The value of incumbency in heterogenous networks*

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*Acknowledgments.

Abstract

We study the dynamics of competition in a model with network effects, an incumbent and entry. We propose a new equilibrium concept which takes formally into account the strategic advantages of incumbency in a static model. We then embed this static analysis in a dynamic framework with heterogeneous consumers. We completely identify the conditions under which inefficient equilibria with two networks will emerge at equilibrium; explore the reasons why these inefficient equilibria arise; and compute the profits of the incumbent when there is only one network at equilibrium.

1 Introduction

> "A standard concern is that if a platform becomes dominant, there may be dynamic inefficiencies because users are coordinated and locked-in to a single platform. It may be difficult for an innovative new platform to gain market share, even if its underlying attributes and technology are better. This concern has helped to motivate antitrust actions in industries such as operating systems and payment cards.... Indeed, even in industries such as social networking, where one might expect positive feedback effects to generate agglomeration, it is easy to point to examples of successful entry (Twitter) or rapid decline (MySpace)."
>
> Levin (2013)

Although the importance of competition for the market has been stressed by economists studying the new information technologies, relatively little work has been done to explore the reasons why, in the presence of network externalities, a platform would be able to maintain a dominant position, or the value of this dominant position.

The main aim of this paper is to explore the values of incumbency in a dynamic market with network effects and free entry. We are in effect analyzing the following conversation between a platform and policy makers: "Because of the network externalities in your industry, you yield enormous market power and can extract large profits from your clients"; "You forget that if I try to use this market power, entrants will be able to convince my consumers to join their network"; "This is your standard argument, but according to your reasoning, these entrants will be scared of entry in future periods, and therefore will not be aggressive."

In order to examine the argument, we construct a model in which there is at the outset a single firm who controls the market. There are positive market externalities so that consumers like to be with other consumers. We study dynamic competition, assuming that there are potential entrants in each of an infinite sequence of periods. We derive bounds on the profit of the incumbent and show that it is always smaller than the discounted value of the profit in a one period model. When consumers are heterogenous, we completely characterize the conditions under which there one network or (inefficiently) multiple networks at equilibrium.

Although most of our analysis focuses on the dynamic model, we also contribute to the analysis of incumbency advantage in static environments. Indeed, while economists often argue as if network effects provided a strong advantages to incumbents, the formal models of competition between networks do not naturally lead to this conclusion. Absent switching costs, there is no reason for all the members of an incumbent network to purchase from a new entrant who would offer better conditions. Consider, for instance, the case where there is an incumbent network who offers its service at price 20€ and an entrant who offers its service at a price of 10€. There is a mass 1 of consumers and each of them is willing to pay a network a price equal to 20 times the mass of (other) consumers who have joined the network. It is clearly an equilibrium for the consumers to all purchase from the incumbent, but it is also a (Pareto better from their point of view) equilibrium for all of them to purchase from the entrant. The characterization of network effects as "social switching costs" is due to the fact that our intuition leads us to believe that the most likely equilibrium is for the consumers to purchase from the incumbent.

However, very little work has been done to formalize this intuition, and without formalization it is impossible to study the constraints that bear on incumbents. Such issues as the constraints that potential entry puts on the pricing strategies of incumbents. Much of the work which has been conducted on this issue, tackles the issue by modelling belief formation of consumers. This approach is not feasible in our case, as we would have to model the belief formation about the whole sequence of future periods. It is also not clear how one handle heterogenous consumers.

We approach the problem by explicitly modeling the migration process between the two platforms. We present this model in section 2. As the reader will notice, this leads to a selection criterion that gives lots of power to the incumbent. In fact, one could view our solution concept as a refinement of the set of equilibria that chooses the best equilibrium from the incumbent's point of view. This makes our results that its profit are limited in the dynamic model more striking. Furthermore, we demonstrate that in the subgames where consumers choose which network to join, given a set of prices offered by the networks, our equilibrium always exists and up to network name the equilibrium is unique.

In section 3 we apply our selection equilibrium concept to simple one period games with one incumbent. First, we assume that the outside opportunity of the consumers is to join no network and assuming that there are two types of consumers compute the cases where the incumbent want to offer one or two networks. We then turn our attention to cases where there is entry.

Starting in section 4, we turn to the study of an infinite horizon model an infinite horizon model with free entry. Our focus is to determine how to value the incumbency advantage in the market and whether the market will be efficient. It is efficient for all consumers to be on the same network, due to network effects. Despite the efficiency of a single network and the advantage that an incumbent possesses over its rivals, we identify conditions where the incumbent will lose some consumers to an entrant. The incumbent chooses to give up the consumers because it will be able to extract higher profits from the remaining consumers. In section 7, we derive equilibrium profits when there is only a single network equilibrium. In all equilibria, whether there are two network or a single network, we find a limited value to the incumbent's advantage. In fact, the present discounted value of the incumbent's profits never exceeds the static equilibrium price times the mass of consumers in the market. We discuss the literature in section 8 and then conclude.

2 Modeling incumbency

Although our main interest below will be on dynamic models, in this section we propose a way to model the advantages of incumbency in a one period model.

To see the source of the difficulties which we face, consider the following example. There is a mass $\alpha > 1/2$ of consumers, and the utility that they derive from belonging to a network is equal to the mass of consumers on the network multiplied by $v \in (0, 4/3)$. If network 1 charges $v/2$ and network 2 charges 0, there are three equilibria in the game played by the consumers: one in which they all join network 1; one in which they all join network 2; and a third one in which $(1 + 2\alpha)/4$ consumers join network 1 while the others join network 2.[1] If the consumers are unattached — the market is just opening up — most economists would presume that they would coordinate on the Pareto-optimal equilibrium (from their point of view) and that they all join network 2. However, if told that network 1 is the incumbent, most economists would predict that the consumers would all stay on that network, despite the fact that incumbency, by itself, does not change in any way the formal description of the game that the consumers are playing. In this section, we propose a way to systematically describe the incumbency advantage, in this simple example and in more complicated environments where there can be heterogenous consumers and several incumbents.

We will show that our modeling of incumbency advantage generates an essentially unique equilibrium, in the consumer subgame and argue that this is the best equilibrium from the incumbent's point of view — in that sense our predictions are not very different from those of Caillaud and Jullien (2003), but our concept is easier to use in the dynamic game that we will be considering. Starting in section 4, we will embed this static model in an infinite horizon stationary dynamic model.

[1] Indeed, $(1 + 2\alpha)/4 \times v - v/2 = [\alpha - (1 + 2\alpha)/4] \times v$.

We use the following strategy. We first define "unattached consumers" (UC) equilibria, in which there are no incumbency advantage (because consumers move easily from network to network) — these are the standard equilibria Nash equilibria of the games played by the consumers. We then define "attached consumers" (AC) equilibria, which are the outcome of a migration process between the incumbent(s) and the entrants. We show that AC equilibria are UC equilibria satisfying additional constraints.

In order to avoid introducing useless complications, we assume that there are two types of consumers, but it should be clear that the results extend to models with any number of types. We also assume that the only choice that the consumers face is which network to join. There is no difficulty extending the definitions where one of the choices is to join no network (and this is indeed what we do in section 3).

There is a mass α_h of High Network Effects HNE consumers and a mass α_ℓ of Low Network Effects (LNE) consumers.[2] We will refer to h or ℓ as the types of the consumers. A consumer of type θ derives utility $u_\theta(\gamma_{i\theta}, \gamma_{i\theta'})$ from belonging to network i when $\gamma_{i\theta}$ consumers of the same type and $\gamma_{i\theta'}$ consumers of the other type also do. The functions u_θ are strictly increasing in both arguments.

Even though consumers like to have more consumers of both types on the network to which they belong, they prefer consumers of their own type:[3]

$$\partial u_{i\theta}(\gamma_{i\theta}, \gamma_{i\theta})/\partial \gamma_{i\theta} > \partial u_{i\theta}(\gamma_{i\theta}, \gamma_{i\theta'})/\partial \gamma_{i\theta'} \geq 0. \tag{1}$$

(Here, as in the rest of the paper, θ' will always be taken to be "the other type", different from θ.) We will need no other hypothesis on the utility functions, in particular no concavity or convexity assumption.

Assume that there are m networks, indexed by i. An allocation γ of consumers is a $2 \times m$ vector of nonnegative numbers $\{\gamma_{ih}, \gamma_{i\ell}\}_{i=1,\dots,m}$ with $\sum_i \gamma_{i\theta} = \alpha_\theta$ and $\sum_i \gamma_{i\ell} = \alpha_\ell$ where γ_{ih} and $\gamma_{i\ell}$ are the number of HNE and LNE consumers on network i. Let p_i be the price charged by network i. An allocation γ is an *unattached consumers* (UC) equilibrium (that is an equilibrium in which incumbency plays no role) if and only if

$$\gamma_{i\theta} > 0 \Longrightarrow u_\theta(\gamma_{i\theta}, \gamma_{i\theta'}) - p_i = \max_j u_\theta(\gamma_{j\theta}, \gamma_{j\theta'}) - p_j.$$

The definition of UC equilibrium treats all networks in the same way and is the standard definition of equilibrium in networks: there is no incumbency

[2]For the purpose of this section, the fact that some consumers derive more utility than the others from the presence of other consumers play no role.

[3] We believe that nearly all our results (lemma 2 is an exception) hold true without this assumption. This will be clarified in a future version of this paper.

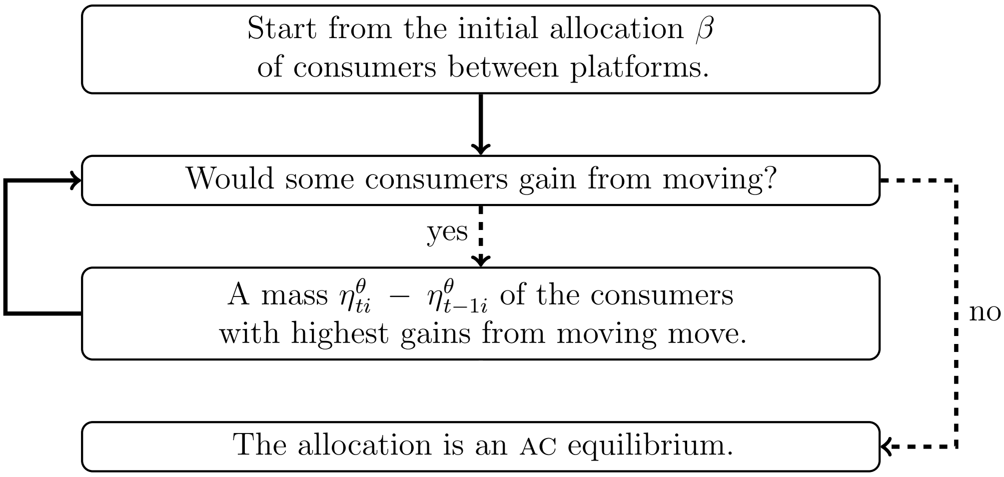

Figure 1: This figure represents in algorithmic form the definition of AC equilibria.

advantage. We now turn to a definition of equilibrium which depends on the initial allocation of consumers, and which we will show in lemma 1 that it selects a UC equilibrium. This definition is illustrated on figure 1.

Let β be an allocation of consumers among the networks — think of β as the initial allocation of consumers. The allocation γ is on a *migration path* from β if there exists an integer $T \geq 0$ and sequences $\{\eta^t_{i\theta} \geq 0\}_{t=0,1\ldots,T}$ of allocations of consumers to networks ($\sum_i \eta^t_{i\theta} = \alpha_\theta$ for all $t = 1, 2, \ldots$ and all θ) which lead from β to γ:

$$\eta^0_{i\theta} = \beta_{i\theta} \text{ and } \eta^T_{i\theta} = \gamma_{i\theta}, \text{ for all } i \text{ and } \mu,$$

and which satisfy the following property: for each $t = 1, 2, \ldots, T-1$ there exists a *type transferred* $\theta(t)$, a *source network* $s(t)$, and a *destination network* $d(t)$ such that

$$\eta^t_{d(t)\theta(t)} - \eta^{t-1}_{d(t)\theta(t)} = \eta^{t-1}_{s(t),\theta(t)} - \eta^t_{s(t)\theta(t)} > 0,$$

$$\eta^t_{i\theta} = \eta^{t-1}_{i\theta} \text{ if } \{i, \theta\} \neq \{d(t), \theta(t)\} \text{ and } \{i, \theta\} \neq \{s(t), \theta(t)\},$$

and

$$\begin{aligned}\Big[u_{\theta(t)}\left(\eta^{t-1}_{d(t)\theta(t)}, \eta^{t-1}_{d(t)\theta'(t)}\right) - p_{d(t)}\Big] - \Big[u_{\theta(t)}\left(\eta^{t-1}_{s(t)\theta(t)}, \eta^{t-1}_{s(t)\theta'(t)}\right) - p_{s(t)}\Big] \\ = \max_{\widetilde{\theta},j,j'}\left\{\Big[u_{\widetilde{\theta}}\left(\eta^{t-1}_{j\widetilde{\theta}}, \eta^{t-1}_{j,\widetilde{\theta}'}\right) - p_j\Big] - \Big[u_{\theta'}\left(\eta^{t-1}_{j'\widetilde{\theta}}, \eta^{t-1}_{j',\widetilde{\theta}'}\right) - p_{j'}\Big]\right\} > 0.\end{aligned} \tag{2}$$

It may be worthwhile restating what we are doing. At each step all the consumers are buying from one network or the other. We check whether some consumer would find it optimal “on its own” to move from one network to an other.[4] If this is the case, we move a strictly positive mass of the consumers who have the greatest incentives to move.

Equation (2) expresses the fact that it is the consumer with the highest gain in utility who changes network, if this *strictly* increases his utility.[5]

The allocation γ is a *final allocation* if there is no migration path leading from γ to another allocation.

Definition 1. *An allocation γ is an* UC *equilibrium if it is on a migration path from the original allocation and it is a final allocation.*

If an initial allocation is a final allocation, then there can be no other allocation that can be reached by a migration path of length 1. It is straightforward to see that this implies that the allocation is a AC equilibrium and therefore proves the following lemma.

Lemma 1. *All* AC *equilibria are also* UC *equilibria. An initial allocation is an* AC *equilibrium if and only if it is a* UC *equilibrium. Furthermore, if an initial allocation is a* UC *allocation, it is the only* AC *equilibrium.*

We will call a migration path a *migration path through large steps* if at every step all the consumers of type $\theta(t)$ in network $s(t)$ migrate to network $d(t)$: $\eta^t_{s(t)\theta(t)} = 0$ for all t. The following lemma makes easier both the identification and proof of existence of AC equilibria.

Lemma 2. *The set of* AC *equilibria is not changed if we impose the restriction that the migration path is a migration path through large steps.*

[4] An alternative assumption would have any group of consumers with a strictly positive gain from moving move, *i.e,* (2) would be rewritten under the simpler form

$$\left[u_{\theta(t)}\left(\eta^{t-1}_{d(t)\theta(t)}, \eta^{t-1}_{d(t)\theta'(t)}\right) - p_{d(t)}\right] - \left[u_{\theta(t)}\left(\eta^{t-1}_{s(t)\theta(t)}, \eta^{t-1}_{s(t)\theta'(t)}\right) - p_{s(t)}\right] > 0.$$

This is not sufficient to prove our results. Indeed, we have built an example using this relaxed assumption where in the initial allocation the HNE consumers are on one network and the LNE consumers on another. The migration leads to an UC equilibrium where all the HNE consumers and 11/16 of the LNE consumers migrate to one network and the rest of the LNE consumers to another one. Lemma 3 does not hold.

[5] This assumption considerably simplifies the reasoning below. In a AC equilibrium, consumers will stay on the incumbent network when they are indifferent between doing so and joining an entrant. We could do without the assumption, but this would require that when studying competition between networks, we use the type of limit pricing arguments standard in, for instance, the study of Bertrand competition with different marginal costs. In our framework, this would make the proofs much more complicated. We relax this condition for equilibria in subsection 7.2.

Proof. Assume that a migration path which leads to AC equilibrium is not a migration path through large steps. Then, there exists a t such that $\eta^t_{\theta(t)s(t)} > 0$. It is easy to see that $\{\theta(t+1), d(t+1), s(t+1)\} = \{\theta(t), d(t), s(t)\}$: at step $t+1$, the migration will involve the same type of consumers moving from the same network to the same network as at step $t+1$. Indeed, in all networks the utility of agents of type $\theta(t)$ is the same at the end and at the beginning of step t, except it is strictly higher in $d(t)$ and strictly smaller in $s(t)$. For agents of the other type, the same property holds true; however by (1), the increase in the utility they derive from $d(t)$ and the the decrease in the utility they derive from $s(t)$ are smaller than for agents of type $\theta(t)$. Hence, condition (2) holds true for $\{\theta(t), d(t), s(t)\}$ when the superscript $t-1$ is replaced by t.

We can therefore construct a new migration path, which will lead to the same final allocation by replacing steps t and $t+1$ by one "larger" step with the same θ, d and s. Iterating on this procedure will lead to a migration path through large steps which leads to the same allocation as the original path. □

From the proof of Lemma 2 it is clear that the following corollary holds:

Corollary 1. *If at any stage t of a large step migration path all consumers of a given type belong to the same network ($\eta^t_{i\theta} = \alpha_\theta$ for some i, θ and t), then they will remain on the same network on all the rest of the migration path.*

This implies the following lemma, which we will use extensively in the sequel.

Lemma 3. *If $\beta_{i\theta} = \alpha_\theta$ for some i and some θ, then in any* AC *equilibrium all consumers of type θ will belong to the same network.*

The following lemma, whose proof can be found in the appendix on page 34, will not be useful in the proofs that follow, but shows that we can also interpret our migration paths as a sequence of "individual moves".

Lemma 4. *The set of* AC *equilibria is not changed if we add the restriction that $\mu(t)$ must be smaller than some $\epsilon > 0$ for all t.*

Lemma 4 is proved by "cutting" each step of a large step algorithm into smaller steps with the same source and destination networks and the same migrating type. It shows that we can think of migration paths as approximating a process in which the consumers move "one by one" from one network to the other; in each stage it is the consumer with the greatest gain from moving who moves.

It is easy to show, and we prove formally on page 34 in the appendix, that large step migrations must eventually stop at a AC equilibrium, which proves the following lemma.

Lemma 5. *Whatever the initial allocation* $\{\beta_{iH}, \beta_{iL}\}_{i=1,\dots,m}$ *and prices* p_i *charged by the networks, there exists a* AC *equilibrium.*

3 One Period Games

In the competition models which we are studying below, we will consider scenarios in which at the start there is a an incumbent network from which all consumers purchased in the past. By lemma 3, this implies that in any period consumers of the same type will always joint the same network, and the following shorthand notation will prove very useful:

$$\begin{aligned} u_h &= u_h(\alpha_h, 0), & u_\ell &= u_\ell(0, \alpha_\ell) \\ v_h &= u_h(0, \alpha_\ell), & v_\ell &= u_\ell(\alpha_h, 0) \\ w_h &= u_h(\alpha_h, \alpha_\ell), & w_\ell &= u_\ell(\alpha_h, \alpha_\ell). \end{aligned}$$

(Note that we use the same symbol u_θ to denote both the utility function of consumers of type θ and their utility if there are all on the same network in which there are no consumers of the other type — this will create no confusion).

We assume

$$w_i > u_i > v_i \text{ for } i \in \{h, \ell\}. \tag{3}$$

We are representing the fact that the HNE consumers value network effects more than LNE consumers by the following conditions:

$$w_h > w_l \text{ and } u_h > u_\ell. \tag{4}$$

Finally, unless we explicitly state the opposite, we assume

$$w_\ell < u_h - v_h. \tag{SmallCE}$$

The right hand side is the difference between what HNE consumers are willing to pay to be on the same network with only the other HNE consumers and what they are willing to pay to be on the same network as with only the LNE consumers. The left hand side is what LNE consumers are willing to pay to be on the same network as all other consumers. Written $v_h < u_h - w_\ell$, it puts an upper bound on v_h, hence the name "Small Cross Effects". When it does not hold there is only one network in the equilibrium in a model of any length of time (see section 5.2).

In the rest of this section, we study the consequences of the equilibrium selection criterion of section 2 in static models.

First, as a benchmark, suppose that there is no entry: the incumbent announces a price and the consumers decide whether or not to stay in the network (in the latter case, their utility is 0). If the incumbent charges w_ℓ, all the consumers stay on the network and its profit is $(\alpha_h + \alpha_\ell)w_\ell$. If the incumbent charges u_h, which is greater than w_ℓ by (SmallCE), its only clients are the HNE consumers and its profit is $\alpha_h u_h$. It is straightforward that all other prices are dominated by one of these two, and we have therefore proved the following lemma.

Lemma 6. *In a static model with no entrants, the incumbent will sell access only to* HNE *consumers if $\alpha_h u_h > (\alpha_h+\alpha_\ell)w_\ell$, and to all consumers otherwise. Its profit will be $\alpha_h u_h$ in the first case and $(\alpha_h + \alpha_\ell)w_\ell$ in the second.*

Let us now add free entry. Although it is easy to see that Nash timing would give exactly the same results, for simplicity, we assume Stackelberg timing where the incumbent first chooses its price p_I followed by the entrants; then the consumers decide which networks to join. Entrants will never charge less than 0, and competition among them implies that any entrant who attract consumers will do so at a price of 0. The incumbent either charge w_l and keep all the consumers or $u_h - v_h$ and keep only the HNE consumers. This implies the following lemma.

Lemma 7. *In the one period model with free entry, the incumbent sells only to the* HNE *customers, at a price $u_h - v_h$, if and only if*

$$u_h - v_h > \frac{\alpha_h + \alpha_\ell}{\alpha_h} w_\ell. \tag{5}$$

In this case its profit is $\alpha_h(u_h - v_h)$. Otherwise, it sells to all consumers at a price of w_ℓ and its profit is $(\alpha_h + \alpha_\ell)w_\ell$.

Lemmas 6 and 7 together show that free entry makes the separation of LNE and HNE consumers less likely. Without entry, the incumbent can generate more surplus from the HNE consumers, since their outside option is 0 rather than v_h. Therefore, more competition, under the form of entry, leads to inefficiencies, since it is efficient for all consumers to be on the same network.

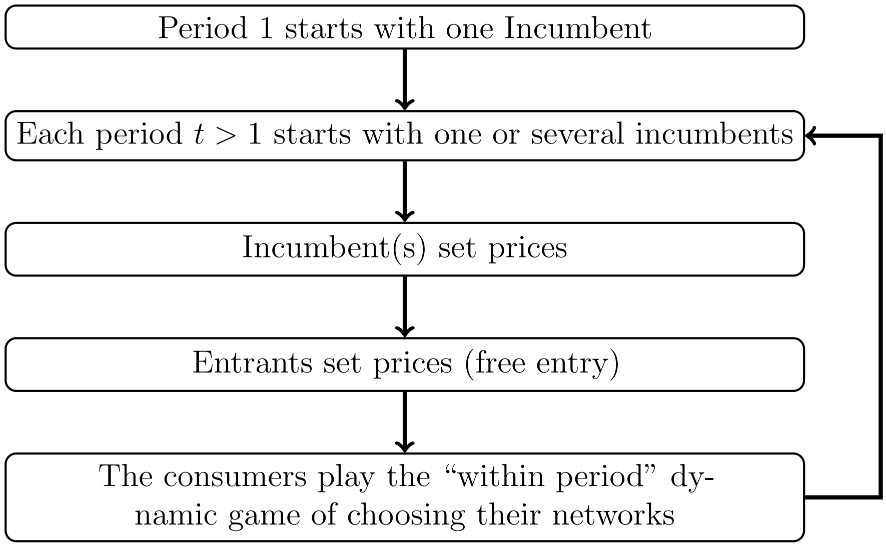

Figure 2: The dynamic model.

4 The dynamic model

4.1 Stationary equilibria with entry

The dynamic model which we will use is represented in Figure 2. At the beginning of period 1 there is one incumbent, which we will denote the "Incumbent". In each subsequent period, there will be one or more incumbents, the firms that sold to a strictly positive measure of consumers in the previous period. There will also be $n_E \geq 2$ entrants.[6] For simplicity, we assume Stackelberg timing where all the incumbents set prices simultaneously and then the entrants, having seen these prices, choose their own prices.[7] Consumers then choose their networks, and then the game moves to period $t + 1$. For simplicity, we assume that firms with no consumer at the end of a period "drop out"of the game.[8]

[6]Assuming the presence of at least two entrants avoid inessential technical difficulties. See Biglaiser, Crémer, and Dobos (2013) for an extended discussion of this point in the context of a switching cost model.

[7]We can also get the same basic results using a Nash timing, where firms simultaneously set prices. In that case, there will only be mixed strategy equilibria, but the equilibrium profits will be the same as with Stackelberg timing; see Biglaiser et al. (2013) for similar issues in a model of switching costs.

[8]Formally, in any period $\tau > t$, their strategy set is a singleton, and the consumers never choose to join these firms whatever the prices charged by the firms which are active in period τ.

Note that there are two dynamics in the game which we are describing: the "large scale" dynamics from period to period and the "small scale" or "within period" dynamics, when consumers choose which network to join according to the process described in section 2. We assume, as in the one period model, there is no discounting within a period, but we assume a common discount factor $\delta < 1$, between periods. We interpret this as players move very quickly and then have a full period to consume the network benefits.

4.2 Within period equilibrium and the myopia principle

In each period, the equilibrium of the game played by the consumers will satisfy the AC equilibrium selection introduced section 3, and we turn to an extension of its definition to the dynamic case. In each period t, there is a set of incumbents $\{1, 2, \ldots, n_I^t\}$ (in equilibrium, n_I^t will actually be equal to either 0 or 1), and a set of entrants $\{1, 2, \ldots, n_E\}$. Incumbent i has some consumers from the previous period, β_{iH}^t and β_{iL}^t with $\beta_{iH}^t + \beta_{iL}^t > 0$.

The purchasing decisions of the consumers depend on the $\beta_{j\theta}^t$s, on the prices charged by the firms, and on their expectations of the decisions of other consumers. As in section 2, the game between consumers will in general have several equilibria. We solve this indeterminacy by extending the notion of AC equilibrium to this dynamic framework. Let us call $W_{i\theta t}\left(\beta^{t+1}\right)$ the expected discounted utility of a consumer of type θ, measured before incumbents have chosen their prices, if he has purchased from incumbent network i in period t. It is important to stress that in any equilibrium, a type θ consumer must receive the same continuation utility at the start of period $t+1$ as any other type θ consumer no matter which network they belong to in period t. This is because in any equilibrium, if a consumer with a lower utility in a period would move to a network that provided them with a higher utility. This move will always happen due to two features of our model First, since all consumers are "small", they do not change the state of the market. Second, the consumers have no switching costs. If the equilibrium allocation of consumers in period t has γ_{jh}^t HNE consumers and $\gamma_{j\ell}^t$ LNE consumers in network j, his utility if he purchases from network i which charges p_i^t will be

$$u_\theta(\gamma_{ih}, \gamma_{i\ell}) - p_i^t + \delta W_{\theta,t+1}\left(\{\gamma_{jh}, \gamma_{j\ell}\}_{j \in \mathcal{N}(t+1)}\right),$$

where $\mathcal{N}(t+1)$ is the set of incumbents at stage t.

We can apply the same reasoning as in section 2 to define migration paths within period t. At each step τ, the consumers who change networks are those consumers of type $\theta(\tau)$ such that there exists source and destination networks, $s(\tau)$ and $d(\tau)$, which are solution of

$$\max_{\theta',i,i'} \left\{ \left[u_{\theta'}(\eta^{\tau-1}_{i\theta'}, \eta^{\tau-1}_{i,-\theta'}) + W_{\theta' t}\left(\left\{ \eta^{\tau-1}_{j\theta'}, \eta^{\tau-1}_{j,-\theta'} \right\}_{j\in\mathcal{N}(t)} \right) - p^t_i \right] \right.$$
$$\left. - \left[u_{\theta'}(\eta^{\tau-1}_{i'\theta'}, \eta^{\tau-1}_{i',-\theta'}) + W_{\theta' t}\left(\left\{ \eta^{\tau-1}_{j'\theta'}, \eta^{\tau-1}_{j',-\theta'} \right\}_{j\in\mathcal{N}(t)} \right) - p^t_{i'} \right] \right\}, \quad (6)$$

as long as the value of this solution is strictly positive. The same W term appears in both terms of this expression and therefore solving (6) is equivalent to (2). We obtain the following "myopia principle" which will play a very important role in the sequel.

Lemma 8 (Myopia principle)**.** *In the continuation game played by the consumers in any period t of a dynamic game, the set of equilibria is the same as if the game were a one period game.*

It is important to note that the myopia principle *does not imply* that the prices charged by the networks will be the same in a multi-period game as in a one period game — it is only the consumers who are "myopic", not the firms. The literature on switching costs often assumes that consumers are myopic; in that case this an assumption on the bounded rationality of consumers who do not take into account their long run interests when taking decisions. In the present paper, the myopia principle is not an assumption but *a result*, which stems from the fact that agents are "very small" and a consumer does not change the state. Furthermore, since there are no switching costs, thus each consumer knows that it will get the same utility in the future as any consumer of its same type.

Notice that the myopia principle is very general in our framework, and in particular it does not depend on the Markov property which we will introduce shortly. The fact that it holds is a fundamental difference with switching cost models, where, as shown in Biglaiser et al. (2013) high switching cost consumers try to "hide among" low switching cost consumers who induce firms to charge low prices.

4.3 Markov equilibria

We focus our attention on Markov equilibria, defined as follows. Along the equilibrium path, the price charged by incumbent i depends only on the $\beta^t_{j\theta}$s and not on t; the prices charged by entrants depend only on the $\beta^t_{j\theta}$s and the prices charged by the incumbents; the equilibrium of the game played between the consumers depend only of $\beta^t_{j\theta}$s and the prices charged by the firms. Furthermore, all that matters is the $\beta^t_{j\theta}$s and not the name of the network.

By Corollary 1 and Lemma 8, consumers of the same type will all "stay together"; therefore along the equilibrium path, there will be only either one or two incumbents in every period. Therefore, we need only distinguish the following prices for the incumbents: p_h, the price charged by a firm whose clients in the previous period were (only) the HNE consumers; p_ℓ, the price charged by a firm whose clients in the previous period were (only) the LNE consumers; p_2, the price charged by a firm who sold to the *2* types of clients in the previous period. We will distinguish three types of equilibria:

- Two network equilibria: along the equilibrium path, consumers buy from two different networks (which implies that in the first periods, at least one type of consumers purchase from an entrant). If after a deviation, there is only one incumbent network, consumers will reallocate themselves among two different networks. We study these equilibria in section 6.
- One network equilibria where "consumers stay separated after they split" and along the equilibrium path consumers always buy from the Incumbent. If after a deviation, consumers have purchased from two different networks, then there are two networks in all subsequent periods (barring, of course, a further deviation). We study these equilibria in 7.1.
- One network equilibria where "consumers get back together after they split". If after a deviation, there are two different networks, in the subsequent period, the consumers all purchase from the same network. We study these equilibria in 7.2.

Before turning to the study of these equilibria, we study some simple versions of our model.

5 Simple infinite horizon models with free entry

In this section, we examine a series of very simple dynamic models with free entry. They both help to illustrate how our solution concept works in dynamic models and show that the value of incumbency can be quite limited.

5.1 Identical Consumers

We begin by examining the equilibrium when there is only one type of consumers, for definitiveness assume that they are the HNE consumers. Our main result will be that the profits in the infinite horizon are exactly the same as in the static model.

Let Π denote the equilibrium profits when all consumers are on the same network. Entrants are willing to price down to $-\delta\Pi/\alpha_h$ and no further, as any lower price would yield negative profits. The Incumbent chooses the largest price p_I which enables him to keep all the consumers:

$$w_h - p_I = \frac{\delta\Pi}{\alpha_h},$$

Because $\Pi = \alpha_h p_I/(1-\delta)$, this proves the following proposition.

Proposition 1. *If all consumers are* HNE *consumers, then the unique equilibrium has a single network. The Incumbent keeps all consumers on its network and charges $p_I = (1-\delta)w_h$ in each period. Its profit is $\alpha_h w_h$, the same as in a static model.*

The result on the equality of static and dynamic profit are reminiscent of those of Biglaiser et al. (2013) in the case of switching costs. Competition from the entrants prevents the incumbent from enjoying the rents of incumbency more than once: it can take only one bite from the apple. Notice that the results would also hold in a model with a finite number of periods. In the infinite horizon case, this is dependant on the stationarity assumption (see Biglaiser and Crémer (2011)).

5.2 Equilibrium when condition (SmallCE) does not hold

We will now show that if condition (SmallCE) does not hold, *i.e.*, if $w_l > u_h - v_h$, then the only equilibrium is a one network equilibrium. We begin by proving two preliminary lemmas. First, because $w_l < w_h$, in the first period it is impossible for an entrant to attract the HNE consumers and not the LNE consumers. We therefore have the following lemma.

Lemma 9. *In the first period of a two network equilibrium, the* HNE *consumers purchase from the Incumbent and the* LNE *consumers from an entrant.*

Consider now a two network equilibrium. In the first period, the incumbent charges p_2 and the lowest priced entrant charges p_E. The fact that the

LNE consumers purchase from the entrant imply $w_\ell - p_2 < -p_E$ and the fact that the HNE consumers choose to purchase from the incumbent imply $u_h - p_2 \geq v_h - p_E$. This implies that (SMALLCE) holds. We state this formally in the following lemma.

Lemma 10. *Condition* (SMALLCE) *is necessary for the existence of a two network equilibrium.*

If follows trivially from Lemma 10, that if (SMALLCE) does not hold, then any equilibrium has to be a single network equilibrium. Therefore, in any period there is an incumbent who charges p_2. Let p_E be the lowest price charged by any entrant. If $-p_E \leq w_\ell - p_2 < w_h - p_2$, then both the LNE and HNE consumers will choose to purchase from the incumbent. On the other hand, if $-p_E > w_\ell - p_2$, the LNE consumers will migrate to the entrant (or one of the entrants) who charges p_E. Once they have done so, HNE consumers will follow, as their utility at the incumbent $u_h - p_2$ is strictly smaller than their utility at the entrant $v_h - p_E$. Therefore, if $p_2 - p_E \leq w_\ell$ the incumbent keep all the consumers; otherwise it will them all.

The lowest price that entrants are willing to price to attract all consumers is $-\delta\Pi_2/(\alpha_l + \alpha_h)$, where Π_2 is the discounted intertemporal profit of the incumbent in any period. Therefore the profit maximizing prize for the incumbent satisfies $p_2 = -\delta\Pi_2/(\alpha_h + \alpha_\ell) + w_\ell$. Because $\Pi_2 = (\alpha_h + \alpha_\ell)p_2/(1 - \delta)$, we have proved the following proposition.

Proposition 2. *If* (SMALLCE) *does not hold, then the only equilibrium is a single network equilibrium. The Incumbent charges* $(1-\delta)(\alpha_h + \alpha_\ell)w_\ell$ *in every period and its discounted profit is* $(\alpha_h + \alpha_\ell)w_\ell$ *the same as in the static model.*

The last part of the proposition is a direct consequence of Lemma 7. Given that the HNE consumers will follow whenever the LNE consumers migrate to an entrant, there is essentially only one type of consumers, and we obtain essentially the same results as in 5.1.

5.3 The LNE consumers do not derive any utility from belonging to a network[9]

We now turn to a case where the LNE consumers derive no utility from belonging to a network: $w_\ell = u_\ell = v_\ell = 0$. In the first part of the subsection

[9]Formally, the results of this subsection are a special case of those of section 6; our aim here is to bring out some of the economics of competition between the incumbents and the entrants which might not be as transparent in the analysis of the general case.

we also assume that HNE consumers only derive utility from the presence of other HNE consumers and do not care about the presence of LNE consumers: $v_h = 0$ and $w_h = u_h > 0$. We will show that even under these circumstances, LNE consumers affect the equilibrium by dampening the aggressiveness of entrants.

As will be proved formally in section 6, there will be two networks at equilibrium: this is intuitively obvious as the Incumbent would have to charge a price of 0 in order to keep the LNE consumers while he can make a strictly positive profit by selling only to the HNE consumers.

Let Π be the discounted profits of the incumbent measured from the start of a period (in section 6, we show that this profit is the same whether all the consumers or only the HNE consumers were its clients in the last period). If an entrant attracts[10] the HNE consumers, it will also attract the LNE consumers and the lowest price that it is willing to offer is $-\delta\Pi/(\alpha_h + \alpha_\ell)$. To keep the HNE consumers, the incumbent chooses a price p_I that makes them just indifferent between staying on its network and purchasing from the entrant at that price: we must have

$$u_h - p_I = \delta\Pi/(\alpha_h + \alpha_\ell), \tag{7}$$

and therefore

$$\Pi = \frac{p_I}{1-\delta} = \frac{(\alpha_h + \alpha_\ell)\alpha_h u_h}{\alpha_h + (1-\delta)\alpha_\ell}.$$

Notice that the profit of the incumbent is increasing in the number of LNE consumers: because they accept the offers at a negative price, they make the entrants less aggressive. This is specially striking when δ converges to 1, as Π converges to $(\alpha_h + \alpha_\ell)u_h$. A LNE consumer is worth as much to the incumbent as a HNE consumer, despite the fact that LNE consumers never purchase from the incumbent.[11] It may be worthwhile noting that in a one period model, the incumbent would charge u_h and its profit would be $\alpha_h u_h$. With δ close to 1, it is as if LNE consumers had been transformed into HNE consumers. [12] When α_ℓ is equal to zero, the derivative of this profit with respect to α_ℓ or to α_h is u_h.[13]

[10] This assumes that the LNE consumers all coordinate on the same entrant. They need not do so if there are indifferent to network effects. This coordination must either be assumed or the results which we present here can be considered as limit results when LNE consumers are close to indifferent to network effects.

[11] This result is similar to the result of Biglaiser et al. (2013) in a switching cost model, where with δ close to 1, a 0 switching cost consumer was worth as much to the incumbent as a consumer with positive switching cost.

[12] This result will also arise in the more general model below. See equation (11).

[13] The reasoning of the preceding paragraph has assumed that u_h is not affected by

6 Two network equilibria

6.1 Main results

We are looking under which conditions two networks will coexist at equilibrium. Given our Markov assumption, and given the results of section 2 which show that consumers of the same type will always purchase from the same network if they are initially together, such an equilibrium would look as follows. In the first period, the Incumbent charges p_2.After the first period, there will be two networks on the equilibrium path. This can only happen if in the first period an entrant charges p_E and attracts one type of consumers (it is clear that the Incumbent must sell to at least one type of consumers, otherwise its profits would be equal to 0). Because $w_\ell < w_h$, the LNE consumers gain the most from moving to any entrant as $[-p_E - (w_\ell - p_2)] > [-p_E - (w_h - p_2)]$. Therefore the entrant will attract the LNE consumers and the Incumbent will sell to the HNE consumers. In subsequent periods, along the equilibrium path, there will be two incumbents: the Incumbent who sells to the HNE consumers in all periods and the successful first period entrant who sells to LNE consumers in every period.

Off equilibrium, in some period all the consumers could buy from one firm. In this case, by the Markov hypothesis, in the next periods the consumers would again split among two networks as described in the previous paragraph.

The equilibrium will therefore be characterized by three prices, p_h, the price charged by the "H incumbent" (which along the equilibrium path will be the Incumbent), p_ℓ the price charged by the L incumbent, and p_2 (along the equilibrium path, this will only be used by the Incumbent in the first period). Let Π_h, Π_ℓ and Π_2 be the corresponding profits. We have $\Pi_h = p_h/(1-\delta)$,

the change in α_h. If we make explicit the dependence of the utilities on the size of the networks, the profit Π when δ is close to 1 becomes equal to $(\alpha_h + \alpha_\ell)u_h(\alpha_h, 0)$. When $\alpha_\ell = 0$, the derivative of this profit with respect to α_ℓ is $u_h(\alpha_h, 0)$ as $\partial u_h/\partial \alpha_\ell = 0$. The derivative with respect to α_h is equal to $u_h(\alpha_h, 0) + \alpha_h \partial u_h(\alpha_h, 0)/\partial \alpha_h$. The second term, $\alpha_h \partial u_h(\alpha_h, 0)\partial \alpha_h$, is the increase in the value of the network for the other HNE consumers. A LNE consumer is worth as much as the "direct" effect of a HNE consumer. If u_h is concave, then $\alpha_h \partial u_h(\alpha_h, 0)/\partial \alpha_h > u_h(\alpha_h, 0)$ and therefore a LNE consumer is worth *less* than half of a HNE consumer.

Very similar results hold if the HNE consumers do care about the presence of LNE consumers; *i.e.*, if we let v_h to be strictly positive. Entrants will still be willing to price down to $-\delta\Pi/(\alpha_h + \alpha_\ell)$, but (7) becomes $u_h - p_I = v_h + \delta\Pi/(\alpha_h + \alpha_\ell)$: the presence of the LNE consumers increase the attractiveness of the entrant. Then Π is given by (11). An increase in v_h decreases profits: it makes it easier for entrants to attract the HNE consumers by first attracting the LNE consumers. It may also be worthwhile noticing that the separation of the consumers in two different networks is now inefficient. Otherwise, the same comments as in the previous two paragraphs hold.

$\Pi_\ell = p_\ell/(1-\delta)$ and $\Pi_2 = p_2 + \delta\Pi_h$.

The following proposition summarizes our results.

Proposition 3. *There exists a two network equilibrium if and only*

$$u_h - v_h \geq \frac{(1-\delta)\alpha_\ell + \alpha_h}{(1-\delta)\alpha_h}(w_\ell - \delta u_\ell). \tag{2NtwCond}$$

In this equilibrium

$$p_\ell = u_\ell(1-\delta) \tag{8}$$

and

$$\Pi_\ell = \alpha_l u_\ell. \tag{9}$$

L incumbents charge the same price and have the same profit as if there were only LNE *consumers.*

H incumbents and firms which, after a deviation, have sold to every consumers in the preceding period charge the same price,

$$p_2 = p_h = \frac{(1-\delta)(\alpha_h + \alpha_\ell)(u_h - v_h)}{(1-\delta)\alpha_\ell + \alpha_h}, \tag{10}$$

and have the same profit,

$$\Pi_2 = \Pi_h = \frac{\alpha_h(\alpha_h + \alpha_\ell)(u_h - v_h)}{(1-\delta)\alpha_\ell + \alpha_h}. \tag{11}$$

This profit, which is also the profit of the Incumbent,

1. *is greater than the profit of the Incumbent in the one period model two network equilibrium, $\alpha_h(u_h - v_h)$, and smaller than the value of a flow of this one period profit, $\alpha_h(u_h - v_h)/(1-\delta)$;*
2. *is less than the profit where all the consumers pay the one period price in the two network equilibrium, $(\alpha_l + \alpha_h)(u_h - v_h)$;*
3. *is increasing in u_h, decreasing in v_h and independent of w_h, w_ℓ, u_ℓ and v_ℓ;*
4. *is increasing in α_h and in α_ℓ;*

The difficult part of the proof is proving that (2NtwCond) is a necessary and sufficient condition and that equations (9) to (11) hold — this is done in 6.2. The rest of the theorem is an immediate consequence of (11).

As we will see later, the binding deviation for the existence of a two network equilibrium is the attempt by the Incumbent to keep all the consumers. By (9) the lowest price that entrants are willing to charge is $-\delta u_\ell$. Because the LNE consumers are the more eager to change network, to keep all the consumers, the incumbent can charge at most $w_\ell - \delta u_\ell$ and the resulting profits from repeating this strategy forever is

$$\Pi_D = \frac{(\alpha_\ell + \alpha_h)(w_\ell - \delta u_\ell)}{1-\delta}.$$

Condition (2NtwCond) simply states that this profit is smaller than Π_2.

Points 1 and 2 are similar to the case when $w_l = 0$; as in Biglaiser et al. (2013) switching cost paper, LNE consumers have some value for the Incumbent even though it does not sell to them, since they get in the way of an entrant trying to attract high value consumers. On the other hand, they do not provide much more value to the incumbent than the static equilibrium payoff. In particular from point 2, the Incumbent's profit does not exceed the static profit maximizing price times the entire measure of consumers. Thus, the incumbent generates profits from having the LNE consumers in the market, but that value is limited to what the Incumbent obtained from the HNE consumers in the static model. We will see when examining the single network equilibrium that this also holds.

From point 3, the fact that profits are independent of the LNE consumers preferences are due to the fact that the constraint on the pricing of the incumbent is the fear of losing the HNE consumers, once the LNE consumers have already been attracted. The LNE consumers preferences dictate when a two network equilibrium exists.

6.2 Existence of two network equilibria: comparing static and dynamic models

The aim of this subsection is to discuss in some detail the circumstances in which there exist a two network equilibrium. We begin by the following, easy to prove, corollary which shows that there is a fundamental difference between the cases where $w_\ell > u_\ell$ and $w_\ell = u_\ell$, this is the case where the presence of HNE consumers add utility to LNE consumers when they are all on the same network and the case where it does not. Corollary 3 provides more details for the case $w_\ell > u_\ell$.

Corollary 2. *If* (5) *holds, then a two network equilibrium exists in the static model. If $w_\ell > u_\ell$ and* (5) *holds, then there exists a $\delta_2 < 1$ such that there exists a two network equilibrium if and only if $\delta < \delta_2$. If $w_\ell = u_\ell$, there exists a two network equilibrium if and only if $\delta \geq \delta_1$, where this δ_1 is equal to 0 if and only if* (5) *holds.*

The first part of the corollary states that if $w_\ell > u_\ell$ and there exists a two network equilibrium in the static case, then there cannot exist a two network equilibrium when the discount factor is close to 1. On the other hand, when $w_\ell = u_\ell$, which is compatible in the static case both with the existence and non existence of a two network equilibrium, there exists a two network equilibrium only if δ is large enough. Indeed, in this case $\Pi_D = (\alpha_h + \alpha_\ell)w_\ell$ is independent of δ, whereas Π_2 grows in δ.

The next corollary demonstrates that for intermediate values of $u_h - v_h$ and small cross effects for the LNE consumers, that an equilibrium may exist in the dynamic model when it does not exist in the static model.[14]

Corollary 3. *If $\alpha_h w_l \geq u_\ell(\alpha_h + \alpha_\ell)$, then*

- *if $u_h - v_h < (1 + \alpha_\ell/\alpha_h)\, w_\ell$ there is no two network equilibrium either in the static or the dynamic model;*

On the other hand, if $\alpha_h w_l < u_\ell(\alpha_h + \alpha_\ell)$, then there exists a $\widetilde{V}$ such that

- *if $u_h - v_h < \widetilde{V}$, there is no two network equilibrium either in the static or the dynamic model;*
- *if $\widetilde{V} \leq u_h - v_h < (1 + \alpha_\ell/\alpha_h)\, w_\ell$, there does not exist a two network equilibrium in the static model, but there is one in the dynamic model for $\delta \in [\delta_1, \delta_2]$ with $\delta_1 > 0$ and $\delta_2 < 1$;*

The second part of the corollary, when $\alpha_h w_l < u_\ell(\alpha_h + \alpha_\ell)$ is illustrated on Figure 3, while the proof is presented in the appendix on page 35.

6.3 Condition (2NtwCond) is a necessary condition for existence of a two network equilibrium

It seems intuitive that in any equilibrium, the net surplus of the HNE consumers must be larger than the net surplus of the LNE consumers:

$$u_\ell - p_\ell < u_h - p_h. \tag{12}$$

This implies that, along the equilibrium path, an entrant cannot attract the LNE consumers without having first attracted the HNE consumers. In the main text, we assume that (12) holds and we show in appendix E that this must indeed be the case whenever a two network equilibrium exists.

We first show that we can strengthen (12). The proof of all the claims that follow can be found in the appendix.

[14]We note that our results from Corollary 2 still hold for large $u_h - v_h$.

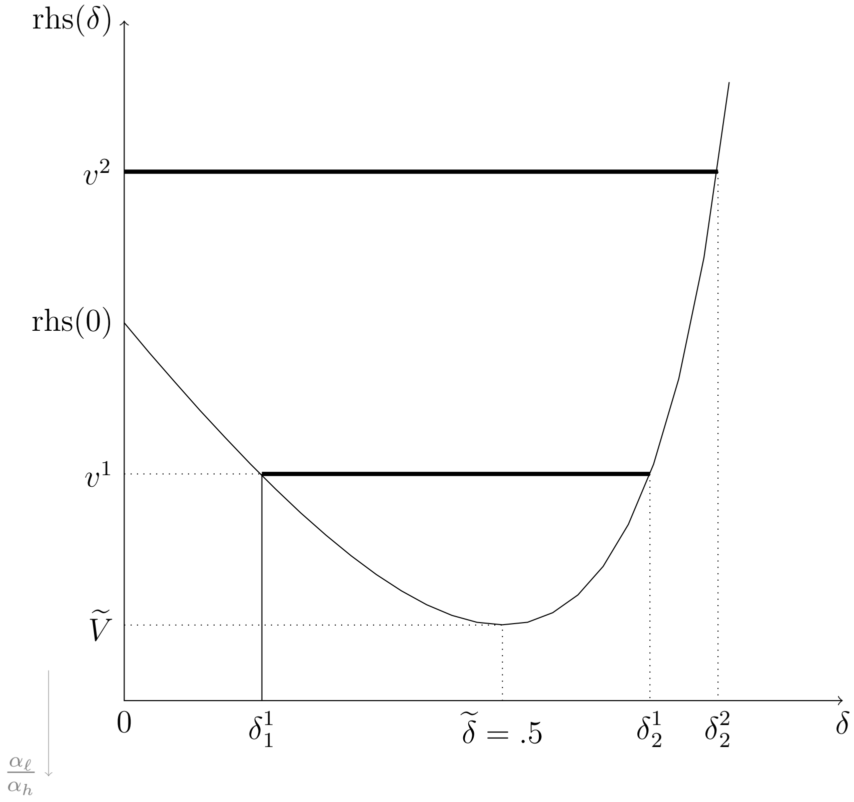

Figure 3: This figure illustrate corollary 3 for $\alpha_\ell = \alpha_h$, $w_\ell = 1$ and $u_\ell = .8$. The right hand side of (2NtwCond) is then to $(2-\delta)(1-.8\delta)/(1-\delta)$, which is equal to 2 when $\delta = 0$. Its minimum, $\widetilde{V}$ is equal to 1.8 and is obtained for $\widetilde{\delta} = .5$. For $u_h - v_h = v^1 \in [1.8, 2]$, there exists an interval $[\delta_1^1, \delta_2^1]$ such that there exists a two network equilibrium.

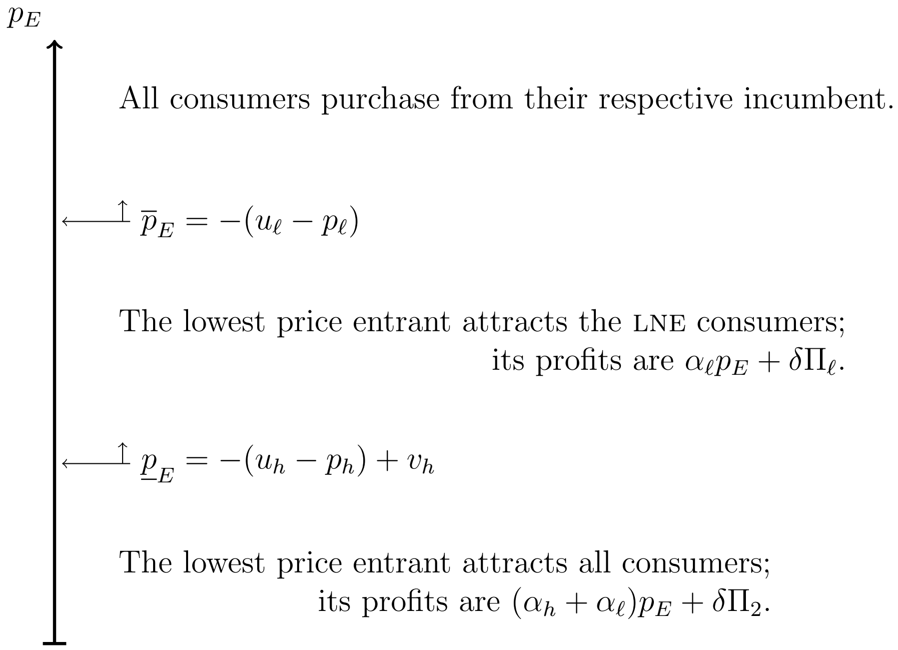

Figure 4: The response of consumers to entry when (13) holds.

Claim 1. *If* (12) *holds, then in a two network equilibrium*

$$v_h + u_\ell - p_\ell < u_h - p_h. \tag{13}$$

An entrant attracts the HNE consumers without attracting the LNE consumers only if $u_h - p_h < -p_E \leq u_\ell - p_\ell$. By (12), this is impossible. Then, (13) implies that the continuation equilibria in the consumers' game as a function of p_E are as represented on Figure 4.

From the definition of prices in Figure 4, there is no profitable entry when consumers start the period on separate networks only if

$$\alpha_\ell \overline{p}_E + \delta\Pi_\ell \leq 0,$$
$$(\alpha_h + \alpha_\ell)\underline{p}_E + \delta\Pi_2 \leq 0.$$

The first equation states that an entrant cannot profitably attract only the LNE consumers, while the second states that an entrant cannot profitably attract all consumers, noting that the LNE consumers would always migrate first since $u_\ell - p_\ell < u_h - p_h$. Using the fact that the Incumbent will choose the highest p_h that is consistent with preventing from profitably attract all the consumers and a successful entrant who attracts the LNE consumers will choose the highest price that prevents new entrants from attracting its consumers we have

Claim 2. *If* (12) *holds, and there is a two network equilibrium, then*

$$-(\alpha_h + \alpha_\ell)[u_h - p_h - v_h] + \delta\Pi_2 = 0, \tag{14}$$

$$-\alpha_\ell(u_\ell - p_\ell) + \delta\Pi_\ell = 0. \tag{15}$$

It is relatively intuitive, and proved in claim D 1 in the appendix, that (15) is binding at equilibrium: otherwise, the L incumbent could raise its price and keep its consumers. Because $\Pi_\ell = \alpha_\ell p_\ell/(1-\delta)$, equations (8) and (9) hold. Thus, once the two groups are separated, the L incumbent behaves in the same way and obtain the same profit as if it where the incumbent with only the LNE consumers present (see proposition 1). Similarly, the fact that (14) is binding follows, otherwise the Incumbent could raise its price.

The reasoning which precedes show the "necessity" part of the following lemma. The sufficiency part, which is quite straightforward, is proved in the appendix.

Lemma 11. *Equations* (8) *and 10 are sufficient and necessary for the fact that once* LNE *and* HNE *consumers have purchased from different networks then will continue to do so in the continuation equilibrium.*

This lemma provides conditions for the fact that once there are two networks in a period, there are also two networks in subsequent periods. Its proof relies on deviations in period 2 and after, when the consumers have already split between the two networks. We now turn to the study of the first period and on the incentives of the agents to create two networks out of one.

First, we must have

$$-(\alpha_h + \alpha_\ell)\left[p_2 - (u_h - v_h)\right] + \delta\Pi_2 \leq 0. \tag{16}$$

Otherwise, in the first period an entrant could attract all the consumers by charging a price "slightly below" $p_2 - (u_h - v_h)$ and make strictly positive profit. Claim D 2, presented on page 37 in the Appendix, shows that in equilibrium this constraint must be binding: otherwise the Incumbent could profitably increase its price in period 1. Because (14) and (D 5) are both binding, we have $p_2 = p_h$ and therefore $\Pi_2 = \Pi_h$. Along with $\Pi_h = \alpha_h p_h/(1-\delta)$, this proves (10) and (11).

Summarizing the discussion so far, we have proved that if there is an equilibrium satisfying (13), then the prices must satisfy equations (8) and (10). Furthermore, if the prices satisfy these equations, then there is no profitable entry by (14) and (15).

Finally, we need to make sure that the incumbent charges a price when it has all the consumers for which an entrant finds it profitable to attract the LNE consumers. If the Incumbent charges p_2', an entrant needs to charge less than $p_2' - w_\ell$ to attract only the LNE consumers and the upper bound on its profit would be $-\alpha_\ell(w_\ell - p_2') + \delta\Pi_\ell = \alpha_\ell(p_2' - (w_\ell - \delta u_\ell))$. Therefore, an entrant will be willing to attract only the LNE consumers if and only if $p_2' > w_\ell - \delta u_\ell$. This implies that $p_2 > w_\ell - \delta u_\ell$ is a necessary condition for the existence of a two network equilibrium. If this inequality holds, we need to ensure that the Incumbent would not find it optimal to price down to $w_\ell - \delta u_\ell$. By the one deviation principle, this will be the case if $(\alpha_h + \alpha_\ell)(w_\ell - \delta u_\ell) \leq \alpha_h p_2$, which is equivalent to (2NtwCond). This condition also implies $p_2 \geq w_\ell - \delta u_\ell$.

We have used (12) and (SMALLCE) to prove that (2NtwCond) is necessary for the existence of a two network equilibrium. It is easy to show that both of these conditions hold whenever (2NtwCond) hold (see claim D 3 on page 38 in the appendix). Thus, we have derived the necessary conditions for a two network equilibrium. We turn now to showing that (2NtwCond) is also a sufficient condition for the existence of a two network equilibrium

6.4 Condition (2NtwCond) is a sufficient condition for a two-network equilibrium.

We show that (2NtwCond) is sufficient for the existence of a two network equilibrium. Much of the construction of the equilibrium in the preceding subsection can be used in this proof. We will proceed by going through the possible deviations and showing that they are not profitable.

First period

Incumbent: By the reasoning showing equation (16) on page 23, if it increased its price an entrant would find it profitable to attract all the consumers. Decreasing the prices to $p_2' \geq w_\ell - \delta u_\ell$ would not change demand and hence would lead to lower prices. Lowering the price below $w_\ell - \delta u_\ell$ would enable the Incumbent to keep all the consumers, but, because (2NtwCond) hold at the cost of lower profits, as shown on page 24.

Entrants: Competition between the entrants will lead them to charge a price equal to $-\delta\Pi_\ell/\alpha_\ell = -\delta u_\ell$. At that price, LNE consumers find it profitable to purchase from the entrants as shown on page 24. The proof of proposition 3 shows that no entrant will find it profitable to attract all the consumers.

Subsequent periods

Incumbents The same reasoning as in period 1 shows that the H incumbent has no incentive to deviate. The incentives of the L incumbent are the same as in 5.1 as far as competing for the LNE consumers. In order to attract the HNE consumers it would have to choose a price inferior to $p_2 - (u_h - v_h)$, which is unprofitable for the same reason that it would be unprofitable for an entrant to attract all the consumers in the first period.

Entrants For the same reasons as in 5.1 they cannot profitably attract the LNE consumers. For the same reason as in the first period, they cannot attract profitably all the consumers.

We have derived all the results up to this point, assuming that equation (12) held. In appendix E, we show that this equation must hold if there exist a two network equilibrium.

7 Analysis of equilibria with one network

In this section, we study the stationary equilibria with only one network, that is equilibria such that, along the equilibrium path, the Incumbent sells to both LNE and HNE consumers in every period. We first present the following corollary which holds for all single network equilibria .

Corollary 4. *There exists a $\overline{\delta}$ such that for any $\delta \geq \overline{\delta}$ a) there exists at least one single network equilibrium and b) in any single network equilibrium the profit of the incumbent is $(\alpha_h + \alpha_\ell)(u_h - v_h)$.*[15]

We pursue the analysis by distinguishing between four types of one network equilibria which differ along two dimensions. The first dimension describes what happens off the equilibrium path if the consumers ever get "separated" in two different networks: consumers can either stay separated in subsequent periods - the S (for *S*eparated) equilibria - or they can all purchase from the same network in the period after they have split so that two networks exist for only one period - the T (for *T*ogether) equilibria.

As long as it sells to both types of consumers, the Incumbent network faces two entry constraints: preventing profitable entry which would attract only the LNE consumers and preventing profitable entry which would attract all consumers. The second dimension describes which of these two constraints is binding. Figure 5 represents the four types of equilibria.

In 7.1, we first examine the S type equilibria in which consumers keep on purchasing from different networks after out-of-equilibrium moves in which they do so. In 7.2, we study T type equilibria. We complete the section, by determining the incumbency advantage and examine comparative static properties of the equilibria.

[15] This corollary is an easy consequence of lemmas E 1, E 2, E 4 and E 3. It is easy to see that, for δ close to 1, an equilibrium of type T1 exists, an equilibrium of type S1 exists if $(u_h - v_h) < (\alpha_h + \alpha_\ell)u_\ell/\alpha_\ell$, that there is no equilibrium of type T2 or S2, and that when an equilibrium exists the profit is $(\alpha_h + \alpha_\ell)(u_h - v_h)$.

		binding constraint	
		LNE consumers	both types
After separation	keep separated	Sℓ	S2
	back together	Tℓ	T2

Figure 5: The type of equilibria in the one network case.

7.1 S type equilibria: consumers stay separated after they split

In S type equilibria, if, off the equilibrium path, HNE and LNE consumers join different networks in some period, then they stay on these networks in subsequent periods. As discussed on page 23, claim D 1 is valid in the continuation path after two networks have been formed when condition (12) holds and we have $\Pi_\ell = \alpha_\ell u_\ell$ and $p_\ell = u_\ell(1-\delta)$.[16]

Along the equilibrium path, in order to attract only the LNE consumers, an entrant must charge a price p_E which satisfies $-p_E > w_\ell - p_2$ as well as $v_h - p_E \leq u_h - p_2$ — such a p_E exists by (SMALLCE). This is profitable if $\alpha_\ell p_E + \delta\Pi_\ell > 0$, which is equivalent to $-p_E < \delta\Pi_\ell/\alpha_\ell = \delta u_\ell$. To make this type of entry impossible the incumbent must ensure that at the price $p_E = -\delta u_\ell$ the LNE consumers choose not to purchase from the entrant. Therefore, it must choose p_2 such that $w_\ell - p_2 \geq \delta u_\ell$, that is

$$p_2 \leq w_\ell - \delta u_\ell. \tag{17}$$

To attract both types of consumers, an entrant must charge a p_2 which satisfies $v_h - p_E > u_h - p_2$.[17] This is profitable if and only if $(\alpha_h + \alpha_\ell)p_E + \delta\Pi_2 > 0$, which is equivalent to $-p_E < \delta\Pi_2/(\alpha_h + \alpha_\ell)$. To make this type of entry impossible the incumbent must choose a price such that for all profitable p_E we have $u_h - p_2 \geq v_h - p_E$. We must therefore have $\delta\Pi_2/(\alpha_h + \alpha_\ell) \leq u_h - v_h - p_2$. Because $\Pi_2 = (\alpha_h + \alpha_\ell)p_2/(1-\delta)$, this is equivalent to

$$p_2 \leq (1-\delta)(u_h - v_h). \tag{18}$$

[16] We focus attention on equilibrium where off the equilibrium path, $u_h - p_h \geq u_\ell - p_\ell$. It is possible to have $u_h - p_h < u_l - p_l$ for small δ. This is possible only if $u_\ell > \delta u_h$. Indeed, if $u_h - p_h < u_\ell - p_\ell$, then by the same argument as in the proof of claim D 1, $p_h = u_h(1-\delta)$ and therefore $u_h - p_h = \delta u_h$. Since $p_\ell \geq 0$, $u_h - p_h < u_l - p_l$ is possible only if $u_l > \delta u_h$.

[17] By (SMALLCE), this condition is sufficient for the entrant to attract first the LNE consumers and then the HNE consumers.

Along the equilibrium path, both of the constraints (17) and (18) must be met and at least one binding. This implies the following lemma.

Lemma 12. *In the "consumers stay separated after they split" single network equilibria the profit of the incumbent is*

$$(\alpha_h + \alpha_\ell) \min \left[\frac{w_\ell - \delta u_\ell}{1 - \delta}, u_h - v_h \right].$$

When (17) is binding, we have a Sl equilibrium; when (18) is binding we have a $S2$ equilibrium. In the Appendix, we find conditions under which either an Sl or $S2$ equilibrium exists. The details are rather complicated, but we find that both equilibrium require small $u_h - v_h$ and that an Sl equilibrium can not exist when a two network equilibrium exists, while for a small slice of parameters, an $S2$ equilibrium may exist when a two network equilibrium exists.

7.2 T type equilibria: consumers come back after they split

In T type equilibria, any entrant who, out of equilibrium, attracts consumers in period t looses them in period $t + 1$.

On the equilibrium path, an entrant is not able to attract the LNE consumers at a strictly positive price, and therefore we must have

$$w_\ell \geq p_2. \tag{19}$$

So, that an entrant cannot profitably give LNE consumers some utility in a period.

An entrant must also find it unprofitable to attract all the consumers. This occurs if and only if

$$u_h - p_2 \geq v_h + \frac{\delta \Pi_2}{\alpha_\ell + \alpha_h},$$

which, because $\Pi_2 = (\alpha_h + \alpha_\ell) p_2 / (1 - \delta)$, is equivalent to

$$p_2 \leq (1 - \delta)(u_h - v_h). \tag{20}$$

The following lemma states the profits of the T equilibrium.

Lemma 13. *In the "consumers come back together if ever separated" single network equilibria the profit of the incumbent is to*

$$(\alpha_h + \alpha_\ell) \min \left[\frac{w_\ell}{1 - \delta}, u_h - v_h \right].$$

When (19) is binding, we have a Tl equilibrium; when (20) is binding we have a $T2$ equilibrium. We prove the lemma in the Appendix, where we also find existence conditions. A subtlety that arises for the existence of the T equilibrium is due to how we defined AC equilibria. In our definition, we assumed that a consumer only left its current network if it strictly prefers to go to a rival network. If we maintain the strictness assumption, no T equilibrium exists. This is because when the incumbent attracts back consumers it will choose the highest possible price that will give the LNE consumers a strict preference to come back to its network. This price does not exist when there are continuum of prices; this would also happen in the standard Bertrand competition model with different marginal costs. So, for the existence of only these equilibria, we relax the strictness assumption. Alternatively, assuming a finite set of prices, where the distance between any consecutive pair of prices goes to 0 will implement the equilibrium profits of Lemma 13.

7.3 Profits and Comparative Statics in Single Network Equilibria

We now compare the single network equilibrium in the static and the dynamic model to demonstrate that the incumbency gain is quite limited when going to the dynamic model. We then discuss comparative statics of both single and two network equilibrium.

Comparing Lemmas 7, 12 and 13 we find that the profit of the Incumbent is greater in the dynamic than in the static model.

Corollary 5. *If there is an single network equilibrium in both the static and dynamic model, then the profits are higher in the dynamic model.*

While the profits are weakly higher in the dynamic model than in the static model when there is a single network equilibrium, we also find

Corollary 6. *The profits in the single network equilibrium dynamic model are smaller than the value of a flow of one period profit.*

These two corollaries are in the same spirit as Proposition 3 for the two network equilibrium. They are part of the story that the incumbent profit is quite limited in the dynamic model with free entry. Furthermore, when looking over all equilibrium of the game we have:

Corollary 7. *The incumbent's equilibrium profit in the dynamic model does not exceed*

$$(\alpha_h + \alpha_l) \min \left[u_h - v_h, \frac{w_l}{1-\delta} \right].$$

Thus, the incumbent can collect the *minimum* from the total measure of consumers of either the single period profit from the two network equilibrium and the present discounted surplus from the LNE consumers. Thus, the incumbent's profits are quite limited in the dynamic model.

It is interesting to compare the profits in the single network equilibrium to those in the two network equilibrium when both exist. We find that the incumbent always prefers the one network equilibrium.

Corollary 8. *When both a single network equilibrium and a two network equilibrium exist, then the profit of the incumbent is larger in the single network equilibrium.*

While there are many possible profits that an incumbent can achieve, the comparative statics are relatively stable across equilibria in both the single network and two network equilibria..

Corollary 9. *The incumbent's equilibrium profits are:*

- *Always increasing in α_l and a_h.*
- *Increasing in u_h and decreasing in v_h a two-network equilibrium and in a S1 and T1 single network equilibrium.*
- *Increasing in δ and w_l in a two network equilibrium and in a S2 and T2 single network equilibrium.*
- *Decreasing in u_l in a S2 single network equilibrium.*
- *Otherwise, independent of all parameters*

8 Literature

There are two main themes of this paper. One is that we want to identify the incumbency value in a dynamic market model with network externalities. The U.S. Department of Justice and others argued that Microsoft in the 1990's was able to make surpranormal profits due to the fact of the network externalities effects generated by consumers wanting to have a platform (computer) with many applications written for them. Farrell and Klemperer (2007) survey the literature on switching costs and network effects, which can sometimes be thought of as social switching costs in the sense that the cost of a consumer changing goods depends on other consumers' purchases. As can be seen from their survey, most of the focus on network goods is on symmetric competition, but there is a brief discussion on incumbency advantages. For example, Fudenberg and Tirole (2000) examine an overlapping generation model where an incumbent uses limit pricing and an incompatible good to detour entry when there are network effects.

It is well known that there are multiple equilibria in games with network effects, see Katz and Shapiro (1985, 1994). This leads to the second theme of identifying which equilibrium consumers coordinate on when choosing which network to belong to.

We identify equilibria in markets with network externalities, and choose a solution concept that gives incumbent(s) an advantage with their current customer base. Many papers with network externalities have examined the issue of coordination among market participants. There have been a variety of approaches to deal with the multiplicity of equilibria. One set of literature relies on the global games approach as pioneered by Carlsson and Van Damme (1993), where agents receive a small piece of private information. The private information can lead to a large amount of strategic uncertainty and can induce a unique equilibrium. The equilibrium that is chosen has the agents choosing a risk dominant action (Harsanyi and Selten, 1988). Sakovics and Steiner (2012), uses the global games approach where consumer preferences are known and a monopoly platform can make consumer specific offers to resolve the coordination problem. Jullien and Pavan (2013) analyze a single period global game where two two-sided platforms compete. Users receive private signals about their own preferences and about the quality of the platforms. There are also some dynamic global games, see Angeletos, Hellwig, and Pavan (2007).

Ochs and Park (2010) analyze a dynamic model where consumers have private information about the value of switching to an entrant network and once they switch, they cannot go back. Firms are not active players. They find a unique equilibrium for sufficiently high discount factor where consumers gradually switch to the entrant network over time. We have active firms and consumers can switch in every period.

As in the global games literature, most work on platform competition deals with static models. Caillaud and Jullien (2003) analyzed a static model where two two-sided platforms compete. They introduce a solution concept that is based on "pessimistic beliefs" towards an entrant in a static duopoly model, which maximizes the incumbent's profit in any equilibrium. In a paper that also examines two-sided markets and focuses on whether agents have pessimistic or optimistic beliefs regarding whether to join a platform, Halaburda and Yehezkel (2013) analyze a model where agents on each side of the market obtain private information about their valuations or cost and observe contracts that platforms are offering before they join. Our solution concept is related to the idea of pessimistic beliefs and can be thought of as generalization for any number of firm and types of consumers and consumers have private information about their valuations before joining the platforms. Ambrus and Argenziano (2009) analyze a two-sided network model and uses the concept of coalitional rationizability to solve the consumer coordination problem. They can generate asymmetric networks with heterogeneous consumers. Cabral (2011) analyzes a dynamic Hotelling style model of one sided networks, where in each period firms compete for a new consumer and other consumers are not allowed to change networks. Weyl (2010) analyzes a static monopoly multi-sided platform and where the firm can offer "insulating tariffs" which are utility offers to consumers to solve the multiplicity problem with network effects,while Weyl and White (2012) demonstrate the effects of insulated tariffs when platforms compete. Much of the literature on two-sided markets focusses on media markets starting with Anderson and Coate (2005).

9 Conclusion

In this paper, we created a new solution concept for models where consumer network externalities are present. The equilibrium that we identify is very favorable for incumbent networks. Despite this fact, we found in a dynamic free entry model their increased profits from moving from a static to an infinite horizon model are quite limited. In particular, the profits never exceed the static equilibrium price times the total mass of consumers in the market.

One of the results that made going from the static equilibrium to the dynamic model was the myopia principle that we identified: consumers rationally choose which network to join based only on their current payoff and not on future payoffs. This was due to two assumptions that we make: consumers are small and so do not affect the state of the market and that there are no switching costs. If either of these two assumptions do not hold, then the myopia principal is not valid.

In a companion paper, Biglaiser et al. (2013), we examine models with an free entry model with an incumbent and consumer switching costs. As in our current paper,we found that the incumbent's profit do not grow very much when expanding the time horizon from 1 period to an infinite number of periods. In future work, we want to investigate a model with both network externalities and switching costs. Some preliminary work on this model, demonstrate that the equilibria can be quite different than with a model with either switching costs or network effects. are

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Appendices

A Proofs of lemmas in section 2

Proof of lemma 4. It is easy to see that a migration path through large steps can be replaced by a migration path with $\mu(t) < \varepsilon$ for all t. Let $\bar{\theta}(t)$, $\bar{\mu}(t)$, $\bar{s}(t)$, and $\bar{d}(t)$ be respectively the type of transferred customers, the mass of migrating customers, the source network and the destination network in the large step migration path. We construct a new migration path in the following way. Let $\underline{\theta}(1) = \bar{\theta}(1)$, $\underline{d}(1) = \bar{d}(1)$, $\underline{s}(1) = \bar{s}(1)$, and $\underline{\eta}$ such that

$$0 < \underline{\eta}^1_{\underline{\theta}(1)\underline{d}(1)} - \underline{\eta}^0_{\underline{\theta}(1)\underline{d}(1)} = \underline{\eta}^0_{\underline{\theta}(1)\underline{s}(1)} - \underline{\eta}^1_{\underline{\theta}(1)\underline{s}(1)} < \varepsilon.$$

At the end of step 1 on the new migration path, by the same reasoning as in the proof of lemma 2,

$$u_{\underline{\theta}(1)}\left(\underline{\eta}^1_{h\underline{d}(1)}, \underline{\eta}^1_{\ell\underline{d}(1)}\right) - p_{\underline{d}(1)} > u_{\theta'}\left(\underline{\eta}^1_{j\theta'}, \underline{\eta}^1_{j,-\theta'}\right) - p_j - \left[u_{\theta'}\left(\underline{\eta}^1_{j'\theta'}, \underline{\eta}^1_{j',-\theta'}\right) - p_{j'}\right]$$

for all $(\theta', j, j') \neq (\underline{\theta}(1), \underline{d}(1), \underline{s}(1))$. Therefore

$$\{\underline{\theta}(2), \underline{d}(2), \underline{s}(2)\} = \{\underline{\theta}(1), \underline{d}(1), \underline{s}(1)\},$$

and by an easy recurrence it is possible to build a new migration path which after a finite number t_1^* of steps will rejoin the original migration path: $\underline{\eta}^{t_1^*}_{\theta j} = \bar{\eta}^1_{\theta j}$ for all θ and j. We can then take $\underline{\theta}(t_1^* + 1) = \bar{\theta}(2)$, $\underline{d}(t_1^* + 1) = \bar{d}(2)$ and $\underline{s}(t_1^* + 1) = \bar{s}(2)$.

By the same reasoning as in the previous paragraph there will exist t_2^* such that after t_2^* steps the new migration path will have the same allocation as the original migration path at $t = 2$. The result is proved by noticing that we can repeat the process until convergence to the final allocation along the original path. □

Proof of lemma 5 (page 8). We have defined migration paths by the fact that they lead from one initial allocation to a final allocation. To show that there exists a final allocation define the following procedures, inspired by large steps migration paths, but without guarantee that they lead to a final allocation. At every step, check whether there exist a $\{\theta(t), d(t), s(t)\}$ satisfying (2). If there is move all the consumers of type $\theta(t)$ from $s(t)$ to $d(t)$. If there is not, we have identified a AC equilibrium. To finish the proof, we only need to show that any such procedure will eventually find itself at a stage when this happens.

At every step, either the destination network already has clients or it charges a strictly lower price that the source network, or both. To each network which has a strictly positive mass of consumers, associate an index equal to the number of networks which charge strictly lower prices multiplied by either 1 if it has a positive mass of only one type of consumers and 2 if it has a positive mass of both types of consumers. The sum of these network indexes decreases by at least one at each stage of the migration. Given that this sum cannot be smaller than 1, the result is proved. □

B Proof of Lemma 9

Proof of lemma 9, page 14. Clearly, the Incumbent will have some sales in the first period. If there is an equilibrium where in period 1 the Incumbent sells only to the LNE consumers, then there must exist p_2 and p_E such that

$$w_h - p_2 \leq -p_E \text{ (HNE consumers purchase from the entrant),}$$
$$u_\ell - p_2 \geq v_\ell - p_E \text{ (LNE consumers purchase from the incumbent).}$$

This implies $w_h \leq p_2 - p_E \leq u_\ell - v_\ell$, which contradicts (3) and (4). □

C Proofs of Corollaries 2 and 3

Proof of corollary 2. Condition (2NtwCond) is equivalent to

$$h(\delta) \stackrel{\text{def}}{=} (1-\delta)\alpha_h(u_h - v_h) - ((1-\delta)\alpha_\ell + \alpha_h)(w_\ell - \delta u_\ell) > 0.$$

If $\alpha_h(u_h - v_h) > (\alpha_h + \alpha_\ell)w_\ell$, the function h is positive for $\delta = 0$ and strictly negative for $\delta = 1$. Furthermore, it is a quadratic function for which the coefficient of δ^2 is equal to $-\alpha_\ell u_\ell$; it is therefore concave and has exactly one zero for $\delta \in (0, 1)$, which proves the first part of the corollary.

When $w_\ell = u_\ell$ condition (2NtwCond) is equivalent to $\alpha_h(u_h - v_h) \geq \big((1-\delta)\alpha_\ell + \alpha_h\big)w_\ell$, which, together with (SMALLCE), proves the result. □

Proof of corollary 3. The derivative of the right hand side of (2NtwCond) with respect to δ has the same sign as

$$\begin{aligned}[-\alpha_\ell(w_\ell - \delta u_\ell) + ((1-\delta)\alpha_\ell + \alpha_h)(-u_\ell)](1-\delta) \\ + ((1-\delta)\alpha_\ell + \alpha_h)(w_\ell - \delta u_\ell) \\ = -\alpha_\ell u_\ell (1-\delta)^2 + \alpha_h(w_\ell - u_\ell).\end{aligned}$$

If $\alpha_h(w_\ell - u_\ell) > \alpha_\ell u_\ell$, this derivative is positive for all δ; the maximum of the right hand side of (2NtwCond) is obtained for $\delta = 1$ and is equal to $\alpha_\ell(w_\ell - u_\ell)$. The first part of the corollary is a direct consequence of these facts.

If $\alpha_h(w_\ell - u_\ell) < \alpha_\ell u_\ell$, this derivative is negative for all

$$\delta < \widetilde{\delta} \stackrel{\text{def}}{=} 1 - \sqrt{\frac{\alpha_h(w_\ell - u_\ell)}{\alpha_\ell u_\ell}} < 1.$$

Calling $\widetilde{V}$ the corresponding value of the right hand side, the second part of the corollary follows. (See figure 3.) □

D Proof of Proposition 3

Proof of claim 1. Assume that the H incumbent charges a price p'_h such that

$$v_h + u_\ell - p_\ell \geq u_h - p'_h. \tag{D 1}$$

If an entrant charges $p_E \geq -(u_\ell - p_\ell)$, by (12) we have $-p_E \leq u_\ell - p_\ell < u_h - p_h$, and no consumer wants to migrate "on his own" to the entrant who attracts no consumer.

On the other hand if the entrant charges $p_E < -(u_\ell - p_\ell)$, it attracts all the consumers: the LNE consumers as $-p_E > u_\ell - p_\ell$, and, once it has attracted the LNE consumers, the HNE consumers as $v_h - p_E > v_h + u_\ell - p_\ell \geq u_h - p'_h$: HNE consumers strictly prefer to be on the same network as the LNE consumers and pay p_E rather than to be on the same network as the other HNE consumers and pay p'_h.

Therefore, for all p'_h which satisfy (D 1), the response of entrants does not depend on p'_h. The H incumbent would increase its price rather than charge a p_h which satisfies (D 1). □

Claim D 1. *If* (12) *holds, then*

$$-\alpha_\ell(u_\ell - p_\ell) + \delta\Pi_\ell = 0. \tag{D 2}$$

Proof. Assume (D 2) did not hold. Then, $-\alpha_\ell(u_\ell - p_\ell) + \delta\Pi_\ell < 0$. In any period after the first, the L incumbent could increase its profit by charging $p'_\ell \in (p_\ell, u_\ell - \delta\Pi_\ell/\alpha_\ell)$. Indeed, any entrant who would want to attract the LNE consumers would have to charge t most $\overline{p}'_E = -(u_\ell - p'_\ell) < -\delta\Pi_\ell/\alpha_\ell$ and would make negative profits, $\alpha_\ell\overline{p}'_E + \delta\Pi_\ell$. □

Proof of claim 2. By (14), if (14) does not hold, $-(\alpha_h + \alpha_\ell)[u_h - p_h - v_h] + \delta\Pi_2 < 0$. By (13), it is then possible to find a price $p'_h > p_h$ which satisfies both

$$-(\alpha_h + \alpha_\ell)(u_h - p'_h - v_h) + \delta\Pi_2 < 0 \tag{D 3}$$

and

$$u_h - p'_h > v_h + u_\ell - p_\ell \Longrightarrow u_h - p'_h > v_h - p_\ell. \tag{D 4}$$

We will show that a deviation by the H incumbent to such a p'_h would be profitable.

The H and L incumbents announce their prices simultaneously; therefore the deviation by the H incumbent would not affect p_ℓ. By (D 4), after such a deviation the LNE consumers would respond by purchasing either from the lowest price entrant or from the L incumbent, as in figure 4 (replacing, of course, p_h by p'_h). Therefore, the deviation would be unprofitable for the H incumbent only if an entrant could profitably attract *all* the consumers. It could do this only by charging a price p'_E which satisfies $v_h - p'_E > u_h - p_h$, which by (D 3) implies $p'_E \leq -(u_h - p'_h - v_h) < -\delta\Pi_2/(\alpha_h + \alpha_\ell)$. The profits of the entrant, $(\alpha_h + \alpha_\ell)p'_E + \delta\Pi_2$, would be strictly negative, which proves the result. □

Proof of lemma 11. Only the sufficiency part is left to prove. From figure 4 an entrant could try either to a) attract only the LNE consumers by charging a price strictly smaller than $-(w_\ell - p_\ell)$, but this is not profitable by claim D 1 as

$$\begin{aligned}\alpha_\ell(-(w_\ell - p_\ell)) + \delta\Pi_\ell \leq \alpha_\ell(-(u_\ell - p_\ell)) + \delta\Pi_\ell \\ = \alpha_\ell u_\ell(-1 + (1-\delta) + \delta) = 0,\end{aligned}$$

or b) attract all consumers by charging a price strictly smaller that $-(u_h - p_h) + v_h$, but this is not profitable by (16). □

Claim D 2. *If* (12) *holds, then* $-(\alpha_h + \alpha_\ell)(u_h - p_2 - v_h) + \delta\Pi_2 = 0$.

Proof. Because (16) holds, it is sufficient to show that if $-(\alpha_h + \alpha_\ell)(u_h - p_2 - v_h) + \delta\Pi_2 < 0$, then a deviation by the period 1 incumbent to a price $p'_2 > p_2$ satisfying

$$-(\alpha_h + \alpha_\ell)(u_h - p'_2 - v_h) + \delta\Pi_2 < 0 \tag{D 5}$$

would be profitable.

At the original p_2, there was profitable entry by attracting only the LNE consumers; *a fortiori*, it will also be profitable to attract the LNE consumers when the price is p'_2. Therefore, the deviation by the period 1 incumbent is unprofitable only if an entrant could profitably attract *all* the consumers when the price is p'_2. By (D 5), in order to attract the HNE consumers as well as the LNE consumers, a entrant needs to charge a price p'_E which satisfies $p'_E < -(u_h - p'_2) + v_h < -\delta\Pi_2/(\alpha_h + \alpha_\ell)$. The profits of the entrant, $(\alpha_h + \alpha_\ell)p'_E + \delta\Pi_2$, would be strictly negative, which proves the result. □

Claim D 3. *If* (2NtwCond) *holds, then a)* (SMALLCE) *holds and b) the prices defined by* (8) *and* (10) *satisfy condition* (12)*.*

Proof. **a)** Because $w_\ell - \delta u_\ell \geq (1-\delta)w_\ell$, (2NtwCond) implies (SMALLCE).
b)

$$
\begin{aligned}
u_h - p_h = u_h - \frac{(1-\delta)(\alpha_h + \alpha_\ell)(u_h - v_h)}{(1-\delta)\alpha_\ell + \alpha_h} &\geq \delta \frac{\alpha_h}{(1-\delta)\alpha_\ell + \alpha_h}(u_h - v_h) \\
&\geq \delta \frac{\alpha_h}{(1-\delta)\alpha_\ell + \alpha_h} \times \frac{(1-\delta)\alpha_\ell + \alpha_h}{(1-\delta)\alpha_h}(w_\ell - \delta u_\ell) \text{ (by (2NtwCond))} \\
&= \frac{\delta}{1-\delta}(w_\ell - \delta u_\ell) \geq \delta u_\ell = u_\ell - p_\ell.
\end{aligned}
$$

□

E The proof that (12) holds

We have assumed in the main text that condition (12) holds. In this appendix, we show that this must indeed be the case whenever a two network equilibrium exists.

We proceed by contradiction. If if did not hold, we would have $u_h - p_h < u_\ell - p_\ell$. As we will see, the results which we obtained based on the stability of the consumers in period 2 and onwards in 6.3 hold with h and ℓ inverted. We then show that these results are incompatible with the separation of the consumers in two different networks in the first period.

The proof of claim 1 can be reproduced with h and ℓ inverted and therefore

$$v_\ell + u_h - p_h < u_\ell - p_\ell. \tag{13 - HL}$$

Similarly, we have

$$p_h = u_h(1-\delta) \tag{8 - HL}$$

which implies

$$\Pi_h = \alpha_h u_h. \tag{9 - HL}$$

Reproducing the reasoning of claim 2:

$$-(\alpha_h + \alpha_\ell)(u_\ell - p_\ell - v_\ell) + \delta\Pi_2 = 0; \tag{cl. 2 - HL}$$

On the other hand, claim D 2 is not affected, as it based on period 1 deviation which attract all consumers

$$-(\alpha_h + \alpha_\ell)(u_h - p_2 - v_h) + \delta\Pi_2 = 0. \tag{cl. D 2}$$

Equations (9 - HL), (cl. 2 - HL) and (cl. D 2) give us three equations in four unknowns: Π_h, p_ℓ, Π_2 and p_2. We add the following

$$\Pi_2 = \alpha_h p_2 + \delta\Pi_h = \alpha_h p_2 + \delta\alpha_h u_h \tag{E 1}$$

From (cl. D 2) and (E 1) we obtain

$$\begin{aligned} &-(\alpha_h + \alpha_\ell)[(u_h - p_2) - v_h] + \delta\left[\alpha_h p_2 + \delta\alpha_h u_h\right] = 0 \\ &\qquad\iff [(1+\delta)\alpha_h + \alpha_\ell]p_2 = (\alpha_h + \alpha_\ell)(u_h - v_h) - \delta^2\alpha_h u_h \\ &\qquad\qquad\iff p_2 = \frac{(\alpha_h + \alpha_\ell)(u_h - v_h) - \delta^2\alpha_h u_h}{(1+\delta)\alpha_h + \alpha_\ell}. \end{aligned} \tag{E 2}$$

By (cl. 2 - HL), we have

$$u_\ell - p_\ell - v_\ell = \frac{\delta\Pi_2}{\alpha_h + \alpha_\ell}, \tag{E 3}$$

and, by (8 - HL) and (cl. D 2)

$$\begin{aligned} &-(\alpha_h + \alpha_\ell)[u_h - p_h + u_h(1-\delta) - p_2 - v_h] + \delta\Pi_2 = 0 \\ &\qquad\implies u_h - p_h = \frac{\delta\Pi_2}{\alpha_h + \alpha_\ell} - u_h(1-\delta) + p_2 + v_h. \end{aligned} \tag{E 4}$$

Equations (13 - HL), (E 3) and (E 4) imply that we must have

$$-u_h(1-\delta) + p_2 + v_h < 0.$$

By (E 2), this is equivalent to

$$-u_h(1-\delta) + \frac{(\alpha_h + \alpha_\ell)(u_h - v_h) - \delta^2\alpha_h u_h}{(1+\delta)\alpha_h + \alpha_\ell} + v_h < 0.$$

Multiplying by $(1+\delta)\alpha_h + \alpha_\ell$, this yields

$$(u_h + v_h)\delta\alpha_\ell < 0$$

which establishes the contradiction.

We have therefore shown the following proposition:

Proposition E 1. *In any two network equilibrium, condition* (12) *must hold.*

E.1 Existence of S2 equilibria

Lemma E 1. *If $\delta\alpha_l - (1-\delta)\alpha_h > 0$, then an S2 equilibrium exists if and only if*

$$u_h - v_h \le \min\left[\frac{w_\ell - \delta u_\ell}{1-\delta}, \frac{(\alpha_h + \alpha_\ell)(\delta u_\ell - v_\ell)}{\delta\alpha_l - (1-\delta)\alpha_h}\right].$$

If $\delta\alpha_l - (1-\delta)\alpha_h < 0$, then an S2 equilibrium exists if and only if

$$\frac{(\alpha_h + \alpha_\ell)(\delta u_\ell - v_\ell)}{\delta\alpha_l - (1-\delta)\alpha_h} \le u_h - v_h \le \frac{w_\ell - \delta u_\ell}{1-\delta}.$$

In both cases, the profit of the incumbent is $(\alpha_h + \alpha_\ell)(u_h - v_h)$.

Proof. Condition (18) is binding if and only if $p_2 = (1-\delta)(u_h - v_h)$ and $\Pi_2 = (\alpha_h + \alpha_\ell)(u_h - v_h)$, and therefore

$$u_h - v_h \le \frac{w_\ell - \delta u_\ell}{1-\delta}. \tag{E 5}$$

By the discussion on page 23, Claim 2 holds and this implies $p_h = p_2$ and $\Pi_h = \alpha_h(u_h - v_h)$.

Off the equilibrium path, in order to attract the LNE consumers the H incumbent would have to charge a price p'_h which satisfies $v_\ell - p'_h > u_\ell - p_\ell = \delta u_\ell$. This is unprofitable only if $\Pi_h \ge (\alpha_h + \alpha_\ell)(v_\ell - \delta u_\ell) + \delta\Pi_2$, which is equivalent to

$$(\alpha_h + \alpha_\ell(\delta u_\ell - v_\ell) \ge (\delta\alpha_l - (1-\delta)\alpha_h)(u_h - v_h),$$

which proves the lemma. □

E.2 Existence of Sℓ equilibria

Lemma E 2. *An Sℓ type equilibrium exists if and only if $u_h - v_h \ge (w_\ell - \delta u_\ell)/(1-\delta)$ and*

$$\frac{(1-\delta)\alpha_\ell + \alpha_h}{(1-\delta)\alpha_h}(w_\ell - \delta u_\ell) \ge u_h - v_h$$
$$\ge \frac{(1-\delta)(\alpha_l + \alpha_h)}{\alpha_h}(v_\ell - \delta u_\ell) + \frac{\delta[\alpha_h(2-\delta) + \alpha_\ell(1-\delta)]}{(1-\delta)\alpha_h}(w_\ell - \delta u_\ell). \tag{E 6}$$

In every period the incumbent charges $w_\ell - \delta u_\ell$ and its equilibrium profit Π_2 is equal to $(\alpha_h + \alpha_\ell)(w_\ell - \delta u_l)/(1-\delta)$.

Proof. The fact that (17) is binding immediately yields the value of p_2 and of the profit of the incumbent.

The left most inequality is the consequence of the fact that the Incumbent must have no incentive to increase the price in such a way that it sells only to the HNE consumers. The lowest price that an entrant would be willing to charge in order to attract all the consumers is $-\delta\Pi_2/(\alpha_h+\alpha_\ell)$, and therefore the incumbent can price up to $(u_h - v_h) - \delta\Pi_2/(\alpha_h+\alpha_\ell)$ and sell to the HNE consumers. In subsequent periods, it will set the same price by claim 2. Therefore, this deviation is unprofitable, only if

$$\Pi_2 \geq \frac{1}{1-\delta} \times \alpha_h \times \left(u_h - v_h - \delta \frac{\Pi_2}{\alpha_\ell + \alpha_h} \right), \tag{E 7}$$

which is equivalent to the left most inequality of (E 6).

Off the equilibrium path, consumers stay separated. In order to attract the LNE consumers away from the L incumbent, the H incumbent must announce a price not larger than $v_\ell - u_\ell + p_\ell = v_\ell - \delta u_\ell$. This is not profitable only if

$$\Pi_h \geq (\alpha_h + \alpha_\ell)(v_\ell - \delta u_\ell) + \delta\Pi_2,$$

where Π_h is the profit of the H incumbent, and equal to the right hand side of (E 7), which is equivalent to the right most inequality in (E 6).[18] Notice first that, when, out of equilibrium, the consumers were separated in two different networks in period $t-1$, it is not the L incumbent which attracts all the consumers in period t. Indeed, this would be profitable only if $p_\ell \geq -\delta\Pi_2/(\alpha_h+\alpha_\ell)$. However, we must also have $p_\ell+(u_h-v_h) < 0$; otherwise the H incumbent would find it profitable to keep the HNE consumers.[19] Therefore $u_h - v_h < -p_\ell \leq \delta\Pi_2/(\alpha_h+\alpha_\ell)$, which contradicts (20) as it implies $\Pi_2 \leq (\alpha_h+\alpha_\ell)(u_h - v_h)$. This implies that (19) is necessary and sufficient for preventing a deviation by an entrant which would attract the LNE consumers. Because (20) is necessary and sufficient to prevent an entrant to attract all the consumers, the lemma is proved. □

E.3 Existence of T2 type equilibria

We start with T2 equilibria and prove the following lemma.

[18] The other possible deviations are not profitable. If the H incumbent increases its price it looses all its consumers. Claim 2 shows that the L incumbent cannot deviate.

[19] It is clear that we must have $p_\ell + (u_h - v_h) \leq 0$. If we had $p_\ell + (u_h - v_h) = 0$, the H incumbent would keep the HNE consumers if it charged 0 and there is no Nash equilibrium in which the L incumbent can attract the HNE consumers profitably.

Lemma E 3. *A T2 equilibrium exists if and only if*

$$\delta\alpha_l - (1-\delta)\alpha_h > 0$$

$$\frac{(\alpha_l+\alpha_h)(u_l-v_l)}{\delta\alpha_l-(1-\delta)\alpha_h} \le u_h - v_h \le w_l/(1-\delta).$$

The equilibrium profit of the incumbent is $(\alpha_h+\alpha_\ell)(u_h-v_h)$.

Proof. In a T2 equilibrium, the Incumbent can never profitably raise its price and keep just the HNE consumers, since $(u_h-v_h) \le w_\ell$. So, we just need to see if the Incumbent is willing to bring back the LNE consumers if the consumers are ever are separated. To attract the LNE consumers, p_h must be less than $u_l - v_\ell$, since the firm with the LNE consumers charges 0.[20] So, it must be the case that $p_h \le v_\ell - u_\ell$.[21] Furthermore, the Incumbent must choose a price that will induce the HNE consumers to stay on its network instead of joining an entrant network. An entrant is willing to price down to $-\delta(u_h - v_h)$ if it can attract all consumers. To keep the HNE consumers instead of joining an entrant if an entrant tries to attract all consumers then $p_h \le v_h + u_h(1-\delta)$. Thus, the binding constraint is $p_h \le v_\ell - u_\ell$. If the H incumbent deviates and just keeps the HNE consumers, it charges $(1-\delta)(u_h-v_h)$. Thus, for the H incumbent to prefer to attract the LNE consumers to deviating and keeping only the HNE consumers we need $(\alpha_\ell+\alpha_h)(v_\ell-u_\ell+\delta(u_h-v_h)) \ge \alpha_h(u_h-v_h)$ or $(u_h-v_h)(\delta\alpha_\ell-(1-\delta)\alpha_h) \ge (\alpha_\ell+\alpha_h)(u_\ell-v_\ell)$. □

E.4 Existence of Tℓ type equilibria

Lemma E 4. *A Tℓ equilibrium exists if and only if*

$$\frac{w_\ell}{1-\delta} \le u_h - v_h \le \min\left[\frac{(\alpha_\ell+\alpha_h)(1-\delta)\left[v_l-u_l+\delta w_\ell/(1-\delta)\right]}{\alpha_h} + \frac{\delta w_\ell}{1-\delta}, \frac{(\alpha_\ell+\alpha_h)\{w_\ell(1+\delta)-\delta(v_l-u_l)\}}{\alpha_h}\right].$$

The equilibrium profit is $(\alpha_\ell+\alpha_h)w_\ell/(1-\delta)$.

[20] We assume that firms do not use weakly dominated strategies and it is clear that if the L firm charged a positive price and lost consumers that it could profitably deviate and lower its price.

[21] Recall that we are using the weak inequality definition of AC equilibria for this class of equilibria.

Proof. Off the equilibrium path, the incumbent must offer a price less than $v_l - u_l$, since the firm with the LNE consumers will price of 0.[22] The lowest price an entrant willing to offer if it attracts all consumers is $\delta w_\ell/(1-\delta)$. The Incumbent's price must not exceed $u_h - v_h - \delta w_\ell/(1-\delta)$ to prevent the HNE consumers from leaving its network if the LNE consumers join an entrant's network. Since $v_l - u_l < 0 < u_h - v_h - \delta w_\ell/(1-\delta)$, the incumbent will price at $v_l - u_l$ to have all the consumers on its network. If the Incumbent were to deviate and have only the HNE consumers on the market, then it will charge $u_h - v_h - \delta w_\ell/(1-\delta)$. Thus, for the Incumbent to prefer to bring all consumers onto its network we must have

$$(\alpha_\ell + \alpha_h)\left[v_l - u_l + \delta w_\ell/(1-\delta)\right] \geq \frac{\alpha_h}{1-\delta}\left[u_h - v_h - \frac{\delta w_\ell}{1-\delta}\right]. \tag{E 8}$$

Or,

$$\frac{(\alpha_\ell + \alpha_h)(1-\delta)\left[v_l - u_l + \delta w_\ell/(1-\delta)\right]}{\alpha_h} + \frac{\delta w_\ell}{1-\delta} \geq u_h - v_h.$$

On the equilibrium path, the Incumbent must prefer to keep all consumers to just keep the HNE consumers and then bringing them back on the network the following period at a price of $v_l - u_l$. So, we need

$$(\alpha_\ell + \alpha_h)w_\ell/(1-\delta) \geq \alpha_h(u_h - v_h) + \delta(\alpha_\ell + \alpha_h)\left[v_l - u_l + \delta w_\ell/(1-\delta)\right],$$

where the l.h.s. is the equilibrium profits, and the r.h.s. is the sum of profits in the defection period plus the discounted l.h.s. of expression E 8. This can be rewritten as

$$\frac{(\alpha_\ell + \alpha_h)w_\ell(1+\delta) - \delta(\alpha_\ell + \alpha_h)\left[v_l - u_l\right]}{\alpha_h} \geq u_h - v_h.$$ □

[22]See proof of Lemma E 3.