Efficient Repeated Implementation: Supplementary Material

Jihong Lee*
Yonsei University and
Birkbeck College, London

Hamid Sabourian†
University of Cambridge

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1 Complete information: the two agent case

Theorem 1. *Suppose that $I = 2$, and consider an SCF f satisfying condition ω''. If f is efficient in the range, there exist a regime R and $\bar{\delta}$ such that, for any $\delta > \bar{\delta}$,(i) $\Omega^{\delta}(R)$ is non-empty; and (ii) for any $\sigma \in \Omega^{\delta}(R)$, $\pi_i^{\theta(t)}(\sigma, R) = v_i(f)$ for any i, $t \geq 2$ and $\theta(t)$. If, in addition, f is strictly efficient in the range then $a^{\theta(t),\theta^t}(\sigma, R) = f(\theta^t)$ for any $t \geq 2$, $\theta(t)$ and θ^t.*

Proof. By condition ω'' there exists some $\tilde{a} \in A$ be such that $v(\tilde{a}) \ll v(f)$. Following Lemma 1 in the main text, let S^i be the regime alternating $d(i)$ and $\phi(\tilde{a})$ from which i can obtain payoff exactly equal to $v_i(f)$. For any j, let $\pi_j(S^i)$ be the maximum payoff that j can obtain from regime S^i when i behaves rationally in $d(i)$. Since S^i involves $d(i)$, Assumption (A) in the main text implies that $v_j^j > \pi_j(S^i)$ for $j \neq i$. Then there must also exist $\epsilon > 0$ such that $v_j(\tilde{a}) < v_i(f) - \epsilon$ and $\pi_j(S^i) < v_i^i - \epsilon$ for any i, j such that $i \neq j$. Next, define $\rho \equiv \max_{i,\theta,a,a'} [u_i(a, \theta) - u_i(a', \theta)]$ and $\bar{\delta} \equiv \frac{\rho}{\rho+\epsilon}$.

Mechanism $\tilde{g} = (M, \psi)$ is defined such that, for all i, $M_i = \Theta \times \mathbb{Z}_+$ and ψ is such that

*School of Economics, Yonsei University, Seoul 120-749, Korea, Jihong.Lee@yonsei.ac.kr
†Faculty of Economics, Cambridge, CB3 9DD, United Kingdom, Hamid.Sabourian@econ.cam.ac.uk

1. if $m_i = (\theta, \cdot)$ and $m_j = (\theta, \cdot)$, $\psi(m) = f(\theta)$;
2. if $m_i = (\theta^i, z^i)$, $m_j = (\theta^j, 0)$ and $z^i \neq 0$, $\psi(m) = f(\theta^j)$;
3. for any other m, $\psi(m) = \tilde{a}$.

Regime $\widetilde{R}$ represents any regime satisfying the following transition rules: $\widetilde{R}(\emptyset) = \tilde{g}$ and, for any $h = ((g^1, m^1), \ldots, (g^{t-1}, m^{t-1})) \in H^t$ such that $t > 1$ and $g^{t-1} = \tilde{g}$:

1. if $m_i^{t-1} = (\theta, 0)$ and $m_j^{t-1} = (\theta, 0)$, $\widetilde{R}(h) = \tilde{g}$;
2. if $m_i^{t-1} = (\theta^i, 0)$, $m_j^{t-1} = (\theta^j, 0)$ and $\theta^i \neq \theta^j$, $\widetilde{R}(h) = \Phi^{\tilde{a}}$;
3. if $m_i^{t-1} = (\theta^i, z^i)$, $m_j^{t-1} = (\theta^j, 0)$ and $z^i \neq 0$, $\widetilde{R}|h = S^i$;
4. if m^{t-1} is of any other type and i is lowest-indexed agent among those who announce the highest integer, $\widetilde{R}|h = D^i$.

We next prove the theorem via the following lemmas below, which characterize the equilibrium set of $\widetilde{R}$.

Lemma 1. *Fix any $\sigma \in \Omega^\delta(\widetilde{R})$. For any $t > 1$ and $\theta(t)$, if $g^{\theta(t)} = \tilde{g}$, $\pi_i^{\theta(t)} \geq v_i(f)$.*

Proof. Suppose not; then at some $t > 1$ and $\theta(t)$, $g^{\theta(t)} = \tilde{g}$ but $\pi_i^{\theta(t)} < v_i(f)$ for some i. Let $\theta(t) = (\theta(t-1), \theta^{t-1})$. Given the transition rules, it must be that $g^{\theta(t-1)} = \tilde{g}$ and $m_i^{\theta(t-1),\theta^{t-1}} = m_j^{\theta(t-1),\theta^{t-1}} = (\tilde{\theta}, 0)$ for some $\tilde{\theta}$. Consider i deviating at $(\mathbf{h}(\theta(t-1)), \theta^{t-1})$ such that he reports $\tilde{\theta}$ and a positive integer. Given ψ, the deviation does not alter the current outcome but, by transition rule 3, can yield continuation payoff $v_i(f)$. Hence, the deviation is profitable, implying a contradiction. □

Lemma 2. *Fix any $\delta \in (\bar{\delta}, 1)$ and $\sigma \in \Omega^\delta(\widetilde{R})$. For any t and $\theta(t)$, if $g^{\theta(t)} = \tilde{g}$, $m_i^{\theta(t),\theta^t} = m_j^{\theta(t),\theta^t} = (\theta, 0)$ for any θ^t.*

Proof. Suppose not; then for some t, $\theta(t)$ and θ^t, $g^{\theta(t)} = \tilde{g}$ but $m^{\theta(t),\theta^t}$ is not as in the claim. There are three cases to consider.

Case 1: $m_i^{\theta(t),\theta^t} = (\cdot, z^i)$ and $m_j^{\theta(t),\theta^t} = (\cdot, z^j)$ with $z^i, z^j > 0$.

In this case, by rule 3 of ψ, $\tilde{a}$ is implemented in the current period and, by transition rule 4, a dictatorship by, say, i follows forever thereafter. But then, by assumption (A) above, j can profitably deviate by announcing an integer higher than z^i at such a history; the deviation does not alter the current outcome from $\tilde{a}$ but switches dictatorship to himself as of the next period.

Case 2: $m_i^{\theta(t),\theta^t} = (\cdot, z^i)$ and $m_j^{\theta(t),\theta^t} = (\theta^j, 0)$ with $z^i > 0$.

In this case, by rule 2 of ψ, $f(\theta^j)$ is implemented in the current period and, by transition rule 3, continuation regime S^i follows thereafter. Consider j deviating to another strategy identical to σ_j everywhere except at $(\mathbf{h}(\theta(t)), \theta^t)$ it announces an integer higher than z^i. Given rule 3 of ψ and transition rule 4, this deviation yields a continuation payoff $(1-\delta)u_j(\tilde{a}, \theta^t) + \delta v_j^j$, while the corresponding equilibrium payoff does not exceed $(1-\delta)u_j(f(\theta^j), \theta^t) + \delta \pi_j(S^i)$. But, since $v_j^j > \pi_j(S^i) + \epsilon$ and $\delta > \bar{\delta}$, the former exceeds the latter, and the deviation is profitable.

Case 3: $m_i^{\theta(t),\theta^t} = (\theta^i, 0)$ and $m_j^{\theta(t),\theta^t} = (\theta^j, 0)$ with $\theta^i \neq \theta^j$.

In this case, by rule 3 of ψ, $\tilde{a}$ is implemented in the current period and, by transition rule 2, in every period thereafter. Consider any agent i deviating by announcing a positive integer at $(\mathbf{h}(\theta(t)), \theta^t)$. Given rule 2 of ψ and transition rule 3, such a deviation yields continuation payoff $(1-\delta)u_i(f(\theta^j), \theta^t) + \delta v_i(f)$, while the corresponding equilibrium payoff is $(1-\delta)u_i(\tilde{a}, \theta^t) + \delta v_i(\tilde{a})$. But, since $v_i(f) > v_i(\tilde{a}) + \epsilon$ and $\delta > \bar{\delta}$, the former exceeds the latter, and the deviation is profitable. □

Lemma 3. *For any $\delta \in (\bar{\delta}, 1)$ and $\sigma \in \Omega^\delta(\widetilde{R})$, $\pi_i^{\theta(t)} = v_i(f)$ for any i, $t > 1$ and $\theta(t)$.*

Proof. Given Lemmas 1-2, and since f is efficient in the range, we can directly apply the proofs of Lemmas 3 and 4 in the main text. □

Lemma 4. *For any $\delta \in (\bar{\delta}, 1)$, $\Omega^\delta(\widetilde{R})$ is non-empty.*

Proof. Consider a symmetric Markov strategy profile in which the true state and zero integer are always reported. At any history, each agent i can deviate in one of the following three ways:

(i) Announce the true state but a positive integer. Given rule 1 of ψ and transition rule 3, such a deviation is not profitable.

(ii) Announce a false state and a positive integer. Given rule 2 of ψ and transition rule 3, such a deviation is not profitable.

(iii) Announce zero integer but a false state. In this case, by rule 3 of ψ, $\tilde{a}$ is implemented in the current period and, by transition rule 2, in every period thereafter. The gain from such a deviation cannot exceed $(1-\delta)\max_{a,\theta}[u_i(\tilde{a},\theta) - u_i(a,\theta)] - \delta\epsilon < 0$, where the inequality holds since $\delta > \bar{\bar{\delta}}$. Thus, the deviation is not profitable. □

2 Complexity-averse agents

One approach to sharpen predictions in dynamic games has been to introduce refinements of the standard equilibrium concepts with players who have preferences for less complex strategies (Abreu and Rubinstein [1], Kalai and Stanford [5], Chatterjee and Sabourian [2], Sabourian [7], Gale and Sabourian [3] and Lee and Sabourian [6], among others). We now introduce complexity considerations to our repeated implementation setup. It turns out that only a minimal refinement is needed to obtain repeated implementation results from period 1.

Consider *any* measure of complexity of a strategy under which a Markov strategy is simpler than a non-Markov strategy.[1] Then, refine Nash equilibrium lexicographically as follows: a strategy profile $\sigma = (\sigma_1, \dots, \sigma_I)$ constitutes a Nash equilibrium with complexity cost, NEC, of regime R if, for all i, (i) σ_i is a best response to σ_{-i}; and (ii) there exists no σ'_i such that σ'_i is a best response to σ_{-i} at every history and σ'_i is simpler than σ_i.[2] Let $\Omega^{\delta,c}(R)$ denote the set of NECs of regime R with discount factor δ. The following extends the notions of Nash repeated implementation to the case with complexity-averse agents.

Definition 1. *An SCF f is payoff-repeated-implementable in Nash equilibrium with complexity cost if there exists a regime R such that (i) $\Omega^{\delta,c}(R)$ is non-empty; and (ii) every $\sigma \in \Omega^{\delta,c}(R)$ is such that $\pi_i^{\theta(t)}(\sigma, R) = v_i(f)$ for all i, t and $\theta(t)$; f is repeated-implementable in Nash equilibrium with complexity cost if, in addition, $a^{\theta(t),\theta^t}(\sigma, R) = f(\theta)$ for any t, $\theta(t)$ and θ^t.*

Let us now consider the canonical regime in the complete information setup with $I \geq 3$, R^*.[3] Since, by definition, a NEC is also a Nash equilibrium, Lemmas 2-4 in the

[1] There are many complexity notions that possess this property. One example is provided by Kalai and Stanford [5] who measure the number of *continuation strategies* that a strategy induces at different periods/histories of the game.

[2] Note that the complexity cost here concerns the cost associated with implementation, rather than computation, of a strategy.

[3] Corresponding results for the two-agent complete information as well as incomplete information cases can be similarly derived and, hence, omitted for expositional flow.

main text remain true for NEC. Moreover, since a Markov Nash equilibrium is itself a NEC, $\Omega^{\delta,c}(R^*)$ is non-empty. In addition, we obtain the following.

Lemma 5. *Every $\sigma \in \Omega^{\delta,c}(R^*)$ is Markov.*

Proof. Suppose that there exists some $\sigma \in \Omega^{\delta,c}(R^*)$ such that σ_i is non-Markov for some i. Then, consider i deviating to a Markov strategy, $\sigma_i' \neq \sigma_i$, such that when playing g^* it always announces (i) the same positive integer and (ii) the state announced by σ_i in period 1, and when playing $d(i)$, it acts rationally. Fix any $\theta^1 \in \Theta$. By part (ii) of Lemma 3 in the main text and the definitions of g^* and R^*, we have $a^{\theta^1}(\sigma_i', \sigma_{-i}, R^*) = a^{\theta^1}(\sigma, R^*)$ and $\pi_i^{\theta^1}(\sigma_i', \sigma_{-i}, R^*) = v_i(f)$. Moreover, we know from Lemma 4 in the main text that $\pi_i^{\theta^1}(\sigma, R^*) = v_i(f)$. Thus, the deviation does not alter i's payoff. But, since σ_i' is less complex than σ_i, such a deviation is worthwhile for i. This contradicts the assumption that σ is a NEC. □

This immediately leads to the following result.

Theorem 2. *If f is efficient (in the range) and satisfies conditions ω (ω'), f is payoff-repeated-implementable in Nash equilibrium with complexity cost; if, in addition, f is strictly efficient (in the range), it is repeated-implementable in Nash equilibrium with complexity cost.*

Note that the notion of NEC does not impose any payoff considerations *off the equilibrium path*; although complexity enters players' preferences only at the margin it takes priority over optimal responses to deviations. A weaker equilibrium refinement than NEC is therefore to require players to adopt minimally complex strategies among the set of strategies that are best responses *at every history*, and not merely at the beginning of the game (see Kalai and Neme [4]).

In fact, the complexity results in our repeated implementation setup can also be obtained using this weaker notion if we limit the strategies to those that are finite (i.e. can be implemented by a machine with a finite number of states). To see this, consider again the three-or-more-agent case with complete information, and modify the mechanism g^* in regime R^* such that if two or more agents play distinct messages then one who announces the highest integer becomes a dictator for the period. Fix any equilibrium (under this weaker refinement) in this new regime. By the finiteness of strategies there is a maximum bound z on the integers reported by the players at each date. Now, for any player i and any history (on- or off-the-equilibrium) starting with the modified mechanism g^*, compare the equilibrium strategy with any Markov strategy for i that always announces a number exceeding z and acts rationally in mechanism $d(i)$. By a similar argument as in the proof of Lemmas 2-4 in the main text, it can be shown that i's equilibrium continuation payoff beyond period 1 is exactly the target payoff. Also, since the Markov strategy makes i the dictator at that date and induces S^i or D^i in the continuation game, the Markov strategy induces a continuation payoff at least equal to the target utility. Therefore, by complexity considerations the equilibrium strategy must be Markov.

3 Non-exclusion vs. condition ω

Consider the following two examples. First, consider $\mathcal{P}$ where $I = \{1, 2\}$, $A = \{a, b\}$, $\Theta = \{\theta', \theta''\}$, $p(\theta') = p(\theta'') = 1/2$ and the agents' state-contingent utilities are given below:

	θ'			θ''		
	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$
a	1	3	2	3	2	1
b	3	2	1	1	3	2
c	2	1	3	2	1	3

Here, the SCF f such that $f(\theta') = a$ and $f(\theta'') = b$ is efficient but fails to satisfy condition ω because of agent 1. But, notice that f is non-exclusive.

Second, consider $\mathcal{P}$ where $I = \{1, 2, 3\}$, $A = \{a, b, c, d\}$, $\Theta = \{\theta', \theta''\}$ and the agents' state-contingent utilities are given below:

	θ'			θ''		
	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$
a	3	2	0	1	2	1
b	2	1	1	2	3	0
c	1	3	1	3	1	1
d	0	0	0	0	0	0

Here, the SCF such that $f(\theta') = a$ and $f(\theta'') = b$ is efficient and also satisfies condition ω, but it fails to satisfy non-exclusion because of agent 3.

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