Optimal Apportionment

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Abstract

This paper provides a theoretical foundation which supports the *degressive proportionality principle* in apportionment problems. The core of the argument is that each individual derives utility from the fact that the collective decision matches her own will with some frequency, with marginal utility decreasing with respect to this frequency. Then classical utilitarianism at the social level recommends decision rules which exhibit degressive proportionality. The model is applied to the case of the 27 states of the European Union.

1 Introduction

1.1 Background

Consider a situation in which repeated decisions have to be taken under the (possibly qualified) majority rule by representatives of groups (e.g. countries) that differ in size. In this case, the principle of equal representation translates into a principle of proportional apportionment. In other words, if we require each representative to represent the same number of individuals,

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the number of representatives of a group should be proportional to its population. Arguments have been raised against this principle and in favor of a principle of *degressive proportionality*, according to which the ratio of the number of representatives to the population size should decrease with the population size rather than be constant.

The degressive proportionality principle is endorsed by most politicians and actually enforced, up to some qualifications, in the European institutions (Duff 2010a, 2010b, TEU 2010). It is sometimes termed the Lamassoure-Severin requirement, following the European Parliament Resolution on "Proposal to amend the Treaty provisions concerning the composition of the European Parliament," which was adopted on October 11, 2007 after the report by Lamassoure and Severin (2007). On that occasion, it was noted that the treaties and amendments of the European Union has been referring to degressive proportionality "without defining this term in any more precise way." The October 2007 Resolution stated:

> [The European Parliament] considers that the principle of degressive proportionality means that the ratio between the population and the number of seats of each Member State must vary in relation to their respective populations in such a way that each Member from a more populous Member State represents more citizens than each Member from a less populous Member State and conversely, but also that no less populous Member State has more seats than a more populous Member State.

It is known that, in the case of a Parliament, in which each member must have one and only one vote, the degressive proportionality requirement is impossible to satisfy exactly, due to unavoidable rounding problems (see for instance Cichocki and Życzkowski, 2010). But if one seeks to respect the principle "up to one", or "before rounding", then many solutions become available, among which one has to choose (Ramírez-González, Palomares and Marquez 2006; Martínez-Aroza and Ramírez-González 2008; Grimmet *et al.* 2011). Such is also the case (rather obviously) if one allows for fractional weights.

The same principles formally apply to the case where a country is represented by a number of representatives, each of whom is given one vote, and to the case where a country is represented by a single delegate who is given a weight in relation to the country size. We shall refer to the two cases as a *Parliament* and a *Council*.

This paper applies Normative Economics to Politics. Its aim is to justify the principle of degressive proportionality by an optimality argument and thereby to suggest the computation of optimal weights in specific instances, optimal weights which will be degressively proportional. Here is a sketch of the argument.

1.2 Illustration of the argument

The argument in favor of degressively proportional apportionment is based on the maximization of an explicit utilitarian social criterion. To evaluate a constitutional rule at the collective level, one has to describe how the society evaluates the fact that the will of each citizen is reflected in the social decision under the rule. Assume that a series of collective decisions (issues) are taken independently under the rule, and that each individual's utility, say $u(t)$, is a function of the number of issues for which the decision matches her own will. Since the issues are independent, von Neumann-Morgenstern expected utility derived from the decision rule is a function of the frequency with which her will is implemented under the rule, say $\psi(p)$.[1] We assume that u has positive and decreasing marginal values. Then, it is straightforward to show that ψ is also increasing and concave. Asymptotically, concavity of these two functions are equivalent. The intuition is that, when the number of issues T is large, the expected payoff is approximated by $u(Tp)$.

The social objective is simply the sum of such individual utilities. The argument can be explained with a very simple example.

Suppose there are only two countries, of size n_1 and n_2, with $n_1 < n_2$. Then, the majority rule gives full power to the big country. When the two countries agree on which decision to take, they are both satisfied, but when they disagree, country 1, the small one, is never satisfied. Intuition, in that case, recommends that the power to decide should be occasionally given to the small country. To be more specific, suppose that binary decisions have to be taken according to the same decision rule. Among these decisions, a fraction α is controversial in the sense that the two countries disagree. Suppose also, for the simplicity of the example, that the citizens within each country always agree on their best choice.

Under the majority rule, a citizen of country 2 is satisfied with probability 1, and a citizen of country 1 is satisfied with probability $1 - \alpha$. The social welfare, defined as the sum of individual utilities, is:

$$n_1\psi(1-\alpha) + n_2\psi(1),$$

[1] This frequency p is determined as a function of the decision rule. In the power-measurement literature, it is called the Rae Index (Rae (1969)).

because the will of the small (resp. big) country's citizens is fulfilled with probability $1-\alpha$ (resp. 1).

Imagine, for the sake of illustration, that when they disagree, the decisions are delegated at random to one or the other country with respective probabilities q_1 and $q_2 = 1 - q_1$. Then the frequency of a decision opposed to country 1's will is αq_2, and the social value is:

$$U(q_1) = n_1 \psi(1 - \alpha q_2) + n_2 \psi(1 - \alpha q_1).$$

If ψ is linear then the maximum of utility is achieved for $q_1 = 0$, that is the majority rule, but if ψ is concave the maximum may be achieved at some interior point $0 < q_1 < 1$. More exactly, the condition for an interior optimum is that the marginal social benefit at point $q_1 = 0$,

$$U'(0) = \alpha \left(n_1 \psi'(1-\alpha) - n_2 \psi'(1) \right),$$

be positive, that is:

$$\frac{n_1}{n_2} > \frac{\psi'(1)}{\psi'(1-\alpha)}.$$

Such a condition is satisfied if the two countries are not too different in size, or if the marginal utility ψ' is rapidly decreasing with the probability p. In that case the optimal value of q is some number between 0 and 1 such that:

$$n_1 \psi'(1 - \alpha q_2) = n_2 \psi'(1 - \alpha q_1).$$

Thus majority rule is sub-optimal. The optimal voting rule in this two-country example involves randomization. In what follows, randomization will be required only in case of exact equality between the total weights of countries in favor of the two alternatives, an event which is unavoidable but occurs rarely among many countries.[2] The point here is that, when marginal utilities are decreasing with p, the optimal organization entails giving way to a smaller country relatively more often than proportionality would suggest. We will build a stochastic model to render the above ideas and apply it to the 27 countries of the European Union.

1.3 Adjacent literature

Most of the existing literature on the subject deals with the measurement of voting power and the tricky combinatorics arising from the different ways

[2] In the numerical example that we will take, the 27 Member States of the European Union, we will obtain that the probability of a tie is on the order of 10^{-7}.

to form a winning coalition with integer-weighted votes; see the books by Felsenthal and Machover (1998) and Laruelle and Valenciano (2008). Our focus is different, as can be seen from the two-country example above. The point made in the present paper rests on the non-linearity of ψ. It should be contrasted with the other contributions which derive an optimal rule from an explicit social criterion.

The first, and now classical, argument proposed in favor of degressive proportionality rests on statistical reasonings leading to the Penrose Law, which stipulates that the weight of a country should be proportional to the square root of the population rather than to the population itself, a pattern that exhibits degressive proportionality (Penrose 1946). The mathematical reason why the square root appears in this literature is linked to the assumption made that, within each country, voters' opinions are independent random variables[3] (see Felsenthal and Machover (1998); Ramirez *et al.* (2006); Słomczyński and Życzkowski (2010); Maaser and Napel (2011)). The political argument is that, in a world where frontiers have no link with the citizens' opinions, the representatives may as well be selected at random with no reference to these countries, but if representatives have to be chosen country-wise, then the focus should be on the statistical quality of the representation of the country by its constituents as a function of the size of the country. This argument is different from the one put forth in the present paper.

In Theil (1971), the objective is to minimize the average value of $1/w_{c(i)}$, where $w_{c(i)}$ is the weight of the country to which individual i belongs. This objective is justified as follows by Theil and Schrage (1977): "...let us assume that when such a citizen expresses a desire, the chance is w_i that he meets a willing ear. This implies that, in a long series of such expressed desires, the number of efforts per successful effort is $1/w_i$. Obviously, the larger this number, the worse the Parliament is from this individual's point of view. Our criterion is to minimize its expectation over the combined population." Minimizing this objective yields weights which are proportional to the square root of the country size.

In Felsenthal and Machover (1999), the objective is the mean majority deficit, that is the expected value of the difference between the size of the majority camp among all citizens and the number of citizens who agree with the decision. In Le Breton, Montero and Zaporozhets (2010) the objective is to get as close as possible to a situation in which all citizens have the same

[3]The realized sum of n independent random variables is approximated by its mathematical expectation up to statistical fluctuations of the order $\sqrt{n}$.

voting power, as measured by the nucleolus of the voting game, a concept derived from cooperative game theory. Feix et. al. (2011) focuses on the majority efficiency, which is known as Condorcet efficiency in Social Choice Theory.

In Barberà and Jackson (2006), and Beisbart and Bovens (2007) the optimality is with respect to a sum of individual utilities, as in the present paper. The basic message of these papers is that country weights should be proportional to the importance of the issue for the country as a whole. In simple settings this provides weights which are simply proportional to the population size. In these contributions, the individual utilities to be summed at the collective level are, by assumption, linear in p. Such is also the case of Beisbart and Hartman (2010), who study the influence of inter-country utility dependencies for weights proportional to some power of the population sizes. This argument in favor of proportionality, called *pure majoritarian* in Laslier (2012), is different from what we wish to highlight here. If we could know in advance the importance for the various countries of the various issues to be voted upon, then we should change the countries' weights accordingly. Of course this is not possible at the constitutional stage, but notice that part of this intuition is endogenized in the setting we propose, along the following reasoning.

Start from weights strictly proportional to the population. Larger countries are more often successful in that game. Therefore the outcome of the system is that a citizen (with concave utility) of a larger country is in a situation of lower marginal utility than a citizen of a smaller country. It may therefore be efficient to distort the weights in favor of the smaller countries if the small loss of the many citizens in the larger countries is more than compensated by the larger benefit for the citizens of the small countries. The optimal weights should thus exhibit degressive proportionality.

1.4 This paper

Many existing apportionment rules show degressive proportionality, in principle and in fact.[4] The main contribution of this paper is to provide a theoretical foundation for the principle of degressive proportionality, which is not sensitive to knife-edge assumptions such as independence or linearity.

[4] Leading examples are the US Electoral College, the European Union Council of Ministers, and the European Parliament. In countries with bicameral legislature, the upper house often uses equal representation while the lower house uses proportional representation. In combination, legislative power can be considered to be distributed with degressive proportionality.

Penrose's square-root law is not robust in the following two aspects. First, it hinges on the assumption that the voters' preferences are independent random variables. Common sense suggests that this assumption is far from plausible. Even if not perfectly correlated, citizens of a country tend to have common interest because of geography, culture, economy, etc. The independence assumption is also empirically rejected by Gelman, Katz and Bafumi (2004). To see that the independence assumption is crucial to obtaining Penrose Law, consider a group with population n. If there is a slight correlation in the preferences, we can show by an elementary computation[5] that the standard deviation of total utility in the group grows by the order of n. As will be shown later, the optimal weights which maximize the utilitarian social welfare are not proportional to the square-root of the population in such a case[6]. Only in the situation where voters' opinions are perfectly independent, the standard deviation grows by the order of $\sqrt{n}$.

Second, and maybe more critically, it is commonly assumed in the apportionment literature that each individual's utility is additively separable over the issues, that is, the utility is a linear function of the number of successes. This assumption leads to convenient properties. For example, expected utility is an affine transformation of the Banzhaf-Penrose index, which allows us to employ various results obtained in the power-measurement literature. However, a more reasonable assumption may be that the marginal utility decreases as the utility level increases, instead of being constant.

Our model brings the decreasing marginal utility assumption commonly used in Economics into Political Science. When the marginal utility is decreasing, the marginal importance of the issue at stake for big countries is relatively smaller, since they have higher chances of winning. Degressive proportionality is obtained as the result of equalizing the marginal utility of the individuals across the countries with heterogeneous sizes so that the utilitarian social welfare is maximized. In Barberà and Jackson (2006), the optimal weights are shown to be proportional to the stake, which depends solely on the utility distribution, exogenously given, independent of the decision rule. In our model, the stake for each country is determined endogenously since it depends on the frequency of successes, which in turn depends on the decision rule. Indeed, we show that the optimal weight is proportional to the endogenously determined stake. In this sense, our result

[5]Suppose that $var\left(u_i\right) = \sigma^2$ for $\forall i$ and $cov\left(u_i, u_j\right) = \sigma_\varepsilon^2$ for $\forall i, \forall j \neq i$. Then $var\left(\sum_i u_i\right) = n\sigma^2 + n\left(n-1\right)\sigma_\varepsilon^2$ increases by the order of n^2 iff $\sigma_\varepsilon \neq 0$.

[6]This is in accordance with the findings of Beisbart and Hartmann (2010), who show by simulation that the interest group model (perfect correlation) of Beisbart and Bovens (2007) is stable, while the aggregate (independent) model is not.

is consistent with Barberà and Jackson (2006). However, precisely because of this endogeneity, we show that the optimal weight should exhibit degressive proportionality. As a consequence, we provide a theoretical foundation for the principle of degressive proportionality, which is not sensitive to either the linearity of the preference or the independence assumption of the preference distribution, as is in Penrose Law.

In section 2, we describe the model and we discuss in details the relevance of the concavity assumption. Our main theorem is given in Section 3, where we first compute the optimal weights for two extreme, benchmark, cases, and then show that the general cases fall between them. Section 4 applies the theory to the EU case. Section 5 concludes. All proofs are provided in the Appendix.

2 The Model

2.1 Objective

There are C countries, and country $c \in \mathcal{C} = \{1, \cdots, C\}$ has a population of n_c individuals. Let $n = \sum_c n_c$ be the total population. We consider binary decision problems. The status quo is labeled as 0, and the alternative decision is labeled as 1. Each individual i announces her favorite decision $X_i \in \{0, 1\}$, and the final decision is denoted by $d \in \{0, 1\}$.[7] A voting rule is used to take all such decisions so that, from the opinions stated by the voters, the final decision is in accordance with i's preference with some frequency:

$$p_i = \Pr[X_i = d].$$

This frequency is the object of preference for individual i, and we denote by $\psi(p_i)$ the utility she attaches to p_i. We suppose that all individuals share the same utility function ψ and make the usual assumption of decreasing marginal utility. Notice that this last assumption has, in our setting, the following natural interpretation.

Imagine that, given the voting rule, the society is going to face a sequence of independent issues. Let T be the total number of independent decisions which will matter for individual i. Imagine that i receives 1 unit of money transfer every time the collective decision matches her will, and let t be the total payoff of this individual. A risk-averse individual evaluates the possible

[7]Since there are only two alternatives, voting for the favorite decision is a dominant strategy. The voting game is dominance-solvable and truthful voting is the unique admissible strategy.

payoffs using a von Neumann and Morgenstern utility function $u(t)$, concave in t. The expected utility is then a function of p_i, the frequency with which each social decision matches with the individual's will:

$$\psi(p_i) = \sum_{t=0}^{T} \binom{T}{t} p_i^t (1 - p_i)^{T-t} u(t).$$

Proposition 1 *Suppose u is increasing and concave. Then, ψ is increasing and concave.*

From now on, we will directly use the function ψ, and we do not need to refer to v. The social goal is defined from the individuals' satisfaction in an additive way:

$$U = \sum_i \psi(p_i).$$

This means that the collective judgment is based only on individual satisfaction with no complementarity at the social level. Notice that, because ψ is concave, the maximization of U tends to produce identical values for the individual probabilities p_i. Here the egalitarian goal is not postulated as a collective principle but follows from the assumption on individuals' utility.[8]

2.2 Egalitarianism

The concavity of ψ can as well be interpreted as the expression of the aversion to inequality of the social planner (the constitutionalist). If the numbers u_i are money-metric measurements of i's welfare, the social planner may have, as her social objective, the maximization of a Kolm-Atkinson index of the form:

$$W = \sum_i \psi(u_i).$$

The social objective W is egalitarian if any Pigou-Dalton transfer increases its value. We recall without proof the following result, well-known from the theory of inequality measurement (see Dutta 2002). The social objective is egalitarian if and only if the function ψ is concave, for instance $\psi(u_i) = u_i^\alpha$ for $0 < \alpha < 1$.

Proposition 2 *W is egalitarian if and only if ψ is concave.*

[8]One exception is allowed later in this paper. In Subsection 3.1, we consider the egalitarian case as a benchmark, where U is defined by the Rawlsian criterion.

As put forth by Bentham (1822)[9]:

> All inequality is a source of evil – the inferior loses more in the account of happiness than the superior is gained.

This Social Welfare point of view can be philosophically grounded on an intrinsic inequality aversion of the social planner reflected in the formula $W = \sum_i \psi(u_i)$, as well as on a purely utilitarian preference that takes into account decreasing marginal utility. These two concepts deserve a unified name, and it is called the *utilitarian-egalitarian argument* in Laslier (2012).

An extreme, degenerated case is the Rawlsian objective of maximizing the well-being of the worst-off individual. This case is obtained when α tends to 0, and we will show that it implies identical weights for all countries; see Proposition 5 below.

Another possible interpretation of the concavity of ψ, based on the submodularity of the underlying preferences, is given in the Appendix.

2.3 Probabilistic Opinion Model

In order to model the correlations between individual opinions, we use a probabilistic opinion model. More precisely, we assume that individual preferences $(X_i)_{i=1}^n \in \{0,1\}^n$ are drawn from a joint distribution $f(X_1, \cdots, X_n)$. We focus our attention on a class of distributions with the following two properties: (i) all individuals are ex ante unbiased with respect to the two alternatives,[10] and (ii) the preferences are positively correlated within countries, but independent across countries. We use the parameter μ to describe the intra-country correlation.

Suppose that voters in country c receive a country-specific signal $Y_c \in \{0,1\}$, and each voter i in her country $c(i)$ forms an opinion conditionally on $Y_{c(i)}$. The conditional probability μ for a voter to follow her country-specific signal is the same for every voter in every country, and for both alternatives:

$$\mu = \Pr\left[X_i = x | Y_{c(i)} = x\right], \ x = 0, 1.$$

We assume that μ is larger than 1/2, so that Y_c can indeed be interpreted as the general opinion in country c. We could have started the other way

[9] Quoted by Trannoy (2011).

[10] If there is a known bias to one of the two alternatives, welfare-maximizing decision is rather obvious, since the right choice is the preferred alternative and thus the society has little interest to take a vote. Most interesting are the cases in which the voters are unbiased ex ante so that voting works as a device to aggregate preferences.

round and, instead of taking the country's general opinion as a primitive, we could have specified a probability distribution for the correlated opinions of the citizens of country c. Then Y_c would be defined as the majority value of the variables X_i for $i \in c$. But since we are dealing with large numbers of individuals (between .4 and 100 millions per country), it is much simpler to take Y_c as the primitive.

The variables $Y_c \in \{0, 1\}$ are assumed to be randomly distributed and independent across countries. This assumption, which is in line with standard assumptions in the literature, captures the idea that the coalitions of countries which share a common view on a question show no systematic pattern. This point can be defended in two ways. First, the way some countries' interests are aligned is itself variable: on some issues larger countries are opposed to smaller ones, other issues divide rich countries against poor ones, East against West, etc. Second, in the spirit of constitutional design, one may wish by principle to be blind to current correlations of interest among some countries and give a strong interpretation to the idea that countries are independent entities. (See Laruelle and Valenciano, 2005, and Barr and Passarelli, 2009.) We will discuss in the conclusion the consequences of relaxing this assumption.

2.4 Weighted Voting Rules

Each country c has a weight w_c. Without loss of generality we can normalize the weights so that

$$\sum_{c=1}^{C} w_c = 1.$$

We introduce two weighted decision models. In the **Council** model, the country has in fact a unique representative, who votes according to the country's general opinion Y_c. Then the decision $d = 1$ is taken if the total weight of the countries who voted for the proposition is strictly larger than a threshold t, and the decision $d = 0$ is taken if the total weight of the countries who voted against the proposition is strictly larger than $1 - t$. For $Y = (Y_c)_{c \in \mathcal{C}}$,

$$d^{\text{council}}(Y|w,t) = \begin{cases} 1 \text{ if } \sum_c w_c Y_c > t \\ 0 \text{ if } \sum_c w_c Y_c < t \end{cases} .$$

When the threshold is exactly met, $d = 1$ is taken with a pre-specified probability that depends on the realization of Y.

In the **Parliament** model, the country c has w_c representatives, who vote in proportion of the voters' opinions. Then, the number of votes at the parliament in favor of $d = 1$ is $w_c\mu$ for a country such that $Y_c = 1$, and is $w_c(1-\mu)$ for a country such that $Y_c = 0$. Here, the decision $d = 1$ is taken if the total weight of the representatives who voted for is larger than the threshold t:

$$d^{\text{parliament}}(Y|w,t) = \begin{cases} 1 \text{ if } \sum_c w_c\left(\mu Y_c + (1-\mu)(1-Y_c)\right) > t \\ 0 \text{ if } \sum_c w_c\left(\mu Y_c + (1-\mu)(1-Y_c)\right) < t \end{cases}.$$

Indeed, these two models are equivalent up to the threshold.

Proposition 3 $d^{parliament}(Y|w,t) = d^{council}\left(Y \middle| w, \frac{t-(1-\mu)}{2\mu-1}\right)$.

Proof is immediate.[11] If $t < 1-\mu$ or $t > \mu$ in the Parliament model, the decision is either $d = 0$ or $d = 1$ regardless of the realized values of Y. It is as if $t < 0$ or $t > 1$ in the Council model.

Note that if the threshold is $1/2$, the two models are identical. When a weighted voting rule has the threshold $t = 1/2$, we call it a *weighted majority rule*. Weighted majority rules keep the symmetry between the two alternative decisions, up to in the limit case where votes are exactly split. Notice that some voting rules are not even weighted (e.g. UN Security Council). However, it will be proven that the optimal voting rules are indeed weighted majority rules, i.e. weighted, with threshold 1/2.

The central idea of this paper is the degressive proportionality.

Definition 1 *Weights are said to exhibit degressive proportionality to the population if*

$$n_c < n_{c'} \Rightarrow w_c \leq w_{c'} \text{ and } \frac{w_c}{n_c} \geq \frac{w_{c'}}{n_{c'}}.$$

2.5 Questions

The same question can be asked for the Council model and for the Parliament model. The objective is to maximize the expected collective welfare. Given are: the population figures ($n = (n_c)_{c\in\mathcal{C}}$), the intra-country homogeneity (μ), and the utility function (ψ). For each model $M \in \{\text{Council, Parliament}\}$, the expected social welfare is:

$$U(w,t) = \sum_i \psi(p_i) = \sum_c n_c \psi\left(\pi_c^M(w,t)\right), \tag{1}$$

[11] $\sum_c w_c\left(\mu Y_c + (1-\mu)(1-Y_c)\right) > t \Leftrightarrow \sum_c w_c Y_c > \frac{t-(1-\mu)}{2\mu-1}$.

with

$$\pi_c^M(w,t) = \Pr\left[X_i = d^M(w,t)\right] \tag{2}$$

for any citizen i in country c. Therefore, our problem is to choose optimal weights w and the threshold t:[12]

$$\max_{(w,t)} U(w,t). \tag{3}$$

3 Optimal weights in theory

In this Section, we first characterize the optimal weights for two extreme cases: linear utility and the Rawlsian social welfare in Section 3.1. Our main result, obtained in Section 3.2, is stated in the general framework of probabilistic simple games. This class of games contains precisely describes how ties are broken, and also contains non-weighted games. We prove that the optimal games in that class are weighted, with weights which exhibit degressive proportionality.

3.1 Two benchmarks

The linear case

Suppose that the function ψ is linear; without loss of generality we can take $\psi(p) = p$. Then the optimal weights are simply proportional to the population.

Proposition 4 *If $U = \sum_i p_i$ the optimal decision rule is a weighted majority, with weights w_c proportional to the population.*

This result is compatible with the existing models, such as Barberà and Jackson (2006) or Fleurbaey (2008). Notice that the result applies to any μ strictly larger than 1/2. If we allow $\mu = 1/2$, then the model is equivalent to the aggregate (independence) model of Beisbart and Bovens (2007), in which the optimal weights are proportional to the square-root of the population. But even a slight degree of correlation in the distribution of preferences implies that the optimal weights are proportional to the population. Proposition 4 gives evidence which indicates that Penrose's square-root law hinges on the independence assumption when the utility function is assumed to be a linear function of the number of successes.

[12] For the description to be complete, the tie-breaking rule should be specified, although our main focuses are the weight vector and the threshold. See Section 4 for further discussion.

The Rawlsian case

On the other hand, suppose that the social criterion gives absolute priority to the worst-off individual, what is sometimes called the MaxMin, or Rawls's criterion. Then the optimal weights are independent of country populations.

Proposition 5 *For any* $\mu > 1/2$*, if* $U = \min_i p_i$ *the optimal decision rule is the simple majority among countries: all countries have equal weight.*

The Rawlsian case corresponds to the limit where the concavity of ψ goes to infinity. Obviously, equal weight is an extreme example of degressive proportionality, where w_i/n_i decreases most rapidly among all degressively proportional rules.

3.2 Optimal Apportionment and Simple Games

We introduce the concept of weighted probabilistic simple games corresponding to the weighted voting rules. For each realization of C Bernoulli variables $(Y_1, Y_2, ..., Y_C)$, we can naturally associate the subset of countries (or coalition) for which the Bernoulli variable takes the value 1: $\{c|Y_c = 1\}$. For any of the 2^C possible coalitions, the social decision can be either to accept or to reject the proposal. The problem can thus be viewed as the selection of a subset $\Gamma \subset \mathcal{P}(\mathcal{C})$ of winning coalitions, the coalitions for which the proposal is accepted. For any Γ, the pair $(\mathcal{C}, \Gamma)$ is called a *simple game*.[13] In the corresponding voting rule d^Γ, the decision $d = 1$ is taken if and only if the coalition of countries which vote in favor of the proposal belongs to Γ:

$$d^\Gamma = 1 \Leftrightarrow \{c|Y_c = 1\} \in \Gamma.$$

Here we generalize the concept of simple games to allow probabilistic decision rules. For any coalition S, we define the probability $q(S)$ that the society accepts the proposition ($d = 1$) when the countries in coalition S vote for it. We call the corresponding function $q : \mathcal{P}(\mathcal{C}) \rightarrow [0, 1]$ a *probabilistic simple game*.

Denote by $\mathcal{PSG}$ the set of probabilistic simple games. Notice that any q in $\mathcal{PSG}$ can be uniquely assimilated to a vector in $[0, 1]^{2^C}$ (and vice versa). A simple game Γ is a probabilistic simple game q such that $q(S) = 1$ for any $S \in \Gamma$ and $q(S) = 0$ for any $S \notin \Gamma$. When the deterministic decisions in a probabilistic simple game can be represented by a system of weights, we say that it is a *weighted probabilistic simple game*:

[13] In what follows, we will omit $\mathcal{C}$ and simply write Γ.

Definition 2 *A probabilistic simple game q is weighted if there exists a vector of weights $w \in \mathbb{R}^C$ and a threshold $t \in [0,1]$ such that for any $S \subset \mathcal{C}$,*

$$\sum_{c \in S} w_c > t \;\Rightarrow\; q(S) = 1,$$
$$\sum_{c \in S} w_c < t \;\Rightarrow\; q(S) = 0.$$

The subset of coalitions for which the total weight equals the threshold is called the tie set: $T(w,t) = \{S \subset \mathcal{C} \mid \sum_{i \in S} w_i = t\}$. The restriction of q on $T(w,t)$ is called the tie-breaking rule.

The benefit of considering a probabilistic simple game is two-fold. First, as illustrated in the two-country example given in the Introduction, it may be optimal for the society to take a probabilistic decision. Second, if we consider only the (deterministic) simple games, we face a maximization problem in which we choose the set of winning coalitions Γ. Providing an analytical solution to such discrete problems is quite demanding, and computation for large values of C is practically impossible in general. Instead, by considering a larger set of games over the continous space $[0,1]^{2^C}$, we can provide an analytical solution.

Of course, the advantage is obtained at an expense of certain cost. A potential problem may be that by considering the entire set of probabilistic simple games, the optimal game may lie outside of the set of all weighted games. Our original motivation is to find the optimal weights, and indeed there exist many probabilistic simple games which are not weighted. However, in the following we show that the optimal games chosen over the entire set of probabilistic simple games are indeed weighted, and the weights exhibit degressive proportionality, provided that ψ is concave.

For any vector of weights $w \in \mathbb{R}^C$ and any threshold $t \in [0,1]$, we denote by $\mathcal{PSG}(w,t)$ the corresponding set of weighted probabilistic simple games. As is clear by definition, any weighted decision rule can be described as a weighted probabilistic simple game and vice versa. Especially, any weighted voting rule in the Council model can be described as a weighted probabilistic simple game $q \in \mathcal{PSG}(w,t)$. For the Parliament model, any weighted voting rule with weight vector w and threshold t can be described as a weighted probabilistic simple game $q \in \mathcal{PSG}\left(w, \frac{t-(1-\mu)}{2\mu-1}\right)$.[14]

For any vector of population $n = (n_c)_{c \in \mathcal{C}}$, intra-country homogeneity μ, concave utility function ψ, and the model M, we denote by (n, ψ, μ, M)

[14] See Proposition 3.

the corresponding utilitarian problem over the set of all probabilistic simple games:

$$\max_{q \in \mathcal{PSG}} \sum_{c \in \mathcal{C}} n_c \psi\left(\pi_c\left(q\right)\right) \tag{4}$$

where $\pi\left(q\right) = \left(\pi_1\left(q\right), ..., \pi_C\left(q\right)\right)$ is a function defined over $\mathcal{PSG}$, exactly in the same way as (2).

Proposition 6 *Let q^* be any solution of the problem $\left(n, \psi, \mu, M\right)$.*

(i) For any two countries c and c' in $\mathcal{C}$, $n_c < n_{c'} \Rightarrow \pi_c^* \leq \pi_{c'}^*$.

(ii) The associated vector of the frequency of success, $\pi^* = \pi\left(q^*\right)$ is the same for any q^*.

We now state the main result of the paper. We show that any solution to the utilitarian apportionment problem is weighted majority rule (i.e. threshold is 1/2), with the vector of weights which exhibits degressive proportionality.

Theorem 1 *Define the weight vector w^* so that w_c^* is proportional to $n_c \psi'\left(\pi_c^*\right)$, where π^* is uniquely determined in Proposition 6. Any solution q^* of $\left(n, \psi, \mu, M\right)$ is a weighted probabilistic simple game with the weights w^* and the threshold $t^* = 1/2$. Moreover, q^* is unique up to the tie-breaking rule.*

Since ψ is concave, an immediate corollary is the following:

Corollary 2 *The optimal weight w^* exhibits degressive proportionality.*

Moreover, since the optimal threshold is 1/2, the Council model and the Parliament model are equivalent (Proposition 3). We thus obtain another Corollary.

Corollary 3 *Given n, ψ and μ, the optimal weights are the same in both the Council model and the Parliament model.*

As we mentioned above, some probabilistic simple games cannot be described by any weighted voting rule, but any weighted voting rule can be described as a probabilistic simple game. Therefore, we have

$$\max_{(w,t)} U\left(w, t\right) \leq \max_{q \in \mathcal{PSG}} \sum_{c \in \mathcal{C}} n_c \psi\left(\pi_c\left(q\right)\right).$$

Theorem 1 implies that $\left(w^*, 1/2\right)$ is a solution of our original problem (3). It also provides a formula that characterizes the optimal weights $\left(n_c \psi'\left(\pi_c^*\right)\right)_{c \in \mathcal{C}}$

and threshold (1/2), although it is silent about the tie-breaking rule. Unfortunately, even with this formula, obtaining the exact values of the optimal weights is challenging, because the probabilities of success π_c^* depend, themselves, on the weights. In the next section, we propose a method to compute the optimal weights numerically.

4 Numerical results

4.1 Methodology

If the optimal tie-breaking rule is constant, say $q^*_{|T(w^*,1/2)} = \bar{q}$, the optimal vector of weights is a fixed point of the following application:

$$\Phi_{\bar{q}} : \; w \;\mapsto\; \left(n_c \psi' \left(\pi_c \left(w, \frac{1}{2}, \bar{q} \right) \right) \right)_{c=1,\ldots,C}$$

where $(w, 1/2, \bar{q})$ denotes the weighted probabilistic simple game associated to the vector of weights w, the threshold 1/2 and the constant tie-breaking rule $\bar{q}$.

The true optimal tie-breaking rule is unknown. (In fact, as can be deduced from the next proposition, any kind of tie-breaking rule can happen at the optimum.) However, we conjecture that fixing a given (possibly suboptimal) tie-breaking rule would not affect the results of the optimization of the weights. Therefore, we simply chose to limit our attention to the tie-breaking rule q^- that assigns a probability 0 to any coalition in the tie set. The solution to the apportionment problem is now a fixed point of a well defined application, Φ_{q^-}, which can be found numerically. Let $\hat{w}$ denote (when it exists) such a solution.

In order to support our conjecture, we provide a way to test, ex post, that the fixed point $\hat{w}$ is indeed very close to the true optimum w^*. First, we show the following proposition:

Proposition 7 *For any vector of weights w and tie-breaking rule $q_{|T(w,1/2)}$, let:*

$$n_c = \frac{w_c}{\psi' \left(\pi_c \left(w, \frac{1}{2}, q_{|T(w,\frac{1}{2})} \right) \right)}.$$

This vector of population n is such that the weighted probabilistic simple game $(w, 1/2, q_{|T(w,1/2)})$ is a solution to the utilitarian problem (n, ψ, μ, M).

This proposition provides an indirect way to check the quality of a numerical procedure; compare the actual vector of population to the vector of population for which the candidate game $(\hat{w}, 1/2, q^-)$ is, with certainty, the optimum. The results below confirm that adopting a fixed arbitrary tie-breaking rule does not affect the results of the optimization.

4.2 The European Union

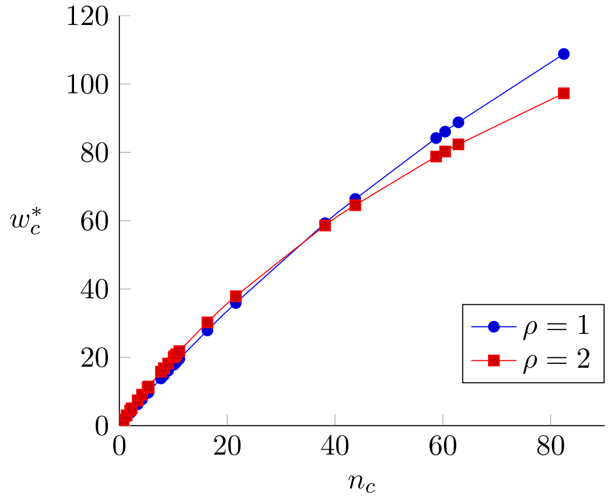
We apply our methodology to the European Union. There are $C = 27$ countries and 751 seats to be allocated[15]. We set the intra-country heterogeneity μ to 1. We use a constant relative risk aversion utility function (CRRA):

$$\psi(\pi_c) = \frac{\pi^{1-\rho} - 1}{1 - \rho}.$$

We perform the numerical optimization for $\rho = 1$ and $\rho = 2$. In each case we use the fixed point methodology to find the optimum weights and apply the test to verify the quality of the procedure, as described in the previous section. In both instances, we find that the vector of population for which our solution is optimal is very close to the actual vector of population. For each country, the difference is less than 1000 inhabitants in the first case ($\rho = 1$) and 3000 inhabitants in the second case ($\rho = 2$). These figures, which are clearly smaller than the typical measurement error in population estimates, indicate that our procedure performs very well. Moreover, we observe that the probability to be in the tie set at the optimum is extremely low (smaller than 10^{-7}), which gives further confidence that, and helps understand why, the restriction to a particular tie-breaking rule does not affect the optimization.

[15] This number of 751 seats is fixed by the Lisbon Treaty and will be applied in 2014.

Figure 1: *Optimal Weights*



We compare the optimal weights (both for $\rho = 1$ and $\rho = 2$) to the current weights at the European Parliament and the weights proposed by Pukelsheim (2010). The apportionment method used by Pukelsheim (2010), often called the Base+Prop method, uses a base of 6 seats per country, with the remaining 589 seats for proportional apportionment using standard rounding methods. The four vectors of weights are presented in Table 1 along with the vector of populations. The last column of Table 1 indicates the value of the probability $p_i = \Pr[X_i = d] = \pi_{c(i)}$ that the collective decision matches individual i's will. Naturally, the optimal utilitarian weights are such that this probability depends on the country and is larger in larger countries.

5 Conclusion

This paper gives a theoretical foundation for the principle of degressive proportionality in the optimal apportionment problem. We consider that the individual utility is a function of the frequency of success in binary decisions, and assume that marginal utility is decreasing. By doing so, we provide a proof which does not hinge on the independence assumption on the distribution of the individual preferences. We believe that our paper provides fundamental support for the degressive proportionality which is currently practiced in many apportionment problems.

	pop.	Lisbon	B+P	$\hat{w},\ \rho = 1$	$\hat{w},\ \rho = 2$	$\pi_c^*,\ \rho = 1$	$\pi_c^*,\ \rho = 2$
Germany	82.4	96	96	108.78	97.27	.712	.698
France	62.9	74	83	88.77	82.32	.666	.662
United Kingdom	60.4	73	80	86.04	80.22	.660	.658
Italy	58.76	73	77	84.16	78.77	.656	.654
Spain	43.76	54	59	66.33	64.51	.620	.624
Poland	38.16	51	52	59.24	58.57	.605	.612
Romania	21.6	33	32	35.90	37.88	.566	.572
Netherlands	16.3	26	26	27.89	30.22	.550	.557
Greece	11.1	22	20	19.54	21.82	.535	.541
Portugal	10.6	22	19	18.62	20.87	.533	.539
Belgium	10.5	22	19	18.52	20.77	.533	.539
Czech Republic	10.3	22	18	18.09	20.32	.532	.538
Hungary	10.1	22	18	17.80	20.01	.532	.538
Sweden	9.0	20	17	16.08	18.20	.529	.534
Austria	8.3	19	16	14.75	16.80	.526	.532
Bulgaria	7.7	18	15	13.82	15.80	.525	.530
Denmark	5.4	13	13	9.85	11.46	.518	.522
Slovak Republic	5.4	13	13	9.79	11.38	.518	.521
Finland	5.3	13	12	9.55	11.12	.517	.521
Ireland	4.2	12	11	7.70	9.04	.514	.517
Lithuania	3.4	12	10	6.26	7.40	.511	.514
Latvia	2.3	9	9	4.25	5.07	.508	.510
Slovenia	2.0	8	8	3.72	4.45	.507	.508
Estonia	1.3	6	8	2.50	3.02	.505	.506
Cyprus	0.8	6	7	1.43	1.74	.503	.503
Luxembourg	0.5	6	7	0.86	1.05	.502	.502
Malta	0.4	6	6	0.76	0.92	.501	.502

Table 1: Population (M.), Base+Prop rounded weights, optimal weights and individual probabilities

Our result includes two important benchmark cases in the literature: in the limit where the concavity diminishes (linear utility), the optimal weights are proportional to the population (except the knife-edge case of zero interdependence); e.g. Barberà and Jackson (2006), Fleurbaey (2008), and the interest group model in Beisbart and Bovens (2007). To the contrary, in the limit where the concavity goes to infinity (MaxMin utility), the optimal weights are equal for all countries. Obviously these two weight profiles are the extreme examples of degressive proportionality, and all the utility functions between the two examples above induce degressive proportionality in between.

These results have been obtained under the assumption that opinions are independent between countries. It should be clear that allowing for any kind of correlation between countries would destroy the result. For instance, suppose that the independence assumption holds except for a given subset of countries, which are, on the contrary, perfectly correlated. Then the above model applies if we treat this set of countries as one large country, summing the populations. Then the optimal weights per country have no reason to be degressively proportional. Nevertheless, it is true that if the optimal values of the probabilities p_i are increasing with the populations, then the optimal weights are degressively proportional. This point is proven in the appendix, as a remark in the proof of the main theorem. Such a paradoxical situation, where a larger country is satisfied less often than a smaller one, cannot happen under independence or if correlations between countries are small.

The next step is to investigate more general conditions which would support the degressive proportionality principle. For example, double correlation within the countries and within the political parties across the countries is a substantial issue in European politics. Integrating these aspects would be in the future research agenda.

A Appendix

A.1 Concavity of ψ (continued)

In addition to the arguments in Section 2.2, we introduce another way to interpret the concavity of ψ.

Facing the sequence of issues under a fixed voting rule, suppose that the individual utility is defined over the sequence of successes. The issues come in a sequence $t = 1, 2, \cdots$. Let $z^t \in \{0, 1\}$ be the success ($z^t = 1$) or failure ($z^t = 0$) at period t, and let $z = \left(z^t\right)_{t\in\mathbb{N}}$. Now utility u is defined on the equivalence class over the sequences z such that the following limit exists and is equal to p:

$$\lim_{T\to\infty} \frac{1}{T}\left(\sum_{t=1}^{T} z^t\right) = p. \tag{5}$$

This is equivalent to saying that the individual is indifferent to the order of successes/failures in the sequence, and only the frequency matters. Define ψ by $\psi(p) = u(z)$ for any sequence which satisfies (5). Then, we have the following proposition.

Proposition 8 *If u is submodular, then ψ is concave.*

Submodularity of u can be interpreted as the substitutability of different issues. An increase in the frequency of successful issues is more favorable when there are less successes among the other issues.

A.2 Proofs

Proof of Proposition 1. It is straightforward to show that:

$$\psi'(p) = T\sum_{t=0}^{T-1}\binom{T-1}{t}p^t(1-p)^{T-1-t}\left\{v(t+1)-v(t)\right\},$$

$$\psi''(p) = T(T-1)\sum_{t=0}^{T-2}\binom{T-2}{t}p^t(1-p)^{T-2-t}\left[\begin{array}{c}\{v(t+2)-v(t+1)\}\\ -\{v(t+1)-v(t)\}\end{array}\right].$$

Since v is increasing and concave, $v(t+1) - v(t) > 0$ and

$$\{v(t+2)-v(t+1)\} - \{v(t+1)-v(t)\} < 0.$$

Hence, $\psi'(p) > 0$ and $\psi''(p) < 0$ for $p \in (0, 1)$. ■

Proof of Proposition 4. The objective is $U = \sum_i \Pr[X_i = d]$. Conditionally on a realization of the vector of variables $(Y_c)_{c\in C} \in \{0,1\}^C$, the social utility of taking decision $d = 0$ or 1 is

$$U(d=0) = \sum_{c:Y_c=0} \mu n_c + \sum_{c:Y_c=1} (1-\mu) n_c,$$
$$U(d=1) = \sum_{c:Y_c=1} \mu n_c + \sum_{c:Y_c=0} (1-\mu) n_c,$$

so that $d = 1$ is strictly better if and only if $(2\mu - 1)\sum_{c:Y_c=1} n_c > (2\mu - 1)\sum_{c:Y_c=0} n_c$. Since $\mu > 1/2$, we know which decision d maximizes the criterion, that is majority rule: $d = 1$ if $\sum_{c:Y_c=1} n_c > \sum_{c:Y_c=0} n_c$ and $d = 0$ otherwise. This optimal rule is indeed a weighted majority rule with weight $w_c = n_c / \sum_{c'} n_{c'}$ and threshold 1/2. ■

Proof of Proposition 5. By Proposition 4, if $n_c = 1$ for all c, the simple majority rule with the equal weight maximizes the sum of the frequencies. That is, for any rule, $\sum_c \pi_c \leq Cp^{eq}$, where p^{eq} is the probability of winning under the equal weight. Now, suppose that $p^{eq} < \min_c \pi_c$. Then, $p^{eq} < \pi_c$ for all c, implying $Cp^{eq} < \sum_c \pi_c$, a contradiction. Therefore, $\min_c \pi_c \leq p^{eq}$ for any rule. Hence, $\max \min_c \pi_c \leq p^{eq}$. The maximum is attained by the equal weight. ■

Proof of Proposition 6. Let us remind that $\pi : \mathcal{PSG} \to [0,1]^C$ is the function defined in (4). For any individual i in country c,

$$\begin{aligned}\pi_c(q) &= \Pr[X_i = Y_c]\Pr[Y_c = d(q)] + \Pr[X_i \neq Y_c]\Pr[Y_c \neq d(q)] \\ &= 1 - \mu + (2\mu - 1)\Pr[Y_c = d(q)]. \end{aligned} \tag{6}$$

Given a probabilistic simple game q, $q(S)$ is the probability that $d = 1$ is chosen. Therefore,

$$\begin{aligned}\Pr[Y_c = d(q)] &= \sum_S \Pr(S)\left(q(S)\mathbf{1}_{\{c\in S\}} + (1-q(S))\mathbf{1}_{\{c\notin S\}}\right) \\ &= \sum_{\{S|c\in S\}} \Pr(S)\, q(S) + \sum_{\{S|c\notin S\}} \Pr(S)\,(1-q(S)) \end{aligned} \tag{7}$$

where $\Pr(S)$ denotes the probability that the set $\{c|Y_c = 1\}$ coincides with $S \subset \mathcal{C}$. Notice that π_c is affine in q. Hence, the image $\pi(\mathcal{PSG})$ is convex in $[0,1]^C$.

Since ψ is strictly concave, the maximization problem $\sum_c n_c \psi(\pi)$ subject to $\pi \in \pi(\mathcal{PSG})$ has a unique solution π^*. Any solution q^* of the problem (n, ψ, μ, M) satisfies $\pi^* = \pi(q^*)$.

Suppose now that there exists $c, c' \in \mathcal{C}$ with $n_c < n_{c'}$ and $\pi_c^* > \pi_{c'}^*$. Consider then $\widehat{q}$ defined by $\widehat{q}(\sigma_{cc'}(S))$, where $\sigma_{cc'}$ is the permutation of $\mathcal{C}$ that exchanges c and c'. We get $\pi_c(\widehat{q}) = \pi_{c'}^*$, $\pi_{c'}(\widehat{q}) = \pi_c^*$ and $\pi_k(\widehat{q}) = \pi_k^*$, $\forall k \neq c, c'$. Then, $\sum_{c \in \mathcal{C}} n_c \psi(\pi_c(\widehat{q})) > \sum_{c \in \mathcal{C}} n_c \psi(\pi^*)$, which contradicts the optimality of π^*. ■

Proof of Theorem 1. Let q^* be a solution, and $\pi^* = \pi(q^*)$ be the corresponding vector of frequency of success. By (6) and (7), we can explicitly compute the first order condition for $q(S)$:

$$\frac{\partial}{\partial q(S)} \sum_{c \in C} n_c \psi(\pi_c(q)) = (2\mu - 1) \Pr(S) \left(\sum_{c \in S} n_c \psi'(\pi_c(q)) - \sum_{c \notin S} n_c \psi'(\pi_c(q)) \right).$$

Hence, $\forall S \subset \mathcal{C}$,

$$\begin{aligned} \sum_{c \in S} n_c \psi'(\pi_c(q)) > \sum_{c \notin S} n_c \psi'(\pi_c(q)) \quad &\Rightarrow \quad q^*(S) = 1, \\ \sum_{c \in S} n_c \psi'(\pi_c(q)) < \sum_{c \notin S} n_c \psi'(\pi_c(q)) \quad &\Rightarrow \quad q^*(S) = 0, \end{aligned}$$

which is equivalent to:

$$\begin{aligned} \sum_{c \in S} n_c \psi'(\pi_c(q)) > \frac{1}{2} \sum_{c \in \mathcal{C}} n_i \psi'(\pi_c(q)) \quad &\Rightarrow \quad q^*(S) = 1, \\ \sum_{c \in S} n_c \psi'(\pi_c(q)) < \frac{1}{2} \sum_{c \in \mathcal{C}} n_i \psi'(\pi_c(q)) \quad &\Rightarrow \quad q^*(S) = 0. \end{aligned}$$

Defining the vector of weights w^* by $w_c^* = \dfrac{n_c \psi'(\pi_c^*)}{\sum\limits_{c' \in \mathcal{C}} n_{c'} \psi'(\pi_{c'}^*)}$ $\forall c \in \mathcal{C}$, we conclude that:

$$\begin{aligned} \sum_{c \in S} w_c^* > \frac{1}{2} \quad &\Rightarrow \quad q^*(S) = 1, \\ \sum_{c \in S} w_c^* < \frac{1}{2} \quad &\Rightarrow \quad q^*(S) = 0, \end{aligned}$$

meaning that the probabilistic simple game q^* is weighted and can be represented by the vector w^* and the threshold $1/2$: $q^* \in \mathcal{PSG}(w^*, 1/2)$. Furthermore, by Proposition 6, we know that for any $c, c' \in C$ with $n_c < n_{c'}$, $\pi_c^* \le \pi_{c'}^*$, which implies in turn that $w_c^*/n_c = \psi'(\pi_c^*) \le \psi'(\pi_{c'}^*) = w_{c'}^*/n_{c'}$ because of the concavity of ψ.

The last thing we need to show is that the vector w^* is increasing. Let c and c' be two countries such that $n_c \le n_{c'}$, and assume that $w_c^* = n_c\psi'(\pi_c^*) > n_{c'}\psi'(\pi_{c'}^*) = w_{c'}^*$.

As a first step, let us show that there always exists a coalition S such that $c \in S$, $c' \notin S$ and $q^*(S) < q^*(\sigma_{cc'}(S))$. By contradiction, assume that for any S which contains c but not c', $q^*(S) \ge q^*(\sigma_{cc'}(S))$. By (7),

$$\begin{aligned}
&\Pr[Y_c = d(q^*)] \\
&= \sum_{\{S|c,c' \in S\}} \Pr(S)\, q^*(S) + \sum_{\{S|c,c' \notin S\}} \Pr(S)\,(1 - q^*(S)) \\
&+ \sum_{\{S|c \in S, c' \notin S\}} \Pr(S)\, q^*(S) + \sum_{\{S|c \notin S, c' \in S\}} \Pr(S)\,(1 - q^*(S)).
\end{aligned}$$

Then,

$$\begin{aligned}
&\Pr[Y_c = d(q^*)] - \Pr[Y_{c'} = d(q^*)] \\
&= \sum_{\{S|c \in S, c' \notin S\}} \{\Pr(S)\,(2q^*(S) - 1) + \Pr(\sigma_{cc'}(S))\,(1 - 2q^*(\sigma_{cc'}(S)))\} \ge 0.
\end{aligned}$$

Note that $\Pr(S) = \Pr(\sigma_{cc'}(S))$. Using (6), this implies $\pi_c^* \ge \pi_{c'}^*$. By Proposition 6, we know that $\pi_c^* \le \pi_{c'}^*$. Therefore, $\pi_c^* = \pi_{c'}^*$, which implies $w_c^* < w_{c'}^*$, a contradiction.

Now, pick a coalition S containing c but not c' with $q^*(S) < q^*(\sigma_{cc'}(S))$. We define another game q' by:

$$\begin{aligned}
q'(S) &= q^*(S) + \varepsilon \\
q'(\sigma_{cc'}(S)) &= q^*(\sigma_{cc'}(S)) - \varepsilon \\
q'(T) &= q^*(T),\ \forall T \ne S, \sigma_{cc'}(S)
\end{aligned}$$

Then, we have $\pi'_c = \pi_c^* + \kappa$, $\pi'_{c'} = \pi_{c'}^* - \kappa$ and $\pi'_k = \pi_k^*$ for any $k \ne c, c'$ where $\kappa = 4(2\mu - 1)\Pr(S)\,\varepsilon$. Hence, $U(q') > U(q^*)$, contradicting the optimality of q^*. ■

Proof of Proposition 7. Let $w = (w_i)_{1 \le i \le c}$ be any vector of positive weights and $q^0_{|T(w,1/2)}$ any associated tie-breaking rule. The corresponding

probabilistic simple game is denoted $q^0 = q(w, \frac{1}{2}, q^0_{|T(w,1/2)})$.
We show that it is always possible to find a vector of populations n such that q^0 is optimal for the utilitarian problem (n, ψ, μ, M).
As a first step, we observe that the set of weighted probabilistic simple games $\mathcal{PSG}(w, \frac{1}{2})$ can be characterized as the solution to the following linear maximization problem:

$$\max_{q \in \mathcal{PSG}} \sum_{c \in \mathcal{C}} w_c \pi_c(q).$$

If we expand this expression we get:

$$\begin{aligned}
\forall q \in \mathcal{PSG}, \qquad \sum_{c=1}^{C} w_c \pi_c(q) &= \sum_{c=1}^{C} w_i \left(\frac{1}{2^C} \sum_{S|c \in S} q(S) + \frac{1}{2^C} \sum_{S|c \notin S} (1 - q(S)) \right) \\
&= \frac{1}{2^C} \sum_{S} \Big[q(S) \sum_{i \in S} w_c + (1 - q(S))(\sum_{c \notin S} w_c) \Big] \\
&= \frac{1}{2^C} \sum_{S} \sum_{c \notin S} w_c + \frac{1}{2^C} \sum_{S} q(S) (\sum_{c \in S} w_c - \sum_{c \notin S} w_c).
\end{aligned}$$

In order to maximize this quantity, it is indeed optimal to choose $q(S) = 1$ whenever $\sum_{c \in S} w_c > \frac{1}{2}$ and $q(S) = 0$ whenever $\sum_{i \in S} w_c < \frac{1}{2}$. The choice of $q(S)$ for the coalitions in the tie set $T(w, \frac{1}{2})$ does not affect the quantity. Therefore, we conclude that the solution set of the above maximization problem corresponds exactly to the set of weighted probabilistic simple games with weights w and threshold $\frac{1}{2}$: $\mathcal{PSG}(w, \frac{1}{2})$.

We define $P = \max_{q \in \mathcal{PSG}} \sum_{c} w_c \pi_c(q)$ and $\mathcal{D} = \{\pi \in (\mathbb{R}_+)^C | \sum_{c} w_c \pi_c \leq P\}$.

We remark that the maximum value of U over $\mathcal{D}$ is necessarily greater or equal to the maximum value of U over $\mathcal{PSG}$:

$$\max_{q \in \mathcal{PSG}} \sum_{c=1}^{C} n_c \psi(\pi_c(q)) \leq \max_{\pi \in \mathcal{D}} \sum_{c=1}^{C} n_c \psi(\pi_c).$$

We now exhibit a vector of populations n for which q^0 is solution to the second maximization problem. We know that the solution of this problem verifies the following conditions:

$$\begin{cases} \frac{n_c \psi'(\pi_c)}{n_{c'} \psi'(\pi_{c'})} = \frac{w_c}{w_{c'}} \quad \forall c, c' \in \mathcal{C} \\ \sum_{c \in \mathcal{C}} w_c \pi_c = P \end{cases}.$$

Setting $n_c = \frac{w_c}{\psi'(\pi_c(q^0))}$ for all $c \in \mathcal{C}$, $\pi(q^0)$ becomes solution to the maximization of U over $\mathcal{D}$. By the previous remark, we conclude that q^0 is solution to the utilitarian problem (n, ψ, μ, M). This achieves the proof. ■

Proof of Proposition 8. We first show that for any $p_1, p_2 \in \mathbb{Q}$ with $p_1 < p_2$, we have $u(p_1) + u(p_2) \leq 2u((p_1 + p_2)/2)$. To see that, let z be a sequence which repeats $\left(\underbrace{1, \cdots, 1}_{(p_1+p_2)/2}, 0, \cdots, 0\right)$, and let z' be a sequence which repeats

$$\left(\underbrace{0, \cdots, 0}_{(p_2-p_1)/2}, \underbrace{1, \cdots, 1}_{(p_1+p_2)/2}, 0, \cdots, 0\right).$$

Then, $u(z) = u(z') = (p_1 + p_2)/2$. Now, obviously $z \vee z'$ is a sequence which repeats

$$\left(\underbrace{1, \cdots, 1}_{p_2}, 0, \cdots, 0\right),$$

and $z \wedge z'$ is a a sequence which repeats

$$\left(\underbrace{0, \cdots, 0}_{(p_2-p_1)/2}, \underbrace{1, \cdots, 1}_{p_1}, 0, \cdots, 0\right).$$

By assumption, u is submodular: $u(z) + u(z') \geq u(z \vee z') + u(z \wedge z')$. Hence, $\psi(p_1) + \psi(p_2) \leq 2\psi((p_1 + p_2)/2)$. By continuity and monotonicity, ψ is concave. ■

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