Choice and individual welfare

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Abstract

We propose an abstract method of systematically assigning a "rational" preference to non-rationalizable choice data. We define an *individual welfare functional* as a mapping from stochastic choice functions into weak orders. A stochastic choice function (or *choice distribution*) gives the empirical frequency of choices for any possible opportunity set (framing factors may also be incorporated into the model). We require that for any two alternatives x and y, if our individual welfare functional recommends x over y given two distinct choice distributions, then it also recommends x over y for any mixture of the two choice distributions. Together with some mild technical requirements, such an individual welfare functional must weight every opportunity set and assign a utility to each alternative x which is the sum across all opportunity sets of the weighted probability of x being chosen from the set. It therefore requires us to have a "prior view" about how important a choice of x from a given opportunity set is.

1 Introduction

1.1 Behavioral welfare debates

In economics, the concept of individual *welfare* is used to guide the economist in making recommendations as well as in making comparisons of alternative situations. The word "welfare" can be interpreted in many different ways. For example, it might refer to a revealed preference. It also might refer to some measure of well-being, such as income.

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Standard welfare economics is grounded on the assumption that each individual has a single and consistent preference ranking over social outcomes or her own consumption, which can be identified from choice.[1] In such a context, the concepts of "better" or "worse" for a given consumer are easily defined. Given basic consistency postulates, such a notion of welfare aids in making predictions out of sample, which in turn allows us to understand the implications of certain institutional changes.

However, welfare might also refer to some other measure of well-being, which may not be entirely determined by choice alone. For example, welfare might refer to income, life span, or other measures of happiness.

We propose in this paper a systematic method of discussing welfare based on choices. It can be applied when all other relevant factors affecting welfare are assumed to be held constant.

There are several reasons for defining such a notion. As mentioned, the concept of welfare is used in helping to guide us in making decisions. The economic, choice-based notion of revealed preference might be reasonably applied in most situations. Indeed, Thaler and Sunstein [37, 38] have recently proposed the controversial notion of "libertarian paternalism" (see also Mitchell [27]). The notion suggests we do nothing if individual choices are consistent. However, the assumption that individuals rank outcomes consistently is controversial. Many types of behavioral anomalies and biases have been found.[2] The community of economists has recently addressed such evidence; and the research program of "behavioral economics" (or psychology and economics) has become established as an accepted framework of descriptive analysis. In this context, libertarian paternalism states that when individuals exhibit inconsistency, *something* must be done, and in this sense paternalism is somehow necessary. However, we have as of yet no systematic method of studying what *should* be done.

The following example is meant to clarify the nature of our exercise.

Example 1 You are going to give a gift to your friend. You know that they choose x out of set $\{x, y\}$, and they choose y out of set $\{x, y, z\}$.
Which one of x, y and z *should* you give them?

Two natural arguments could be put forth. Perhaps y should be given, as it is chosen from a larger set. On the other hand, perhaps x is better, as it beats y in a direct compar-

[1] In general, such a preference need not be complete or transitive.

[2] This literature is vast. Some reference works include Kahneman, Slovic and Tversky [16], Kahneman and Tversky [17], and Thaler [36].

ison. Within the standard framework of revealed preference, we can say nothing. Which view should be taken, if either, cannot be answered with the tools of classical economics. Any ranking of the two outcomes amounts to an outside value judgment, and thus falls squarely in the domain of normative economics. We propose a formal framework for investigating such outside value judgments, allowing us to bring the tools of axiomatic economics to work on the problem.

Even if choice is rationalizable by means of a single consistent preference ranking, as social planners, we may be interested in other concepts of welfare. A number of authors claim that economists should make a distinction between 'decision utility' that governs choice, and 'experienced utility' that determines subjective level of welfare (see for example Kahneman, Wakker, and Sarin [18]). Clearly, retrospective evaluation of outcomes is not a choice-based procedure and thus cannot be tested using classical economic models; but it may be a factor which we believe is relevant. Our model will also help us to understand what can be done when this is the case.

A number of studies in neighboring disciplines such as neuroscience claim that choices have rather little to do with subjective perception of welfare, and that it is determined by (or at least related to) the amount of certain chemicals in brain [6].

We need notions of welfare because without such a notion, it becomes difficult, if not impossible, to make choices *for* an individual. If you decide to give x to the individual, should that be treated as a choice from $\{x, y, z\}$? Or should it be treated simply as a choice from $\{x\}$? And in either case, was it appropriate to have chosen x? Such discussions become more complicated when more individuals are involved, and the constraining factors become part of some equilibrium concept. Moreover, without notions of better or worse, the most basic concepts of welfare economics (for example, Pareto optimality and envy-freeness) are without meaning.

1.2 Our approach

In this paper, we imagine that an economist must determine an individual's welfare as a function of choice data. The *method* the economist uses to determine welfare might itself be influenced by data other than choice (such as empirical observations about retrospective evaluation or brain activities), and we believe it natural to allow sufficient freedom in the set of possible methods. However, we study the *restriction* of these methods to observable choice data; holding all other relevant features fixed.

We imagine that the economist follows a given rule in providing welfare prescriptions for inconsistent choice data. Such a rule is modeled as a function mapping arbitrary choice data into a ranking over alternatives. The data is taken to be a list of empirical distributions of choices indexed by choice situations. In general, these empirical distributions may or may not be rationalized either in the deterministic or stochastic sense. Let us call such a rule an *individual welfare functional*.

We interpret the output of an individual welfare functional for a given set of choice data as a "suitable" proxy for individual welfare, not a "true" or "correct" preference. Philosophically, it is not clear what a "true" preference would look like; even if choices are consistent, revealed preference may not be the same as "true" preference. In particular, we do not imagine a procedure by which our recommended preference induces choice. This marks the departure of our work from work on revealed preference.

We now discuss our axioms. We abstain from discussing axioms which attempt to define welfare. It is our belief that the definition of welfare must necessarily be subjective, depending on both the decision maker and the modeller. Rather, our main ideas are in consistency of welfare across data. This allows our work to be sufficiently broad, so that it may be applied to many different problems in which we may be interested.

Our basic axiom is called Combination. It is a consistency requirement in aggregating situation dependent choices. It roughly says, if two choice situations both support x over y, then the choice data obtained by joining the two also supports x over y.

Our main result is as follows. The combination axiom, combined with other technical axioms, implies a form of linear aggregation. The ranking of alternatives is determined by a utility function. The utility of any alternative in our representation is merely a linear function of the choice probabilities of *all* alternatives. In general, the utility of x can depend on the probability that some alternative y is chosen, where y may not even be present in any situation where x is present. This implies that the choice of any alternative z in any situation has some "weight" in influencing the utility of x.

While the general family of individual welfare functionals may be interesting for many environments, oftentimes it may be unnatural to allow the utility of an alternative to depend on the probability that unrelated alternatives are chosen. To remedy this situation, we study the imposition of additional axioms. One such axiom is binary independence, which specifies that the ranking between two alternatives should not be affected by empirical choice probabilities of other alternatives. A related axiom, binary monotonicity, specifies that utility of an alternative should respond positively to the probability that it is chosen. The additional axioms can deliver a more structured class of individual welfare functionals. In this class, every pair consisting of a situation and a choice from that situation has its own weight. This weight specifies a “prior view” about how important it is to choose that alternative in that situation. The utility of the alternative is then just the sum across all situations of the corresponding weight times the probability the alternative was chosen in that situation.

The paper is organized as follows. Section 2 presents the main results. Section 3 strengthens these results to discuss basic notions of independence, whereby utility of an alternative becomes a weighted sum of the probability of being chosen. Section 4 discusses methods for narrowing down these weights; and Section 5 is devoted to a conclusion and connection of our work to the vast literature studying welfare.

2 Model and basic axioms

Let X be a finite set of alternatives where $|X| \geq 4$. Let $\mathcal{B} \subset 2^X$ be a family of observable opportunity sets. We assume $|B| \geq 2$ for all $B \in \mathcal{B}$. For each $B \in \mathcal{B}$, let $F(B)$ be a set of possible framing factors associate with B, and let $F = \bigcup_{B \in \mathcal{B}} F(B)$.

Example 2 Let $B = \{x, y, z\}$. Then one may for example think of $F(B)$ being a set of linear orderings over $\{x, y, z\}$. Let $> \in F(B)$ be one such order, where $z > y > x$. Then, the pair $(B, >)$ corresponds to presenting the set $\{x, y, z\}$, in the form of an ordered triple (z, y, x). This allows choice to depend on the order of appearance of alternatives.

Example 3 Let $B = \{x, y, z\}$. Then one may for example think of $F(B) = X$ being a set of reference points (in any sense). Let $a \in F(B)$ be one such reference point. Then, the pair (B, a) corresponds to presenting the set $\{x, y, z\}$ together with the reference point a.

Let Θ be the set of observable pairs of choice opportunity and framing factors; namely, let $\Theta \subset \mathcal{B} \times F$ be defined as $\Theta = \{(B, f) \in \mathcal{B} \times F : f \in F(B)\}$.[3] We call elements of Θ *situations*. Each element $\theta \in \Theta$ therefore specifies (B_θ, f_θ) with $f_\theta \in F(B_\theta)$. Let $P = \prod_{\theta \in \Theta} \Delta(B_\theta)$ be the set of all possible empirical distributions of choice, where $p(x|\theta)$ refers to the empirical probability (relative frequency) of $x \in B_\theta$ being chosen in the choice situation $\theta = (B_\theta, f_\theta)$. An element in P is called a *situation-dependent choice distribution*, or simply a choice distribution. Such empirical distributions are arbitrary, and may or may not satisfy any deterministic choice consistency conditions (Chernoff [8], Sen [35], Arrow [2]) or stochastic consistency conditions (Luce [23], Falmagne [9]). They may even violate

[3] Rubinstein and Salant [32] consider a model in which a choice problem comes in the form of such pair, and investigate a rationalizability condition for choice functions defined over the extended domain.

more general conditions (Kalai, Rubinstein and Spiegler [19], Masatlioglu and Ok [28], Rubinstein and Salant [31, 32], Tyson [39]).

Remark 1 There is some loss of generality to taking our data to be situation-dependent choice distributions as the primitive, as opposed to raw choice data. Choosing our primitive in this way amounts to an implicit normalization, where the number of times a particular choice problem has been faced becomes irrelevant. To illustrate, consider data in which the choice problem $\{x, y\}$ appears ten times and the choice problems $\{y, z\}$ and $\{x, z\}$ each appear just once. A simple kind of scoring rule, say, simply counting the number of times each alternative is chosen, likely supports x and y over z simply because z does not appear in choice problems very frequently. Our framework does not allow us to discuss such rules; but we believe the normalization we have chosen to be natural. It avoids a type of bias which can arise due to the number of times a specific problem is faced.

Even after normalizing the choice frequencies for each given θ, there may be more θ's with $x \in B_\theta$ than there are θ's with $y \in B_\theta$. This issue can be resolved by appropriately weighting the pair (x, θ).

An *individual welfare functional* is a function that maps each situation-dependent choice distribution to a complete and transitive binary relation over X. Equivalently, it is given as a family of binary relations indexed by choice distributions, $\{\succsim_p\}_{p \in P}$, where $\succsim_p$ is a complete and transitive ordering over X for each $p \in P$.

Example 4 Suppose $X = \{x, y, z\}$, and that Θ can be identified with the set of subsets of X. That is, for all nonempty $B \subset X$, there exists a unique $\theta \in \Theta$ for which $B_\theta = B$.

A simple example of an individual welfare functional fixes some $u : X \to \mathbb{R}$, and defines $\succsim_p$ as being the ordinal ranking consistent with the function $v : X \to \mathbb{R}$ defined by $v(w) = u(w) \sum_{\theta \in \Theta} p(x|\theta)$.

First, we impose the following three axioms.

Combination: For every $p, q \in P$ and $\lambda \in (0, 1)$,

$$x \succsim_p y \text{ and } x \succsim_q y \text{ imply } x \succsim_{\lambda p + (1-\lambda) q} y,$$

and

$$x \succ_p y \text{ and } x \succsim_q y \text{ imply } x \succ_{\lambda p + (1-\lambda) q} y.$$

The combination axiom requires that the ranking specified by the concatenation of two choice distributions is consistent with the rankings specified by the original choice distributions. It states that when taking two disjoint datasets, each of which recommend that x be ranked over y, then x is ranked over y overall.

Another interpretation of combination is also possible.[4] Every day, the economist observes individual choice and makes a welfare prescription. On days when it is cloudy, the individual chooses according to p, and on days when it is sunny, she chooses according to q. Whether it is cloudy or sunny, x is always at least as good as y. Suppose now that the economist has no way of knowing whether it is cloudy or sunny, but she believes that either could be true with equal probability. The economist may identify this uncertain situation with the choice distribution $(1/2)p + (1/2)q$. Such an identification relies on an implicit reduction assumption which may throw out important data. The axioms state that whether or not it is cloudy or sunny is irrelevant to the ranking of x over y. As x is at least as good as y in either contingency, x should be considered at least as good as y in general.

The following axiom states that the individual welfare functional is non-imposed. The individual has the freedom to effect any preference over a given set of four alternatives.

Diversity: For every list of four distinct alternatives $x, y, z, w \in X$, there is $p \in P$ such that $x \succ_p y \succ_p z \succ_p w$.

Lastly we postulate a continuity requirement.

Continuity: For every $x, y \in X$, the set $\{p \in P : x \succsim_p y\}$ is closed.

Definition 1 A series of vectors in $\mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$, denoted $\{u^x\}_{x\in X}$, is said to be *diversified* if for all distinct four elements x, y, z, w, there do not exist non-negative numbers α, β and a vector $r \in \mathbb{R}^{|\Theta|}$ with the property that

(i)

$$u^w(\theta, b) - u^z(\theta, b) \geqq \alpha(u^x(\theta, b) - u^y(\theta, b)) + \beta(u^y(\theta, b) - u^z(\theta, b)) + r(\theta)$$

and

$$\sum_{\theta\in\Theta} r(\theta) \geqq 0$$

or,

(ii)

$$u^w(\theta, b) - u^z(\theta, b) \leqq \alpha(u^x(\theta, b) - u^y(\theta, b)) + \beta(u^y(\theta, b) - u^z(\theta, b)) + r(\theta)$$

[4] This interpretation was suggested to us by Mark Machina

and

$$\sum_{\theta \in \Theta} r(\theta) \leqq 0.$$

The definition of diversification is somewhat complicated; however, after the statement of our main theorem, its role should become clear.

The following is our main theorem. It states that whether or not x is ranked at least as good as y depends linearly on the choice probabilities of every alternative in every situation. That is, for every alternative x, for every situation θ and every alternative $b \in B_\theta$ there is a real number $u^x(\theta, b)$ by which the probability that b is chosen in situation θ is multiplied. Thus, the choice probabilities of alternatives other than x and y in arbitrary choice situations may be relevant for determining the ranking between x and y. In a later section, we will explore conditions which free our individual welfare functional from such dependence.

Theorem 1 An individual welfare functional $\{\succsim_p\}_{p \in P}$ satisfies Combination, Diversity and Continuity if and only if there exists a diversified series of vectors in $\mathbb{R}^{\sum_{\theta \in \Theta} |B_\theta|}$, denoted $\{u^x\}_{x \in X}$, such that for every $x, y \in X$ and $p \in P$,

$$x \succsim_p y \iff \sum_{\theta \in \Theta} \sum_{b \in B_\theta} u^x(\theta, b) p(b|\theta) \geqq \sum_{\theta \in \Theta} \sum_{b \in B_\theta} u^y(\theta, b) p(b|\theta).$$

Moreover, when another series of vectors $\{v^x\}_{x \in X}$ satisfies the above condition, there is a positive scalar λ and a series of vectors $\{\beta^x\}_{x \in X}$ in $\mathbb{R}^{\sum_{\theta \in \Theta} |B_\theta|}$

$$v^x(\theta, b) = \lambda u^x(\theta, b) + \beta^x(\theta, b),$$

where for all $x, y \in X$:
(i) $\beta^x(\theta, b) - \beta^y(\theta, b) = \beta^x(\theta, b') - \beta^y(\theta, b')$ for all $\theta \in \Theta$, $b, b' \in B_\theta$;
(ii) $\sum_{\theta \in \Theta} \sum_{b \in B_\theta} \beta^x(\theta, b) = \sum_{\theta \in \Theta} \sum_{b \in B_\theta} \beta^y(\theta, b)$.

Diversification of vectors can now be understood as a condition that directly states that for all x, y, z, w, $x \succsim_p y \succsim_p z$ need not imply that $w \succsim_p z$ (Similarly, $z \succsim_p y \succsim_p x$ need not imply $z \succsim_p w$). As an axiom on the individual welfare functional, its role is a technical one. It allows us to complete a critical induction step in the proof. It essentially forces the space of vectors $(u^x - u^y)_{(x,y)}$ to be full-dimensional, so that several important linear equalities can be solved uniquely.

3 Binary Independence

3.1 Binary Independence

Thus far, we have been concerned only with the combination axiom; and our resulting representation theorem is very general. No axiom rules out dependence of the ranking of a pair of alternatives from the choice probabilities of other alternatives. In this section, we propose just such a notion. The axiom of binary independence states formally that a ranking depends only on the probabilities of the alternatives under consideration. Binary monotonicity states that this ranking should somehow depend positively on these probabilities; that is, if x is ranked above y for one distribution, and the probability of x only increases to another distribution (where the probability of y may decrease), then we should not claim that y is ranked above x for the new distribution.

Binary Independence/Monotonicity: For every $x, y \in X$ and $p, q \in P$:

(i) if $p(x|\theta) = q(x|\theta)$ for all $\theta \in \Theta$ with $x \in B_\theta$ and $p(y|\theta) = q(y|\theta)$ for all $\theta \in \Theta$ with $y \in B_\theta$, then $x \succsim_p y$ if and only if $x \succsim_q y$;

(ii) if $x \succsim_p y$ and $p(x|\theta) \leqq q(x|\theta)$ for all $\theta \in \Theta$ with $x \in B_\theta$ and $p(y|\theta) \geqq q(y|\theta)$ for all $\theta \in \Theta$ with $y \in B_\theta$, and at least one of these inequalities are strict, then $x \succ_q y$.

3.2 Result on the domain of multinomial choices

In this subsection, we assume that $\Theta_x = \{\theta \in \Theta : B_\theta \ni x\} \neq \emptyset$ for all $x \in X$, and $|B_\theta| \geqq 4$ for all $\theta \in \Theta$. This assumption is made for technical reasons, as will be discussed in a later section.

The next property states that any pair of alternatives which are never chosen should be ranked as indifferent.

Dummy: For every $x, y \in X$, if $p \in P$ is such that $p(x|\theta) = 0$ for all $\theta \in \Theta_x$ and $p(y|\theta) = 0$ for all $\theta \in \Theta_y$, then $x \sim_p y$.

Definition 2 A series of vectors $\{u^x\}_{x \in X}$, where u^x is an element in $\mathbb{R}^{|\Theta_x|}_{++}$ for each x, is said to be diversified if the series of vectors $\widetilde{u}^x$ in $\mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$, which is given by $\widetilde{u}^x(\theta, x) = u^x(\theta)$ for each $\theta \in \Theta_x$ and zero elsewhere, is diversified.

The next theorem is similar to Theorem 1, however, the utility representation of the induced ranking states that the utility of an alternative depends only on the choice probabilities of that alternative.

Theorem 2 Assume that $\Theta_x \neq \emptyset$ for all $x \in X$, and $|B_\theta| \geqq 4$ for all $\theta \in \Theta$. Then, an individual welfare functional $\{\succsim_p\}_{p \in P}$ satisfies Combination, Diversity, Continuity, Binary Independence/Monotonicity and Dummy, if and only if there exists a diversified series of strictly positive vectors $\{u^x\}_{x \in X}$, where u^x is an element in $\mathbb{R}^{|\Theta_x|}_{++}$ for each x, such that for all $x, y \in X$,

$$x \succsim_p y \iff \sum_{\theta \in \Theta_x} u^x(\theta) p(x|\theta) \geqq \sum_{\theta \in \Theta_y} u^y(\theta) p(y|\theta)$$

for every $p \in P$.
Moreover, $\{u^x\}_{x \in X}$ is unique up to common positive scalar multiplications.

3.3 Result on the domain including choice from triples

Given $x, y, z \in X$, let $\Theta_{\{x,y,z\}} = \{\theta \in \Theta : B_\theta = \{x, y, z\}\}$.

Semi-strong Dummy: For every $x, y, z \in X$, if $p \in P$ is such that $p(x|\theta) = 0$ for all $\theta \in \Theta_x \setminus \Theta_{\{x,y,z\}}$, $p(y|\theta) = 0$ for all $\theta \in \Theta_y \setminus \Theta_{\{x,y,z\}}$ and $p(x|\theta) = p(y|\theta)$ for all $\theta \in \Theta_{\{x,y,z\}}$, then $x \sim_p y$.

Theorem 3 Assume that $\Theta_x \neq \emptyset$ for all $x \in X$, and $|B_\theta| \geqq 3$ for all $\theta \in \Theta$. Then, an individual welfare functional $\{\succsim_p\}_{p \in P}$ satisfies Combination, Diversity, Continuity, Binary Independence/Monotonicity and Semi-strong Dummy, if and only if there exists a diversified series of strictly positive vectors $\{u^x\}_{x \in X}$, where u^x is an element in $\mathbb{R}^{|\Theta_x|}_{++}$ for each x, such that for all $x, y \in X$,

$$x \succsim_p y \iff \sum_{\theta \in \Theta_x} u^x(\theta) p(x|\theta) \geqq \sum_{\theta \in \Theta_y} u^y(\theta) p(y|\theta)$$

for every $p \in P$, for all θ for which $B_\theta \supset \{x, y\}$ and $|B_\theta| = 3$, $u^x(\theta) = u^y(\theta)$.
Moreover, $\{u^x\}_{x \in X}$ is unique up to common positive scalar multiplications.

The following example shows that the diversity axiom is independent of the other axioms used in the characterization.

Example 5 Let $X = \{x, y, z, w\}$ and $\Theta = \{\theta_1, \theta_2\}$, where $B_{\theta_1} = \{x, y, z\}$ and $B_{\theta_2} = \{z, y, w\}$. Consider the scoring rule given by

$$U(x) = p(x|\{x, y, z\}) + p(x|\{x, y, w\}),\ U(y) = p(y|\{x, y, z\}) + p(y|\{x, y, w\})$$

and

$$U(z) = p(z|\{x, y, z\}),\ U(w) = p(w|\{x, y, w\}).$$

Then, $x \succsim_p w$ and $y \succsim_p w$ always imply $w \succsim_p z$, which cannot satisfy the diversity axiom.

3.4 Result on the domain including binary choices

For technical reasons, when the choice situations can involve binary choices, a few more axioms are needed to characterize individual welfare functionals as in Theorem 2. To see why this is the case, imagine a domain in which *all* choice situations involve only binary choices. Then binary independence/monotonicity is clearly vacuous, and extra structure is needed. The axioms in this section deliver this extra structure.

Given $p \in P$, $\theta \in \Theta$ and $q_\theta \equiv q(\cdot|\theta) \in \Delta(B_\theta)$, let $(q_\theta, p_{-\theta})$ be the situation dependent choice distribution that coincides with q_θ at θ and with p elsewhere.

Proportionality: For every $x, y, z \in X$ and $p \in X$ such that $y \sim_p z \succ_p x$, any $\theta \in \Theta$ with $B_\theta = \{x, y\}$, and $q_\theta \in \Delta(B_\theta)$,
then if

$$x \sim_{(1-\lambda)p+\lambda(q_\theta,p_{-\theta})} y$$

it follows that

$$x \sim_{\left(1-\frac{\lambda}{2}\right)p+\frac{\lambda}{2}(q_\theta,p_{-\theta})} z.$$

Also, let $\Theta_{\{x,y\}} = \{\theta \in \Theta : B_\theta = \{x, y\}\}$.

Strong Dummy: For every $x, y, z \in X$, if $p \in P$ is such that $p(x|\theta) = 0$ for all $\theta \in \Theta_x \backslash (\Theta_{\{x,y\}} \cup \Theta_{\{x,y,z\}})$, $p(y|\theta) = 0$ for all $\theta \in \Theta_y \backslash (\Theta_{\{x,y\}} \cup \Theta_{\{x,y,z\}})$, and $p(x|\theta) = p(y|\theta)$ for all $\theta \in \Theta_{\{x,y\}} \cup \Theta_{\{x,y,z\}}$, then $x \sim_p y$.

Theorem 4 Assume that $\Theta_x \neq \emptyset$ for all $x \in X$. Then, an individual welfare functional $\{\succsim_p\}_{p \in P}$ satisfies Combination, Diversity, Continuity, Binary Independence/Monotonicity, Proportionality and Strong Dummy if and only if there exists a diversified series of strictly positive vectors $\{u^x\}_{x \in X}$, where u^x is an element in $\mathbb{R}^{|\Theta_x|}_{++}$ for each x, such that for all $x, y \in X$,

$$x \succsim_p y \iff \sum_{\theta \in \Theta_x} u^x(\theta) p(x|\theta) \geqq \sum_{\theta \in \Theta_y} u^y(\theta) p(y|\theta)$$

for every $p \in P$, and for all θ for which $B_\theta \supset \{x, y\}$ and $|B_\theta| = 2$ or 3, $u^x(\theta) = u^y(\theta)$. Moreover, $\{u^x\}_{x \in X}$ is unique up to common positive scalar multiplications.

The following example shows that the diversity axiom may not follow from the others.

Example 6 Let $X = \{x, y, z, w\}$ and $\Theta = \{\theta_1, \theta_2\}$, where $B_{\theta_1} = \{x, y\}$ and $B_{\theta_2} = \{z, w\}$. Consider the scoring rule given by

$$U(x) = p(x|\{x, y\}),\ U(y) = p(y|\{x, y\}),\ U(z) = p(z|\{z, w\}),\ U(w) = p(w|\{z, w\}).$$

Then, we have $x \succsim_p z$ if and only if $w \succsim_p y$, which cannot satisfy the diversity axiom.

4 Narrowing down the weights

Our concern is mostly with a formal consistency requirement or informational efficiency requirement for normative prescriptions, and not with substantive views about *which* weights should be used.[5] This is useful for a normative model, as it leaves us with a large family of rules. A modeller can choose an individual welfare functional that best accomodates her subjective judgment of how choice should be tied to welfare.

Here we propose several ways to narrow down the weights, in order to respect choice and eliminate unnecessary or unjustified paternalism. One suggestion is that one should avoid putting a larger prior weight on a particular alternative without any sufficient reason. Of course, what is meant by 'sufficient' reason depends on what kind of choice frames are in present. We study a few possibilities, which are clearly not exhaustive.

4.1 Weights on opportunities

Our representation theorems allow us to weight situations by how many elements are available. That is, the economist may have a certain prior weight about the importance of being chosen from a large set.

For simplicity, we assume that for all $B \in \mathcal{B}$, $F(B)$ is a singleton; so that Θ may be formally identified with a class of elements of $\mathcal{B}$; without loss of generality, assume $\Theta \subset \mathcal{B}$. For all permutations $\pi : X \to X$ and for all $B \in \Theta$, we require that $\pi^{-1}(B) \in \Theta$. For $p \in P$, the probability of x being chosen from B is written as $p(x|B)$.

Given $p \in P$ and π, a permutation over X, define p^π by

$$p^\pi(x|B) = p(\pi^{-1}(x)|\pi^{-1}(B)).$$

The following axiom is adapted from social choice theory. It formally states that the names of alternatives are irrelevant to the operation of the individual welfare functional

Neutrality: For all $p \in P$, $x, y \in X$ and any permutation π over X,

$$x \succsim_p y \iff \pi(x) \succsim_{p^\pi} \pi(y).$$

[5]Of course, the Diversity and Dummy axioms rule out certain lists of potential weights.

Corollary 1 An individual welfare functional as characterized in Theorem 2 or 4 further satisfies Neutrality if and only if there exists a function $u : \{|B| : B \in \mathcal{B}\} \to \mathbb{R}_{++}$ such that

$$u^x(B) = u(|B_\theta|)$$

for all $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.

One may here give a solution to the problem in the leading example. Being chosen from larger sets is more important whenever u is increasing. Being chosen in direct comparisons is of highest importance when $u(2) > u(k)$ for all $k \neq 2$.

This individual welfare functional respects choice. To see this, for simplicity, assume $\Theta = \{B \in 2^X : |B| \geqq 2\}$. Suppose now that there exists a strict preference $\succ$ such that $x_1 \succ x_2 \succ \cdots \succ x_n$, and that choice is made according to this ranking in a deterministic manner. Then, the score given to alternative x_k, which is the k-th best element, is

$$U(x_k) = \sum_{l=2}^{n-k+1} C(n-k, l-1)u(l).$$

This is obviously decreasing in k, and the assigned ranking agrees with $\succ$, whatever u is. Here u has nothing to say.

4.2 Weights on framing factors

Let $F(B)$ be the set of linear orderings over B, which describe possible orders of appearance. Given $B \in \mathcal{B}$, an order $>\in F(B)$, and $x \in B$, let $r(x, B, >)$ be the rank of x in B with regard to the order $>$. For example, $r(x, B, >) = 1$ if x is the first element to be presented from B according to $>$, and $r(x, B, >) = |B|$ if it is the last one.

In this case, one can impose additional structure to the previous example, so as to incorporate weights on framing, by writing $u^x(\theta)$ as a function of $|B_\theta|$ and $r(x, B_\theta, >_\theta)$, say $u^x(\theta) = u'(|B_\theta|, r(x, B_\theta, >_\theta))$. We believe it is sensible for the function u' to be nondecreasing in its second argument. This might be because it an alternative which is listed in the back is less likely to be chosen, so that being chosen nevertheless is valued more.[6]

4.3 Adjusting to the number of appearance

When some alternative appears in more situations than others, that is, when $|\Theta_x|$ is not equal across x's, one may worry about the possibility that x is valued highly simply because

[6]Though, it also allows for the possibility that u has a sudden drop or spike. This may be the case when being the last one has a focal-point effect.

of this. The class of individual welfare functionals we obtain allows one to adjust weights according to how many choice problems in which an alternative appears, by letting for example

$$u^x(\theta) = \frac{u'(x, B_\theta, f_\theta)}{|\Theta_x|} \quad \text{for all} \quad x \in X,\ \theta \in \Theta_x.$$

5 Conclusion and related literature

In this paper we have discussed how to prescribe a preference ranking for a given individual based on observed choices when these choices may be inconsistent. We consider a function that maps choice data (a situation dependent empirical distribution) into a complete and transitive ordering over alternatives.

The main axiom we impose is Combination, which states that if, in two situations, the data respectively support x over y, then the choice data obtained by joining the two situations also supports x over y.

Together with other mild axioms, we obtain a scoring type rule, where we must form a prior view about the reletive importance of different choices in forming the preference that should be assigned.

The additional axioms, primarily Binary Independence, deliver a more structured class of scoring rules, where the score for a given alternative depends only on its empirical choice probability. It is of interest that when choice is consistent, the rationalizing preference is the preference assigned to the data.

Our framework should be contrasted with the classical approach which seeks to explain choice through preference. One might ask why one should care about complete and transitive preference in the first place. Completeness and transitivity are the classical hallmarks of "rationality," and thus most of welfare economics takes these notions (at the very least as primitive). Secondly, the set of complete and transitive preferences are much smaller than the set of arbitrary choice functions. This simplifies analysis and allows us to represent the individual problem as a classical maximization problem. Lastly, it provides a reasonable method for choosing alternatives out of sample. A different approach would be to construct a normative theory that extends any given choice function to a choice function on the entire domain (which is not necessarily rationalizable).

5.1 Related literature

In this section, we compare our approach to related ideas. Within the basic revealed preference paradigm, it may not be possible to establish a natural consensus on a notion of revealed preference which can rationalize all possible choice functions. To see this, note that a choice model has three dimensions that work to determine welfare: specification, observation of choices, and identification. Let us suppose, for the sake of argument, that the individual's entire choice function can always be uncovered, so that there are no issues of prediction. We desire a model that can explain all possible types of behavior (such a model by definition has no testable implications). This specification of the model itself involves a subjective choice on the part of the modeller. Given data, identification becomes an issue as well. However, even an identified model involves a belief that defining welfare as identified preference is reasonable. When a model is not identified for some data, it can be identified by making more restrictive assumptions on the specification (throwing away possible explanations). When there are competing theories all of which can explain a given choice pattern, it is not meaningful to speak of the "correct" model–model choice is inherently subjective. This is because all economic models of choice are "as if" models, and do not actually model the explicit process driving choice. The key tradeoff in model choice is between identification and explanation. We do not want to rule out potential natural explanations in an ad-hoc fashion, but allowing more possible explanations results in a model which is identified less frequently. Recognizing that there are tradeoffs and subjectivity *even in the choice of a model*, we suggest that explicitly modelling welfare as a subjective construct on the part of the modeller is useful. We discuss the concerns and tradeoffs in more detail through several following examples.

A naive model would simply eschew the requirement of complete and transitive welfare judgments. Aside from the fact that applied welfare economics is based on the assumption that individuals have complete and transitive rankings, the issue with this approach is that welfare judgments are useful for making decisions out of sample. With any theory, we can never get out more than we put in. A good descriptive theory should be able to extrapolate data and bring it to a situation which has not yet been observed.

Usually modellers seek to define welfare as some concept from which choice can be derived through some process or procedure. In the classical revealed preference paradigm, choice must be obtainable as maximization of preference (welfare), where a preference is a complete and transitive relation. Often it is required that the choice function is *equal* to the maximizing correspondence of a rational preference.[7]

A common argument is that "irrational behavior" is observed because the choice setting is misspecified, and that relevant factors have been somehow ignored. This may in fact be true, and there may be a general, extended model of choice in which all empirically observed behavior becomes rationalizable by a complete and transitive relation. To take a simple example, there are many economic instances in which an action is irrational in the short-run but rational in the long-run. In such cases, seemingly irrational or paradoxical behavior can be rationalized by suitably redefining the choice domain. The two main issues with this approach can be explained simply as follows. First, enlarging the choice domain necessitates observing more choices in order to identify the model. This data may simply be unavailable. Secondly, and more importantly, as rationality has testable implications, it is by definition refutable. Therefore, for *any* choice setting, it is at least *conceivable* that there may be a decision maker who does not behave rationally in this model.

Alternatively, many works stay within the classical choice domain and seek to explain choice from welfare, but weaken either the hypothesis that welfare is represented as a complete and transitive relation, weaken the hypothesis that welfare drives choice through maximization, or both. This requires specifying a model consisting of some form of welfare and some procedure by which welfare drives choice. But if two different models explain the same choices, we have no way of identifying which, if either of the two, is the "correct" one in terms of the actual welfare the individual experiences or the actual choice procedure she follows. This is irrelevant for models which seek only to predict behavior, but for models which seek to evaluate welfare, it can be critical.

Our approach dissociates choice from the derived welfare. Welfare can be defined *from* choice, but the converse need not be true. In this dimension at least, our model is closer to an econometric model, which might seek to find a "reasonable approximation" to preference.

5.2 Models weakening classical rationality hypotheses

In an innovative recent paper, Green and Hojman [11] (see also Ambrus and Rozen [1]) treat an individual as a 'society' populated with multiple personalities who may have conflicting preferences. They weaken the assumption that welfare is represented as a complete and transitive relation and instead assume welfare might be represented as a *list* of complete and transitive relations. They discuss maximization, but the object they maximize is now

[7] An alternative notion would require only that the choice function is a subcorrespondence of this maximizing correspondence–this model can always be rationalized by a welfare relation ranking each alternative as indifferent.

an *aggregate preference.* They imagine that a rule is used to aggregate the conflicting preferences. In order to tie choice non-trivially to their welfare concept, the rule must satisfy certain properties. In particular, they assume that it is a *scoring rule.* They show that for almost all scoring rules (specifically all but the Borda score), any choice function can be rationalized by *some* list of preferences.

Their notion of rationalization is quite weak, in that they only require the choice function only to be a *subcorrespondence* of the maximizer correspondence. Typically, the set of such lists may be quite large. Thus, in their framework, there are *two* degrees of freedom: freedom in the choice of a scoring rule and freedom in the list of preferences rationalizing the choice function given the scoring rule. In general, then the model is not identified, so that they cannot make absolute welfare statements. They can only make absolute welfare comparisons over a pair of alternatives when all preferences in the list agree. If; however, they assume the underlying preferences in the list have some cardinal structure (making a stronger identifying assumption), they can make stronger welfare statements.

Bernheim and Rangel [4, 5] also discuss welfare when choice may not be rationalizable. They also argue that welfare should not be defined through a positive model, but rather should be treated as a normative exercise. Their approach is to define several binary relations through choice. They also consider different framing factors, and define a few binary relations. The more important of the two states that an individual is unambiguously better off if she never chooses y when x is available. If choices are defined on a rich domain (including all finite sets), this relation is acyclic. A naive extension of Bernheim and Rangel's theory to an environment in which choices from some budgets may be unobserved may lead to a cyclic relation. This is not problematic, as their intent is to consider environments in which complete data is available. However, their approach cannot be directly applied to field data, unless some positive theory is proposed to fill in the details. While Bernheim and Rangel's theory can be applied to many economic environments, it is a theoretical possibility that many alternatives remain unranked. As they note, there are methods of ranking unranked alternatives. Our work is intended to be an abstract, prescriptive approach which ranks all alternatives from the outset (applying to general choice models).

Kőszegi and Rabin [21] suggest that irrational choice may be due to mistakes or biases. They suggest a choice-based procedure for eliciting these mistakes, inspired by the decision theory literature. Specifically, they ask an individual to place a bet on which alternative they will choose in the future. Failure to select this alternative is interpreted as a mistake which should presumably be corrected. In this context, they define the "true" preference to be the preference induced by the *ex-ante* preference over bets. This is an example of enriching the choice space in order to uncover a rationalizing preference.[8] Of course, this methodology tells us nothing when the choices over *bets* cannot themselves be rationalized.

Following is a partial discussion of the literature in which notions of rationalizability are generalized, but which provide testable implications, and thus cannot be used to evaluate welfare for all possible decision makers. The literature on revealed preference theory characterizing conditions under which choice can be rationalized by the maximization of a preference is vast. Classic papers are by Samuelson [33, 34], who first provided the rationalizability condition in the market setting. For abstract choice functions, Chernoff [8], Arrow [2] and Sen [35] provide conditions in the deterministic setting under which observed choice is rationalizable. In a stochastic choice setting, Luce [23, 24] provides conditions under which the choice probability of an alternative from a set is proportional to some cardinal utility assigned to it. Falmagne [9] offers conditions under which there is a probability measure over preferences such that the choice probability of an alternative is the probability of the set of preferences for which it is maximal.

Kalai, Rubinstein, and Spiegler [19] provide a condition under which observed choice is rationalizable as a maximal element according to multiple criteria, in which choices need not come from a single objective. Tyson [39] provides a condition under which observed choice is explained as if it is obtained by satisficing an underlying preference. This is a milder procedural rationality requirement than maximization. Mariotti and Manzini study choice which can be rationalized by sequentially eliminating alternatives [26] by a list of binary relations. Masatlioglu and Ok [28] consider an extended domain of choice functions in which a choice problem comes in the form of a pair of a set and a reference point. They provide a condition under which observed choice is rationalizable by maximizing preference with a bias toward the reference point. Rubinstein and Salant [31] study choices from lists, providing a generalized rationalizability condition. Further, Rubinstein and Salant [32] studies rationalizability analysis in a general setting in which a choice problem comes in the form of a pair of a set and an arbitrary framing factor. We consider *arbitrary* choice data, which may or may not be rationalizable in any sense. We are interested in how we can assign preference in a normative or prescriptive sense.

Another related literature is the social choice literature which seeks to define choice functions from tournaments. A tournament is an arbitrary binary relation, interpreted as

[8] For this to make complete sense, one must assume that there is no complementarities between the alternatives under consideration and the payoffs to the bets. This seems reasonable in practice. Such a procedure might not identify *all* mistakes. By placing a bet on an alternative, extra incentive is added to choose that alternative in the future.

a social preference. The well-known Condorcet paradox demonstrates that if preference is cyclic, maximizing choices may not exist. A very good early survey is Moulin [29]. One possible approach is to define some kind of metric on the set of all preferences, rational or otherwise. Then for any tournament, the closest rational preference (or preferences) to it can be determined. This then specifies a choice function through maximization. A common metric minimizes the number of preference reversals needed to turn a tournament into a rational preference (this is related to the so-called Kemeny rule, see Young [41]). More closely related is a work of Rubinstein [30] which characterizes a family of rules mapping tournaments to weak orders. Our work differs from these in that a choice function need not be rationalizable by *any* preference, cyclic or not.

Lastly, our formal model and analysis are similar to those in expected utility theory in the context of games (Gilboa-Schmeidler [15]), inductive inference (Gilboa-Schmeidler [14]), case-based decision making (Gilboa-Schmeidler [12]), relative utility (Ashkenazi and Lehrer [3]), and social aggregation of preferences (Young [40]). Most of our axioms are borrowed directly from this literature. One technical difference, though, is that our domain is larger in the sense that choice distributions are conditioned by different situations, which makes it difficult to establish the uniqueness of the obtained individual welfare functional. Moreover, choice distributions do not form a vector space, but are rather elements of some convex set. While the techniques here essentially follow the same ideas as in the previous works, necessary modifications are needed to accomodate these differences.

5.3 Directions for future research

Our work offers a *prescriptive* approach to the problem of evaluating welfare when choice data is inconsistent. There are several obvious paths to follow in this direction.

Firstly, we have throughout assumed that our individual welfare functional satisfies Diversity. This condition requires that the output of an individual welfare functional is non-imposed on the individual who makes choices. It is similar to the citizen's sovereignty requirement of social choice. In the interest of libertarianism, such a requirement is natural. That is, when the classical rationality postulates are satisfied, it seems natural for welfare to coincide with classical revealed preference. However, in order to allow our individual welfare functional to be more paternalistic, we may wish to force certain preferences on individuals. For example, some individuals may have self control problems–the case of addiction is a prime example. Such an individual may choose to consume an addictive substance even though it harms her. Our notion of individual welfare functional would then say that the consumption of this substance makes the individual better off. However, such an individual may not feel better off. We might therefore imagine a program which leads this individual to be addicted to the substance to be a bad one, even though it is one which he himself would choose. Our axioms at this stage do not allow such recommendations.

Secondly, there is a question of time-dependence. Individuals may make repeated choices, say between x and y. In an environment where they have never before consumed these types of alternatives, they must experiment. Suppose we observe an individual who has consumed x in the first five periods, but then consumed y in the last five. It seems plausible to suspect that this individual has learned that they prefer y to x. However, the primitive of our model (a probability distribution over choices) does not allow us to make such conclusions.

Lastly, there is the question as to what rationality actually means. Our work has taken as given that rationality is rationalizability by a complete and transitive preference. However, weaker (and stronger) notions of rationality certainly exist. In more structured environments, for example, in the theory of choice over lotteries, satisfaction of the independence axiom is often understood as a basic tenet of rationality. This is an issue of the specification of the range of our individual welfare functional (related is the issue of domain specification; however it seems natural to allow our individual welfare functional to take as input *any* stochastic choice function).

6 Proofs

6.1 Proof of Theorem 1

Given $p \in P$ and a vector $u \in \mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$, let

$$u \cdot p = \sum_{\theta\in\Theta}\sum_{b\in B_\theta} u(\theta, b)p(b|\theta)$$

Lemma 1 There exists a series of vectors in $\mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$, denoted $\{u^{xy}\}_{x,y\in X, x\neq y}$, such that for every $x, y \in X$,

(i) $\{p \in P : u^{xy} \cdot p \geqq 0\} = \{p \in P : x \succsim_p y\}$;

(ii) $\{p \in P : u^{xy} \cdot p > 0\} = \{p \in P : x \succ_p y\}$;

(iii) $\{p \in P : u^{xy} \cdot p \leqq 0\} = \{p \in P : y \succsim_p x\}$;

(iv) $\{p \in P : u^{xy} \cdot p < 0\} = \{p \in P : y \succ_p x\}$;

(v) neither $u^{xy} \geqq 0$ or $u^{xy} \leqq 0$;

(vi) $u^{yx} = -u^{xy}$.

Moreover, if another vector series of vectors $\{v^{xy}\}_{x,y\in X,x\neq y}$ delivers the above properties, then there is a series of positive numbers $\{\lambda^{xy}\}_{x,y\in X,x\neq y}$ and a series of vectors in $\mathbb{R}^{|\Theta|}$, denoted $\{r^{xy}\}_{x,y\in X,x\neq y}$, such that for every $x, y \in X$,
(1)

$$v^{xy}(\theta,b) = \lambda^{xy}u^{xy}(\theta,b) + r^{xy}(\theta)$$

for all $\theta \in \Theta$ and $b \in B_\theta$;
(2) $\sum_{\theta\in\Theta} r^{xy}(\theta) = 0$;
(3) $\lambda^{xy} = \lambda^{yx}$;
(4) $r^{yx} = -r^{xy}$.

Proof.
Existence: Fix $x, y \in X$, and let $C^{xy} = \{p \in P : x \succsim_p y\}$ and C^{yx} similarly. By Diversity, there exists $p \in P$ for which $x \succ_p y$. Moreover by continuity, $\{p \in P : x \succ_p y\}$ is relatively open. For any $q \in C^{xy}$, by combination, as there exists $p \in P$ for which $x \succ_p y$, for all $\lambda > 0$, $x \succ_{\lambda p+(1-\lambda)q)} y$, so we establish that C^{xy} is the closure of $\{p \in P : x \succ_p y\}$.

Note that $C^{xy} \cup C^{yx} = P$ by completeness.

Let $\Phi^{xy} = \{\lambda(p-q) : \lambda \geqq 0, p \in C^{xy}, q \in C^{yx}\}$, and define a binary relation $\succsim^{xy}$ over P by

$$p \succsim^{xy} q \quad \text{if} \quad p - q \in \Phi^{xy}$$

Since C^{xy} and C^{yx} are compact by continuity, Φ^{xy} is closed, hence $\succsim^{xy}$ is a continuous binary relation.

Claim 1 If $d, e \in \Phi^{xy}$, then $d + e \in \Phi^{xy}$.

Proof. Without loss of generality, let $d, e \in C^{xy} - C^{yx}$, and let $d = p_1 - p_2$ and $e = q_1 - q_2$, where $p_1, q_1 \in C^{xy}$ and $p_2, q_2 \in C^{yx}$. By the combination axiom, $\frac{p_1+q_1}{2} \in C^{xy}$ and $\frac{p_2+q_2}{2} \in C^{yx}$. Therefore $\frac{d+e}{2} = \frac{p_1+q_1}{2} - \frac{p_2+q_2}{2} \in C^{xy} - C^{yx}$. Therefore $d + e \in \Phi^{xy}$. ■

Claim 2 $\succsim^{xy}$ is complete.

Proof. Case 1: If $p \in C^{xy}$ and $q \in C^{yx}$, then it is obvious that $p \succsim^{xy} q$.
Case 2: Suppose $p, q \in C^{xy}$. Let $r \in P$ be such that $y \succ_r x$. Choose λ so that $(1-\lambda)p+\lambda r \in C^{xy}$, $(1-\lambda)q + \lambda r \in C^{xy}$, and at least one of them is in C^{yx}. For example, choose λ to be the supremal λ for which $(1-\lambda)p + \lambda r \in C^{xy}$ and $(1-\lambda)q + \lambda r \in C^{xy}$; that one of them

is in C^{yx} follows by continuity. Without loss of generality, suppose $(1-\lambda)q + \lambda r \in \partial C^{xy}$, then we have

$$[(1-\lambda)p + \lambda r] - [(1-\lambda)q + \lambda r] = (1-\lambda)(p-q) \in \Phi^{xy},$$

which implies $p - q \in \Phi^{xy}$.
Case 3: If $p, q \in C^{yx}$, the proof is similar to Case 2. ■

Claim 3 $\succsim^{xy}$ is transitive.

Proof. Suppose $p \succsim^{xy} q$ and $q \succsim^{xy} r$. Since $p - q \in \Phi^{xy}$ and $q - r \in \Phi^{xy}$, from the previous claim we have $p - r = (p-q) + (q-r) \in \Phi^{xy}$. ■

Claim 4 For every $p, q, r \in P$ and $\lambda \in (0,1)$, $p \succsim^{xy} q$ if and only if $\lambda p + (1-\lambda)r \succsim^{xy} \lambda q + (1-\lambda)r$.

Proof. This follows as $(\lambda p + (1-\lambda)r) - (\lambda q + (1-\lambda)r) = \lambda(p-q)$. ■

Since $\succsim^{xy}$ satisfies the state-dependent version of Anscombe-Aumann subjective expected utility theorem (see Kreps [22], Karni-Schmeidler-Vind [20]), there is a vector $\widetilde{u}^{xy} \in \mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$ such that

$$p \succsim^{xy} q \iff \widetilde{u}^{xy} \cdot p \geqq \widetilde{u}^{xy} \cdot q$$

for every $p, q \in P$.
Moreover, the vector $\widetilde{u}^{xy}$ is unique in the sense that if there is another vector $\widetilde{v}^{xy}$ that delivers the above condition then there exist a positive number λ^{xy} and a vector $\widetilde{r}^{xy} \in \mathbb{R}^{|\Theta|}$ such that $\widetilde{v}^{xy}(\theta, b) = \lambda^{xy}\widetilde{u}^{xy}(\theta, b) + \widetilde{r}^{xy}(\theta)$ for every $\theta \in \Theta$ and $b \in B_\theta$.

Fix any $p^* \in C^{xy} \cap C^{yx}$, and let $\alpha^{xy} = \widetilde{u}^{xy} \cdot p^*$. Define a vector $u^{xy} \in \mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$ by $u^{xy}(\theta, b) = \widetilde{u}^{xy}(\theta, b) - \frac{\alpha^{xy}}{|\Theta|}$ for every θ and $b \in B_\theta$. Then, it satisfies

$$u^{xy} \cdot p \geqq 0 \iff x \succsim_p y$$

for all $p \in P$. This delivers the properties (i)-(iv).

To show property (v), suppose $u^{xy} \geqq 0$. Then, since $u^{xy} \cdot p \geqq 0$ for all $p \in P$, we have $x \succsim_p y$ for all $p \in P$, which is a contradiction to diversity. A similar contradiction is obtained for the case $u^{xy} \leqq 0$.

Uniqueness: Suppose v^{xy} satisfies

$$v^{xy} \cdot p \geqq 0 \iff x \succsim_p y$$

for all $p \in P$. Then, v^{xy} also forms a representation for $\succsim^{xy}$, where $v^{xy} \cdot p^* = 0$ holds for $p^* \in C^{xy} \cap C^{yx}$. By the uniqueness of representation of $\succsim^{xy}$ above, there exist a positive number λ^{xy} and a vector $\widetilde{r}^{xy} \in \mathbb{R}^{|\Theta|}$ such that

$$\begin{aligned} v^{xy}(\theta, b) &= \lambda^{xy}\widetilde{u}^{xy}(\theta, b) + \widetilde{r}^{xy}(\theta) \\ &= \lambda^{xy}\left(u^{xy}(\theta, b) - \frac{\alpha^{xy}}{|\Theta|}\right) + \widetilde{r}^{xy}(\theta) \\ &= \lambda^{xy}u^{xy}(\theta, b) + r^{xy}(\theta) \end{aligned}$$

for every $\theta \in \Theta$ and $b \in B_\theta$, where r^{xy} is defined by $r^{xy}(\theta) = \widetilde{r}^{xy}(\theta) - \frac{\alpha^{xy}}{|\Theta|}$ for each θ. Since $v^{xy} \cdot p^* = u^{xy} \cdot p^* = 0$, we have $\sum_{\theta\in\Theta} r^{xy}(\theta) = 0$. ■

Lemma 2 For every distinct $x, y, z \in X$ and $\varepsilon, \eta \in \mathbb{R}$, if $\varepsilon u^{xy} + \eta u^{yz} \leqq 0$, then $\varepsilon = \eta = 0$.

Proof. Since neither $u^{xy} \geqq 0$ or $u^{xy} \leqq 0$ holds, if either of ε and η is zero, so is the other. Suppose both are non-zero. Without loss of generality, suppose that $\varepsilon, \eta > 0$. Then, we have $\varepsilon u^{yx} \leqq -\eta u^{yz} = \eta u^{zy}$. Therefore, $z \succsim_p y$ implies $y \succsim_p x$, contradicting Diversity. Similar contradictions are obtained for the other cases. ■

Lemma 3 Let $\{u^{xy}\}_{x,y\in X, x\neq y}$ be the series of vectors obtained in Lemma 1. Then, for every three distinct x, y, z, there exist unique numbers $\alpha, \beta > 0$ and $r \in \mathbb{R}^{|\Theta|}$ such that

$$u^{xz}(\theta, b) = \alpha u^{xy}(\theta, b) + \beta u^{yz}(\theta, b) + r(\theta)$$

for all $\theta \in \Theta$ and $b \in B_\theta$, and

$$\sum_{\theta\in\Theta} r(\theta) = 0.$$

Proof. As for all $p \in P$, $\succsim_p$ is transitive, there does not exist $p \in P$ such that

$$u^{xy} \cdot p \geqq 0,\ u^{yz} \cdot p \geqq 0 \text{ and } -u^{xz} \cdot p > 0.$$

For each $\theta \in \Theta$, let $\mathbf{1}_\theta \in \mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$ be that vector whose coordinate is 1 at all $(\theta, \cdot)$ and 0 elsewhere. Also, for each $\theta \in \Theta$ and $b \in B_\theta$, let $\mathbf{1}_{\theta,b} \in \mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$ be that vector whose coordinate is 1 only at (θ, b) and 0 elsewhere.

By a nonhomoegeneous version of the Farkas' lemma (see Proposition 4.2.3 of [10]), there exist non-negative numbers α, β, $\{\lambda_{\theta,b}\}_{\theta\in\Theta,b\in B_\theta}$, $\{\mu_\theta\}_{\theta\in\Theta}$ and $\{\nu_\theta\}_{\theta\in\Theta}$ such that

$$u^{xz} = \alpha u^{xy} + \beta u^{yz} + \sum_{\theta\in\Theta}\sum_{b\in B_\theta} \lambda_{\theta,b}\mathbf{1}_{\theta,b} + \sum_{\theta\in\Theta} \mu_\theta \mathbf{1}_\theta - \sum_{\theta\in\Theta} \nu_\theta \mathbf{1}_\theta$$

and

$$\sum_{\theta\in\Theta}\mu_\theta - \sum_{\theta\in\Theta}\nu_\theta \geqq 0.$$

Hence,

$$u^{xz} \geqq \alpha u^{xy} + \beta u^{yz} + \sum_{\theta\in\Theta}\mu_\theta \mathbf{1}_\theta - \sum_{\theta\in\Theta}\nu_\theta \mathbf{1}_\theta.$$

Note that for all $p \in P$, $\left(\sum_{\theta\in\Theta}\mu_\theta\mathbf{1}_\theta - \sum_{\theta\in\Theta}\nu_\theta\mathbf{1}_\theta\right)\cdot p \geqq 0$.

Consequently, if $\alpha = 0$, then for all $p \in P$, $y \succsim_p z$ implies $x \succsim_p z$, contradicting Diversity. Hence $\alpha > 0$. Similarly, $\beta > 0$.

A symmetric argument applied to $-u^{xy}$, $-u^{yz}$ and u^{xz} delivers $\delta, \gamma > 0$ and nonnegative numbers $\{\rho_\theta\}_{\theta\in\Theta}$ and $\{\tau_\theta\}_{\theta\in\Theta}$ such that

$$u^{xz} \leqq \delta u^{xy} + \gamma u^{yz} + \sum_{\theta\in\Theta}\rho_\theta \mathbf{1}_\theta - \sum_{\theta\in\Theta}\tau_\theta \mathbf{1}_\theta,$$

where

$$\sum_{\theta\in\Theta}\rho_\theta - \sum_{\theta\in\Theta}\tau_\theta \leqq 0.$$

By taking the sum of these inequalities, we have

$$\left(\sum_{\theta\in\Theta}\rho_\theta\mathbf{1}_\theta - \sum_{\theta\in\Theta}\tau_\theta\mathbf{1}_\theta\right) - \left(\sum_{\theta\in\Theta}\mu_\theta\mathbf{1}_\theta - \sum_{\theta\in\Theta}\nu_\theta\mathbf{1}_\theta\right) \geqq (\alpha-\delta)u^{xy} + (\beta-\gamma)u^{yz}.$$

Therefore, for all $p \in P$,

$$0 \geqq (\alpha-\delta)u^{xy}\cdot p + (\beta-\gamma)u^{yz}\cdot p.$$

By the same logic as in Lemma 2, appealing to Diversity, we conclude $\alpha = \delta$ and $\beta = \gamma$. This further implies that

$$\sum_{\theta\in\Theta}\mu_\theta - \sum_{\theta\in\Theta}\nu_\theta = \sum_{\theta\in\Theta}\rho_\theta - \sum_{\theta\in\Theta}\tau_\theta = 0.$$

Then define $r(\theta) = \mu_\theta - \nu_\theta$, for example.

To see that α, β, and r are unique, suppose by means of contradiction that

$$(\alpha-\delta)u^{xy}(\theta,b) + (\beta-\gamma)u^{yz}(\theta,b) + r'(\theta) - r(\theta) = 0$$

where α, δ, β, and $\gamma > 0$, and $\sum_{\theta\in\Theta} r(\theta) = \sum_{\theta\in\Theta} r'(\theta) = 0$.

Suppose without loss of generality that $\alpha-\delta, \beta-\gamma > 0$. If $x \succ_p y$, then $u^{xy}\cdot p > 0$ and as $(r'-r)\cdot p = 0$, we conclude that $u^{yz}\cdot p < 0$, or $x \succ_p y$ implies $z \succ_p y$, a contradiction to diversity. ■

Lemma 4 There exist a series of vectors $\{u^{xy}\}_{x,y\in X, x\neq y}$, as in Lemma 1, such that for every three distinct x, y, z the condition $u^{xy} + u^{yz} = u^{xz}$ holds.
Moreover, when another series of vectors $\{v^{xy}\}_{x,y\in X, x\neq y}$ satisfies the above condition, there is a positive number λ and a series of vectors in $\mathbb{R}^{|\Theta|}$, denoted $\{r^{xy}\}_{x,y\in X, x\neq y}$, such that for every $x, y, z \in X$,
(1)

$$v^{xy}(\theta, b) = \lambda u^{xy}(\theta, b) + r^{xy}(\theta)$$

for all $\theta \in \Theta$ and $b \in B_\theta$;
(2) $\sum_{\theta\in\Theta} r^{xy}(\theta) = 0$;
(3) $r^{yx} = -r^{xy}$;
(4) $r^{xy} + r^{yz} = r^{xz}$.

Proof. The proof of this statement is by induction, and is similar to that of Gilboa-Schmeidler [13]. Specifically, let us order $X = x_1, ..., x_K$. Begin with x_1, x_2, x_3. Lemma 3 guarantees the existence of $u^{x_1x_2}, u^{x_2x_3}$, and $u^{x_1x_3}$ for which

$$u^{x_1x_3}(\theta, b) = \alpha u^{x_1x_2}(\theta, b) + \beta u^{x_2x_3}(\theta, b) + r(\theta),$$

where $\sum_{\theta\in\Theta} r(\theta) = 0$.

Define $\overline{u}^{x_1x_3}(\theta, b) = u^{x_1x_3}(\theta, b)$, $\overline{u}^{x_1x_2}(\theta, b) = \alpha u^{x_1x_2}(\theta, b)$, and $\overline{u}^{x_2x_3}(\theta, b) = \beta u^{x_2x_3}(\theta, b) + r(\theta)$, and verify that $\overline{u}^{x_1x_3} = \overline{u}^{x_1x_2} + \overline{u}^{x_2x_3}$. We also define u functions relating to (x_3x_1), (x_2x_1) and (x_3x_2) as the negatives of the (x_1x_3), (x_1x_2) and (x_2x_3) functions and note that the order structure is preserved (by the fact that $\sum_{\theta\in\Theta} r(\theta) = 0$).

Now, we induct as follows. Let $m < n$ and suppose that for all $j, l < n$, $\overline{u}^{x_jx_l}$ is defined. Further, suppose that for all $j < m$, $\overline{u}^{x_jx_n}$ is defined. If $m = 1$, by taking $\overline{u}^{x_1x_2}$ as given, we may define $\overline{u}^{x_1x_n}$ and $\overline{u}^{x_nx_2}$ as previously, by using Lemma 3. Otherwise, we use Lemma 3 to establish that (defining $\overline{u}^{x_nx_m}$ and $\overline{u}^{x_mx_n} = -\overline{u}^{x_nx_m}$ implicitly)

$$\alpha\overline{u}^{x_1x_n} + \overline{u}^{x_nx_m} = \overline{u}^{x_1x_m}$$

for some $\alpha > 0$.

For any $j < m$, it follows by the induction hypothesis that

$$\overline{u}^{x_1x_n} + \overline{u}^{x_nx_j} = \overline{u}^{x_1x_j}.$$

Applying Lemma 3 to j, n, and m, we obtain

$$\mu\overline{u}^{x_jx_n}(\theta,b)+\eta\overline{u}^{x_nx_m}(\theta,b)+r(\theta)=\overline{u}^{x_jx_m}(\theta,b)$$

for some $\mu,\eta>0$ and $r\in\mathbb{R}^{\Theta}$, where $\sum_{\theta\in\Theta}r(\theta)=0$.

Subtract the last two lines from the first (using the identities $\overline{u}^{x_nx_j}=-\overline{u}^{x_jx_n}$ and $\overline{u}^{x_1x_j}+\overline{u}^{x_jx_m}=\overline{u}^{x_1x_m}$, the latter of which is by the induction hypothesis) and obtain

$$(\alpha-1)\overline{u}^{x_1x_n}+(1-\eta)\overline{u}^{x_nx_m}+(1-\mu)\overline{u}^{x_nx_j}=-r.$$

For all $p\in P$, $r\cdot p=0$, so that for all $p\in P$,

$$((\alpha-1)\overline{u}^{x_1x_n}+(1-\eta)\overline{u}^{x_nx_m}+(1-\mu)\overline{u}^{x_nx_j})\cdot p=0.$$

Consequently, if any of α, η, or μ is not equal to one, a contradiction to diversity is obtained. Conclude that they all equal one, and furthermore that $r=0$, completing the induction hypothesis. We therefore know there exist u^{xy} satisfying the conclusion of the Lemma.

To establish the uniqueness result, suppose v^{xy} also satisfies the conclusion of the lemma. In particular, we know that there exists, for all x,y, some λ^{xy} and $r^{xy}\in\mathbb{R}^{\Theta}$ for which $v^{xy}(\theta,b)=\lambda^{xy}u^{xy}(\theta,b)+r^{xy}(\theta)$, where $\sum_{\theta\in\Theta}r(\theta)=0$ from Lemma 1. Consequently, as $v^{xy}+v^{yz}=v^{xz}$, we know that

$$\lambda^{xy}u^{xy}+r^{xy}+\lambda^{yz}u^{yz}+r^{yz}=\lambda^{xz}u^{xz}+r^{xz}=\lambda^{xz}(u^{xy}+u^{yz})+r^{xz}.$$

Subtracting the right hand side from the left, we obtain

$$(\lambda^{xy}-\lambda^{xz})u^{xy}+(\lambda^{yz}-\lambda^{xz})u^{yz}=r^{xz}-r^{xy}-r^{yz}.$$

As is usual, unless $\lambda^{xy}=\lambda^{xz}$ and $\lambda^{yz}=\lambda^{xz}$, we have a contradiction to diversity. Moreover, this also establishes $r^{xz}=r^{xy}+r^{yz}$. That λ is constant for all pairs now follows trivially. ∎

Proofs about diversity

First we show that the diversity of $\{\succsim_p\}_{p\in P}$ implies the desired property of the series of vectors.

The diversity condition implies that for all distinct $x,y,z,w\in X$, there exists $p\in P$ such that $x\succsim_p y\succsim_p z\succ_p w$; or

$$u^{xy}\cdot p\geqq 0,\ u^{yz}\cdot p\geqq 0,\ u^{zw}\cdot p>0$$

By Proposition 4.2.3 of [10], there do not exist non-negative numbers α, β, $\{\lambda_{\theta,b}\}_{\theta\in\Theta,b\in B_\theta}$, $\{\mu_\theta\}_{\theta\in\Theta}$ and $\{\nu_\theta\}_{\theta\in\Theta}$ such that

$$-u^{zw} = \alpha u^{xy} + \beta u^{yz} + \sum_{\theta\in\Theta}\sum_{b\in B_\theta} \lambda_{\theta,b}\mathbf{1}_{\theta,b} + \sum_{\theta\in\Theta}\mu_\theta \mathbf{1}_\theta - \sum_{\theta\in\Theta}\nu_\theta\mathbf{1}_\theta$$

and

$$\sum_{\theta\in\Theta}\mu_\theta - \sum_{\theta\in\Theta}\nu_\theta \geqq 0.$$

This implies that there cannot exist non-negative numbers α, β and a vector $r \in \mathbf{R}^{|\Theta|}$ such that

$$-u^{zw}(\theta, b) \geqq \alpha u^{xy}(\theta, b) + \beta u^{yz}(\theta, b) + r(\theta)$$

and

$$\sum_{\theta\in\Theta} r(\theta) \geqq 0.$$

Likewise, diversity also implies that for w, z, y, x, there exists some $p \in P$ for which $w \succ_p z \succsim_p y \succsim_p x$; so that $u^{wz}\cdot p > 0$, $u^{zy}\cdot p \geq 0$, and $u^{yx}\cdot p \geq 0$. Similarly to the previous argument, there cannot exist non-negative numbers α, β and a vector $r \in \mathbf{R}^{|\Theta|}$ such that

$$-u^{wz}(\theta, b) \geqq \alpha u^{yx}(\theta, b) + \beta u^{zy}(\theta, b) + r(\theta)$$

or by taking negation:

$$-u^{zw}(\theta, b) \leqq \alpha u^{xy}(\theta, b) + \beta u^{yz}(\theta, b) + r(\theta)$$

where

$$\sum_{\theta\in\Theta} r(\theta) \leqq 0.$$

Thus, the vectors are diversified.

Now, we show that $\{\succsim_p\}_{p\in P}$ satisfies diversity given that the vectors are diversified. By the Farkas' Lemma we have been using, the fact that the vectors are diversified guarantee that for any $x, y, z, w \in X$ which are distinct, there exists $p \in P$ for which $x \succsim_p y \succsim_p z \succ_p w$ and $q \in P$ for whihc $x \succ_q y \succsim_q z \succsim_q w$. Furthermore, there exists $s \in P$ for which $w \succsim_s x \succsim_s y \succ_s z$. So we obtain $u^{xy}\cdot p \geq 0$, $u^{yz}\cdot p \geq 0$, and $u^{zw}\cdot p > 0$, $u^{xy}\cdot q > 0$, $u^{yz}\cdot q \geq 0$, and $u^{zw}\cdot q \geq 0$, and finally $u^{xy}\cdot s \geq 0$ and $u^{yz}\cdot s > 0$. We do not know the sign of $u^{zw}\cdot s$. Consider now $\lambda, \gamma, \eta > 0$ which sum to one, where η is arbitrarily small. Then in particular $u^{xy}\cdot(\lambda p + \gamma q + \eta s) > 0$, $u^{yz}\cdot(\lambda p + \gamma q + \eta s) > 0$ and $u^{zw}\cdot(\lambda p + \gamma q + \eta s) > 0$. Thus, $x \succ_{\lambda p+\gamma q+\eta s} y \succ_{\lambda p+\gamma q+\eta s} z \succ_{\lambda p+\gamma q+\eta s} w$.

Establishing the representation
Now, fix $e \in X$, and define $u^e \equiv \mathbf{0} \in \mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$, and for any other $x \in X$, define $u^x = u^{xe}$.

Given $p \in P$ and $x, y \in X$, we have

$$\begin{aligned} x \succsim_p y \quad &\Longleftrightarrow \quad u^{xy} \cdot p \geqq 0 \quad \Longleftrightarrow \quad (u^{xe} + u^{ey}) \cdot p \geqq 0 \quad \Longleftrightarrow \quad (u^{xe} - u^{ye}) \cdot p \geqq 0 \\ &\Longleftrightarrow \quad u^x \cdot p \geqq u^y \cdot p. \end{aligned}$$

Uniqueness: Take the representation $\{u^x\}_{x\in X}$ constructed above, where $u^e = \mathbf{0}$. Suppose another series of vectors $\{v^x\}_{x\in X}$ satisfies the above condition. Then, the series of vectors $\{v^x - v^y\}_{x,y\in X, x\neq y}$ satisfies the condition in the previous lemma. Hence there is a scalar λ and a a series of vectors in $\mathbb{R}^{|\Theta|}$, denoted $\{r^{xy}\}_{x,y\in X,x\neq y}$, such that for every $x, y, z \in X$,
(1)

$$v^x(\theta, b) - v^y(\theta, b) = \lambda(u^x(\theta, b) - u^y(\theta, b)) + r^{xy}(\theta)$$

for all $\theta \in \Theta$ and $b \in B_\theta$;
(2) $\sum_{\theta\in\Theta} r^{xy}(\theta) = 0$;
(3) $r^{yx} = -r^{xy}$;
(4) $r^{xy} + r^{yz} = r^{xz}$.

Define $\beta^e = v^e$ and $\beta^x = r^{xe} + \beta^e$ for every $x \neq e$.

6.2 Implications of Binary Independence

Lemma 5 For every $\theta \in \Theta$ with $|B_\theta \setminus \{x, y\}| \geqq 2$, $u^{xy}(\theta, b) = u^{xy}(\theta, b')$ for all $b, b' \in B_\theta \setminus \{x, y\}$.

Proof. Let $p^* \in P$ such that $u^{xy} \cdot p^* = 0$, so that $x \sim_p y$. Let $\theta \in \Theta$ for which $|B_\theta \setminus \{x, y\}| \geqq 2$, and consider any $\{p(b|\theta)\}_{b\in B_\theta\setminus\{x,y\}}$, $\{q(b|\theta)\}_{b\in B_\theta\setminus\{x,y\}}$ satisfying $\sum_{b\in B_\theta\setminus\{x,y\}} p(b|\theta) = \sum_{b\in B_\theta\setminus\{x,y\}} q(b|\theta) = \sum_{b\in B_\theta\setminus\{x,y\}} p^*(b|\theta)$. By condition (i) in the monotonicity axiom, replacing $\{p^*(b|\theta)\}_{b\in B_\theta\setminus\{x,y\}}$ by $\{p(b|\theta)\}_{b\in B_\theta\setminus\{x,y\}}$ or $\{q(b|\theta)\}_{b\in B_\theta\setminus\{x,y\}}$ does not change the indifference condition. Hence we have

$$\sum_{b\in B_\theta\setminus\{x,y\}} u^{xy}(\theta, b)(p(b|\theta) - q(b|\theta)) = 0.$$

Since $\{p(b|\theta)\}_{b\in B_\theta\setminus\{x,y\}}$ and $\{q(b|\theta)\}_{b\in B_\theta\setminus\{x,y\}}$ are otherwise arbitrary, we have $u^{xy}(\theta, b) = u^{xy}(\theta, b')$ for all $b, b' \in B_\theta$. ■

Similar arguments deliver the following lemmata.

Lemma 6 $u^{xy}(\theta, x) > u^{xy}(\theta, b)$ for all $\theta \in \Theta$ with $x \in B_\theta$ and $b \in B_\theta \setminus \{x\}$.

Lemma 7 $u^{xy}(\theta,b) > u^{xy}(\theta,y)$ for all $\theta \in \Theta$ with $y \in B_\theta$ and $b \in B_\theta \setminus \{y\}$.

6.3 Proof of Theorem 2

Assume that $\Theta_x \neq \emptyset$ for all $x \in X$, and $|B_\theta| \geqq 4$ for all $\theta \in \Theta$.

For each $\theta \in \Theta$, fix some $b^{-xy}(\theta) \in B_\theta \setminus \{x,y\}$. The above lemma ensures that the choice of $b^{-xy}(\theta)$ does not matter. Define $\widehat{u}^{xy} \in \mathbb{R}^{\sum_{\theta\in\Theta}|B_\theta|}$ by

1. if $x, y \in B_\theta$,

$$\begin{aligned}
\widehat{u}^{xy}(\theta,x) &= u^{xy}(\theta,x) - u^{xy}(\theta,b^{-xy}(\theta)) \\
\widehat{u}^{xy}(\theta,y) &= u^{xy}(\theta,y) - u^{xy}(\theta,b^{-xy}(\theta)) \\
\widehat{u}^{xy}(\theta,b) &= 0 \quad \text{for all} \quad b \in B_\theta \setminus \{x,y\}
\end{aligned}$$

2. if $x \in B_\theta$ and $y \notin B_\theta$,

$$\begin{aligned}
\widehat{u}^{xy}(\theta,x) &= u^{xy}(\theta,x) - u^{xy}(\theta,b^{-xy}(\theta)) \\
\widehat{u}^{xy}(\theta,b) &= 0 \quad \text{for all} \quad b \in B_\theta \setminus \{x\}
\end{aligned}$$

3. if $x \notin B_\theta$ and $y \in B_\theta$,

$$\begin{aligned}
\widehat{u}^{xy}(\theta,y) &= u^{xy}(\theta,y) - u^{xy}(\theta,b^{-xy}(\theta)) \\
\widehat{u}^{xy}(\theta,b) &= 0 \quad \text{for all} \quad b \in B_\theta \setminus \{y\}
\end{aligned}$$

4. if $x, y \notin B_\theta$,

$$\widehat{u}^{xy}(\theta,b) = 0 \quad \text{for all} \quad b \in B_\theta$$

Lemma 8

$$\widehat{u}^{xy} \cdot p = u^{xy} \cdot p$$

for all $p \in P$.

Proof. Let $\Theta_x = \{\theta \in \Theta : x \in B_\theta\}$ and $\Theta_y = \{\theta \in \Theta : y \in B_\theta\}$. By Lemma 5, we have

$$
\begin{aligned}
& \sum_{\theta\in\Theta}\sum_{b\in B_\theta} u^{xy}(\theta,b)p(b|\theta) \\
= \; & \sum_{\theta\in\Theta_x\cap\Theta_y} \big((u^{xy}(\theta,x) - u^{xy}(\theta,b^{-xy}(\theta))p(x|\theta) + (u^{xy}(\theta,y) - u^{xy}(\theta,b^{-xy}(\theta))p(y|\theta))\big) \\
& + \sum_{\theta\in\Theta_x\setminus\Theta_y} (u^{xy}(\theta,x) - u^{xy}(\theta,b^{-xy}(\theta))p(x|\theta) \\
& + \sum_{\theta\in\Theta_y\setminus\Theta_x} (u^{xy}(\theta,y) - u^{xy}(\theta,b^{-xy}(\theta))p(y|\theta) \\
& + \sum_{\theta\in\Theta\setminus(\Theta_x\cup\Theta_y)} u^{xy}(\theta,b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_x\cap\Theta_y} u^{xy}(\theta,b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_x\setminus\Theta_y} u^{xy}(\theta,b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_y\setminus\Theta_x} u^{xy}(\theta,b^{-xy}(\theta))
\end{aligned}
$$

Choose any $q \in P$ for which for all $\theta \in \Theta$, $q(x|\theta) = q(y|\theta) = 0$. Then $u^{xy} \cdot q = 0$ by Dummy. Moreover,

$$
\begin{aligned}
& u^{xy} \cdot q \\
= \; & \sum_{\theta\in\Theta\setminus(\Theta_x\cup\Theta_y)} u^{xy}(\theta,b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_x\cap\Theta_y} u^{xy}(\theta,b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_x\setminus\Theta_y} u^{xy}(\theta,b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_y\setminus\Theta_x} u^{xy}(\theta,b^{-xy}(\theta))
\end{aligned}
$$

However, these are the last four terms in the preceding expression. Conclude that $u^{xy} \cdot p = \widehat{u}^{xy} \cdot p$. ∎

By construction,

$$\widehat{u}^{yx} = -\widehat{u}^{xy}$$

for all $x, y \in X$. Lemma 8 also guarantees that

$$\widehat{u}^{xy} \cdot p + \widehat{u}^{yz} \cdot p = \widehat{u}^{xz} \cdot p$$

for all $p \in P$.

Lemma 9 For all $x, y, z \in X$ and $\theta \in \Theta_x$,

$$\widehat{u}^{xy}(\theta, x) = \widehat{u}^{xz}(\theta, x).$$

Proof. Note that

$$\widehat{u}^{xy} \cdot p = \sum_{\theta \in \Theta_x} \widehat{u}^{xy}(\theta, x) p(x|\theta) + \sum_{\theta \in \Theta_y} \widehat{u}^{xy}(\theta, y) p(y|\theta),$$

$$\widehat{u}^{yz} \cdot p = \sum_{\theta \in \Theta_y} \widehat{u}^{yz}(\theta, y) p(y|\theta) + \sum_{\theta \in \Theta_z} \widehat{u}^{yz}(\theta, z) p(z|\theta)$$

and

$$\widehat{u}^{xz} \cdot p = \sum_{\theta \in \Theta_x} \widehat{u}^{xz}(\theta, x) p(x|\theta) + \sum_{\theta \in \Theta_z} \widehat{u}^{xz}(\theta, z) p(z|\theta).$$

Since $|B_\theta| \geqq 4$, one may vary $p(x|\theta)$ linearly independently of $p(y|\theta)$ and $p(z|\theta)$. Hence we have

$$\widehat{u}^{xy}(\theta, x) = \widehat{u}^{xz}(\theta, x).$$

■

Now, fix $x \in X$ and fix some $e \neq x$, and define for all $\theta \in \Theta_x$, $u^x(\theta) = \widehat{u}^x(\theta, x)$. Lemmas 6 and 7 ensure that u^x is strictly positive; moreover Lemma 9 establishes that u^x is independent of the choice of e.

Now, fix $x, y \in X$ and choose some $e \notin \{x, y\}$. Then in particular $u^x(\theta) = \widehat{u}^x(\theta, x)$ and $u^x(\theta) = \widehat{u}^y(\theta, y)$.

Then, we have (using Lemma 9 for the third equivalence)

$$\begin{aligned}
& \sum_{\theta\in\Theta_x} u^x(\theta)p(x|\theta) - \sum_{\theta\in\Theta_y} u^y(\theta)p(y|\theta) \geqq 0 \\
\Longleftrightarrow\quad & \sum_{\theta\in\Theta_x} u^{xe}(\theta,x)p(x|\theta) - \sum_{\theta\in\Theta_y} u^{ye}(\theta,y)p(y|\theta) \geqq 0 \\
\Longleftrightarrow\quad & \sum_{\theta\in\Theta_x} u^{xe}(\theta,x)p(x|\theta) - \sum_{\theta\in\Theta_y} u^{ye}(\theta,y)p(y|\theta) + \sum_{\theta\in\Theta_e} (u^{xe}(\theta,e) - u^{ye}(\theta,e))p(e|\theta) \geqq 0 \\
\Longleftrightarrow\quad & \sum_{\theta\in\Theta_x} u^{xe}(\theta,x)p(x|\theta) + \sum_{\theta\in\Theta_e} u^{xe}(\theta,e)p(e|\theta) - \left(\sum_{\theta\in\Theta_y} u^{ye}(\theta,y)p(y|\theta) + \sum_{\theta\in\Theta_e} u^{ye}(\theta,e)p(e|\theta)\right) \geqq 0 \\
\Longleftrightarrow\quad & \widehat{u}^{xe}\cdot p - \widehat{u}^{ye}\cdot p \geqq 0 \\
\Longleftrightarrow\quad & \widehat{u}^{xe}\cdot p + \widehat{u}^{ey}\cdot p \geqq 0 \\
\Longleftrightarrow\quad & \widehat{u}^{xy}\cdot p \geqq 0 \\
\Longleftrightarrow\quad & u^{xy}\cdot p \geqq 0 \\
\Longleftrightarrow\quad & x \succsim_p y.
\end{aligned}$$

Uniqueness: Recall the uniqueness result in Lemma 1 that if there is another series of vectors $\{v^{xy}\}_{x\in X}$ satisfies the above condition, then there is a scalar λ and a a series of vectors in $\mathbb{R}^{|\Theta|}$, denoted $\{r^{xy}\}_{x,y\in X, x\neq y}$, such that for every $x,y,z\in X$,

(1)

$$v^{xy}(\theta,b) = \lambda u^{xy}(\theta,b) + r^{xy}(\theta)$$

for all $\theta\in\Theta$ and $b\in B_\theta$;

(2) $\sum_{\theta\in\Theta} r^{xy}(\theta) = 0$;

(3) $r^{yx} = -r^{xy}$;

(4) $r^{xy} + r^{yz} = r^{xz}$.

Let $\{\widehat{v}^{xy}\}_{x\in X}$ be the series of vectors obtained from $\{v^{xy}\}_{x\in X}$ as in the current argument. Since all the terms in r^{xy} are canceled out along the above procedure, we have $\widehat{u}^{xy} = \lambda\widehat{v}^{xy}$ for all $x,y\in X$.

6.4 Proof of Theorem 3

Construction of $\{\widehat{u}^{xy}\}_{x,y,x\neq y}$ is the same as above.

Lemma 10 For all $x,y,z\in X$ and $\theta\in\Theta_x$,

$$\widehat{u}^{xy}(\theta,x) = \widehat{u}^{xz}(\theta,x).$$

Proof. Note again that

$$\widehat{u}^{xy}\cdot p = \sum_{\theta\in\Theta_x}\widehat{u}^{xy}(\theta,x)p(x|\theta)+\sum_{\theta\in\Theta_y}\widehat{u}^{xy}(\theta,y)p(y|\theta),$$

$$\widehat{u}^{yz}\cdot p = \sum_{\theta\in\Theta_y}\widehat{u}^{yz}(\theta,y)p(y|\theta)+\sum_{\theta\in\Theta_z}\widehat{u}^{yz}(\theta,z)p(z|\theta)$$

and

$$\widehat{u}^{xz}\cdot p = \sum_{\theta\in\Theta_x}\widehat{u}^{xz}(\theta,x)p(x|\theta)+\sum_{\theta\in\Theta_z}\widehat{u}^{xz}(\theta,z)p(z|\theta).$$

Pick $\theta\in\Theta_{\{x,y,z\}}$, and let $p(x|\theta)=p(y|\theta)=\alpha>0$ and $p(x|\theta')=p(y|\theta')=0$ for all $\theta'\neq\theta$.

Then by Semi-strong Dummy, we have $x\sim_p y$, which implies $(\widehat{u}^{xy}(\theta,x)+\widehat{u}^{xy}(\theta,y))\alpha=0$. Hence, we obtain

$$\widehat{u}^{xy}(\theta,x)=-\widehat{u}^{xy}(\theta,y)=\widehat{u}^{yx}(\theta,y).$$

In a similar manner, we obtain

$$\widehat{u}^{yz}(\theta,y)=-\widehat{u}^{yz}(\theta,z)=\widehat{u}^{zy}(\theta,z),$$

$$\widehat{u}^{xz}(\theta,x)=-\widehat{u}^{xz}(\theta,z)=\widehat{u}^{zx}(\theta,z).$$

Again fix $\theta\in\Theta_{\{x,y,z\}}$, and let $p(y|\theta)=p(z|\theta)=\alpha>0$ and $q(y|\theta)=q(z|\theta)=\beta>0$, and let p and q coincide outside of θ. Then the relationship $\widehat{u}^{xy}\cdot p+\widehat{u}^{yz}\cdot p=\widehat{u}^{xz}\cdot p$ together with the above deliver

$$3(\alpha-\beta)\widehat{u}^{xy}(\theta,x)=3(\alpha-\beta)\widehat{u}^{xz}(\theta,x).$$

Since α is arbitrary, we obtain the desired result. ■

The rest of the proof goes on in the same manner as in the previous section.

6.5 Proof of Theorem 4

Lemma 11 For all $x,y,z\in X$ and $\theta\in\Theta$ with $B_\theta=\{x,y\}$,

$$\frac{u^{xy}(\theta,x)-u^{xy}(\theta,y)}{2}=u^{xz}(\theta,x)-u^{xz}(\theta,y).$$

Proof. Given $\theta \in \Theta$ with $B_\theta = \{x, y\}$, let q_θ be a degenerate distribution on x. We claim that there exists $p \in P$ be such that

$$u^{yz} \cdot p = 0, \quad u^{xy} \cdot p < 0, \quad u^{xz} \cdot p < 0.$$

Notice that $u^{xz} \cdot p = u^{xy} \cdot p + u^{yz} \cdot p = u^{xy} \cdot p$. Moreover, we can choose $u^{xy} \cdot p = u^{xz} \cdot p = -c < 0$, where $c > 0$ is a sufficiently small number.
To see this, first note that by Diversity and Continuity, there exist p^1, p^2 in the relative interior of P for which $y \succ_{p^1} z \succ_{p^1} x$ and $z \succ_{p^2} y \succ_{p^2} x$. By taking an appropriate convex combination of p^1 and p^2, we guarantee the existence of $p^3 \in P$ for which

$$u^{yz} \cdot p^3 = 0, \quad u^{xy} \cdot p^3 < 0, \quad u^{xz} \cdot p^3 < 0.$$

To see that in fact we may choose c to be arbitrarily small, note that we may also similarly guarantee the existence of $p^4 \in P$ for which

$$u^{yz} \cdot p^4 = 0, \quad u^{xy} \cdot p^4 = 0, \quad u^{xz} \cdot p^4 = 0.$$

The result then follows by taking an appropriate convex combination of p^3 and p^4.

Now, let $\lambda \in (0, 1)$ satisfy

$$(u^{xy}(\theta, x) - u^{xy}(\theta, y))(1 - p(x|\theta))\lambda = c,$$

. Then the previous equality implies that $x \sim_{(1-\lambda)p+\lambda(q_\theta, p_{-\theta})} y$ is the case. By Proportionality, we have $x \sim_{(1-\frac{\lambda}{2})p+\frac{\lambda}{2}(q_\theta, p_{-\theta})} z$, which implies that

$$(u^{xz}(\theta, x) - u^{xz}(\theta, y))(1 - p(x|\theta))\frac{\lambda}{2} = c$$

holds. Thus we obtain the desired result. ■

For each $\theta \in \Theta$ with $B_\theta \neq \{x, y\}$, fix some $b^{-xy}(\theta) \in B_\theta \setminus \{x, y\}$. The above lemma ensures that the choice of $b^{-xy}(\theta)$ does not matter.

Define $\widehat{u}^{xy} \in \mathbb{R}^{\sum_{\theta \in \Theta} |B_\theta|}$ by

1. if $B_\theta = \{x, y\}$,

$$\widehat{u}^{xy}(\theta, x) = (u^{xy}(\theta, x) - u^{xy}(\theta, y))/2, \quad \widehat{u}^{xy}(\theta, y) = (u^{xy}(\theta, y) - u^{xy}(\theta, x))/2$$

2. otherwise, same as before

Lemma 12

$$\widehat{u}^{xy} \cdot p = u^{xy} \cdot p$$

for all $p \in P$.

Proof. Let $\Theta_{\{x,y\}} = \{\theta \in \Theta : B_\theta = \{x, y\}\}$, $\Theta_x = \{\theta \in \Theta : x \in B_\theta\}$ and $\Theta_y = \{\theta \in \Theta : y \in B_\theta\}$. By Binary Independence, we have

$$\begin{aligned}
& \sum_{\theta\in\Theta} \sum_{b\in B_\theta} u^{xy}(\theta, b)p(b|\theta) \\
= & \sum_{\theta\in\Theta_{\{x,y\}}} \left(\frac{u^{xy}(\theta, x) - u^{xy}(\theta, y)}{2} p(x|\theta) + \frac{u^{xy}(\theta, y) - u^{xy}(\theta, x)}{2} p(y|\theta) \right) \\
& + \sum_{\theta\in\Theta_x\cap\Theta_y\setminus\Theta_{\{x,y\}}} \big((u^{xy}(\theta, x) - u^{xy}(\theta, b^{-xy}(\theta))p(x|\theta) + (u^{xy}(\theta, y) - u^{xy}(\theta, b^{-xy}(\theta))p(y|\theta))\big) \\
& + \sum_{\theta\in\Theta_x\setminus\Theta_y} (u^{xy}(\theta, x) - u^{xy}(\theta, b^{-xy}(\theta))p(x|\theta) \\
& + \sum_{\theta\in\Theta_y\setminus\Theta_x} (u^{xy}(\theta, y) - u^{xy}(\theta, b^{-xy}(\theta))p(y|\theta) \\
& + \sum_{\theta\in\Theta\setminus(\Theta_x\cup\Theta_y)} u^{xy}(\theta, b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_{\{x,y\}}} \frac{u^{xy}(\theta, x) + u^{xy}(\theta, y)}{2} \\
& + \sum_{\theta\in\Theta_x\cap\Theta_y\setminus\Theta_{\{x,y\}}} u^{xy}(\theta, b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_x\setminus\Theta_y} u^{xy}(\theta, b^{-xy}(\theta)) \\
& + \sum_{\theta\in\Theta_y\setminus\Theta_x} u^{xy}(\theta, b^{-xy}(\theta))
\end{aligned}$$

By Strong Dummy, the terms from fifth to ninth on the right-hand-side add up to zero. Therefore the right-hand-side is equal to $\widehat{u}^{xy} \cdot p$. ■

By construction, we have

$$\widehat{u}^{yx} = -\widehat{u}^{xy}$$

for all $x, y \in X$. Also, the above lemma guarantees that

$$\widehat{u}^{xy} \cdot p + \widehat{u}^{yz} \cdot p = \widehat{u}^{xz} \cdot p$$

for all $p \in P$.

Lemma 13 For all $x, y, z \in X$ and $\theta \in \Theta_x$,

$$\widehat{u}^{xy}(\theta, x) = \widehat{u}^{xz}(\theta, x).$$

Proof. When θ is such that $B_\theta = \{x, y\}$, it follows from

$$\widehat{u}^{xy}(\theta, x) = \frac{u^{xy}(\theta, x) - u^{xy}(\theta, y)}{2} = u^{xz}(\theta, x) - u^{xz}(\theta, y) = \widehat{u}^{xz}(\theta, x).$$

For the other cases, it follows from the same argument as before. ■

Now, define $u^x \in \mathbb{R}^{|\Theta_x|}$ by

$$u^x(\theta) = \widehat{u}^{xe}(\theta, x)$$

for each $\theta \in \Theta_x$. Then, we have

$$\begin{aligned}
& \sum_{\theta \in \Theta_x} u^x(\theta) p(x|\theta) - \sum_{\theta \in \Theta_y} u^y(\theta) p(y|\theta) \geqq 0 \\
\iff\ & \sum_{\theta \in \Theta_x} u^x(\theta) p(x|\theta) - \sum_{\theta \in \Theta_e} u^e(\theta) p(e|\theta) - \left(\sum_{\theta \in \Theta_y} u^y(\theta) p(y|\theta) - \sum_{\theta \in \Theta_e} u^e(\theta) p(e|\theta) \right) \geqq 0 \\
\iff\ & \widehat{u}^{xe} \cdot p - \widehat{u}^{ye} \cdot p \geqq 0 \\
\iff\ & \widehat{u}^{xe} \cdot p + \widehat{u}^{ey} \cdot p \geqq 0 \\
\iff\ & \widehat{u}^{xy} \cdot p \geqq 0 \\
\iff\ & u^{xy} \cdot p \geqq 0 \\
\iff\ & x \succsim_p y.
\end{aligned}$$

Uniqueness: Recall the uniqueness result in Lemma 1 that if there is another series of vectors $\{v^{xy}\}_{x \in X}$ satisfies the above condition, then there is a scalar λ and a a series of vectors in $\mathbb{R}^{|\Theta|}$, denoted $\{r^{xy}\}_{x,y \in X, x \neq y}$, such that for every $x, y, z \in X$,

(1)

$$v^{xy}(\theta, b) = \lambda u^{xy}(\theta, b) + r^{xy}(\theta)$$

for all $\theta \in \Theta$ and $b \in B_\theta$;

(2) $\sum_{\theta \in \Theta} r^{xy}(\theta) = 0$;

(3) $r^{yx} = -r^{xy}$;

(4) $r^{xy} + r^{yz} = r^{xz}$.

Let $\{\widehat{v}^{xy}\}_{x \in X}$ be the series of vectors obtained from $\{v^{xy}\}_{x \in X}$ as in the current argument. Since all the terms in r^{xy} are canceled out along the above procedure, we have $\widehat{u}^{xy} = \lambda \widehat{v}^{xy}$ for all $x, y \in X$.

6.6 Proof of Corollary 1

Lemma 14 For all $x, y \in X$ and $B \in \Theta$ with $x, y \in B$, $u^x(B) = u^y(B)$.

Proof. Let $p \in P$ be such that $p(x|B) = p(y|B) = \frac{1}{2}$, and x, y are never chosen elsewhere and the distribution there is uniform over the other alternatives.
By applying Neutrality with regard to the permutation between x and y, we have $x \sim_p y$, which implies $\frac{1}{2}u^x(B) = \frac{1}{2}u^y(B)$. Hence $u^x(B) = u^y(B)$. ■

Thus, there exists a function $u : \mathcal{B} \to \mathbb{R}_{++}$ such that

$$u^x(B) = u(B)$$

for all $B \in \mathcal{B}$ with $x \in B$. The remaining is to show that $u(B)$ depends only on the cardinality of B.

Lemma 15 For all $B, C \in \mathcal{B}$ with $|B| = |C|$, $u(B) = u(C)$.

Proof. Pick $x \in B \setminus C$ and $y \in C \setminus B$. Let $p \in P$ be such that
(i) $p(x|B) = 1$, $p(y|C) = 1$, $p(x|\{x, y\}) = p(y|\{x, y\}) = \frac{1}{2}$, and x, y are never chosen elsewhere and the distribution there is uniform over the other alternatives. By applying Neutrality with regard to the permutation between x and y, we have $x \sim_p y$, which implies $u(B) + \frac{1}{2}u(\{x, y\}) = u(C) + \frac{1}{2}u(\{x, y\})$. Hence $u(B) = u(C)$. ■

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