, w \geq \ell], u - V_\ell] \cup [E[c|x, w \geq j], u - V_i] \right\}$$

Since $E[c|x, w \geq \ell]$ and $\frac{f_i(x)}{f_j(x)}$ are increasing in x, we have $E_{(x,p)}[p|(x,p) \in \Omega_i, w = j] \leq E_{(x,p)}[p|(x,p) \in \Omega_i, w = i]$ and $\Gamma_j(\Omega_i) > \Gamma_i(\Omega_i)$, where the inequalities are strict if $i \neq j$. Using (10) it follows that

$$V_i = u - E_{(x,p)}[p|(x,p) \in \Omega_i, w = i] - \frac{s}{\Gamma_i(\Omega_i)} \leq u - E_{(x,p)}[p|(x,p) \in \Omega_i, w = j] - \frac{s}{\Gamma_j(\Omega_i)} \leq V_j$$

where the first inequality follows from the previous inequalities and is strict if $i \neq j$ and the second from the possible suboptimality for buyer $w = j$ of mimicking buyer $w = i$. Therefore, $V_j > V_w$ for all $w > j$.

Part (ii) follows from (33) and he monotonicity of V_w. □

B.2 Proof of Proposition 1 (existence)

Let $\mathcal{N} = \left[1, \frac{u}{s}\right]^m$. We define a mapping from $\mathcal{N}$ to itself whose fixed point is the equilibrium number of expected searches for each type. Given $\mathbf{n} \in \mathcal{N}$, let

$$E[c|x, \mathbf{n}] = \frac{c_m + \sum_{i=1}^{m-1} \frac{\rho_i}{\rho_m} \frac{f_i(x)}{f_m(x)} \frac{n_i}{n_m} c_i}{1 + \sum_{i=1}^{m-1} \frac{\rho_i}{\rho_m} \frac{f_i(x)}{f_m(x)} \frac{n_i}{n_m}}.$$

Given $\mathbf{n}$, payoffs $\{V_i(\mathbf{n})\}_{i=1}^m$ are defined iteratively as solutions to Equation (15), that is,

$$s = \sum_{j=1}^{i} \int_{\underline{x}}^{\xi_j(\mathbf{n}, V_i)} \left(\int_{\max\{E[c|x, \mathbf{n}, w \geq j], u - V_{j-1}\}}^{u - V_j} (u - V_i - p) g(p) dp \right) f_i(x) dx,$$

with $\xi_i(\mathbf{n}, V_i) = \inf \left\{ x \in \left[\xi_{i-1}(\mathbf{n}), \bar{x}\right] | E[c|x, \mathbf{n}] \geq u - V_i \right\}$ and $\inf \emptyset = \bar{x}$, $\xi_0 = \underline{x}$ and $V_0 = u$. Inspection shows that a solution $\{V_i(\mathbf{n})\}_{i=1}^m$ to this system of equations exists, is unique, and continuous in $\mathbf{n}$. For example, for $i = 1$,

$$s = \int_{\underline{x}}^{\xi_1(\mathbf{n}, V_1)} \left(\int_{E[c|x, \mathbf{n}, w \geq 1]}^{u - V_1} (u - V_1 - p) g(p) dp \right) f_1(x) dx.$$

Existence, uniqueness and continuity of $V_1(\mathbf{n})$ follows because the right side is continuous and strictly decreasing in V_1 and because it is strictly smaller than the left side when $V_1 = u$ (it is zero, since then $\xi_i = \underline{x}$) and strictly larger than the left side if $V_1 = 0$ (by Assumption (1)). Analogous arguments establish the claim for $i > 1$.

The payoffs $\{V_i(\mathbf{n})\}_{i=1}^m$ define $\{\Omega_i(\mathbf{n})\}_{i=1}^m$ iteratively, via Equation (13), that is,

$$\Omega_i = \Omega_{i-1} \cup \{(x,p) : x \in [\underline{x}, \xi_i], p \in [E[c|x, \mathbf{n}, w \geq i], u - V_i(\mathbf{n})]\}.$$

with $\Omega_0 = \emptyset$. Again, $\{\Omega_i(\mathbf{n})\}_{i=1}^m$ are uniquely defined and $\Gamma_i(\Omega_i(\mathbf{n}))$ is continuous in $\mathbf{n}$. Finally, let

$$\hat{n}_i(\mathbf{n}) = \frac{1}{\Gamma_i(\Omega_i(\mathbf{n}))}.$$

The function $\hat{n}(\mathbf{n})$ maps $\mathcal{N}$ into itself. In particular, $\hat{n}_i(\mathbf{n}) \leq \frac{u}{s}$ for all i. To see why, suppose that $\hat{n}_i(\mathbf{n}) > \frac{u}{s}$. This implies $V_i(\mathbf{n}) < 0$, by Equation (15). Then, $[\underline{x}, \bar{x}] \times [c_m, u] \subset \Omega_i(\mathbf{n})$. Hence, $\hat{n}_i(\mathbf{n}) \leq \frac{1}{1-G(c_m)}$. Assumption (1) implies in particular that $\frac{s}{1-G(c_m)} \leq u$. Hence, $\hat{n}_i(\mathbf{n}) \leq \frac{u}{s}$, in contradiction to $\hat{n}_i(\mathbf{n}) > \frac{u}{s}$.

Since the set $\mathcal{N}$ is convex and compact and $\hat{n}$ is a continuous function mapping $\mathcal{N}$ into itself, $\hat{n}$ has a fixed point. The values of V_w and Ω_w that correspond to this fixed points through the above construction constitute an equilibrium. $\square$

B.3 Proof of Lemma 2 (Cutoff)

Proof of Lemma 2: We want to show that $\lim_{k\to\infty} \xi_i^k \to \underline{x}$ for all $i < m$. Suppose to the contrary that there is some subsequence for which

$$\lim_{k\to\infty} \xi_1^k = \hat{x} > \underline{x}.$$

Step 1:

$$\lim_{k\to\infty} E^k[c|\hat{x}, w \geq 1] = \lim_{k\to\infty} E^k[c|x, w \geq 1] \qquad \forall x \in (\underline{x}, \hat{x}).$$

Proof of Step 1: From (15) and from $u - V_1^k \geq E^k\left[c|\xi_1^k, w \geq 1\right]$,

$$\begin{aligned} s^k &= \int_{\underline{x}}^{\xi_1^k} \left(\int_{E^k[c|x,w\geq1]}^{u-V_1^k} \left(u - V_1^k - p\right) g(p)\, dp \right) f_1(x)\, dx \qquad (34) \\ &\geq \int_{\underline{x}}^{\xi_1^k} \left(\int_{E^k[c|x,w\geq1]}^{E^k\left[c|\xi_1^k,w\geq1\right])} \left(E^k\left[c|\xi_1^k, w \geq 1\right]) - p\right) g(p)\, dp \right) f_1(x)\, dx \end{aligned}$$

Since the left side goes to 0 and since $\lim \xi_1^k = \hat{x} > \underline{x}$,

$$\lim_{k\to\infty} E^k[c|\hat{x}, w \geq 1] = \lim_{k\to\infty} E^k[c|x, w \geq 1] \qquad \forall x \in (\underline{x}, \hat{x}).$$

Step 2 :

$$\Gamma_\ell\left(\Omega_1^k\right) = \Delta^k + \int_{\underline{x}}^{\xi_1^k} g(\widetilde{p}(x)) \left(E^k\left[c|\xi_1^k, w \geq 1\right] - E^k[c|x, w \geq 1]\right) f_\ell(x)\, dx$$

where $\Delta^k = \max\left\{G\left(u - V_1^k\right) - G(E^k[c|\bar{x}, w \geq 1], 0\right\}$ and $\widetilde{p}(x) \in \left(E^k[c|x, w \geq 1], E^k\left[c|\xi_1^k, w \geq 1\right]\right)$

Proof of Step 2 : From (14) and a mean value theorem for integration

$$\begin{aligned}\Gamma_\ell\left(\Omega_1^k\right) &= \max\left\{G\left(u-V_1^k\right)-G(E^k\left[c|\bar{x},w\geq 1\right],0\right\}+\\ &\quad \int_{\underline{x}}^{\xi_1^k}\left[G\left(E\left[c|\xi_1^k,w\geq 1\right]\right)-G\left(E\left[c|x,w\geq 1\right]\right)\right]f_\ell\left(x\right)dx\\ &= \Delta^k+\int_{\underline{x}}^{\xi_1^k}\left(\int_{E^k[c|x,w\geq 1]}^{E^k\left[c|\xi_1^k,w\geq 1\right]}g\left(p\right)dp\right)f_\ell\left(x\right)dx\\ &= \Delta^k+\int_{\underline{x}}^{\xi_1^k}g\left(\widetilde{p}(x)\right)\left(E^k\left[c|\xi_1^k,w\geq 1\right]-E^k\left[c|x,w\geq 1\right]\right)f_\ell\left(x\right)dx\end{aligned}$$

Step 3:

$$\begin{aligned}&\int_{\underline{x}}^{\xi_1^k}\left(E^k\left[c|\xi_1^k,w\geq 1\right]-E^k\left[c|x,w\geq 1\right]\right)f_\ell\left(x\right)dx \qquad (35)\\ &= \sum_{i=2}^m\frac{\rho_i}{\rho_1}\frac{n_i^k}{n_1^k}(c_i-c_1)\int_{\underline{x}}^{\xi_1^k}\frac{\frac{\partial\left(\frac{f_i(x)}{f_1(x)}\right)}{\partial x}}{\left(\sum\limits_{j=2}^m\eta_{j1}^k(x)+1\right)^2}F_\ell(x)dx\end{aligned}$$

and either (i) $\lim_{k\to\infty}\frac{n_i^k}{n_1^k}=0$, for all $i>1$, or (ii) there is some $i>1$ such that $\lim_{k\to\infty}\frac{n_i^k}{n_1^k}=\infty$.

Proof of Step 3: Recall from (5)

$$E\left[c|x,w\geq 1\right]=\frac{\sum\limits_{i=2}^m\rho_i f_i\left(x\right)n_i c_i}{\sum\limits_{i=2}^m\rho_i f_i\left(x\right)n_i}=\frac{c_1+\sum\limits_{i=2}^m\eta_{i1}(x)c_i}{1+\sum\limits_{i=2}^m\eta_{i1}(x)}$$

where the second equality is obtained by dividing through by $\rho_1 f_1\left(x\right)n_1$ and recalling that $\eta_{i1}(x)=\frac{\rho_i f_i(x)n_i}{\rho_1 f_1(x)n_1}$. Integration by parts yields

$$\begin{aligned}&\int_{\underline{x}}^{\xi_1^k}\left(E^k\left[c|\xi_1^k,w\geq 1\right]-E^k\left[c|x,w\geq 1\right]\right)f_\ell\left(x\right)dx\\ &= \left[\left(E^k\left[c|\xi_1^k,w\geq 1\right]-E^k\left[c|x,w\geq 1\right]\right)F_\ell(x)\right]_{\underline{x}}^{\xi_1^k}-\int_{\underline{x}}^{\xi_1^k}\left(-\frac{\partial E^k\left[c|x,w\geq 1\right]}{\partial x}\right)F_\ell(x)dx\\ &= \int_{\underline{x}}^{\xi_1^k}\frac{\sum\limits_{i=2}^m\frac{\partial\eta_{i1}^k(x)}{\partial x}(c_i-c_1)}{\left(\sum\limits_{j=2}^m\eta_{j1}^k(x)+1\right)^2}F_\ell(x)dx=\sum_{i=2}^m\frac{\rho_i}{\rho_1}\frac{n_i^k}{n_1^k}(c_i-c_1)\int_{\underline{x}}^{\xi_1^k}\frac{\frac{\partial\left(\frac{f_i(x)}{f_1(x)}\right)}{\partial x}}{\left(\sum\limits_{j=2}^m\eta_{j1}^k(x)+1\right)^2}F_\ell(x)dx\end{aligned}$$

By Step 1 the RHS of (35) goes to 0. Therefore, either (i) $\lim_{k\to\infty}\frac{n_i^k}{n_1^k}=0$, for all $i>1$, or (ii) $\lim_{k\to\infty}\sum\limits_{j=2}^m\eta_{j1}^k(x)=\infty$ which means that there is some $i>1$ such that $\lim_{k\to\infty}\frac{n_i^k}{n_1^k}=\infty$. □

Step 4: For all $\ell > 1$, $\lim_{k\to\infty} \frac{\Gamma_\ell(\Omega_1^k)}{\Gamma_1(\Omega_1^k)} \in (0,\infty)$.
Proof of Step 4: From Step 2

$$\Gamma_\ell\left(\Omega_1^k\right) = \Delta^k + \int_{\underline{x}}^{\xi_1^k} g\left(\widetilde{p}(x)\right)\left(E\left[c|\xi_1^k, w \geq 1\right] - E\left[c|x, w\geq 1\right]\right) f_\ell\left(x\right) dx$$

Since g is bounded and bounded away from 0, $\lim_{k\to\infty} \frac{\Gamma_\ell(\Omega_1^k)}{\Gamma_1(\Omega_1^k)} \in (0,\infty)$ iff

$$\lim_{k\to\infty} \frac{\Delta^k + \int_{\underline{x}}^{\xi_1^k}\left(E\left[c|\xi_1^k, w\geq 1\right] - E\left[c|x,w\geq 1\right]\right) f_\ell\left(x\right)dx}{\Delta^k + \int_{\underline{x}}^{\xi_1^k}\left(E\left[c|\xi_1^k, w\geq 1\right] - E\left[c|x,w\geq 1\right]\right) f_1\left(x\right)dx} \in (0,\infty)$$

From Step 3 this is equivalent to

$$\lim \frac{\Delta^k + \sum\limits_{i=2}^{m} \eta_{i1}^k(\xi_1^k)(c_i - c_1)\int_{\underline{x}}^{\xi_1^k} \frac{\frac{\partial\left(\frac{f_i(x)}{f_1(x)}\right)}{\partial x}}{\left(\sum\limits_{j=2}^{m}\eta_{j1}^k(x)+1\right)^2} F_\ell(x)dx}{\Delta^k + \sum\limits_{i=2}^{m} \eta_{i1}^k(\xi_1^k)(c_i - c_1)\int_{\underline{x}}^{\xi_1^k} \frac{\frac{\partial\left(\frac{f_i(x)}{f_1(x)}\right)}{\partial x}}{\left(\sum\limits_{j=2}^{m}\eta_{j1}^k(x)+1\right)^2} F_1(x)dx} \in (0,\infty). \tag{36}$$

Recall from Step 3 that either (i) $\lim_{k\to\infty} \frac{n_i^k}{n_1^k} = 0$, for all $i > 1$, or (ii) $\exists i > 1$ such that $\lim_{k\to\infty} \frac{n_i^k}{n_1^k} = \infty$. In both cases,

$$\lim \frac{\int_{\underline{x}}^{\xi_1^k} \frac{\frac{\partial\left(\frac{f_i(x)}{f_1(x)}\right)}{\partial x}}{\left(\sum\limits_{j=2}^{m}\eta_{j1}^k(x)+1\right)^2} F_\ell(x)dx}{\int_{\underline{x}}^{\xi_1^k} \frac{\frac{\partial\left(\frac{f_i(x)}{f_1(x)}\right)}{\partial x}}{\left(\sum\limits_{j=2}^{m}\eta_{j1}^k(x)+1\right)^2} F_1(x)dx} \in (0,\infty) \tag{37}$$

This is because $\partial\left(\frac{f_i(x)}{f_1(x)}\right)/\partial x > 0$ and is independent of k, $\lim \xi_1^k = \hat{x} > \underline{x}$ and the following arguments. In case (i) both the numerator and denominator are bounded and bounded away from 0; in case (ii) dividing both the numerator and the denominator by the largest $\frac{n_i^k}{n_1^k}$ yields expressions that are bounded and bounded away from 0.

Since each term in the numerator of (36) has a corresponding term in its denominator whose ratio is in $(0,\infty)$, it follows that (36) holds and hence $\lim_{k\to\infty} \frac{\Gamma_\ell(\Omega_1^k)}{\Gamma_1(\Omega_1^k)} \in (0,\infty)$. □

Step 5: $\lim_{k\to\infty} \frac{n_i^k}{n_1^k} = 0$, for all $i > 1$.

Proof of Step 5: Since $\frac{n_i^k}{n_1^k} = \frac{\Gamma_1(\Omega_1^k)}{\Gamma_i(\Omega_i^k)} \leq \frac{\Gamma_1(\Omega_1^k)}{\Gamma_i(\Omega_1^k)}$, Step 4 implies $\lim_{k\to\infty} \frac{n_i^k}{n_1^k} \leq \lim_{k\to\infty} \frac{\Gamma_1(\Omega_1^k)}{\Gamma_i(\Omega_1^k)} < \infty$ ruling out the possibility that $\exists i > 1$ such that $\lim_{k\to\infty} \frac{n_i^k}{n_1^k} = \infty$. Therefore, by Step 3, the remaining possibility is $\lim_{k\to\infty} \frac{n_i^k}{n_1^k} = 0$, for all $i > 1$. □

Step 6: $\lim E\left[p|\,(p,x) \in \Omega_1^k, w\right] = c_1$ for all $w = 1, \cdots, m$ and $\overline{S}_1 = 0$.

Proof of Step 6: $\lim \frac{n_1^k}{n_i^k} = \infty$ implies $\lim E^k\left[c|x, w \geq 1\right] = c_1$ for all $x < \bar{x}$. Therefore, $\lim E\left[p|\,(p,x) \in \Omega_1^k, w\right] = c_1$ for $w = 1, \cdots, m$. Hence, $\lim V_1^k = u - c_1 - \overline{S}_1$, where $\overline{S}_1 = \lim \frac{s^k}{\Gamma_1(\Omega_1^k)}$. Recall that

$$\begin{aligned}
\lim_{k\to\infty} \Gamma_1\left(\Omega_1^k\right) &\geq \lim_{k\to\infty} \int_{\underline{x}}^{\xi_1^k} \left(G\left(u - V_1^k\right) - G\left(E^k\left[c|x, w \geq 1\right]\right)\right) f_1(x)\, dx \\
&= \int_{\underline{x}}^{\hat{x}} \left(G\left(u - \left(u - c_1 - \overline{S}_1\right)\right) - G(c_1)\right) f_1(x)\, dx \\
&= F_1(\hat{x}) \left(G\left(c_1 + \overline{S}_1\right) - G(c_1)\right)
\end{aligned}$$

Suppose now to the contrary that $\overline{S}_1 > 0$. Then, $F_1(\hat{x})\left(G\left(c_1 + \overline{S}_1\right) - G(c_1)\right) > 0$ hence $\overline{S}_1 = \lim \frac{s^k}{\Gamma_1(\Omega_1^k)} \leq \lim \frac{s^k}{F_1(\hat{x})\left(G\left(c_1 + \overline{S}_1\right) - G(c_1)\right)} = 0$ – contradiction. Therefore, $\overline{S}_1 = 0$. □

Step 7: $\lim_{k\to\infty} V_i^k = u - c_1$, for all i.

Proof of Step 7: From optimality,

$$\begin{aligned}
V_i^k &\geq i\text{'s payoff from mimicking 1 (accepting only } p \in B_1^k) \\
&= u - E\left[p|\,(p,x) \in \Omega_1^k, w = i\right] - \frac{s^k}{\Gamma_i\left(\Omega_1^k\right)} \\
&= u - E\left[p|\,(p,x) \in \Omega_1^k, w = i\right] - \frac{s^k}{\Gamma_1\left(\Omega_1^k\right)} \frac{\Gamma_1\left(\Omega_1^k\right)}{\Gamma_i\left(\Omega_1^k\right)}.
\end{aligned}$$

Hence,

$$\lim_{k\to\infty} V_i^k \geq u - \lim_{k\to\infty} E\left[p|\,(p,x) \in \Omega_1^k, w = i\right] - \overline{S}_1 \lim_{k\to\infty} \frac{\Gamma_1\left(\Omega_1^k\right)}{\Gamma_i\left(\Omega_1^k\right)}.$$

From Step 6 $\lim E\left[p|\,(p,x) \in \Omega_1^k, w\right] = c_1$ for all w and $\overline{S}_1 = 0$. From Step 4, $\lim \frac{\Gamma_1(\Omega_1^k)}{\Gamma_i(\Omega_1^k)} < \infty$. Therefore, $\lim_{k\to\infty} V_i^k \geq u - c_1$. Thus, since $E\left[p|\,(p,x) \in \Omega_i^k, w\right] \geq c_1$ for any w, we have $\lim V_i^k = u - c_1$, for all i. □

Step 8: $\lim \frac{n_1^k}{n_i^k} < 1$.

Proof of Step 8: By Step 7, $\lim_{k\to\infty} V_i^k \geq u - c_1$. Therefore, for sufficiently large k, $V_i^k > u - E^k\left[c|x, W^c \subseteq \{2, \cdots, m\}\right]$. By (12), this implies $\Omega_i^k = \Omega_1^k$ for all i. Since the term $\left(G\left(u - V_1^k\right) - G\left(E^k\left[c|x, w \geq 1\right]\right)\right)$ from (14) is decreasing in x (and strictly decreasing on $\left[\underline{x}, \xi_1^k\right]$), the MLRP implies that $\Gamma_i\left(\Omega_1^k\right) < \Gamma_1\left(\Omega_1^k\right)$. Hence,

$$\frac{n_1^k}{n_i^k} = \frac{\Gamma_i\left(\Omega_1^k\right)}{\Gamma_1\left(\Omega_1^k\right)} < 1. \tag{38}$$

$\square$

Steps 8 contradicts Step 5 implying that the initial supposition that $\lim_{k\to\infty}\xi_1^k > \underline{x}$ is false. Therefore, $\lim_{k\to\infty}\xi_1^k = \underline{x}$. Given this, an essentially identical proof establishes $\lim_{k\to\infty}\xi_2^k = \underline{x}$. From (15), $u - V_1^k = E^k\left[c|\xi_1^k, w \geq 1\right]$ and $u - V_2^k \geq E^k\left[c|\xi_2^k, w \geq 2\right]$, the counterpart of (34) for ξ_2^k is

$$
\begin{aligned}
s^k &= \sum_{j=1}^{2}\int_{\underline{x}}^{\xi_j}\left(\int_{\max\left\{E^k[c|x,w\geq j],u-V_{j-1}^k\right\}}^{u-V_j^k}\left(u - V_2^k - p\right)g(p)\,dp\right)f_2(x)\,dx \\
&\geq \sum_{j=1}^{2}\int_{\underline{x}}^{\xi_1}\left(\int_{\max\left\{E^k[c|x,w\geq j],u-V_{j-1}^k\right\}}^{u-V_j^k}\left(u - V_2^k - p\right)g(p)\,dp\right)f_2(x)\,dx \\
&\quad + \int_{\xi_1^k}^{\xi_2^k}\left(\int_{E^k[c|x,w\geq 2]}^{E^k\left[c|\xi_2^k,w\geq 2\right]}\left(E^k\left[c|\xi_2^k, w \geq 2\right] - p\right)g(p)\,dp\right)f_2(x)\,dx
\end{aligned}
$$

Since $\lim s^k = 0$ and $\lim \xi_1^k = \underline{x}$, it follows that, if $\lim \xi_2^k = \hat{x} > \underline{x}$, then

$$\lim_{k\to\infty} E^k\left[c|\hat{x}, w \geq 2\right] = \lim_{k\to\infty} E^k\left[c|x, w \geq 2\right] \qquad \forall x \in (\underline{x}, \hat{x})\,.$$

The above proof for $\lim \xi_1^k = \underline{x}$ can now be retraced verbatim by substituting $w = 2$ for $w = 1$ everywhere. Therefore, $\lim \xi_2^k = \underline{x}$ and the same follows for all $i < m$. $\square$

B.4 Proof of Lemma 3

We want to show that whenever $\Omega_j^k\backslash\Omega_{j-1}^k \neq \varnothing$, then $\lim E\left[p|\,(p,x) \in \left(\Omega_j^k\backslash\Omega_{j-1}^k\right), w = i\right] = \lim E^k\left[c|\xi_j^k, w \geq j\right]$ for all $i \geq j$ (where $\Omega_0^k = \varnothing$).

For $j = m$ this is immediate. Consider $j < m$. Lemma 2 and the definition of ξ_j^k imply that, for any $j < m$ and for k large enough, $V_j^k = u - E^k\left[c|\xi_j^k, w \geq j\right]$. Hence,

$$E\left[p|\,(p,x) \in \left(\Omega_j^k\backslash\Omega_{j-1}^k\right), w = j\right] \leq E^k\left[c|\xi_j^k, w \geq j\right]$$

If $\frac{f_j(\underline{x})}{f_m(\underline{x})} < \infty$ (signals are boundedly informative), then $\lim E^k\left[c|\xi_j^k, w \geq j\right] = \lim E^k\left[c|\underline{x}, w \geq j\right] \;\leq\; \lim E\left[p|\,(p,x) \in \left(\Omega_j^k\backslash\Omega_{j-1}^k\right), w = j\right]$, which establishes the lemma.

Consider next the case of $\frac{f_j(\underline{x})}{f_m(\underline{x})} = \infty$. Recall that $\lim E^k\left[c|\xi_j^k, w \geq j\right] = \lim \frac{c_j + \sum_{i=j+1}^{m}\eta_{ij}^k(\xi_j^k)c_i}{1 + \sum_{i=j+1}^{m}\eta_{ij}^k(\xi_j^k)}$.

If $\lim \eta_{ij}^k(\xi_j^k) = 0$ for every $i > j$ such that $\frac{f_j(\underline{x})}{f_i(\underline{x})} = \infty$, then $\lim E^k\left[c|\xi_j^k, w \geq j\right] = \lim E^k\left[c|\underline{x}, w \geq j\right]$ and the argument is as above. Suppose therefore that $\lim \eta_{ij}^k(\xi_j^k) > 0$ for some $i > j$ such that $\frac{f_j(\underline{x})}{f_i(\underline{x})} = \infty$. This is possible only if $\lim \frac{n_i^k}{n_j^k} = \infty$. Suppose now to the contrary that $\lim E\left[p|\,(p,x) \in \left(\Omega_j^k\backslash\Omega_{j-1}^k\right), w = j\right] < E^k\left[c|\xi_j^k, w \geq j\right]$.

Since

$$
\begin{aligned}
E\left[p|\,(p,x)\in\Omega_j^k, w=j\right] &= \frac{\Gamma_j\left(\Omega_{j-1}^k\right)}{\Gamma_j\left(\Omega_j^k\right)}E\left[p|\,(p,x)\in\Omega_{j-1}^k, w=j\right] \\
&\quad+\frac{\Gamma_j\left(\Omega_j^k\backslash\Omega_{j-1}^k\right)}{\Gamma_j\left(\Omega_j^k\right)}E\left[p|\,(p,x)\in\left(\Omega_j^k\backslash\Omega_{j-1}^k\right), w=j\right]
\end{aligned}
$$

and $E\left[p|\,(p,x)\in\Omega_{j-1}^k, w=j\right] \le E^k\left[c|\xi_{j-1}^k, w\ge j-1\right] \le E^k\left[c|\xi_j^k, w\ge j\right]$, it follows that

$$
\lim_{k\to\infty} E\left[p|\,(p,x)\in\Omega_j^k, w=j\right] < E^k\left[c|\xi_j^k, w\ge j\right]
$$

This together with $V_j^k = u-E\left[p|\,(p,x)\in\Omega_j^k, w=j\right]-S_j^k$ and $V_j^k = u-E^k\left[c|\xi_j^k, w\ge j\right]$ imply

$$
\lim_{k\to\infty} S_j^k = \lim E^k\left[c|\xi_j^k, w\ge j\right] - \lim_{k\to\infty} E\left[p|\,(p,x)\in\Omega_j^k, w=j\right] > 0.
$$

Combining this observation with $\lim \frac{n_i^k}{n_j^k}=\infty$ we get

$$
\lim_{k\to\infty} S_i^k = \lim_{k\to\infty} s^k n_1^k \frac{n_i^k}{n_j^k} = \lim_{k\to\infty} S_j^k \frac{n_i^k}{n_j^k} = \infty.
$$

Thus, $\lim V_i^k = -\infty$, which may not arise in equilibrium. Therefore, the hypothesis $\lim E\left[p|\,(p,x)\in\left(\Omega_j^k\backslash\Omega_{j-1}^k\right), w=j\right] < \lim E^k\left[c|\xi_j^k, w\ge j\right]$ is false. Hence, $\lim E\left[p|\,(p,x)\in\left(\Omega_j^k\backslash\Omega_{j-1}^k\right), w=j\right] = \lim E^k\left[c|\xi_j^k, w\ge j\right]$ as required. $\square$

B.5 Proof of Lemma 4

To reduce the denseness of the notation, assume that $\underline{I}=i$ and $\bar{I}=\ell$. Since $\lim \eta_{ji}^k(\xi_i^k)\in(0,\infty)$ if $i<j\le\ell$ and $=0$ if $j>\ell$, it follows that, for any $x\le\xi_i^k$

$$
\lim E^k\left[c|x, w\ge i\right] = \lim \frac{c_i+\sum\limits_{j=i+1}^{m}\eta_{ji}^k(x)c_j}{1+\sum\limits_{j=i+1}^{m}\eta_{ji}^k(x)} = \lim \frac{c_i+\sum\limits_{j=i+1}^{\ell}\eta_{ji}^k(x)c_j}{1+\sum\limits_{j=i+1}^{\ell}\eta_{ji}^k(x)} = \lim E^k\left[c|x, i\le w\le\ell\right] \tag{39}
$$

Observe that $\lim \frac{\Gamma_i\left(\Omega_{i-1}^k\right)}{\Gamma_i\left(\Omega_i^k\right)}=0$ implies that, for any $\varepsilon>0$, for k large enough $u-V_{i-1}^k = E\left[c|\xi_{i-1}, w\ge i-1\right]\le c_{i-1}+\varepsilon$, while $E^k\left[c|x, w\ge i\right]\ge c_i-\varepsilon$. Therefore, for $j\ge i$ and large k,

$$
\begin{aligned}
\Gamma_j\left(\Omega_i^k\right) &= \int_{\underline{x}}^{\xi_i^k}\int_{E^k[c|x,w\ge i]}^{E^k\left[c|\xi_i^k,w\ge i\right]} g(p)f_j(x)\,dp dx + \Gamma_j\left(\Omega_{i-1}^k\right) \\
&= \int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{p}(x)\right)\left(E^k\left[c|\xi_i^k, i\le w\le\ell\right]-E^k\left[c|x, i\le w\le\ell\right]\right)f_j(x)\,dx+\varepsilon^k
\end{aligned}
$$

where ε^k captures both the omitted $\Gamma_j\left(\Omega_{i-1}^k\right)$ and the change from $E^k\left[c|x,w\geq i\right]$ to $E^k\left[c|x,i\leq w\leq \ell\right]$. It is important to observe that $\lim \frac{\varepsilon^k}{\Gamma_j\left(\Omega_i^k\right)}=0$. This follows from (39) and since by MLRP, for $j\geq i$, $\frac{\Gamma_j\left(\Omega_{i-1}^k\right)}{\Gamma_j\left(\Omega_i^k\right)}\leq \frac{\Gamma_i\left(\Omega_{i-1}^k\right)}{\Gamma_i\left(\Omega_i^k\right)}\to 0$.

Similarly, for large k,

$$\begin{aligned}
& E^k\left[p|\left(p,x\right)\in\Omega_i^k,w=i\right] \\
= & \int_{\underline{x}}^{\xi_i^k}\int_{E^k[c|x,w\geq i]}^{E^k\left[c|\xi_i^k,w\geq i\right]} p\frac{g(p)dp}{\Gamma_i\left(\Omega_i^k\right)}f_i(x)dx+E^k\left[p|\left(p,x\right)\in\Omega_{i-1}^k,w=i\right] \\
= & \int_{\underline{x}}^{\xi_i^k}\int_{E^k[c|x,i\leq w\leq\ell]}^{E^k\left[c|\xi_i^k,i\leq w\leq\ell\right]} p\frac{g(p)dp}{\Gamma_i\left(\Omega_i^k\right)}f_i(x)dx+\delta^k
\end{aligned}$$

where $\lim \frac{\delta^k}{E^k\left[p|(p,x)\in\Omega_i^k,w=i\right]}=0$.

The optimality of i's search implies

$$\frac{s^k}{\Gamma_i\left(\Omega_i^k\right)}=\left(E^k\left[c|\xi_i^k,w\geq 1\right]-E^k\left[p|\left(p,x\right)\in\Omega_i^k,w=i\right]\right).$$

Let $q>i$.Multiplying both sides by $\frac{\Gamma_1\left(\Omega_i^k\right)}{\Gamma_q\left(\Omega_i^k\right)}\frac{F_i\left(\xi_i^k\right)}{F_i\left(\xi_i^k\right)}$, we have

$$\begin{aligned}
& \frac{s^k}{\Gamma_q\left(\Omega_i^k\right)} \qquad (40)\\
= & \frac{F_i\left(\xi_i^k\right)}{F_i\left(\xi_i^k\right)}\frac{\Gamma_i\left(\Omega_i^k\right)}{\Gamma_q\left(\Omega_i^k\right)}\left(E^k\left[c|\xi_i^k,w\geq 1\right]-E\left[p|\left(p,x\right)\in\Omega_i^k,w=i\right]\right) \\
= & \frac{F_i\left(\xi_i^k\right)}{F_i\left(\xi_i^k\right)}\frac{\Gamma_i\left(\Omega_i^k\right)\left(E^k\left[c|\xi_i^k,w\geq 1\right]-E\left[p|\left(p,x\right)\in\Omega_i^k,w=i\right]\right)}{\Gamma_q\left(\Omega_i^k\right)}.
\end{aligned}$$

The denominator of (40)

$$\begin{aligned}
& \frac{1}{F_i\left(\xi_i^k\right)}\Gamma_q\left(\Omega_i^k\right) \qquad (41)\\
= & \int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{p}(x)\right)\left(E^k\left[c|\xi_i^k,i\leq w\leq\ell\right]-E^k\left[c|x,i\leq w\leq\ell\right]\right)\frac{f_q\left(x\right)}{F_i\left(\xi_i^k\right)}dx+\widetilde{\varepsilon}^k
\end{aligned}$$

where where $\widetilde{\varepsilon}^k$ divided by the first term goes to 0 as $k\to\infty$.

The numerator of (40)

$$\begin{aligned}
& \frac{1}{F_i\left(\xi_i^k\right)}\Gamma_i\left(\Omega_i^k\right)\left(\left(E^k\left[c|\xi_i^k,w\geq 1\right]-E\left[p|\left(p,x\right)\in\Omega_i^k,w=i\right]\right)\right) \qquad (42)\\
= & \int_{\underline{x}}^{\xi_i^k}\int_{E^k[c|x,i\leq w\leq\ell]}^{E^k\left[c|\xi_i^k,i\leq w\leq\ell\right]}\left[E^k\left[c|\xi_i^k,i\leq w\leq\ell\right]-p\right]g(p)dp\frac{f_i(x)}{F_i\left(\xi_i^k\right)}dx+\delta^k \\
= & \frac{1}{2}\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{\widetilde{p}}(x)\right)\left(E^k\left[c|\xi_i^k,i\leq w\leq\ell\right]-E^k\left[c|x,i\leq w\leq\ell\right]\right)^2\frac{f_i(x)}{F_i\left(\xi_i^k\right)}dx+\widetilde{\delta}^k
\end{aligned}$$

where $\widetilde{\delta}^k$ divided by the first term goes to 0 as $k \to \infty$.

To keep the expressions more compact, let us adopt the shorthand

$$\Delta E^k(\xi_i^k, x) = E^k\left[c|\xi_i^k, i \le w \le \ell\right] - E^k\left[c|x, i \le w \le \ell\right]$$

Now, observe that

$$\begin{aligned}
&\frac{1}{2}\int_{\underline{x}}^{\xi_i^k}\left(\Delta E^k(\xi_i^k, x)\right)^2 \frac{f_i(x)}{F_i\left(\xi_i^k\right)}dx \qquad (43)\\
=\;& \frac{1}{2}\left[\left(\Delta E^k(\xi_i^k, x)\right)^2 \frac{F_i(x)}{F_i\left(\xi_i^k\right)}\right]_{\underline{x}}^{\xi_i^k} - \int_{\underline{x}}^{\xi_i^k}\Delta E^k(\xi_i^k, x)\left(-\frac{\partial E^k\left[c|x, i \le w \le \ell\right]}{\partial x}\right)\frac{F_i(x)}{F_i\left(\xi_i^k\right)}dx\\
=\;& \int_{\underline{x}}^{\xi_i^k}\Delta E^k(\xi_i^k, x)\frac{\sum_{j=i+1}^{\ell}\frac{\partial \eta_{ji}^k(x)}{\partial x}(c_j - c_i)}{\left(\sum_{j=i+1}^{\ell}\partial\eta_{ji}^k(x) + 1\right)^2}\frac{F_i(x)}{F_i\left(\xi_i^k\right)}dx\\
=\;& \sum_{j=i+1}^{\ell}\int_{\underline{x}}^{\xi_i^k}\Delta E^k(\xi_i^k, x)\frac{\eta_{ji}^k(x)(c_j - c_i)\left(-\frac{-\frac{\partial\left(\frac{f_i(x)}{f_j(x)}\right)}{\partial x}}{\frac{f_i(x)}{F_i(x)}}\right)}{\left(\sum_{j=i+1}^{\ell}\eta_{ji}^k(x) + 1\right)^2}\frac{f_j\left(x\right)}{F_i\left(\xi_i^k\right)}dx
\end{aligned}$$

where the first equality follows from integration by parts, the second uses $\lim E^k\left[c|x, w \ge i\right] = \lim\left[\left(c_i + \sum_{j=i+1}^{\ell}\eta_{ji}^k(x)c_i\right)\Big/\left(1 + \sum_{j=i+1}^{\ell}\eta_{ji}^k(x)\right)\right]$, the third follows from $\frac{\partial\eta_{ji}^k(x)}{\partial x} = \frac{\rho_j}{\rho_i}\frac{n_j^k}{n_i^k}\frac{\partial\left(\frac{f_j(x)}{f_i(x)}\right)}{\partial x} = \frac{\rho_j}{\rho_i}\frac{n_j^k}{n_i^k}\left(-\frac{\partial\left(\frac{f_i(x)}{f_j(x)}\right)}{\partial x}\Big/\left(\frac{f_i(x)}{f_j(x)}\right)^2\right) = \eta_{ji}^k(x)\left(-\frac{\partial\left(\frac{f_i(x)}{f_j(x)}\right)}{\partial x}\frac{f_j(x)}{f_i(x)}\right)$.

Substituting (43) into (42) and then the resulting expression and 41 into (40),

$$\lim \frac{s^k}{\Gamma_q\left(\Omega_i^k\right)} \tag{44}$$

$$= \lim \frac{\sum_{j=i+1}^{\ell} \int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{\widetilde{p}}(x)\right) \Delta E^k(\xi_i^k, x) \frac{\eta_{ji}^k(x)(c_j - c_i)}{\left(\sum_{j=i+1}^{\ell} \eta_{ji}^k(x)+1\right)^2} \left(-\frac{\frac{\partial\left(\frac{f_i(x)}{f_j(x)}\right)}{\partial x}}{\frac{f_i(x)}{F_i(x)}}\right) \frac{f_j(x)}{F_i\left(\xi_i^k\right)} dx + \widetilde{\delta}^k}{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{p}(x)\right) \Delta E^k(\xi_i^k, x) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx + \widetilde{\varepsilon}^k}$$

$$= \lim \frac{\sum_{j=i+1}^{\ell} \int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{\widetilde{p}}(x)\right) \Delta E^k(\xi_i^k, x) \frac{\eta_{ji}^k(x)(c_j - c_i)}{\left(\sum_{j=i+1}^{\ell} \eta_{ji}^k(x)+1\right)^2} \left(-\frac{\frac{\partial\left(\frac{f_i(x)}{f_j(x)}\right)}{\partial x}}{\frac{f_i(x)}{F_i(x)}}\right) \frac{f_j(x)}{f_q(x)} \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx}{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{p}(x)\right) \Delta E^k(\xi_i^k, x) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx}$$

$$\leq \lim \sum_{j=i+1}^{\ell} \frac{\eta_{ji}^k(\xi_i^k)(c_j - c_i)\lambda_{iq}}{\left(\sum_{j=i+1}^{\ell} \eta_{ji}^k(\xi_i^k)+1\right)^2} \frac{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{\widetilde{p}}(x)\right) \Delta E^k(\xi_i^k, x) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx}{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{p}(x)\right) \Delta E^k(\xi_i^k, x) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx}$$

The second equality follows from the negligibility of $\widetilde{\delta}^k$ and $\widetilde{\varepsilon}^k$ relative to the other terms and multiplication by $\frac{f_q(x)}{f_q(x)}$. The inequality follows from

$$\lim -\frac{\frac{\partial\left(\frac{f_i(x)}{f_j(x)}\right)}{\partial x}}{\frac{f_{\underline{i}}(x)}{F_{\underline{I}}(x)}} \frac{f_j\left(x\right)}{f_q\left(x\right)} \leq -\frac{\frac{\partial\left(\frac{f_i(x)}{f_q(x)}\right)}{\partial x}}{\frac{f_{\underline{i}}(x)}{F_{\underline{I}}(x)}} = \lambda_{iq},$$

from (24). Notice that $\lim \eta_{ji}^k(x)(c_j - c_i)\left(\sum_{j=i+1}^{\ell} \eta_{ji}^k(x)+1\right)^{-2}$ can be pulled out of the integral since it is converging to a positive number. Therefore,

$$\lim \frac{s^k}{\Gamma_q\left(\Omega_i^k\right)} = \lambda_{iq} \lim \sum_{j=i+1}^{\ell} \frac{\eta_{ji}^k(\xi_i^k)(c_j - c_i)}{\left(\sum_{j=i+1}^{\ell} \eta_{ji}^k(\xi_i^k)+1\right)^2}$$

where the removal of $g\left(\widetilde{p}(x)\right)$ and $g\left(\widetilde{\widetilde{p}}(x)\right)$ follows from g being bounded and bounded away from 0. This establishes Part (i) of the lemma.

Part (ii) of the lemma is established by the following argument. From the first and third lines of (44), for any $q \leq \ell$,

$$\begin{aligned}
\lim \frac{s^k}{\Gamma_q\left(\Omega_i^k\right)} &\geq \lim \frac{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{\widetilde{p}}(x)\right) \Delta E^k(\xi_i^k, x) \frac{\eta_{qi}^k(x)(c_q - c_i)}{\left(\sum\limits_{j=i+1}^{m} \eta_{ji}^k(x)+1\right)^2} \left(\frac{-\frac{\partial\left(\frac{f_i(x)}{f_q(x)}\right)}{\partial x}}{\frac{f_i(x)}{F_i(x)}}\right) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx}{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{p}(x)\right) \Delta E^k(\xi_i^k, x) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx} \\
&= \lim \frac{\eta_{qi}^k(x)(c_q - c_i)\lambda_{iq}}{\left(\sum\limits_{j=i+1}^{\ell} \eta_{ji}^k(x) + 1\right)^2} \frac{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{\widetilde{p}}(x)\right) \Delta E^k(\xi_i^k, x) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx}{\int_{\underline{x}}^{\xi_i^k} g\left(\widetilde{p}(x)\right) \Delta E^k(\xi_i^k, x) \frac{f_q(x)}{F_i\left(\xi_i^k\right)} dx} \\
&= \lim \frac{\eta_{qi}^k(x)(c_q - c_i)\lambda_{iq}}{\left(\sum\limits_{j=i+1}^{\ell} \eta_{ji}^k(x) + 1\right)^2}
\end{aligned}$$

The first inequality is because the RHS is just the q-th element out of the sum $\sum\limits_{j=i+1}^{\ell}$ on the RHS of the third line of (44). The rest follows from $\left(\frac{-\frac{\partial\left(\frac{f_i(x)}{f_q(x)}\right)}{\partial x}}{\frac{f_i(x)}{F_i(x)}}\right) \to \lambda_{iq}$ just as in (44) above. □

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