An Order-Theoretic Approach to Dynamic Programming: An Exposition*

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Abstract

In this note, we discuss an order-theoretic approach to dynamic programming. In particular, we explain how order-theoretic fixed point theorems can be used to establish the existence of a fixed point of the Bellman operator, as well as why they are not sufficient to characterize the value function. By doing this, we present the logic behind the simple yet useful result recently obtained by Kamihigashi (2013) based on this order-theoretic approach.

Keywords: Dynamic programming, Bellman equation, value function, fixed point.

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1 Introduction

Dynamic programming is one of the most important tools in modern economics, especially dynamic macroeconomics. Recently, Kamihigashi (2013) obtained a simple yet useful result on the existence, uniqueness, and stability of a solution to the Bellman equation, i.e., a fixed point of the Bellman operator. In this note, we present the logic behind this result as well as explain why the order-theoretic approach used in Kamihigashi (2013) is useful but not sufficient to characterize the value function. To present our arguments in precise terms, we start by introducing some definitions and notations.

2 Preliminaries

Let X be a set. Let Γ be a nonempty-valued correspondence from X to X. Let D be the graph of Γ:

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\}. \tag{2.1}$$

Let $u : D \to [-\infty, \infty)$. Let Π and $\Pi(x_0)$ denote the set of *feasible paths* and that of *feasible paths from* x_0, respectively:

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : \forall t \in \mathbb{Z}_+, x_{t+1} \in \Gamma(x_t)\}, \tag{2.2}$$

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\}, \qquad x_0 \in X. \tag{2.3}$$

Let $\beta \in [0, 1)$. Consider the following optimization problem:

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \lim_{n \uparrow \infty} \sum_{t=0}^{n} \beta^t u(x_t, x_{t+1}), \quad x_0 \in X. \tag{2.4}$$

In this note, we assume that the above limit exists in $\overline{\mathbb{R}}$ for any feasible path $\{x_t\} \in \Pi$.[1] The function $v^* : X \to \overline{\mathbb{R}}$ defined in (2.4) is called the *value function*.

The *Bellman operator* B on the space of functions $v : X \to \overline{\mathbb{R}}$ is defined by

$$(Bv)(x) = \sup_{y \in \Gamma(x)} \{u(x, y) + \beta v(y)\}, \qquad x \in X. \tag{2.5}$$

[1] This assumption as well as the assumption that $\beta \in [0, 1)$ are made here to simplify the exposition. In Kamihigashi (2013), it is assumed that $\beta \in [0, \infty)$, and lim in (2.4) is replaced by a general limit that can be either lim inf or lim sup.

It should be noted that Bv may not be well-defined for every $v : X \to \overline{\mathbb{R}}$. In particular, Bv is not well-defined if there exists $(x, y) \in D$ such that

$$u(x, y) = -\infty, \qquad v(y) = \infty. \tag{2.6}$$

In this case, the sum $u(x, y) + \beta v(y)$ is not well-defined; thus Bv is not a well-defined function.

A function $v : X \to \overline{\mathbb{R}}$ satisfying $Bv = v$ is called a fixed point of B. At this point, two fundamental questions arise:

1. Is the value function v^* a fixed point of the Bellman operator B?

2. Is an arbitrary fixed point of the Bellman operator B equal to the value function v^*?

3 Is the Value function v^* a Fixed point of the Bellman operator B?

The short answer to this question is yes with a caveat. More precisely, it follows from Kamihigashi (2008, Theorem 2) that v^* is a fixed point of B if and only if there exists no $(x, y) \in D$ such that (2.6) holds with $v = v^*$. If there exists such $(x, y) \in D$, then v^* is not a fixed point of B in the sense that v^* is not in the domain of B.

One way to avoid this anomaly is by assuming that

$$\forall (x, y) \in D, \qquad u(x, y) > -\infty. \tag{3.1}$$

This assumption is used by Stokey and Lucas (1989, Theorem 4.2). However, there are various economic models violating (3.1).

3.1 An AK Model with Logarithmic Utility

As a simple example violating (3.1), consider the following specification:

$$X = \mathbb{R}_+, \qquad \forall x \in X,\ \Gamma(x) = [0, Ax], \tag{3.2}$$

$$\forall x \in X, \forall y \in \Gamma(x), \qquad u(x, y) = \ln(Ax - y), \tag{3.3}$$

where $A > 0$ is a constant. Then we have $u(x, y) = -\infty$ whenever $y = Ax$.

3.2 Discussion

Instead of (3.1), the following condition can be used to rule out the existence of $(x, y) \in D$ satisfying (2.6) with $v = v^*$:

$$\forall x \in X, \qquad v^*(x) < \infty. \tag{3.4}$$

This can be ensured by imposing a joint restriction on u, β, and Γ.

Alternatively, one can directly conclude from Kamihigashi (2008, Theorem 2) that as indicated above, the value function v^* is a fixed point of the Bellman operator B if and only if Bv^* is a well-defined function from X to $\overline{\mathbb{R}}$. Since this is a minimum requirement for v^* to be a fixed point of B, there is no further issue concerning the first fundamental question.

4 Is a Fixed Point of the Bellman Operator B the Value Function v^*?

The answer to this question is in general no. Let us illustrate this point using a simple counterexample.

4.1 A Simple Counterexample

Suppose that

$$X = \mathbb{Z}_+, \qquad \forall i \in X, \quad \Gamma(i) = \{i+1\},\ u(i, i+1) = 0. \tag{4.1}$$

Figure 1 shows a symbolic diagram describing this example. Note that at each state $i \in X$, there is only one feasible transition with a return of zero. Hence, given any initial state $i \in X$, one can only move to the next state $i + 1$. It is important to emphasize that there is nothing pathological here: indeed, u is continuous and bounded, and Γ is continuous and compact.[2]

Since $u(i, i + 1) = 0$ for all $i \in X$, we can immediately see that

$$\forall i \in X, \qquad v^*(i) = 0. \tag{4.2}$$

In view of (4.1), for any $i \in X$, the Bellman operator B satisfies

$$(Bv)(i) = u(i, i+1) + \beta v(i+1) = \beta v(i+1). \tag{4.3}$$

[2]See Stokey and Lucas (1989, Section 3.3) for definitions related to correspondences.

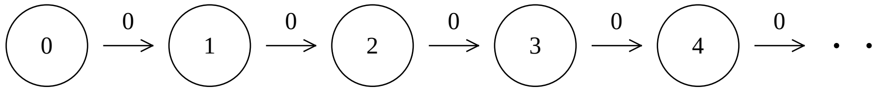

Figure 1: Feasible transitions (arrows) and associated returns (values above arrows) under (4.1)

Hence v is a fixed point of B if and only if

$$\forall i \in X, \qquad v(i) = \beta v(i+1). \tag{4.4}$$

If $v = v^*$, then both sides above are zero; thus v^* is a fixed point of B.

Is there any other fixed point? To answer this, for $\alpha \in \mathbb{R}$ and $i \in X$, define

$$v_\alpha(i) = \alpha\beta^{-i}. \tag{4.5}$$

Then for any $\alpha \in \mathbb{R}$ and $i \in X$, we have

$$\beta v_\alpha(i+1) = \beta\alpha\beta^{-(i+1)} = \alpha\beta^{-i} = v_\alpha(i). \tag{4.6}$$

Thus v_α satisfies (4.4); i.e., v_α is a fixed point of B. Since this is true for any $\alpha \in \mathbb{R}$, it follows that B has a continuum of fixed points. Only when $\alpha = 0$ does v_α coincide with the value function v^*.

4.2 Discussion

The simple counterexample above suggests that in order for a fixed point of the Bellman operator B to equal the value function, the domain of B must be restricted even if u is continuous and bounded and Γ is continuous and compact. Under these conditions, v^* is also continuous and bounded; hence it would be natural to restrict the domain of B to the space of bounded continuous functions. As is well known, the Bellman operator is a contraction on this space, and by the contraction mapping theorem, it has a unique fixed point in this space. Since v^* itself is a fixed point of B by (3.1) or (3.4), the unique fixed point must be the value function (Stokey and Lucas, 1989, Theorem 4.6).

There are various extensions of this result (e.g., Durán 2000; Rincón-Zapatero and Rodríguez-Palmero 2003, 2007, 2009; Martins-da-Rocha and Vailakis 2010; Matkowski and Nowak, 2011). Most of them require a set F of functions on X with the following properties:

1. The Bellman operator B has a fixed point in F.
2. Any fixed point of B in F equals the value function v^*.

In the case of Stokey and Lucas (1989, Theorem 4.6), F is the space of bounded continuous functions. As mentioned above, their approach is based on the contraction mapping theorem, which is a powerful fixed point theorem when it applies. In what follows, we propose an alternative approach based on order-theoretic fixed point theorems.

5 An Order-Theoretic Approach

An inherent property of the Bellman operator B that has not been fully exploited in the economic literature is its monotonicity.[3] To be precise, we define the partial order $\leq$ on the set of functions from X to $\overline{\mathbb{R}}$ as follows:

$$v \leq w \quad \Longleftrightarrow \quad \forall x \in X,\ v(x) \leq w(x). \tag{5.1}$$

It is immediate from (2.5) that B is a monotone operator:

$$v \leq w \quad \Rightarrow \quad Bv \leq Bw, \tag{5.2}$$

provided that both Bv and Bw are well-defined. If $v \leq w$, we define the order interval $[v, w]$ as the set of functions $f : X \to \overline{\mathbb{R}}$ with $v \leq f \leq w$.

There are useful fixed point theorems for monotone maps on partially ordered spaces. Among the best known are the Knaster-Tarski fixed point theorem and Tarski's fixed point theorem (Aliprantis and Border, 2006, pp. 16–17).

5.1 An Application of Tarski's Fixed Point Theorem: An Initial Attempt

The following is an immediate consequence of Tarski's fixed point theorem (or the Kanster-Tarski fixed point theorem).

Lemma 5.1. *Assume (3.1). Suppose that there exist functions $\underline{v}, \overline{v} : X \to \overline{\mathbb{R}}$ with $\underline{v} \leq \overline{v}$ such that B maps $[\underline{v}, \overline{v}]$ into itself.*[4] *Then B has a fixed point in $[\underline{v}, \overline{v}]$.*

[3] See Kamihigashi (2013) for related work by Bertsekas and Shreve (1978).

[4] Since B is monotone, B maps $[\underline{v}, \overline{v}]$ into itself if and only if $\underline{v} \leq B\underline{v}$ and $B\overline{v} \leq \overline{v}$.

Recall that (3.1) ensures that Bv is well-defined for any $v : X \to \overline{\mathbb{R}}$. Under (3.1), if

$$\forall x \in X, \qquad \underline{v}(x) = -\infty, \quad \overline{v}(x) = \infty, \tag{5.3}$$

then B trivially maps $[\underline{v}, \overline{v}]$ into itself. Hence we obtain the following result.

Proposition 5.1. *Under (3.1), B has a fixed point $v : X \to \overline{\mathbb{R}}$.*

Unfortunately, this result is useless. To see this, assume (5.3). Then

$$\forall x \in X, \qquad (B\underline{v})(x) = \sup_{y \in \Gamma(x)} \{u(x,y) + \beta \cdot (-\infty)\} = -\infty. \tag{5.4}$$

Hence $B\underline{v} = \underline{v}$. We similarly obtain $B\overline{v} = \overline{v}$. Since these trivial functions are always fixed points of B, Proposition 5.1 offers no additional information.

5.2 An Application of Tarski's Fixed Point Theorem: A Second Attempt

We can avoid the trivial fixed points discussed above if we require $\underline{v}$ and $\overline{v}$ in Lemma 5.1 to be finite-valued, in which case any fixed point of B in $[\underline{v}, \overline{v}]$ is finite-valued. The following result is once again an immediate consequence of Tarski's fixed point theorem (or the Knaster-Tarski fixed point theorem).

Proposition 5.2. *Suppose that there exist functions $\underline{v}, \overline{v} : X \to \mathbb{R}$ with $\underline{v} \leq \overline{v}$ such that B maps $[\underline{v}, \overline{v}]$ into itself. Then B has a fixed point in $[\underline{v}, \overline{v}]$.*

Unlike Lemma 5.1, this result does not require (3.1) since for any $(x, y) \in D$ and $v : X \to \mathbb{R}$, the sum $u(x, y) + \beta v(y)$ is well-defined, which implies that Bv is well-defined.[5] Since both $\underline{v}$ and $\overline{v}$ are finite-valued, any fixed point of B in $[\underline{v}, \overline{v}]$ is a finite-valued function. Does such a fixed point equal the value function v^*?

The answer is once again no in general. To see this, consider the counterexample in Section 4.1. Recall the definition of v_α in (4.5). Let $\underline{v} = v_{-1}$ and $\overline{v} = v_1$. Then $\underline{v}$ and $\overline{v}$ are finite-valued and, in addition, fixed points of B. Thus B maps $[\underline{v}, \overline{v}]$ into itself, and by Lemma 5.2, B has a fixed point in $[\underline{v}, \overline{v}]$. However, this result provides no additional information since both $\underline{v}$ and $\overline{v}$ are already fixed points of B. In fact, it follows from (4.6) that B has a continuum of fixed points in $[\underline{v}, \overline{v}]$, each of which is a finite-valued function, but only one of them is the value function.

[5]In fact, Bv is well-defined as long as $v : X \to \mathbb{R} \cup \{-\infty\}$.

6 Additional Conditions on $\underline{v}$ and $\overline{v}$

The discussion in the previous section indicates that for an arbitrary fixed point of B in $[\underline{v}, \overline{v}]$ to equal the value function v^*, we need additional conditions on $\underline{v}$ and $\overline{v}$ even when both are finite-valued functions. To identify such conditions, it is useful to note that finite iterations of the Bellman operator are equivalent to finite-horizon approximations of the original infinite-horizon problem. More precisely, $B^n v$ is the value function of the n-period problem with terminal value given by the function v:

Lemma 6.1 (Kamihigashi, 2013, Lemma A.2)**.** *Suppose that there exist functions $\underline{v}, \overline{v} : X \to \mathbb{R} \cup \{-\infty\}$ with $\underline{v} \leq \overline{v}$ such that B maps $[\underline{v}, \overline{v}]$ into itself. Let $v \in [\underline{v}, \overline{v}]$. Then for any $n \in \mathbb{N}$ and $x_0 \in X$, we have*

$$(B^n v)(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \left\{ \sum_{t=0}^{n-1} \beta^t u(x_t, x_{t+1}) + \beta^n v(x_n) \right\}. \tag{6.1}$$

Recalling the definition of the value function v^* in (2.4), one may conjecture that for v to equal v^*, the "extra" term $\beta^n v(x_n)$ should disappear asymptotically. In fact, it is shown in Stokey and Lucas (1989, Theorem 4.3) that under (3.1), a fixed point of B satisfying the following condition is the value function v^*.

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi, \quad \lim_{t \uparrow \infty} \beta^t v(x_t) = 0. \tag{6.2}$$

As a simple application of this condition, consider the continuum of fixed points constructed in Section 4.1. Note from (4.1), (4.4), and (4.5) that for any $\alpha \in \mathbb{R}, t \in \mathbb{N}$, and feasible path $\{x_t\}$, we have

$$\beta^t v_\alpha(x_t) = \beta^{t-1} v_\alpha(x_{t-1}) = v_\alpha(x_0) = \alpha \beta^{-x_0}. \tag{6.3}$$

Therefore, among the continuum of fixed points v_α of B, only the value function $v^* = v_0$ satisfies (6.2), as expected.

Although a fixed point of B satisfying (6.2) is guaranteed to be the value function under (3.1), the value function itself may not satisfy (6.2). For example, in the AK model of Section 3.1, we have $v^*(0) = -\infty$; thus v^* does not satisfy (6.2) for the feasible path identically equal to zero.

To handle such cases, it is useful to decompose the limit condition in (6.2) into two parts:

$$\liminf_{t \uparrow \infty} \beta^t v(x_t) \geq 0, \tag{6.4}$$

$$\limsup_{t\uparrow\infty} \beta^t v(x_t) \leq 0. \tag{6.5}$$

Consider again the AK model of Section 3.1. It can easily be shown that (6.5) with $v = v^*$ is satsified for any feasible path. Hence (6.5) is a property of the value function v^*. In contrast, (6.4) with $v = v^*$ is violated not only for the path identically equal to zero but also for feasible paths converging to zero sufficiently fast. Therefore, the value function v^* satisfies (6.4) only for reasonably "good" paths. It turns out that such paths are provided by the following set:

$$\Pi^0 = \left\{ \{x_t\} \in \Pi : \lim_{n\uparrow\infty} \sum_{t=0}^{n} \beta^t u(x_t, x_{t+1}) > -\infty \right\}. \tag{6.6}$$

The idea to require (6.5) only for the paths in Π^0 is due to Le Van and Morhaim (2002).

The preceding discussion suggests the following conditions:

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi^0, \quad \liminf_{t\uparrow\infty} \beta^t \underline{v}(x_t) \geq 0, \tag{6.7}$$

$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi, \quad \limsup_{t\uparrow\infty} \beta^t \overline{v}(x_t) \leq 0. \tag{6.8}$$

It can be shown that given any fixed point v of B in $[\underline{v}, \overline{v}]$, if v satisfies (6.7), then $v \geq v^*$, and if v satisfies (6.8), then $v \leq v^*$; see Kamihigashi (2013) and Stokey and Lucas (1989, Theorem 4.3). Adding these requirements to Proposition 5.2 and recalling footnote 5, we obtain parts (a) and (b) of the following result.

Theorem 6.1 (Kamihigashi, 2013, Theorem 2.1)**.** *Suppose that there exist functions $\underline{v}, \overline{v} : X \to \mathbb{R} \cup \{-\infty\}$ with $\underline{v} \leq \overline{v}$ such that B maps $[\underline{v}, \overline{v}]$ into itself and such that (6.7) and (6.8) hold. Then the following conclusions hold:*

(a) *The Bellman operator B has a unique fixed point in $[\underline{v}, \overline{v}]$.*

(b) *The unique fixed point is the value function v^*.*

(c) *The increasing sequence $\{B^n \underline{v}\}_{n=1}^{\infty}$ converges to v^* pointwise.*

As in Lemma 5.1 and Proposition 5.2, B has a fixed point in $[\underline{v}, \overline{v}]$ by Tarski's fixed point theorem (or the Kanster-Tarski firxed point theorem). By (6.7) and (6.8), any fixed point of B in $[\underline{v}, \overline{v}]$ is the value function v^*. This in turn implies that B has only one fixed point in $[\underline{v}, \overline{v}]$. Hence parts (a) and (b) follow.

Part (c) is stated here only for the reader's convenience. See Kamihigashi (2013) for discussion of this and other aspects of Theorem 6.1.

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