Price Setting with Interdependent Values

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Abstract

We consider a take-it-or-leave-it price offer game under value interdependence. The main result is a simple but general sufficient condition that ensures existence and uniqueness of a separating equilibrium. The condition amounts to the monotonicity of what we call the *Myerson virtual profit*, and covers quality uncertainty à la Akerlof as a special case. Prices are shown to be higher than under signal disclosure, thus reflecting a *signalling premium.*

Keywords: Adverse selection, lemons, bilateral bargaining, two-sided private information

JEL Codes: D82, D83

1 Introduction

Beginning with Akerlof (1970), economists have long realized that adverse selection may impede trade. However, when there are profitable opportunities for trading, a possibility arises that the seller may transmit information through the price offered to the buyer. This signalling channel, first identified by Spence (1973), may mitigate adverse selection.

In many situations, neither the buyer nor the seller knows the value of the object prior to trade. At the same time, both agents often have certain valuable information concerning their values. Thus, the values are often *interdependent*. Real-life examples of markets with these features are abundant: real estate, financial assets, sales of natural resources such as offshore oil, etc.

These markets feature private information on *both* sides, which exacerbates adverse selection as the "quality" is not perfectly known to either side. Indeed, if trade occurs at a predominantly constant price, buyers may suffer adverse selection as the sellers who choose to trade at this price may have a relatively low quality signal. But with interdependent values, the same effect also obtains on the seller side as the buyers they will trade with may have a higher quality signal, which may drive up the seller's own value (cost).

But in reality prices are never constant, and may transmit seller's information to the buyer, which may mitigate adverse selection. The literature to date, beginning with Myerson (1985), has identified this effect in settings with on one-sided adverse selection, i.e. when the "quality" is perfectly known to the seller, and therefore so is the seller's cost.[1]

In this paper, we investigate signalling in a model with interdependent values. In our model, there is a seller and a buyer of a single unit of the good. The buyer and the seller have ex post valuations of the good that are not known with certainty. Prior to the game, each of them receives information, or *signal*, concerning his or her value. The buyer's signal is never perfectly revealing, and the seller's signal x_S may also be not perfectly revealing.[2] The signals are allowed to be correlated.[3]

The seller makes a take-it-or-leave-it offer to the buyer. The buyer forms a posterior belief of the seller's signal, and then decides whether to accept or reject the offer. We restrict attention to Perfect Bayesian equilibria of this price-setting game, and, further, to equilibria in *monotone strategies*, where the seller's offer is a nondecreasing function of her signal. In this class, an equilibrium is (fully) separating if the offer strategy is *strictly* increasing in the signal.

Our main result is a simple condition for existence, and uniqueness, of a separating equilibrium in pure strategies in this model. The condition amounts to the monotonicity of the *virtual profit* a la Myerson (1981) and Bulow and Roberts (1989), in the buyer's signal.[4] The virtual profit is defined in this paper as the difference between the Myerson virtual value function and the seller's cost. Following Bulow and Roberts (1989), it can be interpreted as the marginal

[1] There is also a literature that examines other types of communication in addition to pure price signalling, such as advertising as in Milgrom and Roberts (1986), inspection as in Bester and Ritzberger (2001) or disclosure as in and Bagwell and Riordan (1991) and more recently, Daughety and Reinganum (2008). This paper, however, only considers price signalling.

[2] However, we not preclude the scenario where the the seller is perfectly informed. As we discuss later in the paper, this corresponds to the "lemons" case, earlier considered in the literature, albeit under assumptions less general than here.

[3] More precisely, we assume that the conditional distribution of the buyer's signal has the monotone likelihood ratio property.

[4] There are other standard conditions such as the MLRP, and the monotonicity assumption on the buyer and seller valuations with respect to the signals.

profit of the seller, assuming the probability of sale is the "quantity".

The result that there is a separating equilibrium in monotone strategies is not self-evident. In order to provide the seller with an incentive to reveal the signal, it is necessary that the probability of the offer acceptance is monotone decreasing in the seller's signal.[5] The monotonicity of seller's offer strategy will ensure this. But, the higher price comes at a cost due to the "winner's curse" effect for the seller. This is because a higher price entails a higher average buyer's signal conditional on acceptance, which in turn leads to a higher seller's cost. This "winner's curse" effect pushes the seller's price down, and it limits the seller's ability to transmit information in a separating equilibrium.

A question then arises, under what conditions a separating equilibrium would exists. Our analysis shows that it does under a condition based the Myerson virtual surplus function, which is frequently invoked in mechanism design. Namely, we show that a (monotone) separating equilibrium exists, and is unique, if this virtual profit is monotone increasing in the buyer's signal.

For intuition, consider first the setting without the winner's curse, where the buyer's information is irrelevant for the seller. This is what we shall call the "lemons" environment, with the seller's signal having the interpretation of "quality". Our assumptions ensure that the seller's cost is increasing in quality. It is known from the literature that the virtual value function is then effectively the seller's marginal revenue, where the probability of sale acts as the "quantity" variable.[6] As the seller's cost increases, incentive compatibility implies that the price is non-decreasing in "quality".

However, if the marginal revenue is *not* monotone, then the expected profit may have multiple local maxima. As the seller's signal (and therefore the cost) increases, the price may exhibit bunching, sticking to one of the maxima, remaining constant over an interval of signals. This may lead to pooling rather than separation. In the "lemons" setting, the marginal cost does not depend on the buyer's information, so if the marginal revenue is monotone in it, the marginal profit is also monotone. In our environment, the seller's cost generally depends on the buyer's information (signal). So in order to avoid the above mentioned possibility of bunching,

[5]See Myerson (1985).
[6]This interpretation is due to Bulow and Roberts (1989).

we instead require that the marginal profit is monotone in the buyer's signal.

The price offered in the separating equilibrium is shown to involve a counter-efficient *signalling premium*, making it above the price that would be offered if the seller's signal were disclosed to the buyer. To get intuition for it, once again consider a "lemons" environment. Suppose that under disclosure, the price is monotonic in the seller's signal.[7] Then without disclosure, the seller with a slightly lower signal would have an incentive to mimic because doing so would convince the buyer that the quality is higher, thereby increasing the probability of acceptance and the expected profit. To countervail this incentive, it is necessary to reduce the acceptance probabilities by increasing the price relative to the full disclosure level.

Although it is widely understood that value interdependance may impede trade, we believe ours is the first general model that allows to quantify the signalling premium for general specifications of the buyer and seller utilities. By deriving a simple, easy to interpret sufficient condition for a separating equilibrium, we clarify the restrictions that would need to be imposed on this model in applied work.

Related literature We are not aware of any similar result in a general setting as in this paper. Our focus on a continuum of quality levels makes our setting closer to the bargaining literature, rather than quality signalling literature that has mostly considered two quality levels (high and low). One notable exception is Myerson (1985), who, among other things, shows existence of a fully separating equilibrium in standard Akerlof (1970)-type model, with mixed strategies on the buyer side.

There is a related paper by Cai, Riley, and Ye (2007) that investigates reserve price signalling in a second-price auction with affiliated *buyer* values. The case of a single bidder corresponds to our model; the reserve price simply becomes the price set by the seller. Cai, Riley, and Ye (2007) also derive a sufficient condition for a separating equilibrium.[8] It turns out that, with a single bidder, their model is a special case of ours, and our condition specializes to theirs. There are two differences from our model. First, in their model there is no "winner's curse" since the seller's cost is assumed to be unaffected by the buyers' signal. Second, they assume

[7]A sufficient condition for this is that the marginal profit function is decreasing in the seller's signal. This condition, however, is not needed in the absence of disclosure; then the price monotonicity would be implied by incentive compatibility.

[8]In an auction with more than one bidder, a corrected version of their condition is given Lamy (2010).

that the seller's signal is independent of the buyers', while we are allowing for correation.

There are some related results in the double auction literature. Satterthwaite and Williams (1989) show existence of an equilibrium in monotone strategies for a k-double auction with independent private values. This result has recently been generalized to affiliated *private* values by Kadan (2007), under additional conditions.[9]

2 Model

There is a buyer and a seller. The seller has a unit of a good to sell, and the buyer has a unit demand. We consider a price-setting game: the seller makes a take-it-or-leave-it offer to the buyer, who may either accept or reject it.

Before the game begins, the buyer and the seller receies *signals* x_B and x_S that affect their ex-post valuations of the good, $u_B(x_B, x_S)$ and $u_S(x_S, x_B)$. The signals x_B and x_S will be sometimes referred to as buyer's and seller's *types*. The buyer and the seller do not observe the signal received by the opposite side, only their own signals. We allow x_B to be correlated with x_S.

The signals are drawn prior to the game from a joint distribution supported on $[0, 1]^2$. We will need the following regularity assumption on the conditional distribution of the buyer's signal, $F_B(x_B|x_S)$.

Assumption 1 (Regularity of density)**.** *The distribution $F_B(x_B|x_S)$ has density $f_B(x_B|x_S)$, continuously differentiable and positive on* $[0, 1]^2$.

To capture the idea that signals are positively associated, we impose the Monotone Likelihood Ratio Property (MLRP).

Assumption 2 (MLRP)**.** *For any $x_B \leq \hat{x}_B, x_S \leq \hat{x}_S$, we have*

$$\frac{f_B(\hat{x}_B|x_S)}{f_B(x_B|x_S)} \leq \frac{f_B(\hat{x}_B|\hat{x}_S)}{f_B(x_B|\hat{x}_S)}.$$

We assume that the buyer's and seller's valuations are increasing in the signals, and are also regular in the sense defined below.

[9] In addition, several papers, in particular Vincent (1989) and, more recently, Deneckere and Liang (2006), have studied dynamic bargaining in a "lemons" environment.

Assumption 3 (Monotonicity and differentiability of valuations). *1. The buyer's valuation $u_B(x_B, x_S)$ is twice continuously differentiable on its domain, with*

$$\frac{\partial u_B(x_B, x_S)}{\partial x_B} > 0, \qquad \frac{\partial u_B(x_B, x_S)}{\partial x_S} \geq 0.$$

2. The seller's valuations $u_S(x_S, x_B)$ is increasing in x_S, nondecreasing in x_B and continuously differentiable.

The above assumption is a natural one. It requires the buyer's valuation to be increasing in both his and partner's signals, while the seller's valuation is only required to be increasing in own signal. This means that, first, the setting is rich enough to incorporate the important *lemons* environment. In the lemons environment, the seller's valuation does not depend on the buyer's signal, and the seller's signal has a clear interpretation as a measure of *quality* of the good. However, we do not allow the *pure* lemons environment, where the buyer's valuation is known to the seller. In our model, in order to sustain a separating equilibrium, it is necessary that the buyer would reject bigger offers with a higher probability. Our assumption that the buyer's utility is strictly increasing in own signal will ensure that the buyer will reject offers not only because the quality is low, but also because of its own signal, which provides a degree of horizontal differentiation.

2.1 Benchmark: Equilibrium under disclosure

As a benchmark, consider a model with one-sided private information: the seller's signal x_S is observable to the buyer. We will refer to this setting as *disclosure*. Then the buyer's responding strategy is obtained as a minimal signal $x_B^*(p, x_S)$ such that the seller's offer p is acceptable to a buyer,

$$x_B^*(p, x_S) = \inf\{x_B \in [0, 1] : u_B(x_B, x_S) \geq p\}$$

Note that the offer p can be so low that it is acceptable to a buyer with any signal if $p \leq u_B(0, x_S)$; in this case, $x_B^* = 0$. Or, it can be so high that it is never acceptable and then x_B^* is not defined. For any price $p \in [u_B(0, x_S), u_B(1, x_S)]$, the buyer's reservation cutoff $x_B^*(p, x_S)$ is determined from

$$u_B(x_B^*(p, x_S), x_S) = p. \tag{1}$$

For $p < u_B(0, x_S)$, the reservation cutoff is $x^*_B(p, x_S) = 0$, while for $p > u_B(1, x_S)$, it is given by $x^*_B(p, x_S) = 1$. Given the buyer's responding strategy $x^*_B(p, x_S)$, the seller's problem is to choose price p to maximize the expected profit.

It turns out to be more convenient to reformulate the problem in terms of q, the induced *probability of sale*.[10] If the marginal buyer type is x_B, then

$$q = 1 - F_B(x_B|x_S) \implies x_B(q) = F_B^{-1}(1 - q|x_S)$$

and the expected seller's profit as a function of q is

$$\pi_S(q, x_s) = q u_B(x_B(q), x_S) - \int_{x_B(q)}^{1} u_S(x_S, \tilde{x}_B) f_B(\tilde{x}_B|x_S) d\tilde{x}_B,$$

so the *marginal profit* is

$$\frac{\partial \pi_S(q, x_S)}{\partial q} = u_B(x_B(q), x_S) + q\frac{\partial u_B(x_B(q), x_S)}{\partial x_B} x'_B(q) - u_S(x_S, x_B(q)) x'_B(q) f_B(x_B(q)|x_S).$$

As

$$x_B(q) = F_B^{-1}(1 - q|x_S) \implies x'_B(q) = -\frac{1}{f_B(x_B(q)|x_S)},$$

after some algebra, the marginal profit simplifies to

$$\frac{\partial \pi_S(q, x_S)}{\partial q} = J_B(x_B(q), x_S) - u_S(x_S, x_B(q)) \equiv J(x_B(q), x_S), \tag{2}$$

where $J_B(x_B, x_S)$ is the *Myerson virtual value*,

$$J_B(x_B, x_S) \equiv u_B(x_B, x_S) - \frac{\partial u_B(x_B, x_S)}{\partial x_B} \cdot \frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)}, \tag{3}$$

and $J(x_B, x_S)$ is what we shall call the seller's *virtual profit*.

Central to our analysis is the following standard monotonicity assumption on the virtual profit function.

Assumption 4 (Monotonicity of virtual profit)**.** *For all $x_B, x_S \in [0, 1]$, we have*

$$\frac{\partial J(x_B, x_S)}{\partial x_B} > 0.$$

[10]This approach is motivated by Bulow and Roberts (1989).

The *iso-marginal profit curve*, or the *J-curve* for short,

$$\{(x_B, x_S) \in [0,1]^2 : J(x_B, x_S) = 0\},$$

separates the square $[0,1]^2$ that represents the set of all possible buyer and seller signals, into two parts. The marginal profit is positive below the J-curve, and negative above it.

Let $x_B^*(x_S) \in [0,1]$ be the minimal buyer type for whom the x_S-seller's optimal offer is acceptable. The optimal price offer is equal to $u_B(x_B^*(x_S), x_S)$. Since $J(x_B, x_S)$ is increasing in x_B, the marginal profit is decreasing in q, and there are three possibilities for $x_B^*(x_S)$.

- If $J(0, x_S) > 0$, then the marginal profit is positive for all price offers that can ever be accepted, and therefore the optimally chosen price is the highest one acceptable to all buyers. This implies that $x_B^*(x_S) = 0$.

- If $J(1, x_S) < 0$, then the marginal profit is negative for all price offers, and there is no trade. In this case, we set $x_B^*(x_S) = 1$ by convention.

- If $J(0, x_S) \leq 0$ and $J(1, x_S) \geq 0$, then $x_B^*(x_S)$ is uniquely determined by $J(x_B^*(x_S), x_S) = 0$, i.e. lies on the J-curve.

From now on, we assume that all seller types trade with a positive probability.

Assumption 5. *We have*

$$x_B^*(x_S) < 1 \quad \forall x_S \in [0,1].$$

Remark 1. More primitive conditions that ensure the monotonicity of the virtual profit J can be provided. First, notice that if the seller's valuation $u_S(x_S, x_B)$ does not depend on x_B, then it is sufficient to require that the virtual value $J_B(x_B, x_S)$ is increasing in x_B and has a positive slope, $\partial J_B(x_B, x_S)/\partial x_B > 0$. This generalizes the condition obtained in Cai, Riley, and Ye (2007) for auctions by allowing the seller's and the buyer's signals to be correlated. Second, if $u_S(x_S, x_B)$ does depend on x_B so that we have genuine value interdependence, then the monotonicity of J in x_B is implied by the following more primitive conditions, most of which are standard in the literature.

1. *Single crossing:* The difference $u_B(x_B, x_S) - u_S(x_S, x_B)$ is increasing in x_B.

2. The buyer's utility function $u_B(x_B, x_S)$ is convex in x_B for any $x_S \in [0, 1]$, so that $\frac{\partial u_B(x_B, x_S)}{\partial x_B}$ is nondecreasing in x_B.

3. The conditional distribution of the buyer's signal, $F_B(x_B|x_S)$ has nondecreasing hazard rate, so that $\frac{1-F_B(x_B|x_S)}{f_B(x_B|x_S)}$ is a *nonincreasing* function of x_B.

It is not hard to provide specific examples of utility functions satisfying (1) and (2). For the condition on the signals (3) above, one needs to ensure that it is compatible with the MLRP, which is the maintained assumption. One such example is the FGM family of densities, introduced to model affiliation in Kosmopoulou and Williams (1998), and applied more recently to double auctions in Kadan (2007) and Gresik (2011). This family is defined as

$$f(x_B, x_S) = 1 + \kappa(1 - 2x_S)(1 - 2x_B) \qquad (\kappa \geq 0),$$

and it can be easily verified that it satisfies the MLRP. Moreover, since the marginals are uniform $[0, 1]$ distributions, the conditional density $f(x_B|x_S) = f(x_B, x_S)$, and it is log-concave in x_B for any fixed $x_S \in [0, 1]$. Since

$$\frac{f_B(x_B|x_S)}{1 - F_B(x_B|x_S)} = -\frac{d \log\big(1 - F_B(x_B|x_S)\big)}{dx_B},$$

this implies that FGM distributions $F(x_B|x_S)$ have hazard rate increasing in x_B.

2.2 Separating equilibrium without disclosure

In this section, we prove our main result: *even when the seller's signal is unobservable to the buyer, there is a unique separating equilibrium in monotone strategies.* That is, in equilibrium the seller fully reveals its signal to the buyer. Moreover, we show that the price will involve a *signalling premium.*

We first show that, in parallel to the disclosure setting, the buyer's best-response responding strategy is still characterized by a cutoff rule.

Lemma 1. *Suppose the seller adopts a (measurable) strategy $S : [0, 1] \to \mathbb{R}_+$. Then the buyer's best-response strategy to a price offer p is characterized by a cutoff $X_B(p)$ such that the buyer accepts the offer if and only if $x_B \geq X_B(p)$.*[11]

[11] As before, we assume that whenever the buyer is indifferent between accepting or not, he will accept.

Proof. The buyer's expected profit upon accepting a price offer p is

$$\Pi_B(x_B, p) = \int_{\{\tilde{x}_S : S(\tilde{x}_S) = p\}} \big(u_B(x_B, \tilde{x}_S) - p\big) dF_S(\tilde{x}_S | x_B), \tag{4}$$

and, because the integrand is increasing in x_B, $\Pi_B(x_B, p)$ is continuous and increasing in x_B under first-order stochastic dominance implied by the MLRP assumption (Assumption 2). So if p is acceptable to a buyer with signal x_B, $\Pi_B(x_B, p) \geq 0$, it is also acceptable for a buyer with a higher signal $x'_B > x_B$, and it follows that the buyer's strategy can be defined as the lowest signal $X_B(p)$ such that the price p is acceptable, $X_B(p) = \inf\{x \in [0,1] : \Pi_B(x, p) \geq 0\}$. □

In this paper, we investigate *fully separating* equilibria (S-equilibria), i.e. those equilibria where (i) there is a unique seller type $X_S(p)$ offering price p, and (ii) there is a unique buyer type $X_B(p)$ such that this type and all types above it find the price offer p acceptable. Moreover, we restrict attention to equilibria in continuous and monotone strategies.

Definition 1 (S-equilibrium)**.** *An S-equilibrium is defined as any perfect-Bayesian equilibrium where the seller's strategy $S(\cdot)$ is increasing, continuous and therefore fully type-revealing.*

This assumption implies that the inverse strategy

$$X_S(p) \equiv \inf\{x \in [0,1] : S(x) \geq p\}$$

is nondecreasing and continuous. We can now define the range of prices that will be offered, $[\underline{p}, \overline{p}]$, where

$$\underline{p} = S(0), \qquad \overline{p} = S(1).$$

A pair of equilibrium strategies is shown in Figure 1. Since $S(\cdot)$ is assumed to be increasing, it is clear that in any Perfect Bayesian equilibrium, the buyer, upon receipt of the offer $p = S(x_S)$, will infer the seller's type x_S and respond accordingly. That is, the offer will be accepted if $u_B(x_B, x_S) \geq p$, and rejected otherwise. So for $p \in [\underline{p}, \overline{p}]$, the equilibrium responding strategy is given by

$$X_B(p) = \inf\{x \in [0,1] : u_B(x, X_S(p)) \geq p\}.$$

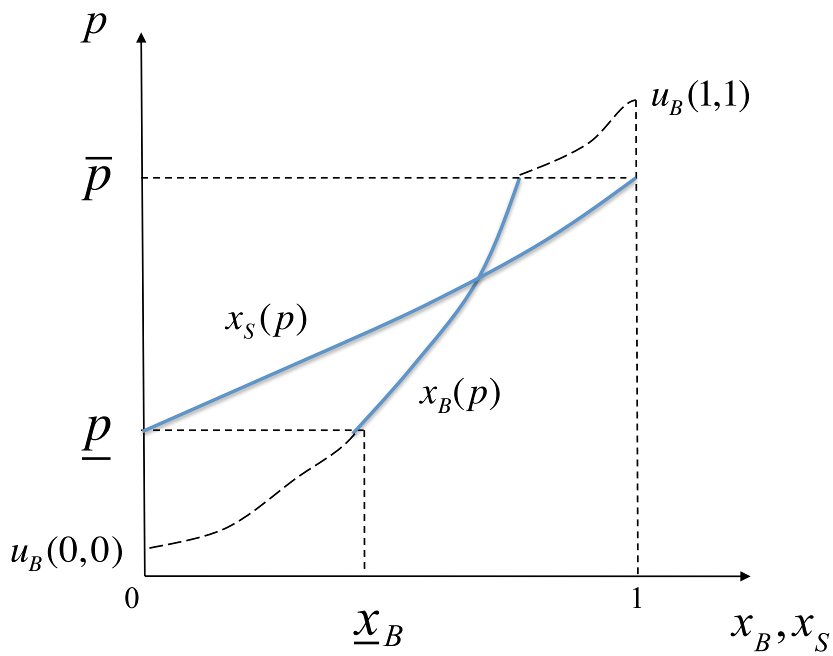

Figure 1: A pair of equilibrium strategies.

In a separating equilibrium, the best-response by the buyer is to accept an offer p if $u_B(x_B, X_S(p)) \geq p$ and reject otherwise, so the best-response strategy $X_B(p)$ for all $p \in [\underline{p}, \overline{p}]$ is given as a (unique) solution to

$$u_B(X_B(p), X_S(p)) = p. \tag{5}$$

In equilibrium, the seller with signal x_S will choose p optimally, maximizing the expected profit function

$$\Pi_S(x_S, p) = \int_{X_B(p)}^{1} [p - u_S(x_S, \tilde{x}_B)] f_B(\tilde{x}_B | x_S) d\tilde{x}_B,$$

resulting in a F.O.C. at all points of differentiability:[12]

$$\frac{\partial \Pi_S(x_S, p)}{\partial p} = -X_B'(p)(p - u_S(x_S, X_B(p))) f_B(X_B(p)|x_S) + 1 - F_B(X_B(p)|x_S) = 0.$$

Using (5) and simplifying, we obtain the differential equation for the *buyer's* inverse strategy

[12] A monotone function is differentiable almost everywhere.

$X_B(p)$:[13]

$$X'_B(p) = \frac{1 - F_B(X_B(p)|X_S(p))}{f_B(X_B(p)|X_S(p))} \frac{1}{u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p))}. \tag{6}$$

It is convenient to totally differentiate (5),

$$\frac{\partial u_B}{\partial x_B} X'_B(p) + \frac{\partial u_B}{\partial x_S} X'_S(p) = 1,$$

and then substitute the slope $X'_S(p)$ from the seller's F.O.C. (6), yielding after some manipulations

$$\begin{aligned} X'_S(p) &= \frac{1}{\frac{\partial u_B}{\partial x_S}} \left(1 - \frac{\partial u_B}{\partial x_B} X'_B(p)\right) \\ &= \frac{1}{\frac{\partial u_B(X_B(p), X_S(p))}{\partial x_S}} \frac{J(X_B(p), X_S(p))}{u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p))} \end{aligned} \tag{7}$$

Equations (6) and (7) were derived for all points of differentiability of $X_B(\cdot)$ and $X_S(\cdot)$. However, as the following lemma shows, both $X_B(\cdot)$ and $X_S(\cdot)$ must in fact be continuously differentiable.

Lemma 2. *The seller's inverse offer strategy $X_S(\cdot)$ and the buyer's responding strategy $X_B(\cdot)$ are continuously differentiable functions.*

Proof. Since $S(\cdot)$ is a strictly increasing, continuous function according to the definition of an S-equilibrium, its inverse $X_S(\cdot)$ is also strictly increasing and continuous. We now show that the definition of the buyer's responding strategy as the function $X_B(p)$ that solves (5) implies that $X_B(\cdot)$ is likewise continuous. We argue by contradiction. If p_0 is a point of discontinuity of $X_B(p)$, then there exist two sequences of prices $\{p_n\}$, $\{p'_n\}$ such that $p_n, p'_n \to p_0$ and $X_B(p_n) \to x_B^1$, $X_B(p'_n) \to x_B^2$ where $x_B^1 \neq x_B^2$. Without loss of generality, assume $x_B^2 > x_B^1$. Since $u_B(\cdot,\cdot)$ is also continuous (indeed, continuously differentiable), passing to the limit in (5) along the sequences p_n and p'_n yields

$$u_B(x_B^1, X_S(p_0)) = p_0 = u_B(x_B^2, X_S(p_0)),$$

but this contradicts our Assumption 3 according to which the buyer's valuation is strictly increasing in x_B. So $X_B(\cdot)$ must be continuous. As a monotone function, $X_S(\cdot)$ is differentiable

[13] In line with usual notation for differential equations, in the r.h.s. we suppress the dependence of x_B and x_S on p.

almost everywhere. Then, equation (5) implies that $X_B(\cdot)$ is also differentiable almost everywhere. Now all the functions appearing on the r.h.s. of (6) and (7), namely $u_B(\cdot,\cdot), u_S(\cdot,\cdot)$ and $F_B(\cdot|\cdot), f_B(\cdot|\cdot)$, are continuous by assumption. Also, $X_S(p)$ is continuous by definition of an S-equilibrium, and we have just shown that $X_B(\cdot)$ is also continuous. It follows that the r.h.s. of (6) and (7) are continuous functions of p, and therefore the derivatives $X_B'(p)$ and $X_S'(p)$ appearing in the l.h.s. can be extended by continuity to the entire interval $[\underline{p}, \overline{p}]$. Therefore in any S-equilibrium, both $X_B(\cdot)$ and $X_S(\cdot)$ are continuously differentiable, not merely continuous. $\square$

So far we have shown that the first-order conditions given by the system of differential equations (6) and (7) are *necessary* for S-equilibrium. In order to complete the characterization of an equilibrium candidate, we need to pin down the initial condition for this system. This is done in the following lemma, which also establishes uniqueness of the equilibrium candidate.

Lemma 3. *For any S-equilibrium, $S(0) = u_B(\underline{x}_B, 0)$, where $\underline{x}_B \equiv x_B^*(0)$ is the point of intersection of the J-curve with the horizontal axis, or $\underline{x}_B = 0$ if $J(x_B, 0) > 0\ \forall x_B \in [0,1]$.*

Proof. Equivalently, we need to show that

$$X_B(\underline{p}) = \underline{x}_B, \quad X_S(\underline{p}) = 0, \tag{8}$$

where $\underline{p} = u_B(\underline{x}_B, 0)$.

Refer to Figure 2. In this figure, the J-curve defined in the previous section separates the feasible signal space $[0,1]^2$ into the upper and lower region. Note that the solution cannot enter the upper region since monotonicity fails there, either $X_S'(p) < 0$ if $u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p)) > 0$, or $X_B'(p) < 0$ if $u_B(X_B(p), X_S(p)) - u_S(X_S(p), X_B(p)) < 0$. So the initial condition for the buyers must be $X_B(\underline{p}) \geq \underline{x}_B$.

We now show that any $X_B(\underline{p}) > \underline{x}_B$ will give the seller the incentive to deviate to a lower price. For an out-of-equilibrium offer $p < \underline{p}$, buyer's beliefs are not pinned down by the Bayes rule. Still, in a Perfect Bayesian equilibrium, the buyer will best-respond given *some* beliefs about the seller's types x_S. So we consider the worst-case scenario when the buyer believes that this offer came from this seller type $x_S = 0$, which is in fact the case. This belief corresponds, for $p < \underline{p}$, to the *highest* buyer's marginal type $X_B(p)$ possible, defined as the unique solution

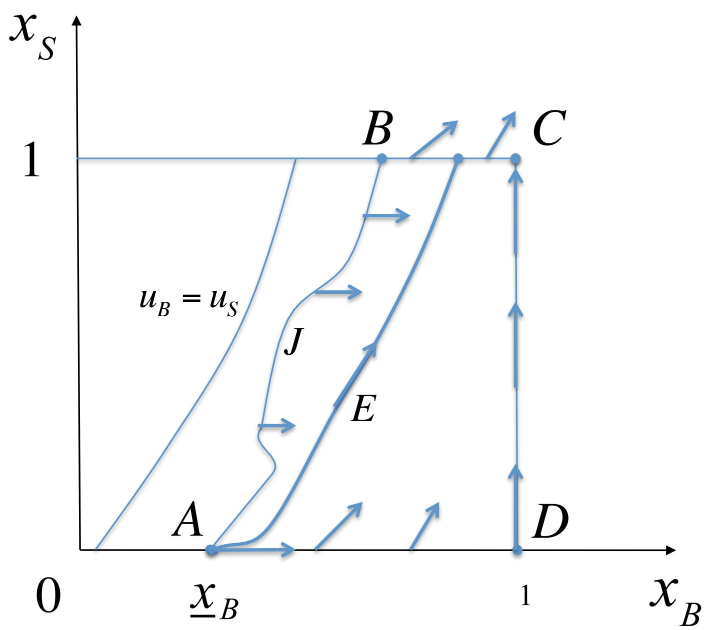

Figure 2: A phase portrait of an equilibrium. The equilibrium curve is marked by E. The monotonicity region corresponds to the part of the square below the $J(x_B, x_S) = 0$ curve, marked as J. The ex-post efficiency region lies below the $u_B(x_B, x_S) = u_S(x_S, x_B)$ curve.

to $u_B(X_B(p), 0) = p$, since with these beliefs, the buyer with any signal x_B assigns the *lowest* possible value to the object, $u_B(x_B, 0)$. So the seller's profit from such a deviation is bounded from below by $\underline{\Pi}_S(X_B(p))$, where

$$\underline{\Pi}_S(x_B) \equiv \int_{x_B}^{1} (u_B(x_B, 0) - u_S(0, \tilde{x}_B)) f_B(\tilde{x}_B|0) d\tilde{x}_B.$$

After some algebra, the slope of this profit function at $x_B = X_B(\underline{p})$ can be shown to be

$$\underline{\Pi}'_S(X_B(\underline{p})) = -J(X_B(\underline{p}), 0) \cdot f_B(X_B(\underline{p})|0).$$

If $X_B(\underline{p}) > \underline{x}_B$, then $J(X_B(\underline{p}), 0) > J(\underline{x}^*_B, 0) \geq 0$ and this slope is *negative*. This means that even under the most unfavourable buyer beliefs, and therefore for any beliefs, the seller's expected profit can be increased by offering a lower price $p < \underline{p}$. This is a contradiction. So the only remaining possibility is $X_B(\underline{p}) = \underline{x}_B$.

$\square$

Let

$$\mathcal{M} \equiv \{(x_B, x_S) \in [0,1]^2 : J(x_B, x_S) \geq 0\}$$

be the domain where the equilibrium candidate is monotone, $X'_S(p), X'_B(p) > 0$. The following lemma shows that the r.h.s. of the differential equations (6) and (7) define a vector field on $\mathcal{M}$.

Lemma 4 (Continuously differentiable vector field)**.** *The mapping*

$$(x_B, x_S) \mapsto \Big(\frac{1 - F_B(x_B)|x_S)}{f_B(x_B|x_S)} \frac{1}{u_B(x_B, x_S) - u_S(x_S, x_B)}, \\ \frac{1}{\frac{\partial u_B(x_B, x_S)}{\partial x_S}} \frac{J(x_B, x_S)}{u_B(x_B, x_S) - u_S(x_S, x_B)} \Big) \tag{9}$$

defines a continuously differentiable vector field on $\mathcal{M}$.

Proof. Assumption 3 implies that u_B,u_S and $\partial u_B / \partial x_B$, $\partial u_B / \partial x_S$ are continuously differentiable. It only remains to verify that the difference $u_B(x_B, x_S) - u_S(x_S, x_B)$ is bounded from below on $\mathcal{M}$ by a positive constant. The mean-value theorem applied to $J(x_B, x_S)$ on $\mathcal{M}$ implies for some $\tilde{x}_B \in [x_B^*(x_S), 1]$:

$$\begin{aligned} J(x_B, x_S) &= J(x_B^*(x_S), x_S) + \frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B}(\tilde{x}_B - x_B^*(x_S)) \\ &\geq 0 + \frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B}(x_B - x_B^*(x_S)). \end{aligned}$$

The definition of J then implies

$$u_B(x_B, x_S) - u_S(x_S, x_B) \geq \frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)} \frac{\partial u_B}{\partial x_B} + \frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B}(x_B - x_B^*(x_S)).$$

Assumption 4 implies that the derivative of J with respect to x_B is bounded from below by a positive constant,

$$\frac{\partial J(\tilde{x}_B, x_S)}{\partial x_B} \geq a > 0.$$

Also, the boundedness of $f_B(x_B, x_S)$ from both above and below on $[0,1]^2$ implies

$$\frac{1 - F_B(x_B|x_S)}{f_B(x_B|x_S)} \geq \frac{\overline{f}_B}{\underline{f}_B}(1 - x_B).$$

while Assumption 3 implies $\partial u_B/\partial x_B$ is bounded from below on $[0,1]^2$ by a positive constant, say $\phi > 0$. Therefore,

$$\begin{aligned} u_B(x_B, x_S) - u_S(x_S, x_B) &\geq \frac{\overline{f}_B}{\underline{f}_B}\phi(1 - x_B) + a(x_B - x_B^*(x_S)) \\ &\geq \min\{\frac{\overline{f}_B}{\underline{f}_B}\phi, a\} \min\{1 - x_B, x_B - x_B^*(x_S)\} \\ &= \min\{\frac{\overline{f}_B}{\underline{f}_B}\phi, a\} \frac{1 + x_B^*}{2} > 0. \end{aligned}$$

So the r.h.s. of (6) and (7) in fact define a continuously differentiable vector field on $\mathcal{M}$. $\square$

Our main result is the following proposition that shows existence and uniqueness of a Perfect Bayesian S-equilibrium.

Proposition 1. *There exists a unique Perfect Bayesian S-equilibrium, characterized by the pair of differential equations* (6) *and* (7)*, with the initial conditions* (8)*. For price offers outside the equilibrium range* $[\underline{p}, \overline{p}]$*, this equilibrium is supported by out-of-equilibrium buyer beliefs as follows:*

- *For* $p < \underline{p}$*, then the buyer believes that the offer originated from the seller with the lowest signal,* $x_S = 0$*;*
- *For* $p > \overline{p}$*, the buyer's beliefs are unrestricted.*

Proof. Step 1: The solution to the system of differential system exists, is unique and monotone, and defines an equilibrium candidate.

By Lemma 4, the right-hand sides of (6) and (7) define a vector field, continuously differentiable on $\mathcal{M}$. Since the initial condition for the system (6) and (7), $(x_B, 0)$, is at the boundary of $\mathcal{M}$, a fundamental result in the theory of differential equations then implies that there is a unique integral curve passing through $(\underline{x}_B, 0)$. This curve is entirely contained in $\mathcal{M}$, and therefore the corresponding $X_B(\cdot)$ and $X_S(\cdot)$ functions are monotone with $X_B'(p), X_S'(p) > 0$. To claim that this integral curve defines a (unique) S-equilibrium candidate, we need to verify that it exits the square through the upper edge $[0,1] \times \{1\}$. Equivalently, this will show that the solution can be extended to the entire interval $[\underline{p}, \overline{p}]$, where $\overline{p} = u_B(\overline{x}_B, 1)$ for some $\overline{x}_B \in [x_B^*(1), 1]$.

The argument is geometric. Refer to Figure 2. Because the vector filed defined by the system is parallel to the horizontal axis on the J-curve, and to the vertical axis on the right side of the square, where $x_B = 1$, the solution curve cannot intersect $\mathcal{M}$ these boundaries. Therefore, the solution will "leave" the $\mathcal{M}$ region only through the segment BC on the upper horizontal side of the square. This intersection point is the required $\overline{x}_B$.

Step 2: Within-equilibrium deviations To show that within-range deviations are not profitable, consider $\hat{p} > p$. We show that the slope of the expected profit function is negative at $\hat{p}$. Denote $x_B = X_B(p), x_S = X_S(p)$ and $\hat{x}_B = X_B(\hat{p}), \hat{x}_S = x_S(\hat{p})$. Then the slope of the seller's expected profit $\Pi_S(x_S, \hat{p})$ is equal to

$$\frac{\partial \Pi_S(x_S, \hat{p})}{\partial p} = f_B(\hat{x}_B|x_S) \cdot \left(-x'_B(\hat{p})(\hat{p} - u_S(x_S, \hat{x}_B)) + \frac{1 - F_B(\hat{x}_B|x_S)}{f_B(\hat{x}_B|x_S)} \right)$$

Now from $u_S(x_S, x_B)$ being strictly increasing in x_S, and the MLRP, which implies

$$\frac{1 - F_B(\hat{x}_B|x_S)}{f_B(\hat{x}_B|x_S)} = \int_{\hat{x}_B}^{1} \frac{f_B(\tilde{x}_B|x_S)}{f_B(\hat{x}_B|x_S)} d\tilde{x}_B \leq \frac{1 - F_B(\hat{x}_B|x_S)}{f_B(\hat{x}_B|\hat{x}_S)},$$

we see that

$$\frac{\partial \Pi_S(x_S, \hat{p})}{\partial p} \leq f_B(\hat{x}_B|x_S) \cdot \left(-x'_B(\hat{p})(\hat{p} - u_S(\hat{x}_S, \hat{x}_B)) + \frac{1 - F_B(\hat{x}_B|\hat{x}_S)}{f_B(\hat{x}_B|\hat{x}_S)} \right)$$

But from our differential equation (6), the r.h.s. of the above inequality is 0. This shows that the expected profit has a non-positive slope for $\hat{p} > p$, and therefore such a deviation is not profitable.

Step 3: Out-of-equilibrium deviations. We now show that outside-range deviations are not profitable. We only need to consider price offers $p \in I \equiv [u_B(0,0), u_B(1,1)]$. This is because $p < u_B(0,0)$ would be acceptable to any buyer type regardless of the beliefs, and is therefore dominated for the seller by the price $p = u_B(0,0)$, while $p > u_B(1,1)$ will never be acceptable to any buyer type.

Our equilibrium does not restrict the buyer's beliefs upon observing a price $p > \overline{p}$. So denote the buyer's perceive type of the seller as $\hat{x}_S$. In general, the buyer's belief may be stochastic. Denote the distribution of $\hat{x}_S$ given price p as $G_B(\hat{x}_S|p)$. Point beliefs are not ruled out, in which case the distribution G_B is degenerate at a point. Given that we only consider a "serious" deviation, not higher than $u_B(1,1)$, there is always an interval of buyer types accepting such an offer. (This interval could be a single point if the offer is made exactly at $u_B(1,1)$.) Given a belief $\hat{x}_S$, we can alternatively reparameterize a deviation to a price $p > \overline{p}$ by the minimal buyer type x for whom such a price is acceptable. The price itself is then $u_B(x, \hat{x}_S)$. Let $\Pi_S(x, x_S, \hat{x}_S)$ denote the seller's profit if the buyer's beliefs were known to the seller

$$\Pi_S^*(x, x_S, \hat{x}_S) \equiv \int_x^1 (u_B(x, \hat{x}_S) - u_S(\tilde{x}_B, x_S)) f_B(\tilde{x}_B|x_S) d\tilde{x}_B,$$

so that the expected seller's profit from a deviation to p is equal to $\int \Pi_S^*(x, x_S, \hat{x}_S) dG_B(\hat{x}_S|p)$. Since $\Pi_S^*(x, x_S, \hat{x}_S)$ is increasing in $\hat{x}_S$, the seller's expected profit is bounded from above by $\Pi_S^*(x, x_S, 1)$.

For any $x_S \in [0,1]$, we have already ruled out within-range deviations, so

$$\Pi(x_S, S(x_S)) \geq \Pi_S(x_S, \underline{p}) = \Pi_S^*(\overline{x}_B, x_S, 1),$$

where the last equality follows by the fact that the offer $\underline{p}$ in equilibrium corresponds to $x_S = 1$: $S(1) = \underline{p}$.

The claim that there is no profitable deviation to $p > \overline{p}$ will follow once we show that the slope of the "most optimistic" profit function $\Pi_S^*(x, x_S, 1)$ with respect to x is nonpositive. Now

$$\frac{\partial \Pi_S(x, x_S, 1)}{\partial x} = -\left(u_B(x,1) - u_S(x_S, x) - \frac{\partial u_B(x,1)}{\partial x_B} \frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \right) \cdot f_B(x|x_S).$$

Since the MLRP implies

$$\frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \leq \frac{1 - F_B(x|1)}{f_B(x|1)},$$

and $u_S(x_S, x) < u_S(1, x)$, we have

$$\begin{aligned} & u_B(x,1) - u_S(x_S, x) - \frac{\partial u_B(x,1)}{\partial x_B} \frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \\ & \geq u_B(x,1) - u_S(1, x) - \frac{\partial u_B(x,1)}{\partial x_B} \frac{1 - F_B(x|1)}{f_B(x|1)} \\ & = J(x,1) \geq 0, \end{aligned}$$

where the last inequality follows because $x > \underline{x}_B$. This implies

$$\frac{\partial \Pi_S^*(x, x_S, 1)}{\partial x} \leq 0,$$

so even for the optimistic beliefs $\hat{x}_S = 1$, a deviation to $p > \overline{p}$ is not profitable to the seller.

For a deviation to a price $p < \underline{p}$, we use the fact the assumed equilibrium belief upon such a deviation is $\hat{x}_S = 0$. We only need to consider the case $\underline{x}_B > 0$: if $\underline{x}_B = 0$, then $\underline{p} = u_B(0,0)$ and the offer will be rejected by the buyer and is thus dominated by the (assumed) equilibrium offer. Also, we have already ruled out within-range deviations, so

$$\Pi(x_S, S(x_S)) \geq \Pi(x_S, \underline{p}) = \Pi_S^*(\underline{x}_B, x_S, 0).$$

A deviation to $p \leq \underline{p}$ is equivalent to picking $x_B = x \in [0, \underline{x}_B]$, with the ensuing expected profit $\Pi_S^*(x, x_S, 0)$. The slope of this profit function is

$$\frac{\partial \Pi_S^*(x, x_S, 0)}{\partial x} = -\left(u_B(x, 0) - u_S(x_S, x) - \frac{\partial u_B(x, 0)}{\partial x_B} \frac{1 - F_B(x|x_S)}{f_B(x|x_S)}\right) \cdot f_B(x|x_S).$$

In this case, the MLRP implies

$$\frac{1 - F_B(x|x_S)}{f_B(x|x_S)} \geq \frac{1 - F_B(x|0)}{f_B(x|0)},$$

while the monotonicity in own signal implies $u_S(x_S, x) \geq u_S(0, x)$, and therefore the slope

$$\frac{\partial \Pi_S^*(x, x_S, 0)}{\partial x} \geq -J(x, 0) \cdot f_B(x|x_S).$$

Since $J(x, 0) \leq J(\underline{x}_B, 0) = 0$, we see that

$$\frac{\partial \Pi_S^*(x, x_S, 0)}{\partial x} \geq 0,$$

so the profit is upward sloping for $x \in [0, \underline{x}_B]$ and a deviation to $p < \underline{p}$ is not profitable.

$\square$

3 Discussion

Some interesting implications of our results are discussed below.

Signalling premium. Recall that for a given x_S, the marginal buyer's type on the J-curve is denoted as $x_B^*(x_S)$, and the price offer by the type x_S seller under disclosure is $S^0(x_S) \equiv u_B(x_B^*(x_S), x_S)$. Inspection of Figure 2 reveals that the solution (or equilibrium) curve, denoted as E in the graph, lies to the right of the J-curve that corresponds to the equilibrium under disclosure. Since the no-disclosure marginal buyer's type is related to the seller's type as $X_B(S(x_S))$, this implies

$$x_B^*(x_S) < X_B(S(x_S)) \quad \forall x_S \in (0, 1].$$

By the strict monotonicity of $u_B(x_B, x_S)$ in x_B, we have

$$u_B(x_B^*(x_S), x_S) < u_B(X_B(S_0(x_S)), x_S) \implies S^0(x_S) < S(x_S).$$

In other words, the no-disclosure equilibrium seller's offer involves a *signalling premium* $S(x_S) - S^0(x_S) > 0$ for $x_S \in (0, 1]$.

No regret. Any S-equilibrium satisfies the no-regret property for both buyers and sellers. That is, no buyer and no seller will obtain a negative surplus. For any price $p \in [\underline{p}, \overline{p}]$, i.e. any price at which trade can occur, $(X_B(p), X_S(p)) \in \mathcal{M}$, and therefore, as we have seen (refer to Figure 2), $u_B(X_B(p), X_S(p)) = p \geq u_S(X_B(p), X_S(p))$. For any seller type $X_S(p)$ who in S-equilibrium trades with buyer types $x \geq X_B(p)$, the surplus is at least $p - u_S(x, X_S(p))$ and is therefore non-negative. Similarly, for any buyer type $X_B(p)$, who in S-equilibrium will trade with sellers having types $x \leq X_S(p)$ and below, the surplus $u_B(X_B(p), x) - p$ is non-negative.

Lemons. Another application of our result is to classical lemons problem, introduced in Akerlof (1970). Can the seller signal quality through prices only, thereby mitigating adverse selection? In a lemons setting, the seller's utility doesn't depend on buyer's signal, only on the seller's signal. At the same time, the buyer's utility depends both on the buyer's signal and the seller's signal. The seller's signal may now be interpreted as a *quality* parameter, with higher quality levels associated with higher cost. There are two interpretations that can be given to the buyer's signal. First, it can be interpreted as a *taste* parameter, in which case it may be (but doesn't have to be) assumed that x_B and x_S are independent. Second, it may be interpreted literally as a signal that provides the buyer additional information about the seller's product, beyond what is known to the seller in terms of quality. Our assumptions allow this signal to be weak; all that is needed is that there is *some* positive dependence of the buyer's utility on x_B. In this setting, our main result implies that there exists a separating equilibrium where seller signals quality to the buyer, so the market for higher quality is not shut down.

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