SEARCH, ADVERSE SELECTION AND MARKET CLEARING

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Abstract. This paper examines a dynamic matching model with adverse selection. The economy is populated by two unit mass of infinitesimal (infinitely-lived) sellers, high type and low type sellers of equal size, and a positive unit mass of infinitesimal (infinitely-lived) buyers. In each period, a seller who knows the quality of the good is randomly matched to a buyer who does not observe the quality, and then randomly draw a price, at which the good is delivered. If either party disagrees, then the two agents return to the pool, waiting for another chance to be matched to another agent. If both parties agree, then the trade occurs and the two agents leave the pool of unmatched agents (but not the economy), generating surplus from trading in each period while the agreement is in place. The long term agreement is dissolved by the decision of either party or by an exogenous shock. Upon dissolution of the long term relationship, both agents return to the respective pool of agents. The objective function of each agent is the expected discounted average payoff. This paper examines stationary equilibria in which trading occurs with a positive probability. It is demonstrated that both unemployment and vacancy persist even in the limit as search friction vanishes. We identify adverse selection as a fundamental source of the coexistence of unemployment and vacancy other than search friction.

Keywords: Matching, Search friction, Adverse selection, Undominated equilibrium, Market clearing

1. Introduction

Persistent coexistence of unemployment and vacancy is a major challenge to general equilibrium theory. Search theoretic models were developed to explain this coexistence as a stationary equilibrium outcome in an economy with non-negligible amount of friction. The source of friction can be the time needed to find a suitable match between a worker and a vacancy (e.g., Mortensen and Pissarides (1994)), or the coordination failure among individual agents in the (directed) search process (e.g., Burdett, Shi, and Wright (2001), Lagos (2000) and Matsui and Shimizu (2005)). This paper demonstrates that adverse section problem can lead to the persistent coexistence of unemployment and vacancy, even in the limit as the search friction vanishes.

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Following Rubinstein and Wolinsky (1985), a large number of dynamic decentralized trading models demonstrated that if search friction is sufficiently small, then the market "almost" clears to approximate the competitive equilibrium outcome (e.g., Gale (1987), Satterthwaite and Shneyerov (2007) and Cho and Matsui (2012)). If the agent has private information about his own type, then the decentralized trading procedure can aggregate the dispersed private information to achieve the competitive market outcome in the limit, as the friction vanishes. We demonstrate that in the presence of adverse selection, however, both unemployment and vacancy may be uniformly bounded away from zero, as search friction vanishes. Unemployment and vacancy can persist, however efficient search technology might become through, say, the use of internet. We demonstrate that the presence of adverse selection is sufficient but also necessary for the persistent equilibrium unemployment and vacancy in an economy where the interaction among agent is very frequent.

We consider an economy which is populated by two unit mass of infinitesimal (infinitely-lived) sellers, high quality and low quality sellers of equal size, and a positive unit mass of infinitesimal (infinitely-lived) buyers. In each period, a seller who knows the quality of the good is randomly matched to a buyer who does not observe the quality. They then randomly draw a price at which the good will be delivered. If either party disagrees, then the two agents return to the pool, waiting for another chance to be matched to another agent. If both parties agree, then the trade occurs and the two agents leave the pool of unmatched agents (but not the economy), generating surplus from trading in each period while the agreement is in place.[1] The long term agreement is dissolved by the decision of either party or by an exogenous shock. Upon dissolution of the long term relationship, both agents return to the respective pool of agents. The objective function of each agent is the expected discounted average payoff. We examine stationary equilibria in which trading occurs with a positive probability. In order to crystallize the impact of the asymmetric information, we examine a sequence of stationary equilibria as the friction, quantified by the time span of each period, vanishes.

We obtain a complete characterization of the equilibrium outcomes in the limit as the friction vanishes. If buyers are on the long side, then equilibrium unemployment and vacancy are *uniformly* bounded away from zero, and low quality sellers grab the entire equilibrium surplus. If buyers are on the short side, the same result holds as long as agents are sufficiently patient relative to the separation rate; otherwise, vacancy disappears in the limit, and the buyers obtain a positive surplus. In particular, if agents are extremely impatient, then the buyers may extract almost all the surplus.

The adverse selection problem is exacerbated by the dynamic trading process from the viewpoint of the uninformed buyers. In a static model of Akerlof (1970), trade can occur only between low quality sellers and buyers. Thus, if buyers are on short side, then they can extract positive surplus from trading. However, the static equilibrium outcome cannot be sustained as an equilibrium in a dynamic model. Instead of trading with a low quality seller, a buyer can wait until most low quality sellers are matched away and trade with

[1]If the true quality is revealed according to Poisson process, then the buyer can decide whether or not to continue the long term relationship, conditioned on the truthfully revealed quality of the good. To simplify notation, however, most of the paper focuses on the case where the quality of the good is not revealed during the long term relationship, as would be true of a market for whole life insurance.

remaining sellers who are likely to have a high quality good and are willing to agree upon any price above their production cost. It turns out that in an equilibrium of the dynamic model, goods are traded at two different price ranges: one below the reservation value of a high quality seller and the other above it. Each price range converges to a single price, as friction vanishes.

Suppose that a buyer and a seller are faced with a price in the low price range, which a buyer knows that only the low quality seller is willing to accept. Note that a low quality seller has an option of waiting for a high price in the future if they cannot reach an agreement today. If a low quality seller is sufficiently patient, she would not accept a price today unless she extracts almost all surplus of trade. As a result, the probability of reaching an agreement becomes so small that the buyers are left out in the pool for an extended amount of time. This force is a decisive factor for an equilibrium outcome, if the sellers are on the short side. A seller should be assertive, since she knows she will meet another buyer almost immediately if she does not reach agreement today. As a result, a positive mass of low quality sellers chooses to stay in the pool for a while. Even if the sellers are on the long side, the low quality sellers remain assertive against buyers rather than reach an agreement immediately as long as they are sufficiently patient. As a result, a positive mass of buyers would stay in the pool for an extended amount of time.

We choose a model with fixed stock of agents instead of a model with a constant inflow of new agents because we are interested in the *rate* of unemployment rather than the *size* of unemployment. We cannot assess the significance of unemployment unless we have a well-defined size of population of workers. One million unemployed people in Singapore would be a national scandal, while the same number of unemployment in China would be a bliss point. In a model with a constant inflow of agents, the rate of unemployment converges to zero over time, if the workers are perpetually employed. If they are employed for finite T period, then the rate of unemployment is determined by the arbitrarily chosen duration time T.

Adverse selection in a search model has recently drawn considerable attention. Guerrieri, Shimer, and Wright (2010) that investigates a static matching model with adverse selection. Chang (2012) embeds Guerrieri, Shimer, and Wright (2010) into Mortensen and Pissarides (1994) to investigate the information revelation in a decentralized financial market under adverse selection. As in most models following the framework of Mortensen and Pissarides (1994), these models are built on matching function, which presume the coexistence of unemployment and vacancy. In our model, we derive, rather than assume, the coexistence.

By letting the friction vanish, we crystallize the impact of asymmetric information on unemployment and vacancy in an equilibrium, in contrast to existing models on the labor market search in which the amount of friction is a free parameter to be specified (e.g., Mortensen and Pissarides (1994)). Existing models to provide the foundation for the matching function (e.g., Burdett, Shi, and Wright (2001)) focus on the coordination failure of the search processes among individual agents. Yet, it is left an open question the coordination problem vanishes as the time span converges to 0. For the same reason, we assume random search as opposed to directed search since coordination failure between firms and workers could be another source of coexistence of unemployment and vacancy.

Matsui and Shimizu (2005) examined the coordination failure among agents, who are searching a particular location to trade out of many markets. But, Matsui and Shimizu (2005) proves that the coordination failure can be extremely resilient by demonstrating that the coordination failure may not vanish, even in the limit as the friction vanishes. As a result, some (sub) market can experience positive unemployment, while some other (sub) market experiences positive vacancy, which could have avoided if the agents can coordinate their search for the trading spots. To exclude the coordination problem arising from the directed search process, we opt for the random search process (e.g., Rubinstein and Wolinsky (1985)), and identify adverse selection as the main source of the failure of market clearing.

This paper differs from the existing papers on the information aggregation under adverse selection in a decentralized dynamic procedure, as we sustain positive *rates* of both unemployment and vacancy in an equilibrium *steady* state. For example, Wolinsky (1990) and Moreno and Wooders (2010) investigated the information aggregation and delay in a model with constant inflow of buyers and sellers, where we cannot calculate the equilibrium *rates* of unemployment and vacancy. Blouin and Serrano (2001) studied the same question, in a model with a fixed mass of agents who leave the market permanently as they reach agreement. Because the population size shrinks as the game continues, the equilibrium steady state does not exist in Blouin and Serrano (2001), which makes it impossible to explain the persistence of the coexistence of unemployment and vacancy.[2]

Section 2 describes the model in which the masses of sellers and buyers are exogenously given. Section 3 presents the preliminary results and concepts. Section 4 formally describes the main results. We first state the prediction of the static model as a benchmark, to which the result of the dynamic model will be compared. After completely analyzing the model, we show that its result is carried over to a model in which a buyer can enter the market after paying a one time fixed cost. Section 5 concludes the paper.

2. Model

2.1. **Static model.** We consider an economy which is populated by 2 unit mass of infinitesimal (infinitely-lived) sellers, high type and low type sellers of equal size, and $x_b > 0$ unit mass of infinitesimal (infinitely-lived) buyers.[3]

High type sellers produce one unit of high quality good at the cost of s_h, while low type sellers produce one unit of low quality good at the cost of s_l. Assume $s_h > s_l$. The goods are indivisible. The marginal utility of the high quality good for a buyer is ϕ_h, while that of the low quality good is ϕ_l, where $\phi_h > \phi_l$. Each seller produces at most one unit of the good, and each buyer consumes at most one unit of the good.

We make the following three standard assumptions on the parameter values, which are critical for capturing the lemons problem.

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[2]Our model shares many common features with Moreno and Wooders (2010). Yet, we prove that the dynamic trading can make the lemons problem "worse" in the sense that the informed low quality seller can extract the entire gain from surplus, even if the buyer is in the short side of the market.

[3]No main result is qualitatively sensitive to the fact that the masses of high and low quality sellers are the same.

*A*1. $\phi_h > s_h > \phi_l > s_l$, which implies that the existence of the gains from trading under each state is common knowledge.

*A*2. $\phi_h - s_h > \phi_l - s_l$ so that it is socially efficient for the high quality sellers to deliver the good to the buyers.

*A*3. $\dfrac{\phi_h + \phi_l}{2} < s_h$ so that the lemons problem is severe in the sense that random transactions lead to a negative payoff either to a buyer or to a high quality seller.

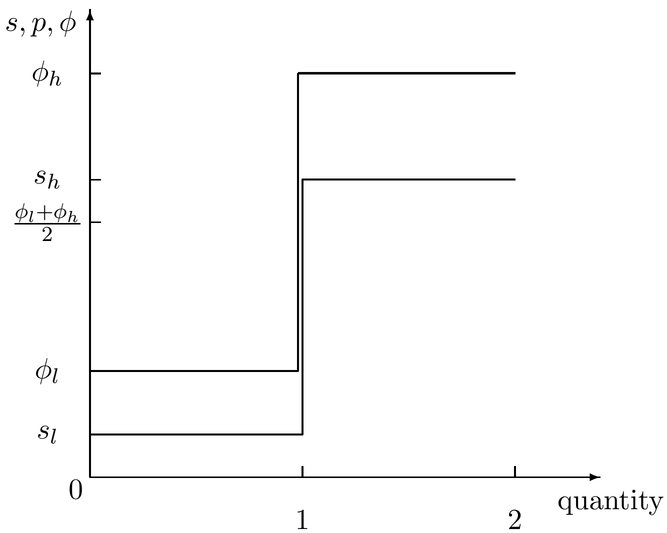

Figure 1. Lemons market

If p is the delivery price of the good, and $y(\in \{h, l\})$ is the quality of the good, seller's profit is $p - s_y$ and buyer's surplus is $\phi_y - p$. Under the assumptions we made, only low quality good is traded in any competitive equilibrium, and the equilibrium price p^* is given by

$$p^* \in \begin{cases} \{s_l\} & \text{if } x_b < 1, \\ [s_l, \phi_l] & \text{if } x_b = 1, \\ \{\phi_l\} & \text{if } x_b > 1. \end{cases}$$

2.2. **Dynamic model.** Let us embed the above static model into a decentralized dynamic trading model. Time is discrete, and the horizon is infinite. When a buyer and a seller is initially matched at period t, conditioned on her type $k \in \{h, l\}$, the seller reports her type as k', possibly in a randomized fashion, to a third party (or mechanism) which draws a price p according to a probability density function $f_{k'}$ over $\mathbb{R}$. We assume that the support of $f_{k'}$ is $[s_l, \phi_h]$.

We assume

$$\forall k' \in \{h, l\},\ \forall p \in [s_l, \phi_h], \quad f_{k'}(p) > 0 \text{ and is continuous.} \tag{2.1}$$

Conditioned on p drawn by the mechanism, each party has to decide whether or not to form a partnership. After forming the partnership, the buyer can purchase the good at the agreed price, and the seller can sell the good at the same price to the buyer. If the good is delivered at p, the seller's surplus is $p - s_k$ and the buyer's surplus is $\phi_k - p$ ($k \in \{h, l\}$).

Then, at the end of the period, either one of two events will occur. The partnership breaks down with probability $1-\delta$, and then, both agents are dumped back to the respective pools. The partnership continues with probability δ without the true quality being revealed.

We assume that the true quality of the good is not revealed to the buyer during the long term relationship, like a life insurance policy, until the partnership dissolves. This assumption is mainly to simplify exposition.[4]

In each period, the buyer and the seller in a partnership can choose to maintain or to terminate it. If one of the agents decides to terminate the partnership, both agents return to the pool, waiting for the next round of matching. If both agents decide to continue the partnership, the partnership continues with probability $\delta = e^{-d\Delta}$ where $d > 0$, and with probability $1 - \delta$, the partnership dissolves, and the two agents are forced to return to the pool.

The objective function of each agent is the long run discounted average expected payoff:

$$(1-\beta)\mathsf{E}\sum_{t=1}^{\infty}\beta^{t-1}u_{i,t}$$

where $u_{i,t}$ is the payoff of agent i in period t and $\beta = e^{-b\Delta}$ is the discount factor.

We focus on undominated stationary equilibrium which is stationary equilibrium where no dominated strategy is used, to exclude "no trading equilibrium" in which every agent refuses to reach agreement. We simply refer to an undominated stationary equilibrium as an equilibrium, whenever the meaning is clear from the context.

To simplify exposition, we assume for the rest of the paper that p is drawn from $[s_l, \phi_h]$ according to the uniform distribution regardless of the report of the seller. The extension to the case where the price is drawn from a general distribution satisfying (2.1) is cumbersome but straightforward (Cho and Matsui (2011)).

3. Preliminaries

Let $W_s^h(p)$, $W_s^l(p)$, and $W_b(p)$ be the continuation game payoffs of a high quality seller, a low quality seller, and a buyer, respectively, after the two agents agree on p. Also, let W_s^h, W_s^l, and W_b be the continuation values of respective agents after they fail to form a long term relationship. Given the equilibrium value functions, let us characterize the optimal decision rule of each agent. In what follows, we write $x \leq O(\Delta)$ if

$$\lim_{\Delta\to 0}\frac{x}{\Delta} < \infty.$$

[4]One can extend the analysis by assuming that the true quality is revealed to the buyer during the long term relationship according to a Poisson distribution with intensity λ_s. Upon the revelation of the true quality, the buyer can decide to continue or terminate the existing long term relationship. As we focus on the case where $\lambda_s = 0$, we essentially examine the worst possible case of the lemons problem. As long as $\lambda_s < \infty$, the lemons problem exists and the main conclusion from the analysis of the baseline model applies. As $\lambda_s \to \infty$, low quality goods are dumped back into the pool more quickly, which alleviates the lemons problem, as the market sorts out lemons from peaches in the limit.

Let z_s^l and z_s^h be the mass of s_l and s_h sellers in the pool. Similarly, let z_b be the mass of buyers in the pool. Since the mass of paired buyers and the mass of paired sellers are of equal size, we have

$$2 - z_s = x_b - z_b, \tag{3.2}$$

where $z_s = z_s^h + z_s^l$. Let

$$\mu_h = \frac{z_s^h}{z_s}$$

be the proportion of high quality sellers in the pool of sellers, and let $\mu_l = 1 - \mu_h$ be the proportion of low quality sellers in the pool.

Our main goal is to find conditions under which

$$\lim_{\Delta \to 0} z_b > 0,$$

and

$$\lim_{\Delta \to 0} z_s > 0$$

hold simultaneously. Throughout the paper, z_s is interpreted as unemployment, while z_b as vacancy.

Because the relative size of buyers and sellers in the pool is an important variable, let us define

$$\rho_{bs} = \frac{z_b}{z_s}.$$

Since ρ_{bs} determines the frequency of meeting the other party with a long side rationed, let us define

$$\zeta = \min\{1, \rho_{bs}\}$$

as the probability that a seller meets a buyer, and

$$\xi = \min\left\{1, \frac{1}{\rho_{bs}}\right\} \tag{3.3}$$

as the probability that a buyer meets a seller. Since the matching is one to one,

$$x_b - z_b = 2 - z_s.$$

Due to (3.2), we have

$$\begin{cases} \zeta = \rho_{bs} < 1 \text{ and } \xi = 1 & \text{if } x_b < 2, \\ \zeta = \rho_{bs} = 1 \text{ and } \xi = 1 & \text{if } x_b = 2, \\ \zeta = \rho_{bs} = 1 \text{ and } \xi > 1 & \text{if } x_b > 2. \end{cases}$$

Let Π_s^h be the set of prices that a high quality seller and a buyer agree to accept, and let $\pi_s^h = \mathsf{P}(\Pi_s^h)$. For $p \in \Pi_s^h$, we can write

$$W_s^h(p) = (1-\beta)(p - s_h) + \beta\left(\delta W_s^h(p) + (1-\delta)W_s^h\right).$$

The first term is the payoff in the present period. At the end of the present period, with probability $1-\delta$, the partnership dissolves, and the high quality seller's continuation payoff is W_s^h. With probability δ, the high quality seller continues the relationship, of which continuation value is given by $W_s^h(p)$.

A simple calculation shows

$$W_s^h(p) = \frac{(1-\beta)(p-s_h)+\beta(1-\delta)W_s^h}{1-\beta\delta}. \tag{3.4}$$

The high quality seller agrees to form a partnership with delivery price p if

$$W_s^h(p) > W_s^h$$

which is equivalent to

$$p > s_h + W_s^h. \tag{3.5}$$

On the other hand, W_s^h is given by

$$W_s^h = \beta\zeta\pi_s^h \mathsf{E}[W_s^h(p)|\Pi_s^h] + \beta(1-\zeta\pi_s^h)W_s^h. \tag{3.6}$$

Substituting (3.4) into (3.6), we obtain, after some calculation,

$$W_s^h = \frac{\beta\zeta\pi_s^h}{1-\beta\delta}\mathsf{E}[p-s_h-W_s^h|\Pi_s^h]. \tag{3.7}$$

Similarly, we obtain

$$W_s^l = \frac{\beta\zeta\pi_s^l}{1-\beta\delta}\mathsf{E}[p-s_l-W_s^l|\Pi_s^l], \tag{3.8}$$

where Π_s^l is the set of prices that a low quality seller and a buyer agree to accept, and $\pi_s^l = \mathsf{P}(\Pi_s^l)$. In any undominated equilibrium, s_l seller accept p if

$$p > s_l + W_s^l.$$

Imitating the behavior of high quality sellers, a low quality seller can always obtain a higher (or equal) continuation value than a high quality seller.[5] Therefore, we have $W_s^l \geq W_s^h$. Now, we would like to claim that the threshold price for a low quality seller is lower than that for a high quality seller.

Lemma 3.1.

$$s_h - s_l > W_s^l - W_s^h.$$

Proof. If a high quality seller imitate a low quality seller, then the long run expected payoff from the deviation is

$$W_s^l - (s_h - s_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l}.$$

Since the deviation payoff is less the equilibrium payoff,

$$W_s^l - W_s^h \leq (s_h - s_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l} < s_h - s_l$$

as desired. □

[5]If the true quality is revealed with a positive probability after the good is delivered, then we cannot invoke the same argument to prove the inequality. Yet, the main result is carried over.

Let Π_s^l and Π_s^h be the set of prices where s_l and s_h sellers trade with a positive probability. Lemma 3.1 says

$$s_l + W_s^l = \inf \Pi_s^l < s_h + W_s^h = \inf \Pi_s^h.$$

In an equilibrium, we can partition the set of prices into three regions, Π_s, Π_p, and the rest:

$$\begin{aligned} \Pi_s &= \Pi_s^l \setminus \Pi_s^h, \\ \Pi_p &= \Pi_s^l \cap \Pi_s^h, \end{aligned}$$

Π_s is the set of the prices at which trade occurs only with low quality sellers (the subscript stands for separating), while Π_p is the set of the prices at which trade occurs with both low and high quality sellers (the subscript stands for pooling). Finally, the remaining region is the one in which no trade occurs. We have yet to show that $\Pi_s \neq \emptyset$ and $\Pi_s \neq \emptyset$. Note that we have

$$\begin{aligned} \Pi_s &\subset [s_l + W_s^l, s_h + W_s^h], \\ \Pi_p &\subset [s_h + W_s^h, \infty). \end{aligned}$$

Let $\pi_s = \mathsf{P}(\Pi_s)$ and $\pi_p = \mathsf{P}(\Pi_p)$. Since we focus on an equilibrium in which trading occurs with a positive probability,

$$\pi_s + \pi_p > 0$$

in an equilibrium.

Definition 3.2. *If $\pi_p = 0$ in an equilibrium, we call such an equilibrium a separating equilibrium. If $\pi_s = 0$, then the equilibrium is called a pooling equilibrium. If $\pi_s > 0$ and $\pi_p > 0$, then it is called a semi-pooling equilibrium.*

Let us calculate the value function of a buyer. In the private value model in which a buyer knows exactly how valuable the objective is (Cho and Matsui (2012)), the informational content of p is irrelevant for a buyer to deciding whether or not to accept p. In contrast, in the common value model like the lemons problem, the expected quality conditioned on p is a critical factor for a buyer to make a decision on p.[6] Let $\phi^e(p)$ be the expected quality if p is the price to be agreed upon. If $p \in (s_l + W_s^l, s_h + W_s^h)$, then only low quality sellers agree to accept the price, and therefore, we have $\phi^e(p) = \phi_l$. On the other hand, if $p > s_h + W_s^h$ holds, then both low and high quality sellers agree to do so, and therefore, we have

$$\phi^e(p) = \phi(\mu_l) \equiv \mu_l \phi_l + \mu_h \phi_h.$$

If a buyer and a seller agree to form a partnership at price p, then the expected continuation value of the buyer is given by

$$W_b(p) = (1-\beta)(\phi^e(p) - p) + \beta \left[\delta W_b(p) + (1-\delta) W_b\right].$$

[6]Even if each individual is infinitesimally small, the informational content of p affects the decision of all buyers. In this sense, each individual is not "informationally small" in the sense of Gul and Postlewaite (1992).

Therefore, we have

$$W_b(p) = \frac{(1-\beta)(\phi^e(p) - p) + \beta(1-\delta)W_b}{1-\beta\delta}.$$

Also, the continuation value after no match is given by

$$W_b = \beta\xi\mu_l\pi_s \mathsf{E}\left[W_b(p)|\Pi_s\right] + \beta\xi\pi_p \mathsf{E}\left[W_b(p)|\Pi_p\right] + \beta(1 - \xi\mu_l\pi_s - \xi\pi_p)W_b.$$

After substitutions and tedious calculation, we obtain

$$W_b = \frac{\beta\xi\mu_l\pi_s}{1-\beta\delta}\mathsf{E}\left[\phi_l - p - W_b|\Pi_s\right] + \frac{\beta\xi\pi_p}{1-\beta\delta}\mathsf{E}\left[\phi(\mu_l) - p - W_b|\Pi_p\right] \tag{3.9}$$

where ξ is the probability that a buyer is matched to a seller, as defined in(3.3).

A buyer is willing to accept p if

$$W_b(p) > W_b,$$

or equivalently,

$$\phi^e(p) - p > W_b.$$

Since $\phi^e(p)$ may change as p changes, the buyer's equilibrium decision rule may not be characterized by a single threshold.

Combining these results and including the endpoints as they are measure zero events, we have

$$\begin{aligned}
\Pi_s &= \begin{cases} [s_l + W_s^l, \phi_l - W_b] & \text{if } s_l + W_s^l \le \phi_l - W_b, \\ \emptyset & \text{otherwise,} \end{cases} \\
\Pi_p &= \begin{cases} [s_h + W_s^h, \phi(\mu_l) - W_b] & \text{if } s_h + W_s^h \le \phi(\mu_l) - W_b, \\ \emptyset & \text{otherwise.} \end{cases}
\end{aligned}$$

The size of population of each type of the agents is determined by the balance equations:

$$1 - z_s^l = \left(\frac{\pi_s\zeta}{1-\delta} + \frac{\pi_p\zeta}{1-\delta}\right) z_s^l \tag{3.10}$$

$$1 - z_s^h = \frac{\pi_p\zeta}{1-\delta} z_s^h \tag{3.11}$$

$$x_b - z_b = \left(\frac{\pi_s\mu_l\xi}{1-\delta} + \frac{\pi_p\xi}{1-\delta}\right) z_b. \tag{3.12}$$

We also rewrite W_s^h, W_s^l, and W_b as

$$W_s^h = \frac{\beta A(\pi_p)^2\zeta}{1-\beta\delta} \tag{3.13}$$

$$W_s^l = \frac{\beta A(\pi_s)^2\zeta}{1-\beta\delta} + \frac{\beta\pi_p\zeta}{1-\beta\delta}\mathsf{E}[p - s_l - W_s^l|\Pi_p] \tag{3.14}$$

$$W_b = \frac{\beta A(\pi_s)^2\mu_l\xi}{1-\beta\delta} + \frac{\beta A(\pi_p)^2\xi}{1-\beta\delta} \tag{3.15}$$

where

$$A = \frac{1}{2}(\phi_h - s_l)$$

under the assumption of uniformity of the density function.

Finally, rewrite π_s and π_p as

$$\pi_s = C[\phi_l - s_l - W_b - W_s^l] \tag{3.16}$$

$$\pi_p = C[\phi(\mu_l) - s_h - W_b - W_s^h] \tag{3.17}$$

where

$$C = \frac{1}{\phi_h - s_l}.$$

An equilibrium in the baseline model is characterized by $(z_b, z_s^h, z_s^l, W_b, W_s^l, W_s^h)$. One of our interests is whether the market clears as the friction vanishes. We say that a market is cleared if

$$z_s = z_b = 0.$$

Since the demand and the supply curves have flat portions, we use a weaker notion of clearing the market.[7]

Definition 3.3. *A market is weakly cleared if*

$$\lim_{\Delta \to 0} z_b z_s = \lim_{\Delta \to 0} z_b(z_s^h + z_s^l) = 0. \tag{3.18}$$

4. Results

4.1. **Static benchmark.** Before stating the main result for the baseline model, let us state, as a benchmark, the prediction from the model populated only with the low quality sellers, and the prediction from the static counterpart.

If the dynamic decentralized market is populated only with the low quality sellers and buyers, then Cho and Matsui (2011) provides complete characterization of stationary equilibria in which trading occurs. If the mass of each part is equal, then the prices at which trading occurs converge to $(\phi_l + s_l)/2$. Otherwise, the prices must converge to the static counter part, depending upon the relative size of the low quality sellers and buyers.

Our exercise is to "add" one unit mass of high quality sellers to the existing low quality sellers, assuming that the buyer does not observe the quality of the good at the time when he decides to form a long term relationship by agreeing on the proposed price. In a static model with lemons problems, satisfying $A1 - A3$, no high quality seller can trade with a positive probability, if one insists that the trading occurs at a single price. Trading will occur between a low quality seller and a buyer and the market clearing price should be $p \in [s_l, \phi_l]$. The multiplicity of the market clearing price is a direct consequence of a feature of the baseline model in which the mass of the low quality seller and the mass of the buyer are equal to 1. Naturally, if the mass of the low quality seller is larger (smaller)

[7]If a single market clearing price exists such that the Walras law holds, then it is the same market clearing condition. We admit that trading can occur at more than a single price. The stated market clearing condition has no reference to a price. Thus, if the market is cleared in the conventional sense, then it is weakly cleared, but not vice versa.

than the buyer's mass, then the trading should occur at s_l (ϕ_l). The equilibrium surplus is

$$\phi_l - s_l$$

which is smaller than the maximum surplus from trading,

$$\phi_h - s_h.$$

Hence, the market outcome is inefficient, but all buyers trade with a seller with probability 1. The market is cleared in the sense that the excess demand is zero, although excess supply of the high quality good remains positive.

By the end of the day, a seller is left in the market if and only if she has a high quality good. "Adding" one unit mass of the high quality sellers to the existing low quality sellers does not change the equilibrium allocation, in the sense that the low quality seller and the high quality seller receives the same payoff as in the case where all sellers have low quality goods. In particular, the buyer can extract a positive surplus from trading with the low quality seller, as long as the buyer is not in the long side of the market.

4.2. **Dynamic Model.** The prediction of the dynamic model is fundamentally different from what the static model predicts. The presence of high quality sellers generates *potentially* additional opportunity for the buyer to trade at a high price, even though the buyer does not observe the true quality of the good. If trading can occur at a higher price, then the low quality seller can benefit by imitating the high quality seller to sell the good at a higher price than the static model predicts. In an equilibrium of the dynamic model, the equilibrium payoff of the low quality seller and the buyer is critically affected by the benefit of the additional trading opportunity for the buyer and the bargaining power of the low quality seller as she has an opportunity to imitate the high quality seller.

Let us state the asymptotic properties of the equilibrium payoffs for the case where $A1 - A3$ hold.

Theorem 4.1. *For any sequence of undominated stationary equilibria,*

$$\begin{aligned} \lim_{\Delta \to 0} W_s^h &= 0 \\ \lim_{\Delta \to 0} W_s^l + W_b &= \phi_l - s_l \end{aligned}$$

Proof. See Appendix A. □

What Theorem 4.1 says is little different from what the static model predicts. In order to understand how the equilibrium surplus $\phi_l - s_l$ is split between a seller and a buyer, we need to investigate the structure of an equilibrium further.

Whether or not a buyer is willing to stay in the pool is affected by the benefit of searching for a lower price, and the cost of having a bad draw repeatedly. If $W_b > 0$ is close to 0, then the cost of searching for a lower price is small. Whether or not a positive mass of buyers stays in the pool is closely related to whether or not a buyer receives a positive payoff in the limit as $\Delta \to 0$. Notice that $\zeta > 0$ if and only if $z_b > 0$.

Lemma 4.2.

$$\lim_{\Delta \to 0} \zeta W_b = 0.$$

Proof. See Appendix B. □

Lemma 4.2 reveals the complementary slackness between z_b and W_b in the limit as $\Delta \to 0$. It would be convenient to analyze the baseline model, conditioned on whether or not $\lim_{\Delta\to 0} z_b > 0$ (or equivalently, $\lim_{\Delta\to 0}\zeta > 0$), which turns out to be equivalent to whether or not

$$\frac{\phi_h - \phi_l}{s_h - s_l + \frac{b}{d}(\phi_l - s_l)} - (2 - x_b) > 0. \tag{4.19}$$

Theorem 4.3. $\lim_{\Delta\to 0} z_b > 0$ *only if* (4.19) *holds.*

By the definition, $z_b \le x_b$ must hold. Thus, one might wonder if

$$\frac{\phi_h - \phi_l}{s_h - s_l + \frac{b}{d}(\phi_l - s_l)} < 2 \tag{4.20}$$

always holds under $A1 - A3$. To verify this inequality, it is sufficient to show that

$$\phi_h - \phi_l \le 2(s_h - s_l).$$

Suppose that

$$\phi_h - \phi_l > 2(s_h - s_l).$$

Then, we have

$$\phi_h - \phi_l > 2(s_h - s_l) > \phi_h + \phi_l - 2s_l$$

where the second inequality follows from $A3$. From the first and the last terms, we conclude that

$$s_l > \phi_l$$

which violates $A1$.

In order to prove the theorem, we need some preliminary results, which reveal the important properties of the equilibrium outcome. From (3.10) and (3.11), we have

$$z_s^h = \frac{1}{1 + \frac{\zeta\pi_p}{1-\delta}}$$

and

$$z_s^l = \frac{1}{1 + \frac{\zeta\pi_p + \zeta\pi_s}{1-\delta}}.$$

Lemma 4.4. *Suppose that* $\lim_{\Delta\to 0} z_b > 0$. *Then*

$$\lim_{\Delta\to 0}\frac{\zeta\pi_p}{1-\delta} = Q_p \equiv \frac{b+d}{d}\left[\frac{\phi_l - s_l}{s_h - \phi_l}\right], \tag{4.21}$$

$$\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\delta} = Q_s \equiv \left[\frac{2s_h - (\phi_h + \phi_l)}{\phi_h - s_h}\right](1 + Q_p). \tag{4.22}$$

Proof. See Appendix C. □

Hence,

$$\lim_{\Delta\to 0} z_s^h = \frac{1}{1+Q_p} = \frac{s_h - \phi_l}{s_h - s_l + \dfrac{b}{d}(\phi_l - s_l)}$$

and

$$\lim_{\Delta\to 0} z_s^l = \frac{1}{1+Q_s+Q_p} = \frac{\phi_h - s_h}{s_h - s_l + \dfrac{b}{d}(\phi_l - s_l)}$$

which are independent of x_b. Thus, if $\lim_{\Delta\to 0} z_b > 0$, then

$$0 < \lim_{\Delta\to 0} z_s = \lim_{\Delta\to 0} z_s^h + z_s^l = \frac{\phi_h - \phi_l}{s_h - s_l + \dfrac{b}{d}(\phi_l - s_l)} < 2 \tag{4.23}$$

where the second inequality follows from (4.20). Note that the right hand side of (4.23) is independent of x_b. From (3.12), we know that z_b is a positive linear function of x_b. An increase of x_b affects z_s in two ways. As x_b increases, z_b increases, which increases the probability ζ of a seller meeting a buyer. One may conclude that if x_b increases, then z_s must decrease, since more sellers are matched away. This observation misses the second way of x_b affecting z_s. As a seller faces a better chance of meeting a buyer, her long run average payoff increases and she bargains more aggressively. As a result, π_s and π_p decreases as a linear function of ζ. In an equilibrium, the two effects of an increase in x_b are perfectly balanced so that z_s remain unaffected.

Recall that

$$z_b = x_b - 2 + z_s.$$

A tedious calculation reveal that

$$\lim_{\Delta\to 0} z_b = \frac{\phi_h - \phi_l}{s_h - s_l + \dfrac{b}{d}(\phi_l - s_l)} - (2 - x_b).$$

Thus,

$$\lim_{\Delta\to 0} z_b > 0$$

only if (4.19) holds, which proves Theorem 4.3.

Since the first term of (4.19) is positive, this condition holds automatically if $x_b \geq 2$.[8] If (4.19) holds, the market does not weakly clear as neither excess demand nor excess supply vanishes in the limit as the friction vanishes.

If $\lim_{\Delta\to 0} \zeta = \lim_{\Delta\to 0} z_b/z_s > 0$, then Lemma 4.4 implies that π_s, π_p vanishes at the rate of $\Delta > 0$. Exploiting Lemma 4.4, we can calculate the rate at which W_b vanishes by substituting π_s, π_p. For any small $\Delta > 0$,

$$W_b = \frac{F}{\zeta^2}\Delta + o(\Delta) \tag{4.24}$$

[8]It is expected, because the buyer is in a long side if $x_b \geq 2$.

where

$$F = A\frac{d^2}{b+d}\left[\frac{\phi_h - s_h}{\phi_h - \phi_l}Q_s^2 + Q_p^2\right],$$

and

$$\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta} = 0.$$

In contrast to the prediction of the static model, where the buyers extract a positive surplus from trading, the low quality sellers extract the entire gain of trading in an equilibrium. The good can be traded either at a high price in Π_p or at a low price in Π_s. The fact that the low quality seller has an option value of waiting to sell at a higher price drawn from Π_s provides a strong bargaining position against a buyer. From (4.19), one can infer that the low quality seller can exercise her bargaining power only if she is sufficiently patient in the sense that discount rate b is small. In fact, a simple comparative static analysis reveals that if $b \to \infty$ so that the agents are extremely impatient, the prediction of the dynamic model converges to what the static analysis shows.

We claim that (4.19) is also a sufficient condition for the results we have obtained so far. Note that (4.19) is violated only when $x_b < 2$, i.e., the buyers are on the short side.

Proposition 4.5. *Suppose that* $\lim_{\Delta\to 0} z_b = 0$. *Then,*

1. $\lim_{\Delta\to 0} z_s = 2 - x_b$.
2. $\frac{\pi_p}{1-\delta} \to \infty$ *and* $\frac{\pi_s}{1-\delta} \to \infty$ *as* $\Delta \to 0$.
3. $\lim_{\Delta\to 0} W_b \geq 0$ *and the equality holds only if* (4.19) *is violated with equality.*
4. (4.19) *is violated.*

Proof. See Appendix D. □

If a buyer fails to reach an agreement, the loss of not reaching agreement is at the order of $1-\delta$. If π_p and π_s vanishes at the rate slower than $1-\delta$, then the chance that the next round's offer is smaller than the present offer by more than $1-\delta$ is large. If the buyer meets a seller very frequently, a lower price will arrive almost immediately, which lets the buyer reach agreement and all the buyers are matched away asymptotically so that all buyers are matched away in the limit as $\Delta \to 0$.

4.3. **Entry.** We have treated $x_b > 0$ as an exogenous parameter. Let us examine the case where a buyer enters the market after paying one time fixed cost $F^* > 0$. Since a buyer will enter the market only if the long run expected average payoff can recover the fixed cost,

$$-(1-\beta)F^* + \beta W_b = 0 \tag{4.25}$$

must hold in any equilibrium. Re-arranging the terms, (4.25) implies that

$$W_b = \frac{(1-\beta)F^*}{\beta}$$

in any equilibrium. We obtain (4.24), and can invoke the same analysis as for the case where (4.19) holds to show

$$\lim_{\Delta\to 0} z_s = \frac{1}{1+Q_p} + \frac{1}{1+Q_p+Q_s} > 0.$$

The limit value of z_b is affected by x_b, which is determined by (4.25):

$$\lim_{\Delta\to 0} z_b = \begin{cases} \left[\frac{1}{1+Q_p} + \frac{1}{1+Q_s+Q_p}\right]\sqrt{\frac{F}{F^*}} & \text{if } F < F^*; \\ \left[\frac{1}{1+Q_p} + \frac{1}{1+Q_s+Q_p}\right]\left[\frac{F}{F^*}\right] & \text{if } F > F^*. \end{cases}$$

5. Concluding Remarks

The excess demand and supply are important objects of investigation in the labor market search models. A typical labor market search model (e.g., Mortensen and Pissarides (1994)) assumes a matching function $m(u, v)$ which specifies the rate at which unemployed workers (u) are matched to the vacant positions (v). Indeed, Blanchard and Diamond (1989) pointed out that the matching function itself presumes the coexistence of a positive excess supply (u) and a positive excess demand (v).

We have demonstrated that if the labor market is subject to adverse selection, then the equilibrium outcome can entail positive excess supply and demand at the same time. We believe it is an important first step to providing a micro foundation for the Beveridge curve, that is, a stable relationship between the unemployment and the job vacancy in the steady state labor market (Blanchard and Diamond (1989)).

Appendix A. Proof of Theorem 4.1

Define $O(\Delta)$ as a function that vanishes at the rate of Δ:

$$\lim_{\Delta\to 0}\frac{O(\Delta)}{\Delta}<\infty.$$

Lemma A.1. $\lim_{\Delta\to 0}(\pi_p)^2\le O(\Delta)$

Proof. The second term of buyer's value function and $W_b<\infty$ imply the statement. □

Lemma A.2. $\lim_{\Delta\to 0}\frac{\zeta\pi_p}{1-\beta\delta}<\infty$.

Proof. Suppose $\lim_{\Delta\to 0}\frac{\zeta\pi_p}{1-\beta\delta}=\infty$. Since $\lim_{\Delta\to 0}W_s^l<\infty$,

$$\frac{\zeta\pi_p}{1-\beta\delta}\mathsf{E}(p-W_s^l-s_l\mid\Pi_p)<\infty.$$

Under the hypothesis of the proof,

$$\lim_{\Delta\to 0}\mathsf{E}(p-W_s^l-s_l\mid\Pi_p)=0.$$

Since $\pi_p>0$ and $\lim_{\Delta\to 0}\pi_p=0$,

$$0<\phi^e-W_b-s_l-W_s^l\to 0.$$

Recall

$$\phi_l<s_h.$$

Thus,

$$\phi_l-W_b<s_h+W_s^h$$

and the gap between the left and the right hand sides does not vanish as $\Delta\to 0$. Since $\pi_s>0$,

$$s_l+W_s^l<\phi_l-W_b<s_h+W_s^h<\phi^e-W_b$$

while

$$\phi^e-W_b-s_l-W_s^l\to 0.$$

This is a contradiction. □

Based upon these two observations, we conclude that the high quality seller's equilibrium payoff vanishes as $\Delta\to 0$, which proves the first part of Theorem 4.1.

Lemma A.3. $\lim_{\Delta\to 0}W_s^h=0$.

Proof. Apply Lemmata A.1 and A.2 to W_s^h. □

Since $\pi_s>0$, an s_l seller and a buyer trades with a positive probability, which imposes an upper bound on $W_s^l+W_b$.

Lemma A.4. $W_s^l+W_b<\phi_l-s_l$.

Proof. A direct implication of $\pi_s>0$. □

The next lemma shows that the low quality seller cannot be completely sorted out in a semi-pooling equilibrium, even in the limit as $\Delta\to 0$. As the pool contains a non-negligible portion of low quality sellers, the buyer needs to sort out the sellers, which is costly for the buyer and for the society as a whole, even if the friction vanishes. On the other hand, the low quality seller has an option to imitate the high quality seller, which provides significant bargaining power to a low quality seller when she is matched to a buyer.

Lemma A.5. $\lim_{\Delta\to 0}\mu_l>0$.

Proof. Suppose $\lim_{\Delta\to 0}\mu_l = 0$. Then $\lim_{\Delta\to 0}\phi(\mu_l) = \phi_h$ holds. Thus, from (3.17), Lemmata A.3 and A.4 together with $W_s^l \geq 0$, we have

$$\lim_{\Delta\to 0}\pi_p = \lim_{\Delta\to 0} C[\phi_h - s_h - W_b - W_s^h] \geq C[(\phi_h - s_h) - (\phi_l - s_l)] > 0,$$

which contradicts with Lemma A.1. □

As in Lemma A.2, we can compute the rate at which $\zeta\pi_s$ vanishes.

Lemma A.6. $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} < \infty$.

Proof. Suppose $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = \infty$. Then from Lemma A.2 and the balance equations of the sellers, $\lim_{\Delta\to 0}\mu_l = 0$ holds, which contradicts to Lemma A.5. □

The next lemma is seller's counter part of Lemma A.1.

Lemma A.7. $\lim_{\Delta\to 0}\pi_s \leq O(\Delta)$.

Proof. This statement is directly implied by Lemma A.5 and (3.15). □

A corollary of Lemma A.7 is that the sum of the long run average payoffs of a buyer and s_l seller converges to $\phi_l - s_l$, which proves the second part of Theorem 4.1.

Lemma A.8. $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - s_l$.

Proof. From Lemma A.7 together with (3.16), we have

$$\lim_{\Delta\to 0}\pi_s = \lim_{\Delta\to 0} C[(\phi_l - s_l) - (W_b + W_s^l)] = 0.$$

□

Appendix B. Proof of Lemma 4.2

From (3.10), (3.11) and (3.12), we know that in order to investigate the asymptotic properties of z_b and z_s, we need to understand the asymptotic properties of $\zeta\pi_p/(1-\delta)$ and $\zeta\pi_s/(1-\delta)$.

Lemma B.1. $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} > 0$

Proof. Suppose that $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = 0$. From the balance equations of the sellers, we have

$$\frac{\mu_l}{1-\mu_l} = \frac{\frac{\pi_p\zeta}{1-\delta}+1}{\frac{\pi_s\zeta}{1-\delta}+\frac{\pi_p\zeta}{1-\delta}+1} \to 1$$

which implies that

$$\mu_l \to \frac{1}{2}.$$

Since the lemons problem is severe (assumption $A3$),

$$\phi(\mu_l) - s_h \to \frac{\phi_h + \phi_l}{2} - s_h < 0.$$

Recall that $W_s^h \to 0$. Since any equilibrium must be semi-pooling, $\pi_p > 0$. For a sufficiently small $\Delta > 0$, however,

$$0 < \phi(\mu_l) - W_b - W_s^h - s_h \leq \phi(\mu_l) - s_h \to \frac{\phi_h + \phi_l}{2} - s_h < 0$$

which is impossible. □

Lemma B.2.

$$0 < \lim_{\Delta\to 0}\frac{\pi_s}{\pi_p} < \infty.$$

Proof. Since we have

$$0 < \lim_{\Delta\to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$

by way of Lemmata A.6 and B.1 , and

$$\lim_{\Delta\to 0} \frac{\pi_p \zeta}{1-\delta} < \infty,$$

by way of Lemma A.2,

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} < \infty.$$

holds. To prove

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} > 0$$

by way of contradiction, suppose that

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} = 0.$$

Since

$$0 < \lim_{\Delta\to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$

$$\lim_{\Delta\to 0} \frac{\pi_p}{\pi_s} = 0$$

implies

$$\lim_{\Delta\to 0} \frac{\pi_p \zeta}{1-\delta} = 0.$$

We claim that $\zeta \to 0$ as $\Delta \to 0$ under the hypothesis of the proof. If

$$\lim_{\Delta\to 0} \zeta > 0,$$

then $\pi_s = O(\Delta)$ and $\pi_p = O(\Delta)$. As a result,

$$\lim_{\Delta\to 0} W_s^l = \lim_{\Delta\to 0} W_b = 0,$$

which is impossible since

$$W_b + W_s^l \to \phi_l - s_l.$$

□

Lemma B.3. $\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\beta\delta} > 0$

Proof. Note

$$\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\beta\delta} = \lim_{\Delta\to 0} \frac{\zeta\pi_s}{1-\beta\delta}\frac{\pi_p}{\pi_s}.$$

The desired conclusion follows from Lemmata B.1 and B.2. □

Lemma B.4. $\lim_{\Delta\to 0} \mathsf{E}[p|\Pi_p] = s_h$.

Proof. Since $\lim_{\Delta\to 0} \pi_p = 0$, $\Pi_p = [s_h + W_s^h, \phi^e(p) - W_b]$ shrinks to a single point. Since $\lim_{\Delta\to 0} W_s^h = 0$, all points in Π_p converge to s_h, from which the conclusion follows. □

Lemma B.5. $\lim_{\Delta\to 0} W_s^l > 0$

Proof. Recall the equilibrium value function of W_s^l, and observe that the second term of the value function is strictly positive, even in the limit as $\Delta \to 0$. □

We are ready to prove Lemma 4.2. Note

$$\frac{W_s^l}{W_b} = \frac{A\zeta\pi_s^2 + \zeta\pi_p \mathsf{E}[p - s_l - W_s^l|\Pi_p]}{A\left(\mu_l\pi_s^2 + \pi_p^2\right)}.$$

Thus,

$$\frac{\mu_l W_s^l}{\zeta W_b} \propto \frac{\mu_l\zeta\pi_s^2 + \mu_l\zeta\pi_p \mathsf{E}(p - s_l - W_s^l|\Pi_p)}{\mu_l\zeta\pi_s^2 + \zeta\pi_p^2} = \frac{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \mu_l \mathsf{E}(p - s_l - W_s^l|\Pi_p)}{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \pi_p}. \tag{B.26}$$

The denominator converges to zero by way of Lemmata A.1, A.7, and B.2, while the numerator converges to a value greater than or equal to $\mu_l(s_h - \phi_l) > 0$ due to Lemma B.4 and $\lim_{\Delta\to 0} W_s^l \leq \phi_l - s_l$. Therefore, since $\lim_{\Delta\to 0}\mu_l W_s^l > 0$, $\zeta W_b \to 0$.

Appendix C. Proof of Lemma 4.4

Suppose $\lim_{\Delta\to 0} z_b > 0$. Then Lemma 4.2 implies $\lim_{\Delta\to 0} W_b = 0$, which in turn implies $\lim_{\Delta\to 0} W_s = \phi_l - s_l$ due to Theorem 4.1. We derive (4.21) from W_s^l by using the fact that the first term converges to zero, and Lemma B.4. As for (4.22), note that $\mu_l = z_s^l/z_s$. Taking the limit of this expression and equating it with $\lim_{\Delta\to 0}\mu_l = \frac{\phi_h - s_h}{\phi_h - \phi_l}$, we derive (4.22).

Appendix D. Proof of Proposition 4.5

Suppose $\lim_{\Delta\to 0} z_b = 0$.

(1) follows from the fact that $2 - z_s = x_b - z_b$.

(2) Note that $\zeta \to 0$ if and only if $z_b \to 0$. Lemma B.1 and Lemma B.3 imply that $\frac{\pi_p}{1-\delta} \to \infty$ and $\frac{\pi_s}{1-\delta} \to \infty$ as $\Delta \to 0$.

(3) To simplify notation, let us write

$$\begin{aligned}
\bar{\mu} &= \lim_{\Delta\to 0}\mu_l = \frac{\phi_h - s_h}{\phi_h - \phi_l} \\
\bar{Q}_s &= \lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\delta} \\
\bar{Q}_p &= \lim_{\Delta\to 0}\frac{\zeta\pi_p}{1-\delta}.
\end{aligned}$$

Under the assumption that $\zeta \to 0$, one can derive from the balance equations that

$$\frac{x_b}{2 - x_b} = \bar{\mu}\bar{Q}_s + \bar{Q}_p$$

and

$$\frac{\bar{\mu}}{1-\bar{\mu}} = \frac{\bar{Q}_p + 1}{\bar{Q}_s + \bar{Q}_p + 1}.$$

From the value function of s_l seller, one can show that

$$\lim_{\Delta\to 0} W_s^l = \frac{\frac{d}{b+d}\bar{Q}_p(s_h - s_l)}{1 + \frac{d}{b+d}\bar{Q}_p}.$$

Since

$$W_s^l + W_b \to \phi_l - s_l,$$

$\lim_{\Delta\to 0} W_b > 0$ if and only if

$$\frac{\frac{d}{b+d}\bar{Q}_p(s_h - s_l)}{1 + \frac{d}{b+d}\bar{Q}_p} < \phi_l - s_l.$$

We know that if $\bar{Q}_p = Q_p$, then

$$\frac{\frac{d}{b+d}\bar{Q}_p(s_h - s_l)}{1 + \frac{d}{b+d}\bar{Q}_p} = \phi_l - s_l.$$

Thus, $\lim_{\Delta\to 0} W_b > 0$ if and only if $\bar{Q}_p < Q_p$. One can show that $\bar{Q}_p$ solves

$$\bar{Q}_p + 1 = \left(1 + \frac{d}{d+b}\bar{Q}_p\right)\left(\frac{\phi_h - \phi_l}{s_h - s_l}\frac{1}{2 - x_b}\right),$$

where we use the balance equations, $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - s_l$, and

$$\bar{\mu}\phi_l + (1-\bar{\mu})\phi_h = s_h + \lim_{\Delta\to 0} W_b.$$

Note that $\bar{Q}_p \leq Q_p$ if and only if (4.19) is violated, and the equality holds only if (4.19) is violated with an equality.

(4) follows from the last part of the proof of (3).

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