Generalizing Envy-Freeness Toward Group of Agents

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Abstract

Envy-freeness is a well-known fairness concept for analyzing mechanisms. Its traditional definition requires that no individual envies another individual. However, an individual (or a group of agents) may envy another group, even if she (or they) does not envy another individual. In mechanisms with monetary transfer, such as combinatorial auctions, considering such fairness requirements, which are refinements of traditional envy-freeness, is meaningful and brings up a new interesting research direction in mechanism design.

In this paper, we introduce two new concepts of fairness called *envy-freeness of an individual toward a group*, and *envy-freeness of a group toward a group*. They are natural extensions of traditional envy-freeness. We discuss combinatorial auction mechanisms that satisfy these concepts. First, we characterize such mechanisms by focusing on their allocation rules. Then we clarify the connections between these concepts and three other properties: the core, strategy-proofness, and false-name-proofness.

1 Introduction

Fairness is an important criterion for analyzing mechanisms, and several concepts of it have been studied so far in economic theory. A concept called *envy-freeness* (or no-envy condition) has especially been widely discussed [Foley, 1967]. A mechanism is envy-free if no individual envies another individual. Recently, this concept has also been discussed in computer science to design fair task scheduling procedures in multi-machines models [Mu'Alem, 2009].

The traditional definition of envy-freeness only considers an individual envy toward another individual. However, in such domains as combinatorial auctions, it is natural to consider different types of envy. For example, assume agent 1 wants two items g_1 and g_2 together, where her valuation of $\{g_1, g_2\}$ is 10. But g_1 is allocated to agent 2 at payment 3, and g_2 is allocated to agent 3 at payment 4. Then agent 1 feels envy; she obtains nothing, and a group of agents, i.e., agents 2 and 3 obtained g_1 and g_2 and paid only 7 in total. She prefers the outcome of this group of agents to her own outcome.

In mechanisms without monetary transfer (e.g., cake-cutting), avoiding such types of envy seems impossible. Certainly, an individual prefers obtaining the entire cake rather than having only a piece of it. On the other hand, in mechanisms with monetary transfer, such as combinatorial auctions or task scheduling, it would be *fair* to allocate a set of items/tasks to an agent who is willing to pay more. Thus, we need stronger concepts for fairness/envy-freeness.

In this paper, we introduce two new concepts of fairness called *envy-freeness of an individual toward a group* (ItG-EFness) and *envy-freeness of a group toward a group* (GtG-EFness), which are natural extensions of traditional envy-freeness. We refer to the traditional one as *envy-freeness of an individual toward an individual* (ItI-EFness).

In a sense, a Pareto efficient allocation is the best, since items are allocated to agents who are willing to pay the most in total. However, in many situations, we cannot achieve a Pareto efficient allocation, e.g., computing a Pareto efficient allocation is too time consuming, or the possibility of false-name bids prevents a Pareto efficient allocation [Yokoo, 2003]. In such cases, some item might remain unsold, or several items might be sold in a single bundle, even though selling them separately would be more efficient. Our new fairness concepts guarantee that the obtained allocation is *locally efficient*, i.e., most efficient within some alternatives.

We discuss combinatorial auction mechanisms that satisfy these concepts. First, we characterize mechanisms that satisfy these concepts by focusing on their allocation rules (Theorems 2 and 3). Our characterization is an extension of the results by Haake *et al.* [2002]. Then we clarify the connections between these concepts and three other properties; the *core*, *strategy-proofness*, and *false-name-proofness*. A part of our obtained results is summarized in Figure 1.

As shown in the above example, an envy toward a group can naturally occur in environments with complementarities. Also, with monetary transfer, it is possible to design a mechanism that prevents such an envy. We believe that our new fairness concepts brings up a new interesting research direction in mechanism design.

1.1 Related Works

There are lots of literature on envy-freeness in the fields of economics and computer science. Haake *et al.* [2002] characterized allocation rules of envy-free mechanisms by a property called *local efficiency*. Mu'Alem [2009] showed an efficient method to check whether an allocation is locally efficient. Othman and Sandholm [2010] discussed the connection between envy-freeness and the core in iterated combinatorial auctions. Cohen *et al.* [2010] provided a complete characterization for the allocation rules of envy-free and strategy-proof mechanisms in conjunction with cycle-monotonicity [Rochet, 1987].

Vind [1971] and Varian [1974] proposed an extension of envy-freeness called *coalition fairness* (or group no-envy), which requires that no group of agents envies any other group of the same size. Lahaie and Parkes [2009] discussed the link between the core and coalition fairness in fair package assignment model.

Coalition fairness (or group no-envy) is an old concept but it has been attracting less attention compared to standard envy-freeness. We suspect that coalition fairness might be too specific since it restricts the scope of envy-freeness within same-sized groups (although this restriction seems inevitable in mechanisms without monetary transfer). Our GtG-EFness puts no restriction on the size of groups. Thus, it can be considered as an extension of coalition fairness[1]. We believe this extension would make the old fairness concept more attractive in social choice and mechanism design.

Zhou [1992] proposed another version of envy-freeness called *strict no-envy* in the literature of fair allocation. Strict no-envy property requires that no agent envies the *average* of the allocation given to any group of agents, while our ItG-EFness requires no agent envies the *sum* of the allocation given to any group.

Besides these works, the structure of our concepts closely resembles *false-name-proofness*, which is an extension of strategy-proofness. The concept considers the possibility that one agent pretends to be multiple agents (see e.g., [Yokoo, 2003]).

2 Preliminaries

We consider a set of items G ($|G| = m$) for sale and a set of agents (bidders) $N = \{1, \ldots, n\}$. Each agent $i \in N$ has her valuation function $v_i \in V_i$ that maps sets of items into $\mathbb{R}$. Here V_i is the set of possible valuation functions. We assume a *quasi-linear, private value* model with *no allocative externality*; the utility of agent i, who obtains a set of items (bundle) $B \subseteq G$ and pays p, is represented as $v_i(B) - p$. v_i is normalized so that $v_i(\emptyset) = 0$ holds.

To describe a mechanism, we sometimes use a concept called *minimal bundle*.

Definition 1 (Minimal Bundle)**.** *For agent i with valuation function v_i, B is a minimal bundle for i, if $\forall B' \subset B$ and $B' \neq B$, $v_i(B') < v_i(B)$.*

[1]Note that coalition fairness does not overlap with our ItG-EFness.

ItI-EF Mechanisms for a Single-Minded Domain

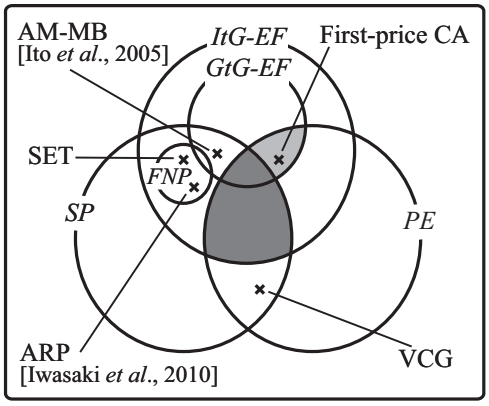

Figure 1: Summary of connections between ItG-EFness, GtG-EFness, the core, SPness, FNPness, and Pareto efficiency. Light shadowed area corresponds to core-selecting mechanisms (Theorem 7). Dark shadowed area is empty (Theorem 11).

We say agent i is *single-minded* if i has at most one minimal bundle with a positive value. We call such a bundle as a *required bundle*. In a general domain, agent i can have multiple minimal bundles. In some parts of this paper, we consider a *single-minded domain*, where all agents are single-minded.

Let $v = (v_1, \ldots, v_n) \in V$ be a valuation profile and $V = \times_{i \in N} V_i$ be a domain of mechanisms. A (direct revelation) combinatorial auction mechanism $M(f, p)$ consists of an *allocation rule* and a *payment rule*. An allocation rule is defined as $f : V \to A$, where A is a set of the possible assignment of items over N. For an assignment $a \in A$, let a_i indicates the bundle allocated to agent i. Note that an assignment $a \in A$ must satisfy *allocation feasibility*; $\bigcup_{i \in N} a_i \subseteq G$ and $\forall i, j$ where $i \neq j$, $a_i \cap a_j = \emptyset$. A payment rule is defined as $p : V \to \mathbb{R}^N$. For a valuation profile v, let $f_i(v)$ and $p_i(v)$ respectively denote the bundle allocated to agent i and the amount agent i must pay in a mechanism $M(f, p)$. We use notations $f(v_i, v_{-i})$ and $p(v_i, v_{-i})$ to represent the allocation and payment when the declared valuation of agent i is v_i and the declared valuation profile of other agents is v_{-i}.

In this paper, we restrict our attention to deterministic mechanisms that satisfy *individual rationality* with *non-negative payment*. Individual rationality (IR) means that no participant obtains negative utility by reporting her true valuation. Formally, $\forall i \in N$, $\forall v$, $v_i(f_i(v)) - p_i(v) \geq 0$. We also assume a mechanism is *almost anonymous* across agents; obtained results from a mechanism are invariant under the permutation of the identifiers of agents except for the case of ties. Furthermore, we assume that a bundle allocated to agent i must be one of her minimal bundles. This assumption does not affect the quality of outcome, e.g., the efficiency or the seller's revenue.

Also, in some part of this paper, we represent a mechanism as a *price-oriented, rationing-free* (PORF) mechanism [Yokoo, 2003]. A PORF mechanism defines a skeleton of a mechanism as follows: (i) for each agent, the price

of each bundle of items is determined independently of her own valuation, and (ii) the mechanism allocates each agent a bundle that maximizes her utility independently of the allocations of other agents. Prices of bundles must be determined so that they satisfy *allocation feasibility*. In a PORF mechanism, when the valuation profile reported by $j \neq i$ is v_{-i}, the price of agent i for a bundle B_i is described as $p(v_{-i}, B_i)$. Yokoo [2003] showed that the PORF representation is a complete characterization of mechanisms that satisfy an incentive property called *strategy-proofness* (discussed in detail in Section 5). Similar price-based representations of strategy-proof mechanisms have also been presented by others, including [Lavi *et al.*, 2003].

Next, let us introduce a fairness concept called *envy-freeness*. To distinguish it from the other fairness concepts that we will introduce, we refer to it as *envy-freeness of an individual toward an individual* (ItI-EFness).

Definition 2 (Envy-Freeness of an Individual toward an Individual)**.** *A mechanism $M(f,p)$ satisfies* envy-freeness of an individual toward an individual *(ItI-EFness) if $\forall i$, $\forall j \neq i$, and $\forall v$, $v_i(f_i(v)) - p_i(v) \geq v_i(f_j(v)) - p_j(v)$.*

In other words, a mechanism satisfies ItI-EFness (or is ItI-EF) if no agent prefers the pair of the bundle and the price of another agent to her own bundle and price. Haake *et al.* [2002] proposed a property called *locally efficient bundle assignment* and characterized the allocation rules of ItI-EF mechanisms. To introduce the property, let us define a notion of permutation. We say that an assignment $a' = (a'_1, \ldots, a'_n) \in A$ is a permutation of another assignment $a = (a_1, \ldots, a_n) \in A$ if $\forall i, \exists j$, such that $a'_i = a_j$ holds.

Definition 3 (Locally Efficient Bundle Assignment)**.** *An allocation a is a* locally-efficient bundle assignment *(LEBA) with respect to v if $\forall a'$, where a' is a permutation of a, $\sum_{i \in N} v_i(a_i) \geq \sum_{i \in N} v_i(a'_i)$ holds.*

Theorem 1 (Haake *et al.* 2002)**.** *An allocation rule f is ItI-EF-achievable, i.e., there exists a non-negative IR payment rule p such that the mechanism $M(f,p)$ is ItI-EF, if and only if $f(v)$ is an LEBA with respect to v for every $v \in V$.*

To evaluate the performance of mechanisms, we focus on a worst-case analysis (*competitive analysis*). Such analysis is commonly used in recent mechanism design literature, especially by computer scientists.

Definition 4 (Competitive Ratio of Efficiency)**.** *A competitive ratio of efficiency for a mechanism $M(f,p)$ is c if*

$$\min_{v \in V} \frac{\sum_{i \in N} v_i(f_i(v))}{\max_{a \in A} \sum_{i \in N} v_i(a_i)} \geq c.$$

3 Envy-Freeness Toward A Group

In this section, we define two extensions of ItI-EFness.

Definition 5 (Envy-Freeness of an Individual toward a Group)**.** *A mechanism $M(f,p)$ satisfies* envy-freeness of an individual toward a group *(ItG-EFness) if $\forall i$, $\forall S \subseteq N$, and $\forall v$,*

$$v_i(f_i(v)) - p_i(v) \geq v_i(\bigcup_{j \in S} f_j(v)) - \sum_{j \in S} p_j(v). \quad (1)$$

In other words, a mechanism satisfies ItG-EFness (or is ItG-EF) if no agent envies a group of agents. Note that in Eq. (1), agent i can be included in S. Thus, this type of envy includes the situation that agent i feels, if I received j's items (as well as my current items) and additionally paid p_j, I would be happier.

Next, First, let us define . Using this notation, let us define the competitive ratio of efficiency.

Definition 6 (Envy-Freeness of a Group toward a Group)**.** *A mechanism $M(f,p)$ satisfies* envy-freeness of a group toward a group *(GtG-EFness) if $\forall v$, $\forall S, S' \subseteq N$,*

$$\sum_{i \in S}(v_i(f_i(v)) - p_i(v)) \geq V^*(S, \bigcup_{j \in S'} f_j(v)) - \sum_{j \in S'} p_j(v). \quad (2)$$

Note that $V^*(S,B)$ is a surplus for a set of agent S when a set of items B are optimally allocated to S:

$$V^*(S,B) = \max \sum_{i \in S} v_i(a_i)$$

where $\bigcup_{i \in S} a_i = B$ and $\bigcap_{i \in S} a_i = \emptyset$.

GtG-EFness requires that no group of agents envies any other group of agents. Note that in Eq. 2, S and S' can overlap. Thus, this type of envy includes all agents feel that if items were allocated in a better way, everybody would be happier. From these definitions, a GtG-EF mechanism satisfies ItG-EFness, and an ItG-EF mechanism satisfies ItI-EFness.

In a single-minded domain, these inclusion relations are strict. The VCG mechanism is ItI-EF but not ItG-EF. If agent i wins bundle B_i, $\forall j \neq i$, where j's required bundle is $B_j \subseteq B_i$, i's payment is at least $v_j(B_j)$. Thus, agent j has no envy toward i. On the other hand, consider the example described in Section 1. If agent 2's valuation for g_1 is 6 and agent 3's valuation for g_2 is 7, the allocation and payment of VCG are exactly the same as this example. Thus, VCG is not ItG-EF.

The SET mechanism, which sells the set of items G to a single agent using the Vickrey (second-price) auction, is ItG-EF but not GtG-EF. In the above example, the SET mechanism allocates g_1 and g_2 to agent 1 at payment 7. Then, agents 2 and 3 as a group envy agent 1. Furthermore, a mechanism called the *first-price combinatorial auction mechanism*, which uses the Pareto efficient allocation rule and collects the winners' reported values as payment, satisfies GtG-EFness.

3.1 Characterizing ItG-EFness and GtG-EFness

In this subsection, we characterize ItG-EF/GtG-EF mechanisms by focusing on their allocation rules. First, let us define the property called *ItG-EF/GtG-EF-achievability*. We say an allocation rule is *ItG-EF/GtG-EF-achievable* if there exists a non-negative IR payment rule p such that the mechanism $M(f,p)$ satisfies ItG-EFness/GtG-EFness.

To characterize ItG-EF/GtG-EF-achievable allocation rules, let us define two additional concepts:

- An assignment a' is a *permutation with merger* of another assignment a if $\forall i, \exists S_i \subseteq N$, $a'_i = \bigcup_{j \in S_i} a_j$.
- An assignment a' is a *permutation with split/merger* of another assignment a if $\bigcup_{i \in N} a'_i = \bigcup_{i \in N} a_i$.

By using these concepts, let us define two properties called *LEBA with merger* (LEBA-M) and *LEBA with split/merger* (LEBA-SM).

Definition 7 (LEBA with Merger)**.** *An allocation a is an* LEBA with Merger *(LEBA-M) with respect to v if $\forall a'$, where a' is a permutation with merger of a, $\sum_{i\in N} v_i(a_i) \geq \sum_{i\in N} v_i(a'_i)$ holds.*

Definition 8 (LEBA with Split/Merger)**.** *An allocation a is an* LEBA with Split/Merger *(LEBA-SM) with respect to v if $\forall a'$, where a' is a permutation with split/merger of a, $\sum_{i\in N} v_i(a_i) \geq \sum_{i\in N} v_i(a'_i)$ holds.*

In other words, an LEBA-SM a allocates a set of items $\bigcup_{i\in N} a_i \subseteq G$ optimally to N, while it allows some item to remain unsold. The idea of LEBA-SM is similar to a property called *consistency*, which was introduced in the literature of fair allocation [Tadenuma and Thomson, 1991]. The following theorems clarify the relation between ItG-EFness/GtG-EFness and LEBA-M/SM. For space reasons, we only show the proof of Theorem 3.

Theorem 2. *If an allocation rule f is ItG-EF-achievable, then $f(v)$ is an LEBA-M with respect to v for every $v \in V$. Also, the converse is true in a single-minded domain.*

Theorem 3. *If an allocation rule f is GtG-EF-achievable, then $f(v)$ is an LEBA-SM with respect to v for every $v \in V$. Also, the converse is true in a single-minded domain.*

Proof (if part). In the definition of GtG-EFness, by choosing $S = S' = N$, we obtain

$$\sum_{i\in N}(v_i(a_i) - p_i(v)) \geq V^*(N, \bigcup_{j\in N} a_j) - \sum_{j\in N} p_j(v).$$

Thus,

$$\sum_{i\in N} v_i(a_i) \geq V^*(N, \bigcup_{j\in N} a_j)$$

holds. From the definition of V^*, $\forall a'$, which is a permutation with split/merger of a,

$$V^*(N, \bigcup_{j\in N} a_j) \geq \sum_{i\in N} v_i(a'_i)$$

holds. Thus,

$$\sum_{i\in N} v_i(a_i) \geq \sum_{i\in N} v_i(a'_i)$$

holds. □

Proof (converse part). Consider an allocation rule f that is LEBA-SM in a single-minded domain. We choose a payment rule p such that $p_i(v) = v_i(f_i(v))$ and show that the mechanism $M(f, p)$ satisfies GtG-EFness.

We derive a contradiction by assuming that $\exists v$, $\exists S, S' \subseteq N$, such that

$$V^*(S, \bigcup_{j\in S'} a_j) - \sum_{j\in S'} p_j(v) > \sum_{i\in S}(v_i(a_i) - p_i(v)),$$

where $a = f(v)$, holds. Since $p_i(v) = v_i(a_i)$, we have

$$V^*(S, \bigcup_{j\in S'} a_j) > \sum_{j\in S'} v_j(a_j).$$

Here, we can assume $\forall i \in S$, $v_i(a_i) = 0$ holds; otherwise, such an agent i does not have a positive valuation in $V^*(S, \bigcup_{j\in S'} a_j)$, since the required bundle a_i is not included in $\bigcup_{j\in S'} a_j$. Thus, we have the same inequality for the set of agents $S \setminus \{i\}$.

Let us consider an assignment a', where $\bigcup_{j\in S'} a_j$ are optimally allocated to S, items $\bigcup_{i\in S} a_i$ are optimally allocated to $S' \setminus S$, and the rest of items are allocated in the same way as a. Note that a' is a permutation with split/merger of a.

From the definition of a', we have

$$\begin{aligned}\sum_{i\in N} v_i(a'_i) &\geq V^*(S, \bigcup_{j\in S'} a_j) + \sum_{l\in X} v_l(a_l) \\ &> \sum_{j\in S'} v_j(a_j) + \sum_{l\in X} v_l(a_l) \\ &= \sum_{i\in N} v_i(a_i),\end{aligned}$$

where $X = N \setminus (S \cup S')$. This violates the assumption that $f(v)$ is an LEBA-SM. □

3.2 Competitive Analysis

The following theorems show that when the domain is general, even if f satisfies LEBA-SM (more specifically, f is Pareto efficient), there exists no payment rule p such that $M(f, p)$ satisfies ItG-EFness.

Theorem 4. *The competitive ratio of efficiency for any ItG-EF mechanism in a general domain is at most $3/4$.*

Proof. We derive a contradiction by assuming that mechanism $M(f, p)$ satisfies ItG-EFness and its efficiency ratio is strictly more than $3/4$. Consider three agents and two items g_1, g_2. Agent 1 values 2 for g_1 and $3 + \epsilon$ for g_2. Agent 2 values 3 only for $\{g_1, g_2\}$. Agent 3 values 2 only for g_2. Since f's efficiency ratio is strictly more than $3/4$, agent 1 wins g_1 and agent 3 wins g_2. From IR, agent 3 pays at most 2. Also, from ItG-EFness for agent 2, the sum of their payments must be at least 3. Thus, agent 1's payment is at least 1. In this case, agent 1 envies agent 3, which contradicts ItG-EFness. Then, the best possible allocation is to allocate g_2 to agent 1, where the efficiency ratio is $(3 + \epsilon)/4$. □

Theorem 5. *The competitive ratio of efficiency for any GtG-EFness mechanism in a general domain is at most $1/2$.*

Proof. We derive a contradiction by assuming that mechanism $M(f, p)$ satisfies GtG-EFness and its efficiency ratio is strictly more than $1/2$. Consider three agents and two items g_1, g_2. Agent 1 values 2 for g_1 and $2 + 2\epsilon$ for g_2. Agent 2 values $4 - \epsilon$ only for $\{g_1, g_2\}$. Agent 3 values 2 only for g_2. Using a similar argument to Theorem 4, allocating g_1 to agent 1 and g_2 to agent 3 is impossible. Also, from GtG-EFness, it is impossible to allocate both items to agent 2; otherwise, the group of agents 1 and 3 will envy. Then, the best possible allocation is to allocate g_2 to agent 1, where the efficiency ratio is $(1 + \epsilon)/2$. □

3.3 Non-trivial GtG-EF Mechanism

The previous subsection shows that the competitive ratio of a GtG-EF mechanism in a general domain is at most $1/2$. This implies that even the first price combinatorial auction mechanism does not satisfy GtG-EFness. Thus, the GtG-EFness condition seems very restrictive and one might imagine that developing a non-trivial GtG-EF mechanism in a general domain is impossible. However, we show that this pessimistic conjecture is not true; there exists a non-trivial GtG-EF mechanism called the *average-max minimal-bundle* (AM-MB) mechanism [Ito *et al.*, 2005]. Quite surprisingly, this mechanism is also strategy-proof, while the first price combinatorial auction mechanism is not.

Definition 9 (Average-Max-Minimal-Bundle (AM-MB) mechanism)**.** *The Average-Max-Minimal-Bundle (AM-MB) mechanism is defined as a PORF mechanism, in which for agent i, the price of bundle B_i is defined as:*

$$|B_i| \cdot \max_{B_j \subseteq G, j \neq i} v_j(B_j)/|B_j|,$$

where $B_i \cap B_j \neq \emptyset$ and B_j is a minimal bundle for agent j.

As stated in [Ito *et al.*, 2005], the AM-MB mechanism is a strategy-proof, greedy allocation mechanism (i.e., easy to compute) and achieves better efficiency and revenue in expectation, although its worst-case efficiency ratio can be arbitrarily small.

We show that the AM-MB mechanism is GtG-EF.

Theorem 6. *The AM-MB mechanism is GtG-EF in a general domain.*

Proof. Let us assume AM-MB is not GtG-EF and derive a contradiction. We assume, $\exists v$, $\exists S, S' \subseteq N$, the following inequality holds:

$$\sum_{i \in S}(v_i(a_i) - p_i(v)) < V^*(S, \bigcup_{j \in S'} a_j) - \sum_{j \in S'} p_j(v), \quad (3)$$

where $a = f(v)$. Here, denote $S'_I = S \cap S'$, and $S'_C = S' \setminus S_I$. In the left-hand side of Eq. (3), each term in the summation is non-negative. Thus,

$$\sum_{i \in S}(v_i(a_i) - p_i(v)) \geq \sum_{i \in S'_I}(v_i(a_i) - p_i(v))$$

holds. Assume agent $i \in S$ receives B_i in $V^*(S, \bigcup_{j \in S'} a_j)$. Then, the right-hand side of Eq. (3) can be re-written as:

$$V^*(S, \bigcup_{j \in S'} a_j) - \sum_{j \in S'} p_j(v)$$
$$= \sum_{i \in S} v_i(B_i) - \sum_{j \in S'_I} p_j(v) - \sum_{j \in S'_C} p_j(v).$$

Thus, we obtain:

$$\sum_{i \in S'_I} v_i(a_i) + \sum_{j \in S'_C} p_j(v) < \sum_{i \in S} v_i(B_i) \quad (4)$$

From the definition of AM-MB, we can rewrite Eq. (4) as:

$$\sum_{i \in S'_I} \sum_{g \in a_i} v_i(a_i)/|a_i| + \sum_{j \in S'_C} \sum_{g \in a_j} p_j(v)/|a_j|$$
$$< \sum_{i \in S} \sum_{g \in B_i} v_i(B_i)/|B_i| \quad (5)$$

Each term within the summation of the right-hand side of Eq. (5) corresponds to an item $g \in \bigcup_{j \in S'} a_j$. Then, for each $g \in \bigcup_{j \in S'} a_j$, consider agent j who receives the item g in $a = f(v)$. Agent j is either in S'_I or S'_C. If $j \in S'_I$,

$$v_j(a_j)/|a_j| \geq v_i(B_i)/|B_i|$$

holds for all $i \in S$ such that $B_i \cap a_j \neq \emptyset$; otherwise, j cannot obtain a_j. Note that $v_j(a_j)/|a_j|$ appears in the left-hand side of Eq. (5). Also, if $j \in S'_C$,

$$p_j(v)/|a_j| \geq v_i(B_i)/|B_i|$$

holds (since $v_i(B_i)/|B_i|$ is used for calculating $p_j(v)$). Note that $p_j(v)/|a_j|$ also appears in the left-hand side of Eq. (5). In summary, for each term $v_i(B_i)/|B_i|$ in the right-hand side of Eq. (5), there also exists a corresponding term in the left-hand side, which is no less than $v_i(B_i)/|B_i|$. Also, these left-hand terms are non-overlapping. This is a contradiction. □

4 Connection with the Core

In this section, we discuss the connection between GtG-EFness and the core. Let us introduce the definition of *core-selecting auctions* [Day and Milgrom, 2008].

Definition 10 (Blocking Coalition)**.** *For a mechanism $M(f, p)$ and a valuation profile v, where $a = f(v)$ and $G_r = G \setminus \bigcup_{i \in N} a_i$, $S \subseteq N$ is a blocking coalition if there exists $S' \subseteq N$, such that the following condition holds:*

$$\sum_{i \in S}(v_i(a_i) - p_i(v)) < V^*(S, G_r \cup \bigcup_{j \in S'} a_j) - \sum_{j \in S \cup S'} p_j(v).$$

Here, G_r is a set of items that are not allocated to any agent. If S is a blocking coalition, the members of S can ask the seller to allocate $G_r \cup \bigcup_{j \in S'} a_j$ to them instead of S' by paying slightly more than $\sum_{j \in S \cup S'} p_j(v)$. Then, both the seller and the members of S are better off. Thus, the outcome of this mechanism is unstable.

Definition 11 (Core-selecting mechanism)**.** *A mechanism $M(f, p)$ is* core-selecting *if $\forall v$, there exists no blocking coalition.*

The next theorem shows that the core selection implies GtG-EFness in a single-minded domain.

Theorem 7. *In a single-minded domain, a mechanism is core-selecting if and only if it is Pareto efficient and GtG-EF.*

Proof sketch. In a single-minded domain, wlog, we can assume each agent i in a blocking coalition S obtains $\emptyset$ (if not, $S \setminus \{i\}$ is also a blocking coalition). Also, we can assume $G_r = \emptyset$ in a core-selecting or Pareto efficient mechanism. Thus, the left-hand side of the blocking coalition condition must be 0 and the condition is simplified to:

$$\sum_{j \in S'} p_j(v) < V^*(S, \bigcup_{j \in S'} a_j).$$

Also, in a single-minded domain, wlog, we can assume each agent $i \in S$ in Eq. (2) obtains $\emptyset$. Then, the left-hand side of Eq. (2) must be 0, and the condition is simplified to:

$$\sum_{j \in S'} p_j(v) \geq V^*(S, \bigcup_{j \in S'} f_j(v)).$$

It is clear that GtG-EFness is equivalent to the fact that there exists no blocking coalition. □

Theorem 8. *In a general domain, there exists no core-selecting ItG-EF mechanism.*

Proof. From Theorem 4, there exists no Pareto efficient ItG-EF mechanism. Also, a core-selecting mechanism must achieve a Pareto efficient allocation. Thus, there exists no core-selecting ItG-EF mechanism. □

5 Connection with SPness

Strategy-proofness (SPness) is a property that deals with incentives of agents. A mechanism satisfies SPness (or is SP) if reporting true valuation v_i is a weakly dominant strategy for any agent i and for any valuation profile v_{-i}. First, let us clarify the connection between ItI-EFness and SPness. We omit the proof since it is almost identical to the proof of Theorem 13.

Theorem 9. *Any SP mechanism is ItI-EF in a single-minded domain, but not vice versa.*

Rochet [1987] proposed a property of allocation rules called *cycle-monotonicity* and characterized SP mechanisms. However, Cohen et al. [2010] showed that even if an allocation rule f is LEBA and cyclic-monotone, there might be no appropriate payment rule p such that $M(f,p)$ is both ItI-EF and SP. Since ItG-EFness is more restrictive than ItI-EFness, this statement is also true for ItG-EFness. Indeed, although the Pareto efficient allocation rule is LEBA-M and cyclic-monotone, there is no appropriate payment rule even in a single-minded domain (see Theorem 11).

Our next objective is to obtain a complete characterization of ItG-EF SP mechanisms.

5.1 Characterization

As we stated in Section 2, Yokoo [2003] gave a complete characterization of SPness by introducing a class of PORF mechanisms. To characterize ItG-Ef SP mechanisms, let us define the following property.

Definition 12 (No-Envy Pricing Rule Toward A Group)**.** *A PORF mechanism satisfies* no-envy pricing rule toward a group *(NEP) if* $\forall v$, $\forall i \in N$, *and* $\forall S \subseteq N$, *if* B_j *maximizes* j*'s utility* $\forall j \in S$, *then*

$$\sum_{j \in S} p(v_{-j}, B_j) \geq v_i(\bigcup_{j \in S} B_j)$$

holds.

By using this property, we obtain the following characterization theorem.

Theorem 10. *In a single-minded domain, an SP mechanism* $M(f,p)$ *is ItG-EF if and only if its prices satisfy NEP.*

Proof sketch. In a single-minded domain, an agent whose utility is positive does not envy any group of agents. Then, the left-hand side of Eq. (1) becomes 0. Thus, the ItG-EFness condition is simplified to:

$$\sum_{j \in S} p_j(v) \geq v_i(\bigcup_{j \in S} (f_j(v))),$$

which is identical to NEP. □

Neither the first-price combinatorial auction mechanism nor the VCG mechanism satisfy NEP. On the other hand, the SET mechanism satisfies NEP in a single-minded domain, since the winner's price equals $\max_{j \in N \setminus \{i\}} v_j(B_j)$, where B_j is j's required bundle.

5.2 Competitive Analysis

In this subsection, we discuss the competitive ratio achieved by an ItG-EF SP mechanism. Putting aside SPness, we already revealed that there exists a Pareto efficient ItG-EF mechanism in a single-minded domain. However, taking into account SPness, there no longer exist such Pareto efficient mechanisms even in a single-minded domain.

As an upper bound, we obtain the following result.

Theorem 11. *The competitive ratio of efficiency for any ItG-EF SP mechanism is at most* $2/3$.

Proof. Consider three agents and two items $\{g_1, g_2\}$. Agents 1 and 3 value 1 for g_1 and g_2, respectively. Agent 2 values 1 for a bundle $\{g_1, g_2\}$. Thus, if a mechanism achieves an efficiency ratio better than $2/3$, both agents 1 and 3 must win. Let us define the payments of agents 1 and 3 as p_1 and p_3, respectively. From ItG-EFness, they must satisfy $p_1 + p_3 \geq 1$. Thus, either p_1 or p_3 must be greater than or equal to $1/2$. Wlog, we assume $p_1 \geq 1/2$.

Next, consider the case where agent 1 has a valuation $p_1 - \epsilon$. From SPness, the payment in which agent 1 wins the item g_1 must be uniquely determined when the other agents' reports are fixed. Thus, from IR, agent 1 with valuation $p_1 - \epsilon$ cannot win the item g_1 and the efficiency is at most 1. In a Pareto optimal allocation, the efficiency is $1 + p_1 - \epsilon$. Thus, efficiency ratio is $1/(1 + p_1 - \epsilon)$, which cannot be strictly more than $2/3$. □

Theorem 12. *In a single-minded domain, the lower bound of the competitive ratio of efficiency for an ItG-EF SP mechanism is* $2/(m+1)$.

Proof. In a single-minded domain, the competitive ratio of a FNP mechanism called *adaptive reserve price* (ARP) mechanism [Iwasaki *et al.*, 2010] is $2/(m+1)$. From Theorem 13, this mechanism is also ItG-EF. □

6 Connection with FNPness

False-name-proofness (FNPness) generalizes strategy-proofness by assuming a bidder can submit multiple bids under fictitious identifiers, e.g., multiple e-mail addresses [Yokoo, 2003]. A mechanism satisfies FNPness (or is FNP) if for each agent, reporting her true valuation using a single identifier (although the agent can use multiple identifiers) is a weakly dominant strategy. From the definition, a FNP mechanism is SP. We show that FNPness implies ItG-EFness in a single-minded domain.

Theorem 13. *Any FNP mechanism is ItG-EF in a single-minded domain, but not vice versa.*

Proof. First, we derive a contradiction by assuming that a FNP mechanism $M(f,p)$ is not ItG-EF. We assume $\exists v$, $\exists i$, and $\exists S \subseteq N$, such that

$$v_i(f_i(v)) - p_i(v) < v_i(\bigcup_{j \in S} f_j(v)) - \sum_{j \in S} p_j(v)$$

holds. Then, the left-hand side of the above equation must be 0; if the left-hand side is positive, the right-hand side cannot exceed the left-hand side.

Next, let us consider the case where a set of agents S drops out and agent k, who has exactly the same valuation as i, joins. Since the mechanism is almost anonymous, the utilities of agents i and j must be the same. Since agents are single-minded, their utilities cannot be positive at the same time. Thus,

$$v_k(f_k(v_k, v_{N \setminus S})) - p_k(v_k, v_{N \setminus S}) = 0,$$

where $v_{N \setminus S}$ indicates the valuation profile reported by a set of agent $N \setminus S$. However, agent k can make the situation identical to the above case by using false identifiers and obtain the set of bundles $\bigcup_{j \in S} f_j(v)$ at payment $\sum_{j \in S} p_j(v)$, which is strictly smaller than $v_k(\bigcup_{j \in S} f_j(v))$. This violates FNPness.

The converse is not true since AM-MB is ItG-EF (GtG-EF) but not FNP even if the domain is single-minded. □

7 Conclusions and Future Works

In this paper, we introduced two new concepts of fairness called *envy-freeness of an individual toward a group* and *envy-freeness of a group toward a group*, which are natural extensions of traditional envy-freeness. We characterized them and clarified their connections with the core, strategy-proofness, and false-name-proofness.

Our new fairness concepts bring up a new interesting direction in mechanism design. Although we obtained a fair amount of initial results, there remain many interesting research questions. For example, we want to obtain the full characterization of allocation rules for ItG-EFness/GtG-EFness with/without SPness in a general domain. Also, we want to narrow the gap between the lower/upper bounds of the competitive ratio. Furthermore, we hope to investigate the envy-freeness of a group toward an individual (GtI-EFness), which would be related to a manipulation that is symmetric to false-name bidding, i.e., a coalition of agents negotiates and sends one representative to a mechanism.

Acknowledgements

This research was partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (A), 20240015, 2008, and Grant-in-Aid for JSPS fellows. The authors would like to thank Daisuke Oyama and anonymous reviewers for helpful discussions.

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