# Middlemen: the bid-ask spread

Makoto Watanabe*
Universidad Carlos III de Madrid†

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**Abstract**

This paper studies the bid-ask spread set in an intermediated market for a homogeneous good with middlemen who hold inventories of the good. Using a directed search approach, I investigate a steady state equilibrium. The middlemen set an ask price for buyers in retail markets and a bid price for sellers in wholesale markets in order to stock their inventories. Middlemen's inventories can provide buyers with immediacy service under market frictions and price competition. The distribution of middlemen's inventories turns out to be a critical determinant of the bid-ask spread, compared to the usual market parameter representing the total demand relative to total supply.

**Keywords**: Search frictions, Bid-ask spread, Intermediation, Inventory holdings
**JEL Classification Number**: D4, D5 G2, L1

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†Correspondence: Makoto Watanabe, Department of Economics, Universidad Carlos III de Madrid, Calle Madrid 126, Getafe (Madrid) 28903, SPAIN. E-mail: mwatanab@eco.uc3m.es

# 1 Introduction

Recent developments in the directed search literature provide a rich set of tools to study the functioning of market economies.[1] Unlike in the traditional random meeting approach, directed search equilibrium incorporates: (i) price competition among sellers so that individual sellers can influence the search-purchase behaviors of buyers through prices; (ii) buyers' choice of where to search so that the meeting rate between buyers and sellers is determined endogenously. This approach can place the extent of competition at the center of studying market behaviors and the implication of market frictions.

The objective of this paper is to study the bid-ask spread set in intermediated markets. Using a standard directed search approach, I consider an economy in which there exist middlemen who hold inventories of a homogeneous good. The middlemen are specialized in buying and selling, and their inventories enable them to stand ready to serve many buyers at a time. They set an ask price for buyers in retail markets and a bid price for sellers in wholesale markets. The gap between these two prices determines a sort of price markup called the bid-ask spread that arises due to the presence of market frictions.

To be specific, I consider an infinite horizon model in which each period consists of two sub-periods. In the first sub-period, retail markets are open where buyers can search for a good. In the second sub-period, wholesale markets are open where middlemen can restock their inventories from sellers who still hold the good. Retail markets are operated by both middlemen and sellers, while wholesale markets are operated only by sellers. Borrowing from a setup recently proposed by Lagos and Wright (2005), which is now familiar in the monetary economics literature, I assume that the retail markets are frictional but the wholesale markets are frictionless. Focussing my attention on a steady state, I investigate a directed search equilibrium in which: (i) buyers are indifferent between searching in the sellers' market where the price is low and the likelihood of finding the good is low, and in the middlemen' market

[1] See, for example, Accemoglu and Shimer (1999), Albrecht, Gautier and Vroman (2006), Burdett, Shi and Wright (2001), Coles and Eeckhout (2003), Faig and Jerez (2005), Julien, Kennes, and King (2000), McAfee (1993), Moen (1997), Montogomery (1991), Peters (1991), and Shi (2002ab).

where both the price and the likelihood are high; (ii) sellers are indifferent between selling to buyers in the frictional retail market at a higher price with a risk of not clearing out their stocks, and selling to middlemen in the frictionless wholesale markets at a lower price with no risk of unsold goods.

At such an equilibrium, the retail price of sellers is smaller than the retail price of middlemen and is greater than the wholesale price. This means, the retail price in the sellers' private market lies within the bid-ask spread of middlemen. This occurs because the middlemen's inventory can provide buyers with a high meeting rate under market frictions, thereby the ask price of middlemen includes a premium for immediacy service to buyers and the bid price includes a premium charged to sellers for guaranteed sale. Further, the bid-ask spread increases with the number of middlemen because it leads to a greater amount of competition in retail markets. This result is driven by the market-tightness effect due to an increase in the total supply that is standard in the directed/competitive search literature.

An increase in inventories of middlemen implies a larger total supply but its effects on the bid-ask spread are not uniform. In addition to the usual market-tightness effect, there are two important effects in our equilibrium. On the one hand, a larger inventory maintained by middlemen creates a demand effect that induces more buyers to search in the middlemen's market rather than in the sellers' market. This effect implies a larger premium of middlemen' inventories, and a lower likelihood of sellers' success in the private market. Therefore, the demand effect of middlemen's inventories pushes up the ask price and pushes down the bid price, thus increases the bid-ask spread. On the other hand, as the inventory of middlemen grows it is less likely that out-of-stock situation occurs for the individual middlemen. This effect, which shall be referred to as a stock-out effect, implies a downward pressure on the ask price (and hence on the bid-ask spread), since buyers know that the unsold inventories yield a lower value to the middlemen. These two conflicting effects cause a non-monotonic response of the bid-ask spread to changes in the inventory of middlemen.

Using a simulation approach I investigate the quantitative importance of these effects. Four robust results are obtained: (i) the demand effect is dominant for relatively small inventories whereas the stock-out effect is dominant for relatively large inventories; (ii) with fixed supply in the middlemen's market, many middlemen, each with few inventories lead to a relatively wider bid-ask spread than few middlemen, each with many inventories; (iii) the stock-out effect accounts for a significantly large part of the reductions in the bid-ask spread in response to increases in inventories; (iv) the bid-ask spread is relatively more sensitive to the distribution of middlemen's inventories than to the population of buyers and sellers. In short, one can conclude that middlemen's inventories affect the bid-ask spread more significantly than the usual market parameter representing the total demand relative to total supply does, due to the demand effect and the stock-out effect.

In the current literature of middlemen, there are two approaches that use random meeting models that are related to my work. One approach is used in Rubinstein and Wolinsky (1987), Li (1998), Shevichenko (2004) and Masters (2007) which emphasize middlemen's high meeting rates, but do not consider price competition among middlemen. In the other approach used in Spulber (1996), Rust and Hall (2003), Hendershott and Zhang (2006) and Loertscher (2007), price competition is emphasized as the middlemen's main role of market-makings, but the meeting rate is exogenous. Watanabe (2006) presents a simple directed search model which allows for both price competition and an endogenous high meeting rate of middlemen. However, as it is altered to myopic agents, the equilibrium established in that paper confines itself to a constant continuation value of buyers not trading and zero wholesale price, thus these elements are not taken into account in the equilibrium price formation. That paper is therefore less suitable to provide novel insights into the bid-ask spread set in intermediated markets, which is the main issue in the current paper.

The rest of the paper is organized as follows. Section 2 shows the existence and uniqueness of a steady state equilibrium. Section 3 provides a characterization of the bid-ask spread of middlemen. Section 4 concludes.

## 2 Model

Consider an economy that has a continuum of buyers, sellers and middlemen, referred to using an index $b$, $s$ and $m$, respectively. Time is discrete and lasts forever. Each period is divided into two subperiods. During the first subperiod, a retail market is open for a homogeneous good to buyers. There are search frictions which I describe in detail below. In the retail market, each buyer wishes to obtain one unit of the good while each seller holds $k_s = 1$ unit and each middleman holds $k_m \geq 1$ units of the good. If a buyer successfully purchases the good at a price $p$, then he obtains a period utility $1 - p$ and exits the market. Otherwise he receives zero utility at that period and enters the next period. A seller or middleman who sells $z$ units at a price $p$ obtains profit $zp$ per period during the first subperiod.

After the retail market is closed, another market opens during the second subperiod. This market is a wholesale market where middlemen can restock their units to sell for the future retail markets. Sellers can sell to one of the middlemen if they still hold the good. In contrast to the retail market, there are no search frictions in the wholesale market. The period is then repeated infinitely. While buyers and sellers leave the market once they complete the trade, middlemen are active all the periods. Agents discount future payoffs at a rate $\beta \in [0, 1)$ across periods, but there is no discounting between the two sub-periods.

The environment in each retail market is the same as in the standard directed search models (see for example Burdett, Shi, and Wright (2001)). It can be described as a simple two-stage game. In the first stage, sellers and middlemen post simultaneously a price which they are willing to sell at given the capacity. Observing the prices and capacities, buyers simultaneously decide which seller or middleman to visit in the second stage. If more buyers visit a seller or middleman than its capacity, then the good or goods are allocated randomly. Assuming buyers cannot coordinate their actions over which seller or middleman to visit, I study a symmetric equilibrium where all buyers use the identical mixed strategy for any configuration of the announced prices. Further, I focus my attention on a steady-state equilibrium where entry of buyers and sellers are exogenous, and the population of agents and the capacity of middlemen

$k_m$ are constant over time.[2] In such an equilibrium, all sellers post the identical price $p_s$ and all middlemen post the identical price $p_m$ every period. In any given period each seller or middleman is characterized by an expected queue of buyers, denoted by $x$. The number of buyers visiting a given seller or middleman who has expected queue $x$ is a random variable, denoted by $n$, which has the Poisson distribution $\text{Prob}(n = k) = \frac{e^{-x}x^k}{k!}$. In a symmetric equilibrium where $x_i$ is the expected queue of buyers at $i$, each buyer visits some seller (and some middleman) with probability $\frac{S}{B}x_s$ (and $\frac{M}{B}x_m$) with assigning an equal probability to each seller (and each middleman), where $B, S, M$ denote the measure of buyers, sellers and middlemen, respectively. They should satisfy the adding-up restriction,

$$Mx_m + Sx_s = B, \tag{1}$$

requiring that the number of buyers at all sellers and middlemen equal to the total number of buyers each period. Finally, the stationarity requires that middlemen restock the identical units from the sellers for all the periods so that they hold $k_m \geq 1$ units at the beginning of every period. This is possible as long as it holds in any given period that

$$Mx_m\eta(x_m, k_m) \leq S(1 - x_s\eta(x_s, 1)) \tag{2}$$

where $x_m\eta(x_m, k_m)$ represents the number of sales by a middlemen at the retail market and $S(1 - x_s\eta(x_s, k_1))$ the number of sellers at the wholesale market. Each period the total restocking units by middlemen must be no greater than the available units.

**Buyers' directed search** Assuming for the moment the existence of a symmetric equilibrium, the following lemma gives the buyer's probability of being served by a supplier who has capacity $k_i$, denoted by $\eta(x_i, k_i)$. The derivation is given in Watanabe (2006).

[2] To guarantee the existence of a steady state, if a buyer (or seller) leaves the market, then assume that another buyer (or seller) enters the market instantly. With a homogeneous agents setup, this is the simplest way to describe a market equilibrium. With heterogeneous agents setups, the other specifications of exogenous inflows of agents are considered in the marriage matching and labor force mobility literature. See Burdett and Coles (1999).

**Lemma 1** *Given $x_i > 0$ and $k_i \geq 1$, the buyer's probability of obtaining a good from a supplier $i$ that has $k_i$ units of the goods, $\eta(x_i, k_i)$, is given by the following closed form expression.*

$$\eta(x_i, k_i) = \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} + \frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)}\right)$$

*where $\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$ and $\Gamma(k, x) = \int_x^\infty t^{k-1} e^{-t} dt$. $\eta(\cdot)$ is strictly decreasing (increasing) in $x_i$ (in $k_i$) and satisfies $\eta(x_s, 1) = (1 - e^{-x_s})/x_s$.*

Given $\eta(\cdot)$ described above, I now characterize the expected queue of buyers. In any equilibrium where $V_b$ is the value of being a buyer, should a seller or a middleman deviate by setting price $p$ in a period, the expected queue of buyers denoted by $x(p, \cdot)$ satisfies

$$V^b = \eta(x(p, \cdot), k_i)(1 - p) + (1 - \eta(x(p, \cdot), k_i))\, \beta V^b. \tag{3}$$

A buyer choosing $p$ is served with probability $\eta(x(p, \cdot), k_i)$ in which case he obtains per-period utility $1 - p$. If not served by the seller or middleman then the buyer enters the next period and obtains the discounted value $\beta V^b$. The situation is the same for all the other buyers. (3) is an implicit equation that determines $x(p, \cdot) = x(p, k_i \mid V^b) \in (0, \infty)$ as a strictly decreasing function of $p$ given $\beta$, $k_i$ and $V^b$.

**Optimal pricing** Given the directed search of buyers described above, the next step is to characterize the equilibrium retail prices. Denote by $V^s$ the value of being a seller. Suppose that middlemen can restock their units at price $\beta V^s$ in the wholesale market where the sellers are just indifferent between selling (leaving the market) and not selling (remaining in the market). Then in any equilibrium where $V^b$ and $V^s$ are the value of a buyer and a seller, respectively, the optimal price of a seller who has a capacity $k_s = 1$, denoted by $p_s(V^b, V^s)$, satisfies

$$p_s(V^b, V^s) = \text{argmax}_p \left[x(p, 1 \mid V^b)\eta(x(p, 1 \mid V^b), 1)p + (1 - x(p, 1 \mid V^b)\eta(x(p, 1 \mid V^b), 1))\beta V^s\right]$$

as the seller sells its good at price $p$ with probability $x(p, 1\cdot)\eta(x(p, 1\cdot), 1)$, and is otherwise guaranteed $\beta V^s$ in the wholesale market. Similarly, the optimal price of a middleman who has capacity $k_m \geq 1$ is given by

$$p_m(V^b, V^s) = \text{argmax}_p \left[(p - \beta V^s)x(p, k_m \mid V^b)\eta(x(p, k_m \mid V^b), k_m)\right]$$

where $x(p, k_m \cdot)\eta(x(p, k_m \cdot), k_m)$ represents the expected number of sales (by law of large number), and the middleman restocks at price $\beta V^s$ in the wholesales market.

Substituting out $p$ using (3), $p = 1 - \frac{1-(1-\eta(\cdot))\beta}{\eta(\cdot)} V^b$, the objective function of a seller or a middleman denoted by $\Pi_s(x)$ or $\Pi_m(x)$ can be written as

$$\begin{aligned}\Pi_s(x) &= x\eta(x,1) - x(1 - (1-\eta(x,1))\beta)V^b + (1 - x\eta(x,1))\beta V^s \\ \Pi_m(x) &= x\eta(x,k_m) - x(1 - (1-\eta(x,k_m))\beta)V^b - x\eta(x,k_m)\beta V^s\end{aligned}$$

where $x = x(p, k_i \mid V^b)$ satisfies (3). The first-order condition is

$$\frac{\partial \Pi_i(x)}{\partial x} = \left(\eta(x,k_i) + x\frac{\partial \eta(x,k_i)}{\partial x}\right)(1 - \beta(V^b + V^s)) = 0$$

for both $i = s, m$.[3] By rearranging it using (3) and $\frac{\partial \eta(x,k_i)}{\partial x} = -\frac{k_i}{x^2}\left(1 - \frac{\Gamma(k_i+1,x)}{\Gamma(k_i+1)}\right)$, one can obtain the optimal price of the seller (if $i = s$) or the middleman (if $i = m$),

$$\begin{aligned}p_i(V^b, V^s) = \varphi^i(x,k_i)(1-\beta V^b) + (1-\varphi^i(x,k_i))\beta V^s \qquad (4)\\ \text{where} \qquad \varphi^i(x,k_i) \equiv -\frac{\partial \eta(x,k_i)/\partial x}{\eta(x,k_i)/x} = \frac{k_i\left(1 - \frac{\Gamma(k_i+1,x)}{\Gamma(k_i+1)}\right)}{x\eta(x,k_i)}.\end{aligned}$$

### Existence and uniqueness of steady-state equilibrium

**Definition 1** *Given the population parameters $B, S, M$, the initial endowments $k_i$, $i = s, m$, and the discount factor $\beta$, a steady state equilibrium is a set of expected values $\{V^j\}$ for $j = b, s, m$, and market outcomes $\{x_i, p_i\}$ for $i = s, m$ such that:*

1. *Buyers' directed search satisfies (1) and (3);*
2. *Sellers' and middlemen's retail price satisfy the first-order conditions (4) for $i = s, m$;*
3. *Middlemen restock their units from sellers at the end of each period at price $\beta V^s$. The middlemen's restocking satisfies the steady state condition (2);*
4. *Agents have rational expectations.*

[3]The second-order condition is satisfied as it holds that for both $i = s, m$

$$\frac{\partial^2 \Pi_i(x)}{\partial x^2} = -\frac{x^{k_i-1}e^{-x}}{\Gamma(k_i)}(1 - \beta(V^b + V^s)) < 0.$$

The analysis above has established the equilibrium prices $p_i(V^b, V^s)$ given $V^b$ and $V^s$. Equilibrium implies buyers are indifferent between any of the equilibrium prices $p_i = p_i(V_b, V_s)$, $i = s, m$, leading to

$$V^b = \eta(x_s, 1)(1 - p_s) + (1 - \eta(x_s, 1))\beta V^b \tag{5}$$

$$= \eta(x_m, k_m)(1 - p_m) + (1 - \eta(x_m, k_m))\beta V^b, \tag{6}$$

where $x_i = x(p_i, k_i \mid V^b)$ is the equilibrium queue of buyers at $i = s, m$. Buyers then successfully purchase the good from the seller or middleman with probability $\eta(x_i, k_i)$ each period. The value of sellers and middlemen are given by

$$V^s = x_s \eta(x_s, 1) p_s + (1 - x_s \eta(x_s, 1))\beta V^s \tag{7}$$

$$V^m = x_m \eta(x_m, k_m)(p_m - \beta V^s)/(1 - \beta), \tag{8}$$

respectively. Middlemen restock at wholesale price $\beta V^s$ each period and sellers are indifferent between selling and not selling at that price. To guarantee the existence of an equilibrium, the set of parameters that satisfy (2) should be identified. To see this, it is important to observe that indifference conditions (5) and (6) can be reduced to the following simple form: applying (4) for $i = s$ to (5) with a rearrangement,

$$\frac{V^b}{1 - \beta V^s} = \frac{\eta(x_s, 1)(1 - \varphi^s(x_s, 1))}{1 - \beta(1 - \eta(x_s, 1)) - \beta \eta(x_s, 1)\varphi^s(x_s, 1)} = \frac{e^{-x_s}}{1 - \beta\left(1 - e^{-x_s}\right)};$$

similarly, applying (4) for $i = m$ to (6) with a rearrangement,

$$\frac{V^b}{1 - \beta V^s} = \frac{\eta(x_m, k_m)(1 - \varphi^m(x_m, k_m))}{1 - \beta(1 - \eta(x_m, k_m)) - \beta \eta(x_m, k_m)\varphi^m(x_m, k_m)} = \frac{\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)}}{1 - \beta\left(1 - \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)}\right)};$$

these two equations imply

$$\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} = e^{-x_s}. \tag{9}$$

As the adding-up restriction (1) and the indifference condition (9) identify a unique allocation $x_s, x_m > 0$ (see the proof of Theorem 1), the steady state condition (2) imposes a parameter restriction on $k_m, B, S, M$, but not on the discount factor $\beta$. Given this parameter restriction, to find an equilibrium is now reduced to a standard fixed point problem.

**Theorem 1 (Steady state equilibrium)** *For any $\beta \in [0,1)$, a steady state equilibrium exists and is unique, satisfying $V^b \in (0,1)$, $x_i \in (0,\infty)$, $p_i \in (0,1)$, and $V^i \in (0,k_i)$, $i = s,m$, given that for all $B, M \in (0,\infty)$:*

1. *if $S \in [B,\infty)$, the steady state condition (2) holds for all $k_m \geq 1 \in \boldsymbol{Z}_+$;*
2. *if $S \in [\bar{S},B)$, (2) holds for $k_m \leq \bar{k}_m$ and (2) does not hold for $k_m > \bar{k}_m$;*
3. *if $S \in (0,\bar{S})$, there is no $k_m \geq 1$ that satisfies (2).*

*$\bar{S} \in (0,B)$ is a unique solution to (2) with equality for $k_m = 1$, and $\bar{k}_m \in [1,\infty) \subset \boldsymbol{R}_+$ is a unique solution to (2) with equality.*

In the Appendix it is shown that $\bar{k}_m = k_m(B,S,M)$ is strictly increasing (or decreasing) in $S$ (or $B, M$), and $\bar{S} = S(B,M)$ is strictly increasing in $B, M$. Hence, the parameter restrictions in the steady state equilibrium can be stated as follows: if the population of middlemen $M$ is relatively large, then the units of each middleman $k_m$ need to be relatively small; if the population of buyers $B$ (sellers $S$) is relatively large (small), then the units of each middleman $k_m$ or the population of middlemen $M$ need to be relatively small.

As mentioned before, the equilibrium allocation of buyers $x_s, x_m$ is determined irrespective of the discount factor $\beta$ each period. Therefore, the results obtained in Watanabe (2006), where I have investigated the case $\beta = 0$, are applicable here for all $\beta \in [0,1)$:

1. For $k_m = 1$ all sellers and middlemen receive the identical number of buyers $x_s = x_m$ and post the identical price $p_s = p_m$;
2. An increase in the capacity of middlemen $k_m$ creates *a demand effect* that induces more buyers to visit middlemen and fewer buyers to visit sellers, resulting in an increase in $x_m$ and a decrease in $x_s$;
3. An increase in the proportion of sellers $S$ or middlemen $M$ decreases $x_s, x_m$, while an increase in the proportion of buyers $B$ increases $x_s, x_m$.

As a lower $x_s$ implies a lower value of sellers $V^s$ and thus a lower wholesale price $\beta V^s$, the above results can be extended to

**Corollary 1** *For all $\beta \in [0,1)$, an increase (decrease) in the population of sellers or middlemen (buyers), or in the capacity of middlemen leads to a lower wholesale price $\beta V^s$.*

# 3 Bid-ask spread

In this section I characterize the behaviors of the bid-ask spread of middlemen, i.e., the difference between the ask price (retail price) and the bid price (wholesale price) set by middlemen, given by

$$p_m - \beta V^s = \varphi^m(x_m, k_m)(1 - \beta(V^b + V^s)).$$

Here $\varphi^m(x_m, k_m)$ represents the middleman's share of the net trading surplus, $1-\beta(V^b+V^s) = (1-\beta)/(1-\beta x_s e^{-x_s})$. In what follows, I shall focus on the case

**Assumption 1** $S \in [B, \infty)$.

As shown in Theorem 1, this assumption guarantees the existence of a steady state equilibrium for all $k_m \geq 1$ and $B, M \in (0, \infty)$. Below I present the results with (i) $\beta = 0$ and (ii) $\beta \in [0, 1)$ in separation. As it turns out, the former (special) case identifies important effects that help understand the behavior of the bid-ask spread in the latter (general) case.

## 3.1 Special Case $\beta = 0$

In this case, the wholesale price is zero and the bid-ask spread of middlemen is identical to their retail market price, i.e.,

$$p_m = \varphi^m(x_m, k_m) = \frac{1 - \frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)}}{x_m \eta(x_m, k_m)/k_m}$$

while the retail market price of sellers is given by $p_s = \varphi^s(x_s, 1) = \frac{1-e^{-x_s}-x_s e^{-x_s}}{1-e^{-x_s}}$.

**Proposition 1** *Suppose $\beta = 0$. An increase (a decrease) in the population of sellers $S$ or middlemen $M$ (buyers $B$) leads to a lower retail market price $p_i$, $i = s, m$, for all $k_m \geq 1$.*

This result represents the usual market-tightness effect which is standard in the directed/competitive search literature - see, for example, Moen (1997) and Acemoglu and Shimer (1999): A larger total supply relative to total demand, i.e., a larger $S/B$ or $M/B$, implies a larger amount of competition in retail markets and thus a lower price $p_i$, $i = s, m$.

An increase in the capacity of middlemen $k_m$ implies an increase in total supply, but its effects on the prices are not uniform. Apart from the (market-tightness) effect that results from an increase in total supply, there are two important effects. On the one hand, a *demand effect* of the middlemen's capacity implies an increase in the number of buyers to visit middlemen, rather than sellers. This effect pushes up $p_m$ and pushes down $p_s$. On the other hand, a larger capacity of a middleman implies it is less likely that excess demand occurs at the middleman. Because buyers know that the middleman receives zero payoff when $\beta = 0$ (or a lower expected payoff in general when $\beta > 0$) from unsold units, the middleman can extract only a smaller fraction of trading surplus per unit when the capacity $k_m$ is larger.[4] The total effects of a change in $k_m$ on the retail price of middlemen $p_m$ are, in general, ambiguous.

To be more precise, the demand effect is relatively strong when $x_m$ increases by relatively more as $k_m$ increases. This is the case when the slope of the adding-up constraint $|-\frac{S}{M}|$ is relatively large. Therefore, the demand effect, which pushes up $p_m$ as $k_m$ increases, is likely to become dominant for high values of $S/M$. The second effect to decrease $p_m$, which shall be referred to as a *stock-out effect*, depends on the behavior of the stock-out probability, which is the probability that the number of buyers appearing at a given middleman $n$ is strictly greater than its capacity $k_m$:[5]

$$\text{Prob.}(n > k_m) = \sum_{n=k_m+1}^{\infty} \frac{e^{-x_m} x_m^n}{n!} = 1 - \frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)}.$$

The price $p_m$ is lower when the stock-out probability is lower. As stock-outs are less likely to occur when $x_m$ is smaller or $k_m$ is larger, the stock-out probability is relatively lower with larger values of $k_m$, or smaller values of $B/S, B/M$ which lead to smaller $x_m$.

[4] The demand effect captures a familiar observation that supermarkets attract more customers when they increase their inventories, while the stock-out effect captures that intermediaries often have a sale when holding many inventories implies a high risk. Indeed, stock-outs are prevalent in retail markets. Aguirregabiria (2005) finds that intermediaries' inventory is a critical variable to explain their pricing patterns, especially when customers trade-off the price against the service rate. Further, Aguirregabiria (1999) shows that a negative effect of inventory ordering on the markups is consistent with data in a supermarket chain. In connection with this, retailers' price increases following supermarket leveraged buy-outs (LBOs) are observed in Chevalier (1995). This evidence can be consistent with the stock-out effect illustrated here, given that high leverage may lead firms to be cash-constrained and hence may force them to reduce their size.

[5] The second equation follows from the series definition of cumulative gamma function, $\sum_{n=0}^{k-1} \frac{e^{-x} x^n}{n!} = \frac{\Gamma(k,x)}{\Gamma(k)}$.

I now confirm the above intuition by numerical examples. The numbers and figures that appear within this subsection are only for an illustrative purpose, and the results presented below do not depend on the choice of parameter values. Figure 1 (a)-(c) illustrate the behaviors of retail market price of middlemen $p_m$ to changes in their capacity $k_m$. The figures show the occurrence of price increases in response to increases in the capacity of middlemen for relatively small $k_m$. Note that the price discrease is impossible unless the demand effect is dominant for the price determination. While a lower likelihood of stock-outs and an increase in the total supply both imply that the retail price of middlemen decreases in sufficiently large $k_m$, the occurrence and degree of price-drops depend on the parameter values.

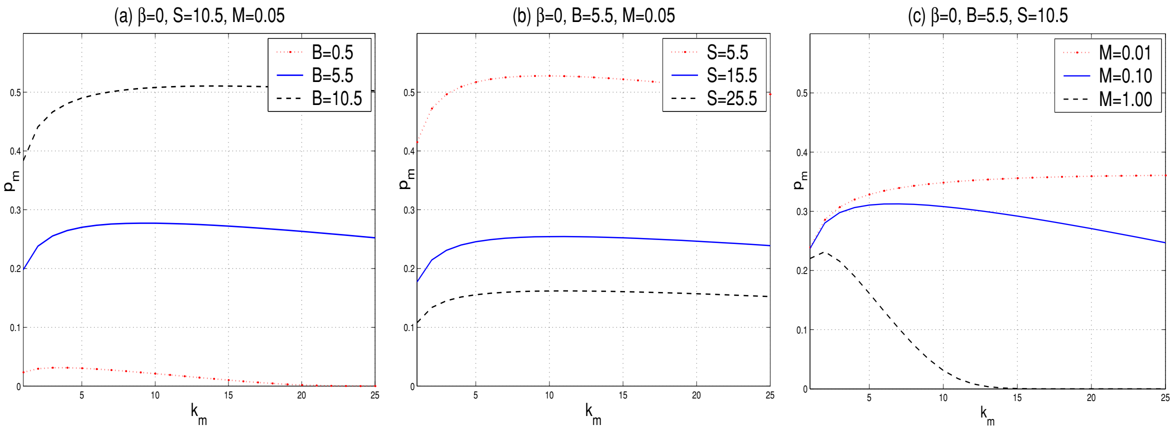


Figure 1: Retail price $p_m$ and capacity of middlemen's capacity $k_m$

To detect exactly the relative magnitude of the demand effect and the stock-out effect described above I should abstract them from the (market-tightness) effect caused by changes in total supply. As Assumption 1 guarantees the existence of steady state equilibrium for all $k_m \geq 1$, one can take either of the population parameters as an instrument to control the per-period total deamnd/supply ratio,

$$X \equiv \frac{B}{Mk_m + S},$$

as $k_m$ changes. Taking $B$ as an instrument, Figure 2 (a) plots the retail price of middlemen $p_m$ for values of $k_m$. Note that to keep the total ratio $X$ constant, the buyer population $B$

must take a larger value as $k_m$ increases. The figure illustrates that the demand effect is likely to become dominant for relatively low values of $k_m$, whereas the stock-out effect is likely to become dominant for relatively high values of $k_m$. As stock-outs are less likely to occur for lower values of $X$, the price drops tend to occur at relatively low capacity levels $k_m$ when $X$ is relatively low (i.e., when $B$ is relatively low).

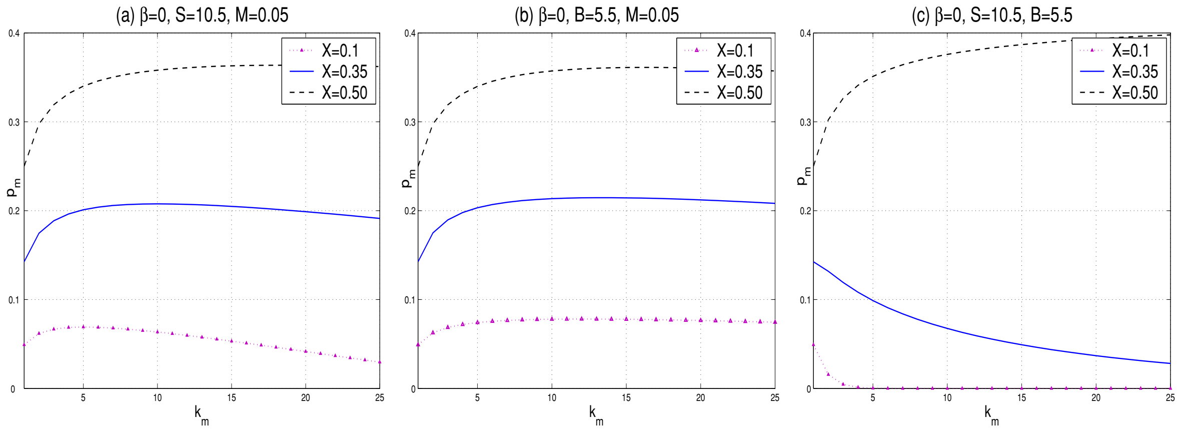


Figure 2: Demand effect and stock-out effect of middlemen's capacity $k_m$ (fixing the total demand/supply ratio $X = \frac{B}{Mk_m+S}$)

Similarly, Figure 2 (b) and (c) are drawn taking $S$ and $M$ as an instrument, respectively. This time, the population of sellers or middlemen must take a smaller value as $k_m$ increases. Remember that a larger $S$ implies both a larger demand effect and a larger stock-out effect which work oppositely on the retail price of middlemen. The conflict of these two effects can be seen in Figure 2 (b) as a relatively small difference, across differing values of $X$, in the range of $k_m$ that the price $p_m$ is increasing, although the demand effect is still dominant for relatively small $k_m$. In contrast, a larger $M$ implies a smaller demand effect and a larger stock-out effect which work in the same direction. In Figure 2 (c), one can therefore observe that the demand effect is likely to become dominant for relatively low $k_m$ and high $X$ (i.e., low $M$), whereas the stock-out effect is likely to become dominant for relatively high $k_m$ and low $X$ (i.e., high $M$). In general, the following property holds.

**Proposition 2** *Suppose $\beta = 0$ and fix the total supply by middlemen, $G = Mk_m \in (0, \infty)$. Then, it holds that $p_m \to 0$ as $k_m \to \infty$ if and only if $G > B$.*

When the middlemens capacity is sufficiently large, the stock-out effect dominates the demand effect for sufficiently large $Mk_m$ relative to the population of buyers $B$.

### 3.2 General Case $\beta \in [0, 1)$

In general the bid-ask spread depends on the wholesale price $\beta V^s$ and the buyers' opportunity cost of trading, given by $1 - \beta V^b$, as well as on the middlemen's share of surplus $\varphi^m(\cdot)$. These terms were constrained to be constant when $\beta = 0$. For $\beta > 0$ their behavior needs to be taken into account and it can be captured by that of the net trading surplus denoted $N(x_s) \equiv 1 - \beta(V^b + V^s)$. Under Assumption 1 the equilibrium number of buyers at each seller or middleman is relatively small, so that the behavior of $N(\cdot)$ is dictated by that of $\beta V^s$. Further, as shown in Corollary 1, a larger total supply relative to demand leads to a lower $\beta V^s$. Therefore, the property established on the retail market price in Proposition 1 under $\beta = 0$ can be extended to the bid-ask spread for all $\beta \in [0, 1)$.

**Proposition 3** *An increase (a decrease) in the population of sellers $S$ or middlemen $M$ (buyers $B$), or in the discount factor $\beta$ leads to a lower bid-ask spread of middlemen, $p_m - \beta V^b$, for all $k_m \geq 1$ and $\beta \in [0, 1)$.*

The market-tightness effect implies an intensified competition in the retail-markets, leading to a lower bid-ask spread of middlemen. The discount factor does not affect the equilibrium allocations, thus a higher $\beta$ implies a higher wholesale price and a lower spread.

The comparative statics result of middlemen's capacity $k_m$ on the bid-ask spread is in general ambiguous. While the stock-out effect is always negative on the spread, the demand effect to increase the trading share of middlemen may be counteracted by a lowering seller's continuation value as $k_m$ increases, which lowers the spread, for $\beta > 0$. In what follows I use numerical simulations to investigate the behaviors of the bid-ask spread. The baseline parameters I have chosen are $B = 1$, $S = 1.14$, $M = 0.13$, $k_m = 9$ and $\beta = 0.96$. This choice reflects an intention to take into accounts the following empirical facts: (i) the average commercial margin in the entire U.S. retail sectors is stable over the past decades and is around 28%; (ii) the retail inventories/sales ratio (seasonally adjusted, monthly) is on average $1.45 - 1.75$ in the U.S in the last ten years.[6] Under the baseline parameter values, the ratio of the bid-ask spread to the retail price (i.e., the retail commercial margin) satisfies

$$\frac{p_m - \beta V^b}{p_m} = 28.8\%$$

and the average inventories/sales ratio of middlemen is given by

$$\frac{k_m}{x_m \eta_m(x_m, k_m)} = 1.52.$$

As the proportion of middlemen's sales should be reasonably high it may be worth mentioning that the trade share of middlemen's sector, in the benchmark case, amounts to

$$\frac{M x_m \eta(x_m, k_m)}{M x_m \eta(x_m, k_m) + S x_s \eta(x_s, 1)} = 0.804.$$

Notwithstanding the effort to generate the equilibrium outcomes close to data, I shall mention that the results presented below is not affected by the parameter choice.

Figure 3 depicts the behaviors of the bid-ask spread of middlemen in response to changes in $k_m$. The blue line shows the demand effect dominates the two other effects (i.e., stock-out effect and market-tightness effect) for relatively low values of $k_m$: the bid-ask spread is increasing in relatively small $k_m$, which is impossible in the absent of the demand effect, and decreasing in relatively large $k_m$. The black dot line nets out the demand effect and the stock-out effect by keeping the total demand/supply ratio $X$ constant at the benchmark value $\bar{X} = 0.43$.[7] Although which effect becomes dominant depends in general on parameter values,

[6] These number are from the Annual Retail Trade Survey published by the Bureau of the Census. In 2006, for example, the total retail gross margin was 28.6% and the retail inventories/sales ratio was 1.49. Faig and Jerez (2005) also use the retail commercial margin, but not the retail inventories/sales ratio, for a calibration study in their model which has imperfect information in retail markets but does not have inventories and endogenous high meeting rates.

[7] In this subsection the population of buyers $B$ is used to net out the market-tightness effect arising from changes in $X$. This seems to be the most natural way as discussed in the previous subsection, although the results presented below will remain unaffected by this choice.

it is important to observe that the bid-ask spread tends to be decreasing in relatively large $k_m$ even in the absent of the competitive effect.

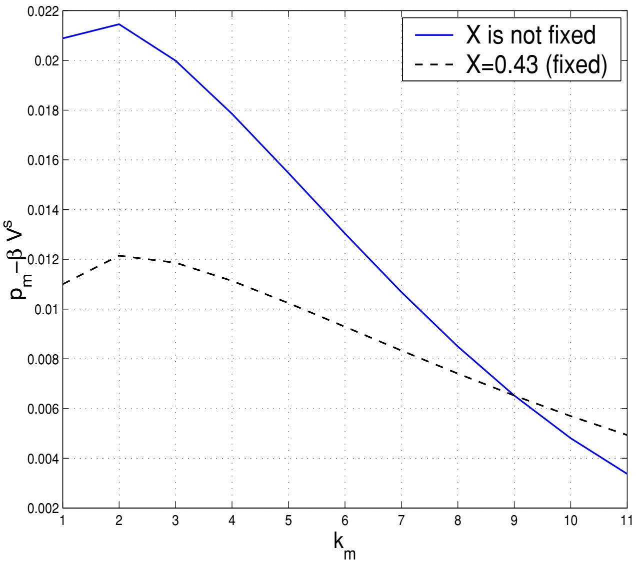


Figure 3: Inventory and bid-ask spread of middlemen

**Result 1** *As middlemen's capacity $k_m$ increases, the bid-ask spread is increasing in relatively small $k_m$ due to the demand effect, while the spread is decreasing in relatively large $k_m$ due to the market-tightness effect and the stock-out effect.*

Figure 4 draws a similar picture with keeping the total supply by middlemen $Mk_m$ fixed at the benchmark. The fixed supply constraint in the middlemen's market uncovers the net impact of the distribution of middlemen's inventories on the bid-ask spread that arises in the absent of the market-tightness effect. For example, the result shows that an economy with $k_m = 1$ and $M = 1.17$ has a spread that is 77.4% larger than that with $k_m = 10$ and $M = 0.117$. When the inventory of middlemen is sufficiently large, the property established in Proposition 2 can be extended to:

**Proposition 4** *Fix the total supply by middlemen, $G = Mk_m \in (0, \infty)$. For all $\beta \in [0, 1)$, it holds that $p_m - \beta V^s \to 0$ as $k_m \to \infty$ if and only if $G > B$.*

For sufficiently large $Mk_m$ (relative to the population of buyers), the middlemen's share of the trading surplus satisfies $\varphi(\cdot) \to 0$ as $k_m \to \infty$ due to the stock-out effect, as shown in Proposition 2, while at the same time the wholesale price satisfies $\beta V^s \to 0$ as $k_m \to \infty$ due to the demand effect. Obviously, $G > B$ as $k_m \to \infty$ is satisfied when $G$ is not fixed, i.e., when the market-tightness effect is present.

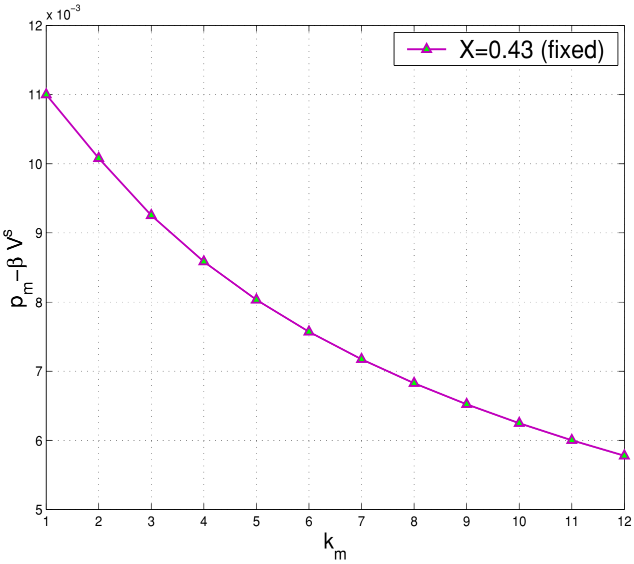


Figure 4: Inventory and bid-ask spread of middlemen (with fixed supply by middlemen $Mk_m$)

**Result 2** *The distribution of middlemen's inventories matters for the determination of the bid-ask spread.*

1. *Fixing total supply by middlemen, the bid-ask spread is relatively higher with a large quantity of small scaled middlemen (i.e. many middlemen, each holding few inventories) than with a small quantity of large scaled middlemen (i.e. few middlemen, each holding many inventories).*

2. *The bid-ask spread converges to zero as middleman's inventories grow sufficiently large, which can occur even without the market-tightness effect if the total supply of goods by middlemen is relatively large.*

To further investigate the quantitative importance of the respective effect, one can decompose a change in the bid-ask spread as follows:

$$\frac{\Delta(p_m - \beta V^b)}{\Delta k_m} |_{\bar{X}} = \frac{\partial \varphi(x_m, k_m)}{\partial x_m} \left( \frac{\Delta x_m}{\Delta k_m} + \frac{dx_m}{dB} \frac{\Delta B}{\Delta k_m} |_{\bar{X}} \right) N(x_s)$$
$$+ \varphi(x_m, k_m) \frac{\partial N(x_s)}{\partial x_s} \left( \frac{\Delta x_s}{\Delta k_m} + \frac{dx_s}{dB} \frac{\Delta B}{\Delta k_m} |_{\bar{X}} \right) + \frac{\partial \varphi(x_m, k)}{\partial k} |_{k=k_m} N(x_s)$$

where $k \in \mathbf{R}_+$ and $N(x_s) \equiv 1 - \beta(V^b + V^s)$ is the net trading surplus of middlemen. The first two terms in the R.H.S. of this expression represent the demand effect, whereas the last term represents the stock-out effect which is negative. Here, the population of buyers $B$ must take a higher value as $k_m$ increases, to keep the total ratio fixed at the benchmark value $X = \bar{X}$. The market-tightness effect of an increase in the total supply, due to an increase in middlemen's capacity $k_m$, is then given by

$$\frac{\Delta(p_m - \beta V^b)}{\Delta k_m} - \frac{\Delta(p_m - \beta V^b)}{\Delta k_m} |_{\bar{X}} = -\left( \frac{\partial \varphi(x_m, k_m)}{\partial x_m} \frac{dx_m}{dB} N(x_s) + \varphi(x_m, k_m) \frac{\partial N(x_s)}{\partial x_s} \frac{dx_s}{dB} \right) \frac{\Delta B}{\Delta k_m} |_{\bar{X}}$$

which is negative. Table 1 summarizes these three effects averaged over a reasonable parameter interval.[8]

| | demand effect | stock-out effect | market-tightness effect | overall effect |
|---|---|---|---|---|
| $p_m$-$\beta V^s$ | 4.55 | -3.59 | -3.21 | -2.24 |
| $(p_m$-$\beta V^s)/p_m$ | 5.64 | -2.88 | -0.89 | 1.87 |

Table 1: The decomposition of changes in the bid-ask spread

In the table, the columns represent the respective effect calculated using the decomposition described above. The rows represent the percent change in the bid-ask spread $p_m - \beta V^s$ and

[8] According to the Annual Retail Trade Survey, the sectoral average of the retail margins in 2006 ranges from 16.4% (gasoline stations) to 48.1% (furniture and home furnishings stores). The target interval of parameter values in Table 1 and 2 are selected so that the model generates the retail margins that fall within this range, which turn out to be $k_m \in [6, 11]$, $B \in [0.77, 1, 14]$, $S \in [1, 4.32]$ and $M \in [0.07, 0.18]$. Again, the result does not depend on the choice of intervals.

the retail margins $(p_m - \beta V^s)/p_m$, respectively, in response to the percent change in parameter $k_m$. The demand effect turns out to be positive for all the parameter values examined. Of the total negative impact of a capacity change on the bid-ask spread, the stock-out effect accounts for 52.9%. The corresponding number for the retail margins is even larger and amounts to 76.4%.

**Result 3** *The demand effect of an increase in middlemen's inventories is positive on the bid-ask spread. The stock-out effect accounts for a significant part of the decreases in the bid-ask spread.*

| | $k_m$ | B | S | M |
|---|---|---|---|---|
| $p_m$-$\beta V^s$ | 4.71 | 2.89 | 1.01 | 4.13 |
| $(p_m$-$\beta V^s)/p_m$ | -3.99 | -1.75 | -0.58 | -2.65 |

Table 2: The sensitivity of the bid-ask spread

Finally, I have calculated the elasticity of the bid-ask spread with respect to the total demand/supply ratio given by

$$\text{elasticity}_i \equiv \frac{\Delta(p_m - \beta V^s)/\Delta_i X}{(p_m - \beta V^s)/X}$$

where $\Delta_i X$ denotes a change in the ratio $X$ due to a change in parameter $i \in \{k_m, B, S, M\}$. Table 2 summarizes the result. The column represents the original parameter change. Comparing this elasticity across parameters allows us to identify the quantitative importance of each parameter on the bid-ask spread in terms of an equal change in $X$. For example, the bid-ask spread is more sensitive to changes in $k_m$ and $M$ than to those in $B$ by 63.0% and 42.9%, and than to those in $S$ by 371.0% and 313.0%, respectively. The quantitative importance of these two parameters $k_m, M$, which represent the distribution of middlemen's inventories, is even more magnified on the retail margins. To sum up,

**Result 4** *The bid-ask spread is relatively more sensitive to the distribution of middlemen's inventories, in terms of the number of middlemen and the individual inventory level, than the population of buyers and sellers.*

## 4 Conclusion

This paper proposed a simple theory of the bid-ask spread set in intermediated markets. Middlemen set an ask price for buyers in retail markets and a bid price for sellers in wholesale markets. They hold inventories of a good and are specialized in buying and selling. Middlemen's inventories can provide buyers with immediacy service under market frictions, thereby the ask price of middlemen includes a premium for immediacy service to buyers and the bid price includes a premium charged to sellers for guaranteed sale. The model generates two important effects of middlemen's inventories that serve as the critical determinant of the bid-ask spread. On the one hand, it allows middlemen to enjoy a simultaneous increase in both their buying and selling power. On the other hand, it puts downward pressure on their retail price. These conflicting effects cause non-monotonic responses of the bid-ask spread to changes in their inventories. The equilibrium constructed here may remind some readers of Galbraith's (1952) observation that large corporations such as a nationwide grocery chain can act as if they use their bulk buying power to receive discounts from the suppliers of products and pass them to final consumers in the form of lower retail prices. It would be interesting to see that a simple search theory like the one presented here can provide novel economic insights into real-life phenomena that have been emphasized generation after generation.

# Appendix

## Proof of Theorem 1

The analysis in the main text has established that (1), (4), (5), (6), (7) and (8) describe necessary and sufficient conditions for an equilibrium given (2) holds. All that remains here is to establish a solution to these conditions, $x_s, x_m, p_s, p_m, V^b, V^s, V^m > 0$, exists and is unique. The proof takes 3 steps. Step 1 establishes a unique solution $x_s, x_m > 0$ for all $k_m \geq 1$ and $B, S, M \in (0, \infty)$ using (1), (4), (5) and (6). With a slight abuse of notation, let $x_i(k_m, B, S, M)$ denote this solution for $i = s, m$. Given this solution, Step 2 identifies the set of parameters that satisfy the steady state condition (2) or

$$Mx_m(k_m, B, S, M)\eta(x_m(k_m, B, S, M), k_m) \leq S(1 - x_s(k_m, B, S, M)\eta(x_s(k_m, B, S, M), 1)). \tag{10}$$

Hence, Step 1 and 2 establish that a solution $x_s, x_m > 0$ exits and is unique for $k_m \geq 1, B, S, M$ satisfying (10). Given this solution, Step 3 then identifies a unique solution $V^j \in (0, 1)$ to (5), (6) and (7) for $j = b, s$. The rest of the equilibrium values are identified immediately: given $V^b$, $V^s$ and $x_i$, (4) determines a unique $p_i \in (0, 1)$ for $i = s, m$; given $V^s$, $x_m$ and $p_m$, (8) determines a unique $V^m \in (0, k_m)$. For all $\beta \in [0, 1)$ and the $k_m \geq 1$, $B, S, M$ satisfying (10), this solution then satisfies (1), (4), (5), (6), (7) and (8) so describes equilibrium.

**Step 1** For any $k_m \geq 1$ and $B, S, M \in (0, \infty)$, a solution $x_i = x_i(k_m, S, M)$ to (1), (4), (5) and (6) exists and is unique for $i = s, m$ that is: continuous in $S, M, k_m \in \mathbf{R}_+$; strictly decreasing (or increasing) in $S, M$ (or in $B$) for all $k_m \geq 1$; strictly increasing (or decreasing) in $k_m$ for $B, S, M \in (0, \infty)$ if $i = m$ (or if $i = s$) satisfying $x_s(1, \cdot) = x_m(1, \cdot) = B/(S + M)$, $x_s(k_m, \cdot) \to 0$ and $x_m(k_m, \cdot) \to B/M$ as $k_m \to \infty$.

**Proof of Step 1.** In the main text, it has been shown that (4), (5) and (6) imply (9). Substituting out $x_m$ in (9) by using (1),

$$\frac{\Gamma\left(k_m, \frac{B - Sx_s}{M}\right)}{\Gamma(k_m)} = e^{-x_s} \tag{11}$$

where $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$ and $\Gamma(k, x) = \int_x^\infty t^{k-1}e^{-t}dt$. The L.H.S. of this equation, denoted by $\Phi(x_s, k_m, B, S, M)$, is continuous and strictly increasing in $x_s$ and $k_m \in \mathbf{R}_+$, satisfying for any $B, S, M \in (0, \infty)$:

$$\Phi(x_s, \cdot) \to \frac{\Gamma\left(k_m, \frac{B}{M}\right)}{\Gamma(k_m)} < 1 \ \text{ as } \ x_s \to 0; \quad \Phi\left(\frac{B}{S + M}, \cdot\right) = \frac{\Gamma\left(k_m, \frac{B}{S+M}\right)}{\Gamma(k_m)} \geq e^{-\frac{B}{S+M}}$$

with equality only when $k_m = 1$;

$$\Phi(x_s, 1, \cdot) = e^{-\frac{B - Sx_s}{M}}; \quad \Phi(x_s, k_m, \cdot) \to 1 \text{ as } k_m \to \infty.$$

Similarly, $\Phi(\cdot)$ is continuous and strictly increasing (decreasing) in $S, M$ (in $B$) for any $x_s \in (0, \frac{B}{S+M})$ and $k_m \geq 1$. It follows therefore that a unique solution $x_s = x_s(k_m, B, S, M) \in$

$(0, \frac{B}{S+M}]$ exists that is: continuous and strictly decreasing in $k_m \in [1, \infty) \subset \mathbf{R}_+$ satisfying $x_s(1, \cdot) = \frac{B}{S+M}$ and $x_s(k_m, \cdot) \to 0$ as $k_m \to \infty$ for any $B, S, M$; continuous and strictly decreasing (increasing) in $S, M$ (in $B$) for all $k_m \geq 1$.

Applying this solution to (1), one can obtain a unique solution $x_m = x_m(k_m, B, S, M) \in [\frac{B}{S+M}, \frac{B}{M})$ that is: continuous and strictly decreasing in $S$ and $M$; continuous and strictly increasing in $k_m$ and $B$ satisfying $x_m(1, \cdot) = \frac{B}{S+M}$ and $x_m(k_m, \cdot) \to \frac{B}{M}$ as $k_m \to \infty$. This completes the proof of Step 1.

<u>**Step 2**</u> (i) For $S \in [B, \infty)$, (10) holds for all $k_m \geq 1$ and $B, M \in (0, \infty)$. (ii) For $S \in [\bar{S}, B)$ and for all $B, M \in (0, \infty)$, (10) holds for $k_m \leq \bar{k}_m$ and (10) does not hold for $k_m > \bar{k}_m$ where $\bar{k}_m = k_m(B, S, M) \in [1, \infty) \subset \mathbf{R}_+$ is strictly increasing (or decreasing) in $S$ (or $B, M$), and $\bar{S} = S(B, M) \in (0, B)$ is strictly increasing in $B, M$. (iii) For $S \in (0, \bar{S})$ and for all $B, M \in (0, \infty)$, there is no $k_m \geq 1$ that satisfies (10).

**Proof of Step 2.** For $B, S, M \in (0, \infty)$ and $k_m \in [1, \infty) \subset \mathbf{R}_+$, define

$$\Psi(k_m, B, S, M) \equiv M x_m(k_m, B, S, M) \eta(x_m(k_m, B, S, M), k_m) - S e^{-x_s(k_m, B, S, M)}$$

where $x_i(\cdot)$, $i = s, m$, satisfies the properties obtained in Step 1. (10) requires $\Psi(\cdot) \leq 0$. Observe that $\Psi(\cdot)$ is continuous and strictly increasing in $k_m$ for any $B, S, M \in (0, \infty)$, and satisfies:

$$\Psi(1, B, S, M) = M(1 - e^{-\frac{B}{S+M}}) - S e^{-\frac{B}{S+M}}; \quad \Psi(k_m, B, S, M) \to B - S \text{ as } k_m \to \infty.$$

Hence, if $S \in [B, \infty)$ then $\Psi(k_m, \cdot) \leq 0$ holds for all $k_m \in [1, \infty)$ and $B, M \in (0, \infty)$, leading to the first claim. Observe further that $\Psi(1, B, S, M)$ is strictly increasing in $B, M$ and strictly decreasing in $S$, satisfying:

$$\Psi(1, B, S, M) \to M(1 - e^{-\frac{B}{M}}) > 0 \text{ as } S \to 0; \quad \Psi(1, B, B, M) = M(1 - e^{-\frac{B}{B+M}}) - S e^{-\frac{B}{B+M}} < 0.$$

Hence, there exists a unique solution $\bar{S} = S(B, M) \in (0, B)$ to $\Psi(1, B, \bar{S}, M) = 0$, which is strictly increasing in $B, M$, such that $\Psi(1, B, \bar{S}, M) > 0$ for $S < \bar{S}$ and $\Psi(1, B, \bar{S}, M) < 0$ for $S > \bar{S}$. Therefore, if $S \in (0, \bar{S})$ it holds that $\Psi(k_m, S, M) > 0$ for all $k_m \in [1, \infty)$ and $B, M \in (0, \infty)$, leading to the third claim.

Notice that if $S \in [\bar{S}, B)$ then $\Psi(1, B, S, M) \leq 0 < \Psi(k_m, B, S, M)$ as $k_m \to \infty$ for all $B, M \in (0, \infty)$. Because $\Psi(\cdot)$ is strictly decreasing in $S$ and is strictly increasing in $B, M$, this implies that if $S \in [\bar{S}, B)$ then there exists a unique solution $\bar{k}_m = k_m(B, S, M) \in [1, \infty) \subset \mathbf{R}_+$ to $\Psi(\bar{k}_m, B, S, M) = 0$, which is strictly increasing (or decreasing) in $S$ (or $B, M$), such that $\Psi(k_m, B, S, M) \leq 0$ for $k_m \leq \bar{k}_m$ and $\Psi(k_m, B, S, M) > 0$ for $k_m > \bar{k}_m$. Hence, the second claim follows. This completes the proof of Step 2.

<u>**Step 3**</u> Given $x_s \in (0, B/(S + M)]$ established in Step 1 and $k_m \geq 1$, $B, S, M > 0$ satisfying (10), there exists a unique solution $V^j \in (0, 1)$, $j = b, s$, to (4), (5), and (7).

**Proof of Step 3.** (4), (5), and (7) imply $V^b$ satisfies

$$V^b = \frac{e^{-x_s}}{1 - \beta x_s e^{-x_s}}.$$

The R.H.S of this equation, denoted by $\Upsilon_b(x_s)$, is strictly decreasing in $x^s \in (0, \infty)$ and satisfies: $\Upsilon_b(\cdot) \to 1$ as $x_s \to 0$; $\Upsilon(\cdot) \to 0$ as $x_s \to \infty$. As equilibrium implies $x_s \in (0, B/(S + M)]$, there exists a unique $V^b \in (0,1)$ that satisfies $V^b = \Upsilon_b(\cdot)$. (4), (5), and (7) also imply

$$V^s = \frac{1 - e^{-x_s} - x_s e^{-x_s}}{1 - \beta x_s e^{-x_s}}$$

and this time, the R.H.S. of this equation, denoted by $\Upsilon_s(x_s)$, is strictly increasing in $x^s \in (0, \infty)$ and satisfies: $\Upsilon_s(\cdot) \to 0$ as $x_s \to 0$; $\Upsilon_s(\cdot) \to 1$ as $x_s \to \infty$, thereby there exists a unique solution $V^s \in (0,1)$. This completes the proof of Step 3. ■

## Proof of Corollary 1

Step 1 in the proof of Theorem 1 showed that $x_s = x_s(k_m, B, S, M)$ is strictly decreasing (or increasing) in $k_m, S, M$ (or in $B$) while in Step 3 in the proof of Theorem 1 that $V^s$ is strictly increasing in $x_s$, implying $V^s$ is strictly decreasing (or increasing) in $k_m, S, M$ (or in $B$). ■

## Proof of Proposition 1

Remember that $\frac{dx_i}{dS} < 0$ and $\frac{dx_i}{dM} < 0$, $i = s, m$. Differentiation yields

$$\frac{d\varphi^i(x_i, \cdot)}{dS} = -\frac{\frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i+1, x_i)}{\Gamma(k_i+1)}\right)}{\eta(\cdot)^2} \frac{d\eta(\cdot)}{dx_i} \frac{dx_i}{dS} + \frac{\frac{d}{dx_i}\left[\frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i+1, x_i)}{\Gamma(k_i+1)}\right)\right]}{\eta(\cdot)} \frac{dx_i}{dS}.$$

The first term in the L.H.S. of this equation is negative. To identify the sign of the second term, note that

$$\frac{d}{dx_i}\left[\frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i+1, x_i)}{\Gamma(k_i+1)}\right)\right] = \frac{d}{dx_i}\left[\sum_{j=k}^{\infty} \frac{x_i^j e^{-x_i}}{j!} \frac{k}{j+1}\right] = \sum_{j=k}^{\infty} \frac{x_i^{j-1} e^{-x_i}(j - x_i)}{j!} \frac{k}{j+1} > 0$$

where the last inequality holds because of the fact that $x_i < 1$, $i = s, m$ if $S \in [B, \infty)$. Hence, the second term is negative and so $\frac{d\varphi^i(x_i, \cdot)}{dS} < 0$, $i = s, m$. Similar steps apply to show $\frac{d\varphi^i(x_i, \cdot)}{dM} < 0$ and $\frac{d\varphi^i(x_i, \cdot)}{dB} > 0$, $i = s, m$. ■

## Proof of Proposition 2

The claims can be shown by using the following property (see Temme (1996) p.285):

$$\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} \to D \quad \text{as} \quad k_m \to \infty \tag{12}$$

where $D \in [0,1]$ satisfies: $D = 1$ if and only if $x_m < k_m$; $D = 0$ if and only if $x_m > k_m$.

Throughout the proof given below, keep in mind that with $G = Mk_m \in (0,\infty)$, we have $x_m \to \infty$ as $k_m \to \infty$. There are three cases to examine. Suppose $G < B$. Then $x_m > k_m$ as $k_m \to \infty$. This leads to $\frac{\Gamma(k_m,x_m)}{\Gamma(k_m)} \to 0$ as $k_m \to \infty$ by (12) and so $x_s \to \infty$ as $k_m \to \infty$ by (11). However, this contradicts to (1) which requires $x_s \in (0, B/S)$. Hence, $G < B$ as $k_m \infty$ is not possible.

Suppose next that $G = B$. Then, $x_m = k_m$ as $k_m \to \infty$, leading to

$$\eta(x_m, k_m) = \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} + \frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)}\right) \to 1 \quad \text{as} \quad k_m \to \infty$$

because $\frac{\Gamma(k_m+1,x_m)}{\Gamma(k_m+1)} - \frac{\Gamma(k_m,x_m)}{\Gamma(k_m)} = \frac{x_m^{k_m} e^{-x_m}}{\Gamma(k_m+1)} \to 0$ as $k_m \to \infty$ for any $\frac{x_m}{k_m} \in (0,\infty)$. Since $\frac{\Gamma(k_m,x_m)}{\Gamma(k_m)} \in (0,1)$ as $k_m \to \infty$ when $x_m = k_m$ as $k_m \to \infty$ by (12), this implies

$$\varphi^m(x_i, k_i) = \frac{\frac{k_m}{x_m}\left(1 - \frac{\Gamma(k_m+1,x_m)}{\Gamma(k_m+1)}\right)}{\eta(x_m, k_m)} \in (0,1) \quad \text{as} \quad k_m \to \infty.$$

Suppose finally that $G > B$. Then, $x_m < k_m$ as $k_m \to \infty$, leading to $\frac{\Gamma(k_m,x_m)}{\Gamma(k_m)} \to 1$ and $\eta(x_m, k_m) \to 1$ as $k_m \to \infty$, which further implies $\varphi^m(x_i, k_i) \to 0$ as $k_m \to \infty$. ■

## Proof of Proposition 3

Differentiating the net trading surplus yields

$$\frac{d}{dS}\left(1 - \beta(V^b + V^s)\right) = \frac{d}{dS}\left(\frac{1-\beta}{1 - \beta x_s e^{-x_s}}\right) = \frac{\beta(1-\beta)e^{-x_s}(1-x_s)}{(1-\beta x_s e^{-x_s})^2}\frac{dx_s}{dS} < 0$$

where I use the expression for $V^b, V^s$ given in the proof of Theorem 1 to derive the first equation, and the last inequality follows from the fact that $x_s < 1$ for all $S \in [B,\infty)$, $M \in (0,\infty)$ and $k_m \geq 1$ and $\frac{dx_s}{dS} < 0$ (see Step 1 in the proof of Theorem 1). Combined with $\frac{d\varphi^m(\cdot)}{dS} < 0$ (as shown in Proposition 1), this proves

$$\frac{d}{dS}(p_m - \beta V^s) = \frac{d\varphi^m(\cdot)}{dS}\left(1 - \beta(V^b + V^s)\right) + \varphi^m(\cdot)\frac{d}{dS}\left(1 - \beta(V^b + V^s)\right) < 0.$$

A similar procedure applies to show $\frac{d}{dB}(p_m - \beta V^s) > 0$, $\frac{d}{dM}(p_m - \beta V^s) < 0$, and $\frac{d}{d\beta}(p_m - \beta V^s) < 0$. ■

## Proof of Proposition 4

The claims are immediate by noting that $\varphi^m(\cdot) \to 0$ as $k_m \to \infty$ if and only if $Mk_m > B$ (as shown by Proposition 2), leading to $p_m - \beta V^s = \varphi^m(\cdot)(1 - \beta(V^b + V^s)) \to 0$ as $k_m \to \infty$. ■

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