Precautionary Principle and the Optimal Timing of Environmental Policy under Ambiguity *

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This Version: June 16, 2008

Abstract

We consider a problem in environmental policy design by applying optimal stopping rules. The purposes of this paper are (1) to provide an economic foundation for the *precautionary principle* and the 1992 Rio Declaration on Environment and Development, (2) to derive the optimal timing rule that governments should adopt in order to deal with emissions of SO_2 or CO_2 and increases in greenhouse gas concentrations under *ambiguity in continuous time*, and (3) to show that this optimal timing rule has a reservation property. Furthermore, we analyze the effect of an increase in ambiguity on the optimal timing of adopting some environmental policy, and show that an increase in ambiguity decreases the optimal timing of adopting the environmental policy.

Journal of Economic Literature Classification Numbers: D81, H23, L51, Q23

Key Words: Precautionary Principle, Environmental Policy, Ambiguity

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*I would like to thank Hiroyuki Ozaki, Koichi Futagami, Akihisa Shibata and Hajime Sugeta for their advice and discussions on this work. I am also grateful to Rieko Ishii for her comments and suggestions. Financial support from Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists is greatly acknowledged.

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1. Introduction

The policy stance of taking preventive action based on the worst case scenario before the resolution of uncertainty about possible environmental damages is referred to as the *precautionary principle*. [1] Principle 15 in the 1992 Rio Declaration on Environment and Development is based on this principle and states that "where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation." Although the precautionary principle is not necessarily theoretically supported, European countries and Canada adopt the precautionary principle when they consider environmental policies. On the other hand, there exist some countries, for example, the US that takes a position that if there exists lack of full scientific certainty, then environmental policies should be postponed. The difference in environmental policy stances among nations without sound theoretical foundations leads to severe conflicts at international organizations where environmental polices are discussed. If the precautionary principle is supported by a sound theoretical foundation, then it is possible to discuss environmental problems fruitfully and to come up with effective measures for global warming. The purpose of this paper is to provide a foundation for the Rio Declaration and the precautionary principle by incorporating three characteristics of environmental problems, *ambiguity*, *irreversibility*, and *the flexibility in deciding the timing of adopting environmental policies* into a continuous-time model of ambiguity in order to discuss environmental policies for global warming based on a sound economic foundation. [2]

The importance of the distinction between risk and ambiguity is pointed out by Ellsberg (1961), which provides some evidence that people tend to prefer to act on known rather than unknown or vague probabilities. Uncertainty captured by a set of probability measures is called *Knightian uncertainty* or *ambiguity* (henthforth, ambiguity). On the other hand, uncertainty captured by a unique probability measure is called *risk*. Ambiguity can be analyzed in the framework of Maxmin Expected Utility (henceforth MMEU). MMEU axiomatized by Gilboa

[1] For example, see Hanley, Shogren and White (2007).

[2] Based on robust control theory, Gonzalez (2007) shows that the policy maker can implement the precautionary principle to regulate a stock pollutant from a point of view of model uncertainty. For the details of robust control theory and model uncertainty, see Hansen and Sargent (2008).

and Schmeidler (1989) states that if a certain set of axioms is satisfied, then decision maker's beliefs are captured by a set of probability measures and her preferences are represented by the minimum of expected utilities over the set of probability measures. MMEU has deepened our understanding of decision maker's behaviors under ambiguity. [3] If governments are assumed to be less confident about climate changes in the future and they are assumed to make decisions very cautiously, then the validity of environmental policies based on the precautionary principle can be well analyzed by MMEU. Since climate changes in the future cannot be easily foreseen and the environment cannot be easily recovered once destroyed, the assumptions that governments are less confident about climate changes in the future and they make decisions very cautiously make sense. Therefore, this paper adapts a continuous-time model of ambiguity proposed by Chen and Epstein (2002) in order to provide an economic foundation for the precautionary principle. [4]

Environmental policies are usually evaluated on the basis of cost-benefit analysis (the net present value (NPV) approach), which states that a policy should be adopted if the present value of the expected flow of benefits exceeds the present value of the expected flow of costs. However, as pointed out by Pindyck (2000, 2002), this standard approach does not consider three significant characteristics of environmental problems. First, there exist *economic uncertainty* over future costs and benefits of adopting environmental policies, and *ecological uncertainty* over the evolution of ecological systems. We do not exactly know costs and effects of adopting environmental policies nor know the economic damages caused by increases or decreases in average temperature. Second, there exist two *irreversibilities* to be considered in environmental policy design. One is the irreversibility with respect to environmental damage. For instance, emissions of CO_2 will increase greenhouse gas concentrations, which is considered to lead to global warming and to damage ecological systems. The damage to environmental systems caused by global warming cannot be reversible, which implies that adopting environmental policies right now rather than waiting has *sunk benefits*. The standard cost-benefit analysis based on the NPV

[3] For example, see Epstein and Wang (1994), and Nishimura and Ozaki (2004).

[4] In static frameworks, Ghirardato et al. (2004), Klibanoff et al. (2005), and Klibanoff et al. (2006) provide more general frameworks than MMEU. In this paper, we adopt a continuous-time model of ambiguity proposed by Chen and Epstein (2002) in order to analyze behaviors under ambiguity whithin a continuous-time framework.

approach ignores this kind of *opportunity benefits*.[5] The other is the irreversibility with respect to economic damage. For example, the installation of scrubbers by coal-burning utilities will be *sunk costs* on society, which implies that adopting environmental policies immediately rather than waiting for the arrival of new information about environmental damage and economic consequences generates *opportunity costs*. [6] Again, the traditional cost-benefit analysis ignores this kind of opportunity costs. Finally, the adoption of environmental policies is *not now-or-never decisions*. That is, the government has the option to postpone the adoption of policies and can wait for the arrival of new information. [7]

By incorporating ambiguity, irreversibility, and the flexibility in deciding the timing of adopting policies into a continuous-time model of ambiguity, this paper shows that an increase in ambiguity *decreases* the value of adopting some environmental policy. Furthermore, this paper shows that an increase in ambiguity *decreases* the optimal timing of the environmental policy above which the environmental policy is immediately adopted and below which the environmental policy is not adopted, which implies that Principle 15 in the 1992 Rio Declaration on Environment and Development and the precautionary principle are theoretically supported within the framework of MMEU. The second claim implies that the government's less confidence in the prospect of the economic situation will *encourage* the government to adopt the optimal environmental policy immediately, and make her *hasten* to adopt the optimal environmental policy. Our results are stark contrasts to Pindyck (2000) that shows that in a continuous-time model of risk, an increase in risk does not affect the value of adopting an environmental policy, and an increase in risk *increases* the optimal timing of adopting the environmental policy, which implies that an increase in risk will *discourage* the government to adopt the optimal environmental policy, and make her *postpone* adopting the environmental policy.

The organization of this paper is as follows. Section 2 provides a continuous-time model under ambiguity, and derives the value of adopting environmental policies under ambiguity in continuous time. In order to analyze the value of adopting the environmental policy under

[5] This notion is pointed out by Arrow and Fisher (1974), and Henry (1974).

[6] For example, we will have the arrival of new information about innovations of new technologies in the near future that might enable us to remove sulfur more inexpensively and efficiently.

[7] For example, we will receive some data about global warming, or some innovation about scrubbers, which will enable us to put off adopting environmental policies and to avoid imposing sunk costs on society immediately.

ambiguity in continuous time, mathematical definitions and results are provided. Section 3 derives the value of the optimal environmental policy under a continuous-time model of ambiguity. Moreover, we show that the optimal timing rule the government should adopt exists and that this optimal timing rule has a reservation property. Section 4 analyzes the effects of an increase in ambiguity on the value of adopting the environmental policy and the optimal timing of adopting the environmental policy, respectively. Section 5 compares this paper with Pindyck (2000, 2002) that analyze optimal environmental policies under risk, and concludes this paper. All proofs are relegated to Appendices.

2. The Value of Adopting Environmental Policies under Ambiguity in Continuous Time

In this section, in order to analyze the government's optimal environmental policies within the framework of *ambiguity* in continuous time, we provide a continuous-time model under ambiguity, and derive the value of adopting environmental policies under ambiguity in continuous time.

Let $(\Omega, \mathcal{F}_T, P)$ be a probability space, and let $(B_t)_{0 \le t \le T}$ be a standard Brownian motion with respect to P. We consider the standard filtration $(\mathcal{F}_t)_{0 \le t \le T}$ for a standard Brownian motion $(B_t)_{0 \le t \le T}$. Let Θ be a set of density generators. [8] For such a set Θ, define the set of probability measures, $\mathcal{P}^{\Theta}$ on $(\Omega, \mathcal{F}_T)$, generated by Θ, by

$$\mathcal{P}^{\Theta} = \left\{ Q^{\theta} \;\middle|\; \theta \in \Theta \right\}, \tag{1}$$

where Q^{θ} is derived from P. [9] In this paper, decision maker's beliefs are captured by not a single probability measure, but the set of probability measures equivalent to a probability measure P. That is, decision maker's beliefs can be deviated from the true probability measure P within probability measures equivalent to P.

[8] For the definition of density generators, see Appendix A.

[9] Let θ be a density generator in Θ. Let $(z^{\theta}_t)_{0 \le t \le T}$ be a stochastic process defined by (22) in Appendix A. Then, a probability measure Q^{θ} on $(\Omega, \mathcal{F}_T)$ defined by

$$(\forall A \in \mathcal{F}_T) \;\; Q^{\theta}(A) = \int_{\Omega} z^{\theta}_T(\omega)\chi_A(\omega)\, dP(\omega) = E^P[\chi_A z^{\theta}_T],$$

is equivalent to P. Conversely, any probability measures equivalent to P can be obtained by a density generator in this way. Note that χ denotes the indicator function.

A set of density generators Θ^{K_t} is *strongly rectangular* if there exist a nonempty compact subset [10] $\mathcal{K}$ of $\mathbb{R}$ and a compact-valued, convex-valued, measurable correspondence [11] $K : [0,T] \twoheadrightarrow \mathcal{K}$ such that $0 \in K_t$ and

$$\Theta^{K_t} = \left\{ (\theta_t) \in \mathcal{L}^2 \,|\, \theta_t(\omega) \in K_t \;\; (m \otimes P)\text{-a.s.} \right\}, \tag{2}$$

where m is the Lebesgue measure restricted on $\mathcal{B}([0,T])$. Any element in Θ^{K_t} satisfies Novikov's condition, which follows since K_t is a subset of a compact subset $\mathcal{K}$ of $\mathbb{R}$ for all t. Note that the set K_t is independent of the state ω, contrary to the set $K_t(\omega)$ in (23). [12]

Let us introduce two concepts; *i.i.d. ambiguity* and *κ-ignorance.* Ambiguity characterized by Θ^K is *i.i.d. ambiguity* if there exists a compact subset K of $\mathbb{R}$ such that $0 \in K$ and

$$\Theta^{K} = \left\{ (\theta_t) \in \mathcal{L}^2 \,|\, \theta_t(\omega) \in K \;\; (m \otimes P)\text{-a.s.} \right\}, \tag{4}$$

where m is the Lebesgue measure restricted on $\mathcal{B}([0,T])$. Note that the set Θ^K is independent of state and time. In the case of κ-ignorance, which is a special case of *i.i.d.* ambiguity Θ^K, we can parameterize the degree of ambiguity, that is, the set K is specified as $K = [-\kappa, \kappa]$ for all $\kappa > 0$. The positive real number κ is considered to represent the degree of ambiguity because the larger κ is, the larger the set of probability measures is. By considering the case of κ-ignorance, we can perform comparative static analyses on the effects of ambiguity.

Let $(M_t)_{0 \le t \le T}$ be a deterministic process of the stock of an environmental pollutant (for example, SO_2 or CO_2 in the atmosphere) and let E_t be the amount of the emitted pollutant at time t. We assume that the evolution of the stock of the environmental pollutant $(M_t)_{0 \le t \le T}$

[10] This assumption corresponds to *uniform boundedness* in Chen and Epstein (2002). Under this assumption, we can show that any $\theta \in \Theta^{K_t}$ satisfies Novikov's condition.

[11] That is, $\{t \in [0,s] \,|\, K_t \cap U \neq \emptyset\} \in \mathcal{B}([0,s])$ holds for any closed subset U of $\mathcal{K}$. Note that $\mathcal{B}([0,s])$ denotes the smallest σ-algebra containing $[0,s]$.

[12] Let $0 \le s \le t \le T$ and let X be an $\mathcal{F}_T$-measurable function. If a set of density generators Θ satisfies the strong rectangularity, then the following recursive structure holds:

$$\min_{\theta \in \Theta} E^{Q^\theta}\left[X \,|\, \mathcal{F}_s\right] = \min_{\theta \in \Theta} E^{Q^\theta}\left[E^{Q^\theta}\left[X \,|\, \mathcal{F}_t\right] \middle| \mathcal{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta}\left[\min_{\theta' \in \Theta} E^{Q^{\theta'}}\left[X \,|\, \mathcal{F}_t\right] \middle| \mathcal{F}_s \right] \tag{3}$$

as long as the minima exist. The first equality holds by the law of iterated integrals. Although the second equality holds by the rectangularity that is weaker than the strong rectangularity, the strong rectangularity is needed in order to show Proposition 1. See Nishimura and Ozaki (2007) in details.

satisfies the following: [13]

$$dM_t = (\beta E_t - \delta M_t)dt, \tag{5}$$

where $\beta \in (0, 1]$ denotes the absorption rate of the environmental pollutant, and $\delta \in [0, 1]$ denotes the natural decay rate of the stock of the environmental pollutant over time.[14] In other words, a fraction β of the emitted pollutant at time t, E_t, goes into the atmosphere, and a fraction δ of the environmental pollutant at time t, M_t, in the atmosphere diffuses into the ocean, the forests, and so forth. [15] We assume that until an environmental policy is adopted, the constant initial level E_0 continues to be emitted, and that once the environmental policy is adopted, the amount of the emitted pollutant is reduced to a new and permanent level E_1 that satisfies $0 \leq E_1 \leq E_0$. For simplicity, we assume that $E_1 = 0$.

We assume that economic uncertainty follows a geometric Brownian motion,

$$dX_t = \alpha X_t dt + \sigma X_t dB_t, \tag{6}$$

where α and σ are constant real numbers. This stochastic process $(X_t)_{0 \leq t \leq T}$ is assumed to capture economic uncertainty over future costs and benefits of policy adoptions. Changes in X_t over time might reflect changes in technologies. For instance, if M_t denotes the stock of CO_2 in the atmosphere, then changes in X_t might reflect the innovation of technologies that would drastically reduce the social cost of M_t, or might reflect population increase that would raise the social cost of M_t. Without loss of generality, it is assumed that $\sigma > 0$. Within the framework of ambiguity in continuous time, by Girsanov's Theorem (see Appendix A), the stochastic differential equation capturing economic uncertainty over future costs and benefits over adopting environmental policies turns out to be

$$dX_t = (\alpha - \sigma\theta_t) X_t dt + \sigma X_t dB_t^\theta$$

for any $\theta \in \Theta$.

[13] For simplicity, we ignore the stochastic fluctuation of M_t.

[14] In addition to the evolution of the stock of the environmental pollutant $(M_t)_{0 \leq t \leq T}$, Nordhaus (1991) considers a particular temperature adjustment process.

[15] For extreme cases, $\beta = 1$ means that 100% of the emitted pollutant at time t remains in the atmosphere, and $\delta = 1$ means that 100% of the stock of the environmental pollutant at time t in the atmosphere depreciates.

We assume that the flow of the social cost associated with the stock variable M_t, $C(M_t, X_t)$, is linear in M_t, [16] that is

$$C(X_t, M_t) = -X_t M_t. \tag{7}$$

We assume that the government is ambiguity averse. In other words, her beliefs are captured by the set of probability measures $\mathcal{P}^{\Theta}$, (1), and she maximizes the infimum of expected returns over $\mathcal{P}^{\Theta}$. Furthermore, we impose the strong rectangularity on Θ, which implies that Θ is equal to Θ^{K_t}. Thus, the value of adopting the environmental policy that is enforced at time t until time T is

$$W(X_t, M_t, t) \equiv \inf_{Q \in \mathcal{P}^{\Theta}} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds \middle| \mathcal{F}_t \right],$$

where $C(X_t, M_t)$ is defined by (7), $r > 0$ is the discount rate, and $E^P[\cdot | \mathcal{F}_t]$ is the expectation with respect to P conditioned on $\mathcal{F}_t$. The government is assumed to decide whether or not to adopt the environmental policy from time t to time T. After time T, the environmental policy will be evaluated, and a new environmental policy will be considered.

Proposition 1. *Suppose that the government is ambiguity averse, and her beliefs are characterized by Θ^{K_t}, where Θ^{K_t} is a strongly rectangular set of density generators defined by* (2) *for some* (K_t). *Then the value of adopting the environmental policy that is enforced at time t until time T is provided by*

$$W(X_t, M_t, t) = -\int_t^T X_t M_t \exp\left(-(r + \delta - \alpha)(s - t) - \int_t^s \sigma(\theta_h)_* dh \right) ds, \tag{8}$$

where $(\theta_t)_*$ is defined by

$$(\forall t \in [0, T]) \quad (\theta_t)_* \equiv \operatorname{argmin} \{ \sigma x \mid x \in K_t \} = \{ \min K_t \}.$$

Proof. See Appendix B. □

[16] As pointed out in Pindyck (2000), for most environmental problems, the damage from a pollutant is considered to rise more than proportionally with the stock of the pollutant. If the cost function $C(X_t, M_t)$ is assumed to be quadratic in M_t, $C(X_t, M_t) = -X_t M_t^2$, then the optimal policy rule will depend on the stock M_t. However, the assumption that the cost function is quadratic in M_t will not affect our main result in this paper. Thus, for simplicity, we assume that the cost function is linear in M_t.

In the next section, based on the value of adopting the environmental policy (8), we derive the value of the optimal environmental policy by solving the Hamilton-Jacobi-Bellman (henceforth, HJB) equation.

3. The Optimal Environmental Policy under the Strong Rectangularity

In this section, we derive the value of the optimal environmental policy under the strong rectangularity. Furthermore, in order to solve the HJB equation analytically and to provide the further characterization of the value of adopting the environmental policy, we have to assume that (1) ambiguity is characterized by *i.i.d.* ambiguity, (2) the government's planning horizon is infinite, and (3) the environmental policy once adopted never expires. [17]

3.1 The Value of the Optimal Environmental Policy under the Strong Rectangularity

The optimal time is the solution to the optimal stopping problem of finding an $(\mathcal{F}_t)$-stopping time, $t' \in [0, T]$ that maximizes the value of the environmental policy at period 0

$$\min_{Q \in \mathcal{P}^{\Theta}} E^{Q} \left[\int_0^T e^{-rs} C(X_s, M_s) ds - e^{-rt'} I \,\middle|\, \mathcal{F}_0 \right],$$

where $I > 0$ denotes the flow of sunk costs associated with policy adoption. It is also assumed that $r > \alpha + \sigma\kappa$, [18] where κ denotes a degree of ambiguity. Thus, the value at t of the optimal environmental policy V_t, is defined by

$$V_t \equiv \max_{t' \in [t,T]} \min_{Q \in \mathcal{P}^{\Theta}} E^{Q} \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} I \,\middle|\, \mathcal{F}_t \right], \tag{9}$$

where $C(X_t, M_t)$ is defined by (7). Appendix B shows that V_t is a solution to the following Hamilton-Jacobi-Bellman equation,

$$V_t = \max \left\{ W_t - I, -X_t M_t dt + \min_{\theta \in \Theta} E^{Q^{\theta}} \left[dV_t \,|\, \mathcal{F}_t \right] + V_t - rV_t dt \right\}. \tag{10}$$

[17] Assumption (3) implies that policy adoption is irreversible, which may be extreme. For example, environmental policies could be enhanced or weakened as time goes by. However, it seems that irreversibility with respect to adoption of environmental policies is common. For example, once a carbon tax is introduced, it cannot be easily abolished. For more discussions about assumption (3) from the mathematical point of view, see Nishimura and Ozaki (2007).

[18] This assumption is stronger than the usual assumption $r > \alpha$. However, the assumption $r > \alpha + \sigma\kappa$ is not so strong for sufficiently small $\kappa > 0$. The crux of the matter is that $\kappa \neq 0$, which differentiates analyses in ambiguity from analyses in risk.

Further assumptions enable us to solve this type of the Hamilton-Jacobi-Bellman equation, otherwise difficult to solve analytically. We discuss this topic in the following subsections.

3.2 The Further Characterization of the Value of Adopting the Environmental Policy

In this subsection, we derive the value of adopting the environmental policy by assuming that ambiguity is characterized by *i.i.d.* ambiguity, the government's planning horizon is infinite, and the environmental policy once adopted never expires.

Under *i.i.d.* ambiguity, it follows that

$$\theta_* = \operatorname{argmin}\{\sigma x |\, x \in K\} = \min K. \tag{11}$$

Note that θ_* is independent of time and state. Under *i.i.d.* ambiguity, (8) reduces to

$$\begin{aligned} W(X_t, M_t, t) &= -\int_t^T X_t M_t \exp\left(-(r+\delta-\alpha+\sigma\theta_*)(s-t)\right) ds \\ &= -\int_t^T X_t M_t \exp(-\eta(s-t)) ds \\ &= -\frac{X_t M_t}{\eta}\left(1-\exp(-\eta(T-t))\right), \end{aligned} \tag{12}$$

where $\eta \equiv r+\delta-\alpha+\sigma\theta_*$. Thus, the value of adopting the environmental policy reduces to

$$W_t \equiv W(X_t, M_t) = -\frac{X_t M_t}{\eta} \tag{13}$$

as T goes to ∞, since $r > \alpha + \sigma\kappa$, $\sigma > 0$, $\kappa > 0$, $\delta > 0$, and $0 \in K$, which implies the positivity of η. Note that (13) does not directly depend on the calendar time t. This stationarity of W_t enables us to solve the HJB equation (10) analytically.

3.3 The Further Characterization of the Value of the Optimal Environmental Policy

In this subsection, we derive the value of the optimal environmental policy V_t under *i.i.d.* ambiguity and the infinite-time horizon. In order to solve the HJB equation (10) analytically, we assume that ambiguity is characterized by *i.i.d.* ambiguity, the planning horizon is infinite, and the environmental policy once adopted never expires, which implies that V_t depends on X_t

and M_t, not on the calendar time t directly. Thus, the value at t of the optimal environmental policy V_t defined by (9) turns out to be stationary, which implies that we can denote V_t as $V_t = V(X_t, M_t)$, where $V : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a real-valued function. Accordingly, the HJB equation (10) turns out to be

$$V_t \equiv V(X_t, M_t) = \max \left\{ W_t - I, -X_t M_t dt + \min_{\theta \in \Theta} E^{Q^\theta} [dV_t \mid \mathcal{F}_t] + V(X_t, M_t) - rV(X_t, M_t)dt \right\}, \quad (14)$$

where $V : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is some real-valued function. From the HJB equation (14), it follows that

$$-X_t M_t dt + \min_{\theta \in \Theta} E^{Q^\theta} [dV_t \mid \mathcal{F}_t] = rV(X_t, M_t)dt, \quad (15)$$

in the continuation region. The left-hand side of this equation is the social cost assosiated with the stock of the pollutant plus the government's expected minimum gain of having the rights to carry out the optimal environmental policy, and the right-hand side is the opportunity cost measured in terms of government's discount rate.

In Appendix B, we show that in the continuation region, [19]

$$\min_{\theta \in \Theta} E^{Q^\theta} [dV_t | \mathcal{F}_t] = \frac{\partial V}{\partial X_t}(\alpha - \sigma\theta_*)X_t dt + \frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2 X_t^2 dt + \frac{\partial V}{\partial M_t}(\beta E_0 - \delta M_t)\, dt.$$

Thus in the continuation region, it follows that

$$\frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V}{\partial X_t^2} + (\alpha - \sigma\theta_*)X_t \frac{\partial V}{\partial X_t} - rV(X_t, M_t) + \frac{\partial V}{\partial M_t}(\beta E_0 - \delta M_t) - X_t M_t = 0.$$

We solve this differential equation with the following boundary conditions,

$$V(0, M_t) = 0, \quad (16)$$

$$V(X^*, M_t) = -\frac{X^* M_t}{\eta} - I, \text{ and} \quad (17)$$

$$\frac{\partial V}{\partial X_t}(X^*, M_t) = \frac{\partial W}{\partial X_t}(X^*, M_t), \quad (18)$$

where X^* is the critical value of X at or above which the optimal environmental policy should be adopted. Condition (16) reflects the fact that if X_t is always zero, then the flow of the social cost associated with the stock variable M_t, $C(X_t, M_t)$, is zero. Thus the value of the optimal

[19] In order to derive this equation, we assume that $\partial V/\partial X_t$ is negative in the continuation region, and V is twice differentiable in the continuation region. We can show that these two assumptions actually hold.

environmental policy will remain to be zero. Condition (17) is the value matching condition, and Condition (18) is the smooth pasting condition. [20] By solving the differential equation with the three boundary conditions, we obtain the value of the optimal environmental policy as follows:

$$V(X_t, M_t) = \begin{cases} AX_t^\gamma - \dfrac{X_t M_t}{r+\delta-(\alpha-\sigma\theta_*)} - \dfrac{\beta E_0 X_t}{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))} & \text{if} \quad X_t < X^* \\ W_t - I & \text{if} \quad X_t \geq X^*, \end{cases}$$

where

$$A = \left(\frac{I}{\gamma-1}\right)^{1-\gamma} \gamma^{-\gamma} \left(\frac{\beta E_0}{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))}\right)^{\gamma},$$

$$X^* = \left(\frac{\gamma I}{\gamma-1}\right)\left(\frac{(r-(\alpha-\sigma\theta_*))(r+\delta-(\alpha-\sigma\theta_*))}{\beta E_0}\right), \tag{19}$$

$$\gamma = \frac{-(\alpha-\sigma\theta_*-\sigma^2/2)+\sqrt{(\alpha-\sigma\theta_*-\sigma^2/2)^2+2r\sigma^2}}{\sigma^2}, \tag{20}$$

and γ is the positive part of the solutions to the quadratic equation $(1/2)\sigma^2 x(x-1) + (\alpha - \sigma\theta_*)x - r = 0$.[21] The uniqueness of the reservation value X^*, the twice-differentiability of V, and the negativity of $\partial V/\partial X_t$ follow from this value of the optimal environmental policy.

The value function in the continuation region consists of three components. The first term is the value of the option to adopt the optimal environmental policy. The second term is the present value of the flow of the social cost from the current stock of the pollutant. The third is the present value of the flow of the social cost from emission E_0. The value function in the stopping region consists of two terms. The first term is the value of adopting the environmental policy defined by (13), and the second is the direct cost resulting from adopting the optimal environmental policy.

4. Sensitivity Analyses

In this section, we analyze the effects of an increase in ambiguity on the value of adopting the environmental policy and the optimal timing of adopting the environmental policy, respectively.

[20] For the value matching condition and the smooth pasting condition, see Dixit and Pindyck (1994).

[21] In Appendix B, we show that $\gamma > 1$.

We assume that the government's beliefs are represented by κ-ignorance in order to analyze the effects of ambiguity on the value of adopting the environmental policy and the optimal timing of adopting the environmental policy, respectively.

Under the assumption of κ-ignorance, θ_* defined by (11) is further characterized by

$$\theta_* \equiv \operatorname{argmin}\{\sigma x \,|\, x \in [-\kappa, \kappa]\} = -\kappa.$$

Thus, (13) and (20) turn out to be

$$W(X_t, M_t) = -\frac{X_t M_t}{r + \delta - (\alpha + \sigma\kappa)}$$
$$\gamma = \frac{-(\alpha + \sigma\kappa - \sigma^2/2) + \sqrt{(\alpha + \sigma\kappa - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2},$$

respectively. In the following proposition, we show that an increase in ambiguity *decreases* the value of adopting the environmental policy, and an increase in ambiguity *decreases* the optimal timing of adopting the environmental policy.

Proposition 2. *We assume that the environmental policy once adopted never expires, the planning horizon is infinite, and that ambiguity is characterized by κ-ignorance. Then, an increase in ambiguity decreases the value of adopting the environmental policy W_t, and an increase in ambiguity decreases the optimal timing of adopting the environmental policy X^* if and only if the following condition is satisfied:*

$$\frac{\gamma}{r + \frac{1}{2}\sigma^2\gamma^2} < \frac{(\gamma - 1)(2r + \delta - 2(\alpha + \sigma\kappa))}{(r - (\alpha + \sigma\kappa))(r + \delta - (\alpha + \sigma\kappa))}. \tag{21}$$

Proof. See Appendix B. □

The condition (21) is always satisfied if we set the rate of depreciation δ to zero. This assumption is appropriate in the case that the pollutant M_t represents CFC (chlorofluorocarbon) or mercury that cause severe damage on the environment and do not depreciate at all once they are released into the atmosphere and the ocean. [22]

Corollary 1. *Suppose that $\delta = 0$. Then, an increase in ambiguity always decreases the optimal timing of adopting the environmental policy X^*.*

[22] Pindyck (2002) sets the rate of depreciation δ to zero in order to derive an analytical solution within the framework of risk.

Proof. See Appendix B. □

The first claim in Proposition 2 states that the government's less confidence in the prospect of the economic situation lowers the value of the effect of adopting the environmental policy. Corollary 1 states that the government's less confidence in the prospect of the economic situation will encourage the government to adopt the optimal environmental policy immediately. This result implies that if governments are less confident about climate change in the future and they make decision very cautiously, then environmental policies to avoid global warming should be adopted immediately. Therefore, it can be considered that this result provides an economic foundation for the Rio Declaration and the precautionary principle.

5. Conclusion

This paper analyzes a problem in environmental policy design by applying optimal stopping rules under *ambiguity* in continuous time. Our result shows that an increase in ambiguity decreases the optimal timing of adopting an environmental policy, which implies that the government's less confidence in the prospect of the economic situation will make the government *hasten* the adoption of the optimal environmental policy. Furthermore, our result provides an economic foundation for the Rio Declaration on Environment and Development and the precautionary principle by incorporating ambiguity, irreversibility, and the flexibility in deciding the timing of adopting the environmental policy into a continuous-time model of ambiguity.

This paper are related to Pindyck (2000, 2002) that analyze a problem in environmental policy design by applying optimal stopping rules under *risk* in continuous time. While this paper shows that an increase in ambiguity decreases the value of adopting the environmental policy, Pindyck (2000) shows that an increase in risk does not affect the value of adopting the environmental policy. It is intuitively appropriate that the government's less confidence in the prospect of the economic situation does have any effects on the value of adopting the environmental policy. Moreover, while this paper shows that an increase in ambiguity decreases the optimal timing of adopting the environmental policy, Pindyck (2000) shows that an increase in risk increases the optimal timing of adopting the environmental policy. It is also intuitively appropriate that the government's less confidence in the prospect of the economic situation will *encourage* the government to adopt the optimal environmental policy immediately, contrary to Pindyck (2000) that implies that an increase in risk will make the government *postpone* the adoption of the optimal environmental policy.

Appendix A.

Let $(\Omega, \mathcal{F}_T, P)$ be a probability space, and let $(B_t)_{0\le t\le T}$ be a standard Brownian motion with respect to P. Let $(\mathcal{F}_t)_{0\le t\le T}$ be the standard filtration for a standard Brownian motion $(B_t)_{0\le t\le T}$. Let $\mathcal{L}$ be the set of real-valued, measurable, and $(\mathcal{F}_t)$-adapted stochastic processes on $(\Omega, \mathcal{F}_T, P)$ with an index set $[0,T]$, and let $\mathcal{L}^2$ be a subset of $\mathcal{L}$ that is defined by

$$\mathcal{L}^2 = \left\{ (\theta_t)_{0\le t\le T} \in \mathcal{L} \,\middle|\, \int_0^T \theta_t^2 dt < +\infty \ P\text{-a.s.} \right\}.$$

Given $\theta = (\theta_t) \in \mathcal{L}^2$, define a stochastic process $(z_t^\theta)_{0\le t\le T}$ by

$$(\forall t \in [0,T]) \ \ z_t^\theta = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s \right). \tag{22}$$

By Ito's lemma, we can define $(z_t^\theta)_{0\le t\le T}$ as a unique solution to the stochastic differential equation, $dz_t^\theta = -z_t^\theta \theta_t dB_t$ with $z_0^\theta = 1$.

A stochastic process $(\theta_t) \in \mathcal{L}^2$ is called a *density generator* if (z_t^θ) is $(\mathcal{F}_t)$-martingale. Novikov's condition is one of the sufficient conditions for (z_t^θ) to be $(\mathcal{F}_t)$-martingale. For exapmle, see Karatzas and Shreve (1991).

A set of density generators $\Theta^{K_t(\omega)}$ is *rectangular* if there exists a set-valued stochastic process $(K_t)_{0\le t\le T}$ such that

$$\Theta^{K_t(\omega)} = \left\{ (\theta_t) \in \mathcal{L}^2 \,|\, \theta_t(\omega) \in K_t(\omega) \ \ (m \otimes P)\text{-a.s.} \right\}, \tag{23}$$

and, there exists a nonempty compact subset $\mathcal{K}$ of $\mathbb{R}$ such that for each t, $K_t : \Omega \twoheadrightarrow \mathcal{K}$ is compact-valued and convex-valued, the correspondence $(t,\omega) \twoheadrightarrow K_t(\omega)$, when restricted to $[0,s]\times\Omega$, is $\mathcal{B}([0,s])\otimes\mathcal{F}_s$-measurable for any $0 < s \le T$, and $0 \in K_t(\omega)$ $(m\otimes P)$-a.s., where m is the Lebesgue measure restricted on $\mathcal{B}([0,T])$. Note that *i.i.d.* ambiguity (4) is a special case of rectangularity (23). Nishimura and Ozaki (2007) proves the following lemma.

Lemma 1. *Let $0 \le s \le t \le T$ and let X be an $\mathcal{F}_T$-measurable function. If a set of density generators Θ satisfies the rectangularity, then the following recursive structure holds:*

$$\min_{\theta\in\Theta} E^{Q^\theta}\left[X \,|\, \mathcal{F}_s\right] = \min_{\theta\in\Theta} E^{Q^\theta}\left[E^{Q^\theta}\left[X \,|\, \mathcal{F}_t\right] \,\middle|\, \mathcal{F}_s \right] = \min_{\theta\in\Theta} E^{Q^\theta}\left[\min_{\theta'\in\Theta} E^{Q^{\theta'}}\left[X \,|\, \mathcal{F}_t\right] \,\middle|\, \mathcal{F}_s \right] \tag{24}$$

as long as the minima exist.

Let θ be a density generator and define a probability measure Q^θ on $(\Omega, \mathcal{F}_T)$ by

$$(\forall A \in \mathcal{F}_T) \ \ Q^\theta(A) = \int_\Omega z_T^\theta(\omega)\chi_A(\omega)\, dP(\omega) = E^P[\chi_A z_T^\theta].$$

Note that Q^θ is equivalent to P. Conversely, any probability measures equivalent to P can be obtained by a density generator in this way.

For any $\theta \in \Theta$, define a stochastic process $(B_t^\theta)_{0 \leq t \leq T}$ by

$$(\forall t \in [0, T]) \ \ B_t^\theta = B_t + \int_0^t \theta_s ds.$$

Since (z_t^θ) is $(\mathcal{F}_t)$-martingale, it follows from Girsanov's theorem that it is a standard Brownian motion with respect to $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}_T, Q^\theta)$. By Girsanov's theorem, the stochastic differential equation capturing economic uncertainty over future costs and benefits of polic adoptions turns out to be

$$dX_t = (\alpha - \sigma\theta_t) X_t dt + \sigma X_t dB_t^\theta \tag{25}$$

for any $\theta \in \Theta$. By (25), and by applying Ito's lemma to $\ln N_t$ considering Q^θ as the true probability measure, it follows that

$$(\forall \theta \in \Theta)(\forall t \in [0, T]) \ \ X_t = X_0 \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t - \sigma \int_0^t \theta_s ds + \sigma B_t^\theta\right). \tag{26}$$

Appendix B: Derivations of Mathematical Results

Proof of Proposition 1

The following lemma is based on Lemma 1 in Nishimura and Ozaki (2007).

Lemma 2. *For any $s \geq t$ and for any $\theta \in \Theta^{K_t}$,*

$$\begin{aligned} & E^{Q^\theta}\left[\exp\left(-\int_t^s \sigma\theta_h dh + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \Bigg| \mathcal{F}_t\right] \\ \leq \ & E^{Q^{\theta_*}}\left[\exp\left(-\int_t^s \sigma(\theta_h)_* dh + \sigma\left(B_s^{\theta_*} - B_t^{\theta_*}\right)\right) \Bigg| \mathcal{F}_t\right], \end{aligned}$$

where $\theta_* \equiv \text{argmin}\,\{\sigma x | x \in K\} = \min K$.

Proof. Let $s \geq t$ and let $\theta \in \Theta^{K_t}$. Then, by the definition of $(\theta_t)_*$, it follows that

$$(\forall \omega) \ \ \exp\left(-\int_t^s \sigma\theta_r dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \leq \exp\left(-\int_t^s \sigma(\theta_r)_* dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right).$$

Thus

$$\begin{aligned} & E^{Q^\theta}\left[\exp\left(-\int_t^s \sigma\theta_r dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \Bigg| \mathcal{F}_t\right] \\ \leq \ & E^{Q^\theta}\left[\exp\left(-\int_t^s \sigma(\theta_r)_* dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \Bigg| \mathcal{F}_t\right] \\ = \ & \exp\left(-\int_t^s \sigma(\theta_r)_* dr\right) \exp\left(\frac{1}{2}\sigma^2(s-t)\right) \\ = \ & E^{Q^{\theta_*}}\left[\exp\left(-\int_t^s \sigma(\theta_r)_* dr + \sigma\left(B_s^{\theta_*} - B_t^{\theta_*}\right)\right) \Bigg| \mathcal{F}_t\right], \end{aligned}$$

where the inequality follows from the monotonicity of conditional expectation.[23] □

[23] For example, see Billingsley (1995, p.447).

Proof of Proposition 1. [24]

$$
\begin{aligned}
& W(X_t, M_t, t) \\
= \;& \inf_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T \exp(-r(s-t)) C(X_s, M_s) ds \,\middle|\, \mathcal{F}_t \right] \\
= \;& \inf_{\theta \in \Theta} E^{Q^\theta} \left[-\int_t^T \exp(-r(s-t)) X_s M_s ds \,\middle|\, \mathcal{F}_t \right] \\
= \;& \inf_{\theta \in \Theta} \int_t^T E^{Q^\theta} \left[-X_s M_s \exp(-r(s-t)) \,\middle|\, \mathcal{F}_t \right] ds \\
= \;& \inf_{\theta \in \Theta} -\int_t^T X_t M_s E^{Q^\theta} \left[\exp(-r(s-t)) \exp\left((\alpha - \sigma^2/2)(s-t) - \sigma \int_t^s \theta_h dh + \sigma (B_s^\theta - B_t^\theta) \right) \,\middle|\, \mathcal{F}_t \right] ds \\
= \;& \inf_{\theta \in \Theta} -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t) \right) E^{Q^\theta} \left[\exp\left(-\sigma \int_t^s \theta_h dh + \sigma (B_s^\theta - B_t^\theta) \right) \,\middle|\, \mathcal{F}_t \right] ds \\
= \;& -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t) \right) E^{Q^{\theta_*}} \left[\exp\left(-\sigma \int_t^s (\theta_h)_* dh + \sigma (B_s^{\theta_*} - B_t^{\theta_*}) \right) \,\middle|\, \mathcal{F}_t \right] ds \\
= \;& -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t) - \int_t^s \sigma (\theta_h)_* dh \right) E^{Q^{\theta_*}} \left[\exp \sigma \left(B_s^{\theta_*} - B_t^{\theta_*} \right) \,\middle|\, \mathcal{F}_t \right] ds \\
= \;& -\int_t^T X_t M_s \exp\left((\alpha - \sigma^2/2 - r)(s-t) - \int_t^s \sigma (\theta_h)_* dh \right) \exp\left(\sigma^2 (s-t)/2 \right) ds \\
= \;& -\int_t^T X_t M_s \exp\left(-(r - \alpha)(s-t) - \int_t^s \sigma (\theta_h)_* dh \right) ds \\
= \;& -\int_t^T X_t M_t \exp\left(-\delta (s-t) \right) \exp\left(-(r - \alpha)(s-t) - \int_t^s \sigma (\theta_h)_* dh \right) ds \\
= \;& -\int_t^T X_t M_t \exp\left(-(r + \delta - \alpha)(s-t) - \int_t^s \sigma (\theta_h)_* dh \right) ds,
\end{aligned}
$$

where the second equality holds by (1), the third equality holds by Fubini's theorem for conditional expectation,[25] the fourth equality holds by (26), the sixth equality holds by Lemma 1, the seventh equality follows from the fact that (θ_*) is a degenerate stochastic process, the eighth equality holds by the fact that $(B_t^{\theta_*})$ is a Brownian motion with respect to Q^{θ_*}, and the tenth equality follows since $M_s = M_t \exp(-\delta(s-t))$ for all $s \geq t$. [26] □

Derivation of HJB in case of Ambiguity

$$
\begin{aligned}
& V_t \\
= \;& \max_{t' \in [t,T]} \min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds - e^{-r(t'-t)} I \,\middle|\, \mathcal{F}_t \right] \\
= \;& \max \left\{ \min_{Q \in \mathcal{P}^\Theta} E^Q \left[\int_t^T e^{-r(s-t)} C(X_s, M_s) ds \,\middle|\, \mathcal{F}_t \right] - I, \right.
\end{aligned}
$$

[24] The idea of the proof is based on Nishimura and Ozaki (2007).

[25] See Ethier and Kurtz (1986) and Nishimura and Ozaki (2007) in details.

[26] The solution to $dM_t/dt = \beta E_0 - \delta M_t$ is $M_s = -\mu + (M_t + \mu)\exp(-\delta(s-t))$ for all $s \geq t$, where $\mu \equiv -(\beta/\delta)E_0$. Since we assume $E_0 = 0$ if the environmental policy is adopted, $M_s = M_t \exp(-\delta(s-t))$ for all $s \geq t$.

$$
\begin{aligned}
& -X_tM_tdt + \max_{t'\geq t+dt}\min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_{t+dt}^T e^{-r(s-t)}C(X_s,M_s)ds - e^{-r(t'-t)}I \,\middle|\, \mathcal{F}_t\right]\Bigg\} \\
= & \max\Bigg\{W_t - I, \\
& -X_tM_tdt + \max_{t'\geq t+dt}\min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_{t+dt}^T e^{-r(s-t)}C(X_s,M_s)ds - e^{-r(t'-t)}I \,\middle|\, \mathcal{F}_t\right]\Bigg\} \\
= & \max\Bigg\{W_t - I, \\
& -X_tM_tdt + \max_{t'\geq t+dt}\min_{\theta\in\Theta} E^{Q^\theta}\left[\int_{t+dt}^T e^{-r(s-t)}C(X_s,M_s)ds - e^{-r(t'-t)}I \,\middle|\, \mathcal{F}_t\right]\Bigg\} \\
= & \max\Bigg\{W_t - I,\, -X_tM_tdt \\
& +e^{-rdt}\max_{t'\geq t+dt}\min_{\theta\in\Theta} E^{Q^\theta}\left[E^{Q^\theta}\left[\int_{t+dt}^T e^{-r(s-t-dt)}C(X_s,M_s)ds - e^{-r(t'-t-dt)}I \,\middle|\, \mathcal{F}_{t+dt}\right] \,\middle|\, \mathcal{F}_t\right]\Bigg\} \\
= & \max\Bigg\{W_t - I,\, -X_tM_tdt \\
& +e^{-rdt}\max_{t'\geq t+dt}\min_{\theta\in\Theta} E^{Q^\theta}\left[\min_{\theta'\in\Theta} E^{Q^{\theta'}}\left[\int_{t+dt}^T e^{-r(s-t-dt)}C(X_s,M_s)ds - e^{-r(t'-t-dt)}I \,\middle|\, \mathcal{F}_{t+dt}\right] \,\middle|\, \mathcal{F}_t\right]\Bigg\} \\
= & \max\Bigg\{W_t - I,\, -X_tM_tdt \\
& +e^{-rdt}\min_{\theta\in\Theta} E^{Q^\theta}\left[\max_{t'\geq t+dt}\min_{\theta'\in\Theta} E^{Q^{\theta'}}\left[\int_{t+dt}^T e^{-r(s-t-dt)}C(X_s,M_s)ds - e^{-r(t'-t-dt)}I \,\middle|\, \mathcal{F}_{t+dt}\right] \,\middle|\, \mathcal{F}_t\right]\Bigg\} \\
= & \max\left\{W_t - I,\, -X_tM_tdt + e^{-rdt}\min_{\theta\in\Theta} E^{Q^\theta}\left[V_{t+dt}\middle|\, \mathcal{F}_t\right]\right\} \\
= & \max\left\{W_t - I,\, -X_tM_tdt + (1-rdt)\left(\min_{\theta\in\Theta} E^{Q^\theta}[dV_t|\, \mathcal{F}_t] + V_t\right)\right\} \\
= & \max\left\{W_t - I,\, -X_tM_tdt + \min_{\theta\in\Theta} E^{Q^\theta}[dV_t|\, \mathcal{F}_t] + V_t - rV_tdt\right\},
\end{aligned}
$$

where the first equality follows from the definition of V_t, the third equality follows from (13), the fourth follows from the definition of $\mathcal{P}^\Theta$, the fifth holds by the law of iterated integrals, the sixth follows from the rectangularity (24), the eighth follows from the definition of V_t, the ninth holds by approximating e^{-rdt} by $(1-rdt)$, and the last equality holds by eliminating higher order terms than dt. □

Derivation of V_t under Ambiguity

By Ito's lemma,

$$
\begin{aligned}
dV_t &= \frac{\partial V}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}dX_t^2 + \frac{\partial V}{\partial M_t}dM_t \\
&= \frac{\partial V}{\partial X_t}\left((\alpha-\sigma\theta_t)X_tdt + \sigma X_t dB_t^\theta\right) + \frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2X_t^2dt + \frac{\partial V}{\partial M_t}\left(\beta E_0 - \delta M_t\right)dt.
\end{aligned}
$$

Thus, it follows that

$$\begin{aligned}
& \min_{Q\in\mathcal{P}^\Theta} E^Q\left[dV_t|\mathcal{F}_t\right] \\
= \; & \min_{\theta\in\Theta^K} E^{Q^\theta}\left[dV_t|\mathcal{F}_t\right] \\
= \; & \min_{\theta\in\Theta^K} E^{Q^\theta}\left[\frac{\partial V}{\partial X_t}\left((\alpha-\sigma\theta_t)X_t dt+\sigma X_t dB_t^\theta\right)+\frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2 X_t^2 dt+\frac{\partial V}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt\,\middle|\,\mathcal{F}_t\right] \\
= \; & \min_{\theta\in\Theta^K}\frac{\partial V}{\partial X_t}(\alpha-\sigma\theta_t)X_t dt+\frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2 X_t^2 dt+\frac{\partial V}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt \\
= \; & \frac{\partial V}{\partial X_t}\max_{\theta\in\Theta^K}(\alpha-\sigma\theta_t)X_t dt+\frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2 X_t^2 dt+\frac{\partial V}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt \\
= \; & \frac{\partial V}{\partial X_t}\left(\alpha+\max_{\theta\in\Theta^K}(-\sigma\theta_t)\right)X_t dt+\frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2 X_t^2 dt+\frac{\partial V}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt \\
= \; & \frac{\partial V}{\partial X_t}\left(\alpha-\min_{\theta\in\Theta^K}(\sigma\theta_t)\right)X_t dt+\frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2 X_t^2 dt+\frac{\partial V}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt \\
= \; & \frac{\partial V}{\partial X_t}(\alpha-\sigma\theta_*)X_t dt+\frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}\sigma^2 X_t^2 dt+\frac{\partial V}{\partial M_t}\left(\beta E_0-\delta M_t\right)dt, \qquad (27)
\end{aligned}$$

where the first equality follows from the assumption of *i.i.d.* ambiguity, the fourth equality holds by the negativity of $\partial V/\partial X_t$ in the continuation region, and the last equality follows from the definition of θ_*. Therefore, in the continuation region, it follows form (15) and (27) that

$$\frac{1}{2}\sigma^2 X_t^2\frac{\partial^2 V}{\partial X_t^2}+(\alpha-\sigma\theta_*)X_t\frac{\partial V}{\partial X_t}-rV(X_t,M_t)+\frac{\partial V}{\partial M_t}(\beta E_0-\delta M_t)-X_t M_t=0,$$

with the following boundary conditions:

$$\begin{aligned}
& V(0,M_t)=0, \\
& V(X^*,M_t)=-\frac{X^* M_t}{\eta}-I, \text{ and} \\
& \frac{\partial V}{\partial X_t}(X^*,M_t)=\frac{\partial W}{\partial X_t}(X^*,M_t).
\end{aligned}$$

We guess the solution to this equation as follows:

$$V(X_t,M_t)=AX_t^\gamma+BX_tM_t+DX_t,$$

where A, B and D are some constants. Then

$$\begin{aligned}
& \frac{1}{2}\sigma^2 X_t^2 A\gamma(\gamma-1)X_t^{\gamma-2}+(\alpha-\sigma\theta_*)\left(A\gamma X_t^{\gamma-1}+BM_t+D\right)X_t \\
& -r\left(AX_t^\gamma+BX_tM_t+DX_t\right)-X_tM_t+(\beta E_0-\delta M_t)BX_t=0. \\
\Leftrightarrow \quad & AX_t^\gamma\left(\frac{1}{2}\sigma^2\gamma(\gamma-1)+(\alpha-\sigma\theta_*)\gamma-r\right) \\
& +((\alpha-\sigma\theta_*-r-\delta)B-1)X_tM_t+(\beta BE_0+D(\alpha-\sigma\theta_*-r))X_t=0.
\end{aligned}$$

Thus,

$$\frac{1}{2}\sigma^2\gamma(\gamma-1)+(\alpha-\sigma\theta_*)\gamma-r=0$$

$$(\alpha - \sigma\theta_* - r - \delta)B - 1 = 0 \Leftrightarrow B = -\frac{1}{r + \delta - \alpha + \sigma\theta_*}$$

$$\beta B E_0 + D(\alpha - \sigma\theta_* - r) = 0 \Leftrightarrow D = -\frac{\beta E_0}{(r - (\alpha - \sigma\theta_*))(r + \delta - (\alpha - \sigma\theta_*))}.$$

By the boundary conditions, the negative part of the solution to $(1/2)\sigma^2\gamma(\gamma-1)+(\alpha-\sigma\theta_*)\gamma-r = 0$ is ruled out. Furthermore, it follows that

$$A = \left(\frac{I}{\gamma-1}\right)^{1-\gamma} \gamma^{-\gamma} \left(\frac{\beta E_0}{(r - (\alpha - \sigma\theta_*))(r + \delta - (\alpha - \sigma\theta_*))}\right)^{\gamma}$$

$$X^* = \left(\frac{\gamma I}{\gamma - 1}\right)\left(\frac{(r - (\alpha - \sigma\theta_*))(r + \delta - (\alpha - \sigma\theta_*))}{\beta E_0}\right). \quad \square$$

Proof of $\gamma > 1$.

This claim follows since

$$\begin{aligned}
\gamma &= \frac{-(\alpha - \sigma\theta_* - \sigma^2/2) + \sqrt{(\alpha - \sigma\theta_* - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \\
&> \frac{-(\alpha - \sigma\theta_* - \sigma^2/2) + \sqrt{(\alpha - \sigma\theta_* - \sigma^2/2)^2 + 2(\alpha - \sigma\theta_*)\sigma^2}}{\sigma^2} \\
&= \frac{-(\alpha - \sigma\theta_* - \sigma^2/2) + |\alpha - \sigma\theta_* + \sigma^2/2|}{\sigma^2} = 1, \qquad (28)
\end{aligned}$$

where the inequality holds by the assumption that $r > \alpha + \sigma\kappa = \alpha - \sigma\theta_*$, and the last equality holds since α, $\sigma > 0$ and $\theta_* = -\kappa < 0$, which implies that $|\alpha - \sigma\theta_* + \sigma^2/2| = \alpha - \sigma\theta_* + \sigma^2/2$. $\square$

Proof of $\partial\lambda/\partial\sigma < 0$.

Recall that λ is the positive part of the solutions to the quadratic equation $(1/2)\sigma^2 x(x-1) + \alpha x - r = 0$. Thus, λ satisfies the following:

$$\frac{1}{2}\sigma^2\lambda(\lambda - 1) + \alpha\lambda - r = 0.$$

By differentiating both sides with respect to σ considering λ as a function of σ, it follows that

$$\sigma\lambda(\lambda-1) + \frac{1}{2}\sigma^2(\lambda-1)\frac{\partial\lambda}{\partial\sigma} + \frac{1}{2}\sigma^2\lambda\frac{\partial\lambda}{\partial\sigma} + \alpha\frac{\partial\lambda}{\partial\sigma} = 0,$$

from which we obtain

$$\begin{aligned}
\frac{\partial\lambda}{\partial\sigma} &= \frac{-\sigma\lambda(\lambda-1)}{\frac{1}{2}\sigma^2(\lambda-1) + \frac{1}{2}\sigma^2\lambda + \alpha} \\
&= \frac{-\sigma\lambda^2(\lambda-1)}{\frac{1}{2}\sigma^2\lambda(\lambda-1) + \frac{1}{2}\sigma^2\lambda^2 + \alpha\lambda} \\
&= \frac{-\sigma\lambda^2(\lambda-1)}{r + \frac{1}{2}\sigma^2\lambda^2} < 0,
\end{aligned}$$

where the inequality follows from $\lambda > 1$ and $\sigma > 0$. $\square$

Lemma 3.

$$\frac{\partial\gamma}{\partial\kappa} = \frac{-\sigma\gamma^2}{\frac{1}{2}\sigma^2\gamma^2 + r} < 0.$$

Proof. Recall that γ is the positive part of the solutions to the quadratic equation: [27]

$$\frac{1}{2}\sigma^2 x(x-1) + (\alpha+\sigma\kappa)x - r = 0.$$

Thus, γ satisfies the following:

$$\frac{1}{2}\sigma^2\gamma(\gamma-1) + (\alpha+\sigma\kappa)\gamma - r = 0. \tag{29}$$

By differentiating both sides with respect to κ considering γ as a function of κ, it follows that

$$\frac{1}{2}\sigma^2\frac{\partial\gamma}{\partial\kappa}(\gamma-1) + \frac{1}{2}\sigma^2\gamma\frac{\partial\gamma}{\partial\kappa} + \sigma\gamma + (\alpha+\sigma\kappa)\frac{\partial\gamma}{\partial\kappa} = 0,$$

from which we obtain

$$\begin{aligned}\frac{\partial\gamma}{\partial\kappa} &= \frac{-\sigma\gamma}{\frac{1}{2}\sigma^2(\gamma-1) + \frac{1}{2}\sigma^2\gamma + \alpha + \sigma\kappa} \\ &= \frac{-\sigma\gamma^2}{\frac{1}{2}\sigma^2\gamma(\gamma-1) + \frac{1}{2}\sigma^2\gamma^2 + (\alpha+\sigma\kappa)\gamma} \\ &= \frac{-\sigma\gamma^2}{\frac{1}{2}\sigma^2\gamma^2 + r} < 0.\end{aligned}$$

□

Proof of Proposition 2

It follows from (13) that $\partial W/\partial\kappa < 0$. Next, differentiating (19) with respect to κ leads to

$$\begin{aligned}&\frac{\partial X^*}{\partial\kappa} \\ =\;& -\frac{I}{(\gamma-1)^2}\left(\frac{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))}{\beta E_0}\right)\frac{\partial\gamma}{\partial\kappa} \\ &+\left(\frac{\gamma I}{\gamma-1}\right)\left(\frac{-\sigma(r+\delta-(\alpha+\sigma\kappa)) - \sigma(r-(\alpha+\sigma\kappa))}{\beta E_0}\right) \\ =\;& -\frac{I}{\beta E_0(\gamma-1)}\left(\frac{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))}{\gamma-1}\frac{\partial\gamma}{\partial\kappa} + \sigma\gamma\left(2r+\delta-2(\alpha+\sigma\kappa)\right)\right).\end{aligned}$$

Thus, $\partial X^*/\partial\kappa < 0$ if and only if

$$\frac{\partial\gamma}{\partial\kappa} > -\frac{\sigma\gamma(\gamma-1)(2r+\delta-2(\alpha+\sigma\kappa))}{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))}.$$

Lemma 2, $\sigma > 0$ and $\gamma > 1$ imply that $\partial X^*/\partial\kappa$ if and only if

$$\frac{\gamma}{r+\frac{1}{2}\sigma^2\gamma^2} < \frac{(\gamma-1)(2r+\delta-2(\alpha+\sigma\kappa))}{(r-(\alpha+\sigma\kappa))(r+\delta-(\alpha+\sigma\kappa))}. \quad \square$$

[27] We are grateful to Hiroyuki Ozaki for his suggestion on this proof.

Proof of Corollary 1

It suffices to show that

$$\frac{\gamma}{r+\frac{1}{2}\sigma^2\gamma^2} < \frac{2(\gamma-1)}{r-(\alpha+\sigma\kappa)}.$$

It follows that

$$\begin{aligned}
& \frac{1}{2}\sigma^2\gamma^2(\gamma-1) - r > \frac{1}{2}\sigma^2\gamma(\gamma-1) - r \\
\Rightarrow\quad & \frac{1}{2}\sigma^2\gamma^2(\gamma-1) - r > -\gamma(\alpha+\sigma\kappa) \\
\Rightarrow\quad & \frac{1}{2}\sigma^2\gamma^2(\gamma-1) + r(\gamma-1) > r\gamma - \gamma(\alpha+\sigma\kappa) \\
\Rightarrow\quad & (\gamma-1)\left(r+\frac{1}{2}\sigma^2\gamma^2\right) > \gamma(r-(\alpha+\sigma\kappa)) > \frac{1}{2}\gamma(r-(\alpha+\sigma\kappa)),
\end{aligned}$$

where the first inequality holds by $\gamma > 1$, and the second holds by (29). Since $r > \alpha + \sigma\kappa$ and $r + (1/2)\sigma^2\gamma^2 > 0$, it follows that

$$\frac{\gamma}{r+\frac{1}{2}\sigma^2\gamma^2} < \frac{2(\gamma-1)}{r-(\alpha+\sigma\kappa)}. \quad \square$$

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