An Equilibrium-Econometric Analysis of Rental Housing Markets with Indivisibilities*

Mamoru Kaneko and Tamon Ito†

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Abstract

We develop an equilibrium-econometric analysis in the context of rental housing markets with indivisibilities. The theory provides some bridge between a (competitive) market equilibrium theory and a statistical/econometric analysis. First, we develop this theory: The listing service of apartments, which we call the housing magazine, provides the information to both households (and landlords) and the econometric analyzer. Our theory explains this double use of the information sources. We apply our theory to the data in the rental housing markets in the Tokyo area, and we examine the law of diminishing marginal utility for the household. It does not hold at a significant degree for the marginal utility with respect to the size of apartment, but it does strictly with respect to the commuting time-distance and consumption other than the housing services.

Key-Words: Rental Housing Market, Indivisibilities, Competitive Equilibrium, Total Sum of Square Residuals, Law of Diminishing Marginal Utility

1. Introduction

1.1. Basic Motivations

We develop an equilibrium-econometric analysis in the context of rental housing markets, and test it with some data in the rental housing markets in the Tokyo area. Our analysis has the following salient features:

(i): An econometric method is developed directly through a market (equilibrium) theory;

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†Institute of Policy and Planning Sciences, University of Tsukuba, Ibaraki 305-8573, Japan. e-mail: (M: Kaneko) kaneko@sk.tsukuba.ac.jp and (T. Ito) taito@sk.tsukuba.ac.jp

(ii): We take statistical components into account both for economic agents in models and the econometric analyzer, which explains the source of error erms. Then we show that a market theory with the elimination of statistical components is regarded as an idealization and can be used for an econometric analysis;

(iii): We define the measure of the discrepancy between the prediction by the theory and the best statistical estimator, and show in the example of the Tokyo area that the prediction could be regarded as more than satisfactory.

Feature (ii) tells why and how we can use an market theory for an econometric analysis. Feature (iii) is a requirement from econometrics. Here, focussing on these features, we discuss our motivations and background of our research.

One fundamental problem involved in an application of a market theory to real economic problems with some econometric method is: What is the source of an error term in the econometric analysis? This question is often ignored, or is answered in the same way as classical statistics: The source is attributed to the partial observations of the econometric analyzer, which is regarded as the kind of measurement errors[1]. On the other hand, in some markets such as rental housing markets, economic agents (households/landlords) may be, at the time of decision making, in the same informational situation with the econometric analyzer. In this case, error terms represent the effects of variables not included in available information to either to economic agents or the econometric analyzer.

We find a nice example in the Tokyo area for the above mentioned situation. In the Tokyo area, the rental housing market is held, day by day, in a highly decentralized manner, i.e., many households (demanders) and many landlords (suppliers) look for better opportunities in places to places[2]. Various weekly magazines, daily news papers and internet services for listing apartments for rents are available as mediums for information transmission of supplied units together with rental prices from suppliers to demanders[3]. With the help of those mediums, the rental housing markets function well, even though prices are not uniform over the "same" category of apartment units. We will call these mediums simply *the housing magazine*.

The housing magazine gives *concise and coarse* information about each listed apartment unit, following a fixed number of criteria, price, size, location, age, geography etc. The number of listed units is large, e.g., $100 - 1,200$ listed units around one railway

[1] In the logit model and its generalizations, error terms are inlcuded in the utility functions of economic agents (e.g., Berry [3]). These may look similar to ours, since they deal also with discrete choices. But in our approach, as described now, error-terms are not included in utility function.

[2] In the city of Tokyo (about 12 millions of residents), the percentage of households renting apartments is about 55% in 2005, and in the entire Japan, this number is about 37%.

[3] There are many still decentralized real estate agents. In our analysis, we do not explicitly count real-estate companies. But we should remember that behind the market description, many real-estate companies are included.

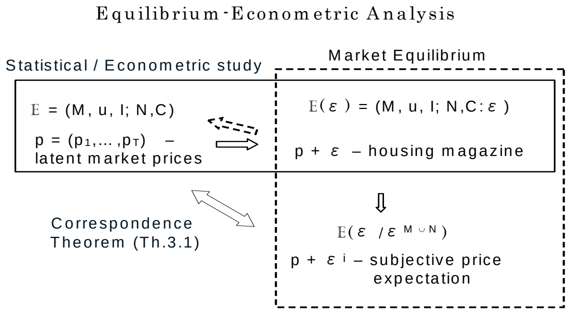

Figure 1.1: Two Faces of the Equi-Economeric Analysis

station are found in our example, which will be given in Section 2.2. This large number has various implications. One is that the information about each unit is concise and coarse, but is far from the description of its full characteristics[4].

While data of rental housing prices show that they are non-uniform over the "same" apartment units, the market can be regarded as a "perfectly competitive market". These statements may appear contradictory, but can be reconciled, which is Feature (ii).

Feature (iii) is about more details of our development. Hence, we will describe more details of our equilibrium-econometric analysis, and then consider Feature (iii).

1.2. Development of the Equilibrium-Econometric Analysis

We treat the attributes listed in the magazine as *systematic* components and the others as *non-systematic* components. We assume that the systematic components are described by a market equilibrium theory, and that the other part is described as error terms. We follow the tradition of econometrics for this division. In our actual numerical treatment, we will choose only two main variables as systematic components.

As the description of the systematic part, we adopt the theory of *assignment markets*, which has come from the tradition from Böhm-Bawerk [22], von Neumann-Morgenstern [25] and Shapley-Shubik [21]. It is a significant difference from the urban economics literature of bid-rent theory from Alonso [1] that housings are treated as indivisible commodities.[5]. More specifically, we adopt a theoretical model given by Kaneko [8][6],

[4] A weekly housing magazine typically has about 500 pages listing 18 units in each page, but is still quite partial in the great Tokyo metoropolitan area.

[5] See Laan-Talman-Yang [15] and its references for recent papers for the literature of assingment markets. For a recent survery on the urban economics literature from Alonso [1], see Arnott [2]

[6] The general theory was also discussed in Kaneko-Yamamoto [13], and is applied to the rental housig

which differs from the formers in that the assumption of no income effects is removed. In the model, apartments units are classified into a finite number, T, of *categories*. Apartment units are traded for rent payments measured by the composite commodity. In this manner, the housing market can be expressed as a market model with indivisible goods.

The *systematic* part of the housing market is summarized as:

$$\mathbb{E} = (M, u, I; N, C), \tag{1.1}$$

where M is the set of households, u the vector of utility functions, I the income distribution for households, and N is the set of landlords, C the vector of cost functions for landlords. The details of (1.1) and the market equilibrium theory will be given in Section 2. In this systematic part, the market price is uniform over each category of apartments.

In the housing magazine, the prices listed are non-uniform over each category of apartments. Those non-uniform prices are resulted by *nonsystematic components* other than the components listed in (1.1). We assume that the effects of non-systematic components are summarized by one random variable $\boldsymbol{\epsilon}_k$ for each category k. That is, the apartment rent for a unit d in category k is determined by $p_k + \boldsymbol{\epsilon}_{kd}$, where p_k is the competitive rent price for category k and $\boldsymbol{\epsilon}_{kd}$ is an independent random variable identical to $\boldsymbol{\epsilon}_k$. Since only $p_k + \boldsymbol{\epsilon}_{kd}$ is observed in the housing magazine, we regard the price p_k as *latent*. The market model with the housing magazine is denoted by $\mathbb{E}(\boldsymbol{\epsilon}) = (M, u, I; N, C; \boldsymbol{\epsilon})$.

From our research point of view, the housing market model $\mathbb{E}(\boldsymbol{\epsilon})$ has two faces:

(1): It is purely the trading place with the mediums for information transmissions.

(2): It is a target of a study of the econometric analyzer.

These faces are described by the dotted square and the solid rectangle in Figure 1.1. In both faces, the housing magazine serves information about rental prices to the households/landlord and the econometric analyzer. However, these two faces are asymmetric.

In (1), households and landlords look at the magazine, while in (2), the econometric analyzer studies the housing magazine. Since the housing magazine lists a very large number of apartment units, households (landlords) behave entirely differently from the econometric analyzer. Typically, each household (landlord) takes a price distribution from the housing magazine in his *subjective* manner. If a household looks at the average of the prices of randomly taken 10 apartment units from one category, its variant becomes 1/10 of the original distribution.

Thus, a household's (landlord's) limited cognitive ability leads to a price distribution with a much smaller variant. Hence, the uniform price assumption for each category

market model in Kaneko [9] and Kaneko-Ito-Osawa [12]. Certain results in [9] and [12] will be used in this paper.

seems to be an approximation. Formally, this interpretation will be expressed by the correspondence theorem (Theorem 3.1) taking the form of a limit theorem. Here, we should interpret a limit as meaning just relatively large number. In our example, perhaps, 10 or 20 units are enough.

Once the correspondence theorem is obtained, we can use a housing market model $\mathbb{E}$ without error terms as representing a market structure. Then we can conduct an econometric study by focusing on the solid rectangle of Figure 1.

Now, we are in a state to discuss our econometric problem. We formulate it in the following manner: Let Γ be a class of market models $\mathbb{E}$ of the form (1.1). Each model $\mathbb{E}$ in Γ has a *competitive equilibrium* (p, x, y), where $p = (p_1, ..., p_T)$ is a rental price vector for categories $1, ..., T$ and (x, y) is an allocation. Then the total sum of square residuals $T_R(P_D, p)$ from $p = (p_1, ..., p_T)$ to the data P_D given by the housing magazine is defined: Then we have the following problem:

$$\begin{array}{l}\text{Minimize } T_R(P_D, p) \text{ subject to } \mathbb{E} \text{ in } \Gamma \text{ and that } (p, x, y) \text{ is} \\ \text{a competitive equilibrium in } \mathbb{E} \text{ compatible with the data.}\end{array} \tag{1.2}$$

This will be formulated in Section 4.

We need still to consider two additional problems:

(A): discrepancy η between the predicted rent vector and the data;

(B): choice of a set of market models Γ.

A discrepancy measure η is developed in Section 4. In our application to the data in Tokyo, the value of the measure will be shown to be $1.025 \sim 1.032$, i.e., only $2.5\% \sim 3.2\%$ of the optimal estimator, by specifying certain classes of market models with homogeneous utility functions.

As an application, we examine the law of diminishing marginal utility for the household. It does not hold at a significant degree for the marginal utility with respect to the size of apartment, but it does strictly with respect to the commuting time-distance and consumption other than the housing services, especially, the degree for consumption is very large. This estimation result is in strong contrast with a result in the hedonic price approach, e.g., Kanemoto-Nakamura **??**. This will be discussed in Section 5.

For problem (B), we will consider two classes of market models. We will show the *Ex Post Rationalization Theorem* in Section 6 that by a choice of Γ that by choosing a set of market models Γ in a slightly general manner, we could make the value of the discrepancy measure to be exactly 1. However, this has no prediction power. The class Γ for this theorem may be regarded as very restrictive from the viewpoint of mathematical economics or game theory, but the theorem itself implies that Γ is too large to make the estimation problem nontrivial. It may be compared with the fact that in linear regression, if the class of linear functions is generalized into the class of piecewise linear functions, it can "explain" the data with the 100% accuracy, but it has no "prediction power". It is simply an *ex post* rationalization. For this reason, a restrictive choice of Γ is important.

We will also consider the status of the standard linear regression in our equilibrium-econometric analysis. The answer is simple: When the households have the common linear utility functions with respect to attributes of housing and consumption, our econometric analysis becomes linear regression.

The paper is organized as follows: In Section 2, the market equilibrium theory of Kaneko [8] will be given, and the example from the Tokyo area will be given. In Section 3, a market equilibrium theory with perturbed prices will be given. In Section 4, statistical/econometric treatments will be developed as well as a measure of discrepancy will be defined. In Section 5, we apply those concepts to a data set in the Tokyo metropolitan area. In Section 6, we will consider two classes of utility functions. Section 8 gives conclusions and concluding remarks.

2. Equilibrium Theory of Rental Housing Markets

In Section 2.1, we describe the market structure $\mathbb{E}$ of (1.1), and state the existence results of a competitive equilibrium in $\mathbb{E}$ due to Kaneko [8]. In Section 2.2, we describe a rental housing market in the Tokyo area.

2.1. Basic Theory: the Assignment Market

In the *rental housing market model without error terms* $\mathbb{E} = (M, u, I; N, C)$, various kinds of apartments are traded with the composite commodity called *money* for some fixed length of time period. The components of $\mathbb{E}$ are as follows:

M1: $M = \{1, ..., m\}$- the set of *households*, and each $i \in M$ has a utility function u_i and an income $I_i > 0$ measured by money;

M2: $N = \{1, ..., T\}$ - the set of *landlords* and each $k \in N$ has a cost function C_k.

Each $i \in M$ looks for (at most) one unit of an apartment, and each k in N supplies some units of apartments to the market. The apartments are classified into *categories*, $1, ..., T$. These categories of apartments are interpreted as potentially supplied. Several units in one category of apartments may be on the market. When no confusion is expected, we use the word "apartment" for either one unit or a category of apartments.

The emphasis of the model $\mathbb{E}$ is rather on the households and their behavior. We simplify the descriptions of landlords: As far as competitive equilibrium is concerned, we can assume without loss of generality that only one landlord k provides apartments of category k (see Kaneko *et al* [12], p.146).

Each household $i \in M$ chooses a consumption bundle from the consumption set $X := \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\} \times \mathbf{R}_+$, where $\mathbf{e}^k$ is the unit T-vector with its k-th component 1 for

$k = 1, ..., T$ and $\mathbf{R}_+$ is the set of nonnegative real numbers. We may write $\mathbf{e}^0$ for $\mathbf{0}$. A typical element $(\mathbf{e}^k, m_i)$ means that household i rents one unit from the k-th category and enjoys the consumption $m_i = I_i - p_k$ after paying the rent p_k for $\mathbf{e}^k$ from his *income* $I_i > 0$. The zero vector $\mathbf{e}^0 = \mathbf{0}$ means that he does not rent any apartment in this market $\mathbb{E}$.

The *initial endowment* of each household $i \in M$ is given as $(\mathbf{0}, I_i)$ with $I_i > 0$. His *utility function* $u_i : X \to \mathbf{R}$ is assumed to satisfy:

Assumption A (Continuity and Monotonicity): For each $x \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$, $u_i(x_i, m_i)$ is a continuous and strictly monotone function of m_i and $u_i(\mathbf{0}, I_i) > u_i(\mathbf{e}^k, 0)$ for $k = 1, ..., T$.

The last inequality, $u_i(\mathbf{0}, I_i) > u_i(\mathbf{e}^k, 0)$, means that going out of the market is preferred to renting an apartment by paying all his income.

The set of landlords is given as $N = \{1, ..., T\}$, where only one landlord k provides apartments of category k $(k = 1, ..., T)$. Each landlord k has a cost function $C_k(y_k) : \mathbf{Z}^*_+ \to \mathbf{R}_+$ with $C_k(0) = 0 < C_k(1)$, where $\mathbf{Z}^*_+ = \{0, 1, ..., z^*\}$ and z^* is an integer greater than the number of households m. The cost of providing y_k units is $C_k(y_k)$. No fixed costs are required when no units are provided to the market.[7] The finiteness of $\mathbf{Z}^*_+$ will be used only in Theorem 3.1.

We impose the following on the cost functions:

Assumption B (Convexity): For each landlord $k \in N$,

$$C_k(y_k + 1) - C_k(y_k) \leq C_k(y_k + 2) - C_k(y_k + 1) \text{ for all } y_k \in \mathbf{Z}^*_+.$$

This is a discrete version of the standard convexity assumption on a cost function, and means that the marginal cost of providing an additional unit is increasing.

We write the set of all economic models $\mathbb{E} = (M, u, I; N, C)$ satisfying Assumptions A and B by Γ_0. For an econometric analysis, we will choose some subclass of Γ_0, which is crucial, as mentioned in Section 1.

Now, we define a competitive equilibrium in a market model $\mathbb{E} = (M, u, I; N, C)$. Let $(p, x, y) = ((p_1, ..., p_T), (x_1, ..., x_m), (y_1, ..., y_T))$ be a triple of $p \in \mathbf{R}^T_+$, $x \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}^m$ and $y \in (\mathbf{Z}^*_+)^T$. We say that (p, x, y) is a *competitive equilibrium* in $\mathbb{E}$ iff

(Utility Maximization Under the Budget Constraint): for all $i \in M$,

(1): $I_i - px_i \geq 0$;

(2): $u_i(x_i, I_i - px_i) \geq u_i(x'_i, I_i - px'_i)$ for all $x'_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$ with $I_i - px'_i \geq 0$;

(Profit Maximization): for all $k \in N$,

[7] The cost functions here should not be interpreted as measuring costs for building new apartments. In our rental housing market, the apartment units are already built and fixed. Therefore, $C_j(y_j)$ is the valuation of apartment units y_j below which he is not willing to rent y_j unit for the contract period. This will be clearer in the numerical example in Section 2.3.

$p_k y_k - C_k(y_k) \geq p_k y'_k - C_k(y'_k)$ for all $y'_k \in \mathbf{Z}^*_+$;

(Balance of the Total Demand and Supply): $\sum_{i \in M} x_i = \sum_{k=1}^{T} y_k \mathbf{e}^k$.

Note $px_i := \sum_{k=1}^{T} p_k x_{ik}$. These conditions constitute the standard notion of competitive equilibrium.

The above housing market model is a special case of Kaneko [8] and Kaneko-Yamamoto [13], where the existence of a competitive equilibrium is proved. Therefore, we have the following.

Theorem 2.1 (Existence): In each $\mathbb{E} = (M, u, I;\ N, C)$ in Γ_0, there is a competitive equilibrium (p, x, y).

The purpose of this paper is to study not general properties of equilibrium in $\mathbb{E} = (M, u, I; N, C)$ but more specific behavior of a competitive rent (price) vector. For this reason, it would be convenient to choose one specific competitive price vector. It is possible to choose a unique maximal competitive rent vector, though a corresponding competitive allocation may not be unique.

We say that $p = (p_1, ..., p_T)$ is a *competitive rent vector* iff (p, x, y) is a competitive equilibrium for some $x \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}^m$ and $y \in (\mathbf{Z}^*_+)^T$. Also, we say that $p = (p_1, ..., p_T)$ is a *maximal competitive rent vector* iff $p \geq p'$ for any competitive rent vector p'. By definition, *a maximal competitive rent vector would be unique* if it ever exists. In a standard equilibrium model, the existence of such a price vector cannot be expected. However, we have the existence of a maximal competitive rent vector in $\mathbb{E} = (M, u, I; N, C)$. This fact has been known in slightly different models since the pioneering work of Shapley-Shubik [21] and Gale-Shapley [5].[8] Also, see Miyake [17].

Theorem 2.2 (Existence of a Maximal Competitive Rent Vector). There is a maximal competitive price vector in each $\mathbb{E} = (M, u, I; N, C)$ in Γ_0.

It is also shown in the same manner that a minimal competitive rent vector exists. Although the subsequent analysis holds for either maximal or minimal competitive rent vector, we focus on the maximal one. In the following, we will use the following function ψ defined over Γ_0 assigning the maximal competitive rent vector p to each each $\mathbb{E} = (M, u, I; N, C)$ in Γ_0.

[8] The models of these papers as well as that of the present paper belong to the literature of assignment markets (see Roth-Sotomayor [20] for an extensive survey). In this literature, it is shown that the core of a two-sided assignment market has the specific geometric structure that the core has the maximal and minimal payoff vectors for one side of players. Also, it holds that the core is equivalent to the set of competitive allocations in a typical assignment market, though this equivalence requires each landlord to produce at most one unit in the model of the present paper (see Kaneko [8]). We can expect the existence of a maximal rent vector from these facts. Our housing market keeps this property for the competitive rent vectors.

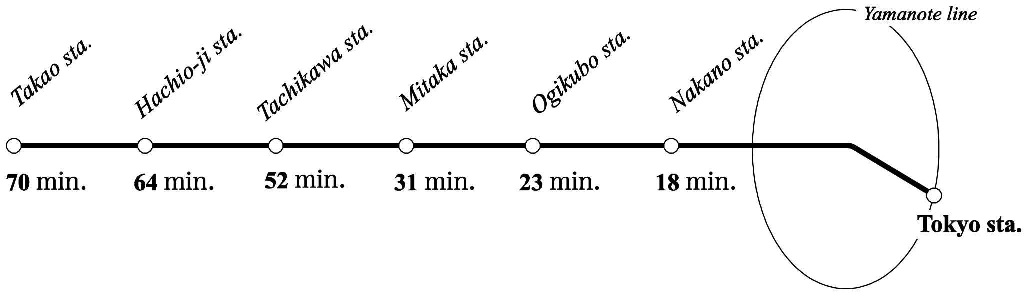

Figure 2.1: The Chuo Line

2.2. Application to a Rental Housing Market in Tokyo (1)

We exemplify the theory described above in a rental housing market in Tokyo. This example will be continued in Section 5.

Consider the JR (Japan Railway) Chuo line running from Tokyo station to the west direction, along which residential areas are spread out. See Figure 2.1. The line has 30 stations from Tokyo station up to Takao station, which is almost on the west boundary of the Tokyo great metropolitan area. Here, we consider only a submarket of the entire market: We take six stations and three types of sizes for apartments. We explain how we formulate this submarket by means of a housing market model $\mathbb{E} = (M, u, I; N, C)$ given in Section 2.1.

Look at Table 2.1: The first column shows the time distance measured by minutes from Tokyo station to each station, i.e., 18, 23, 31, 52, 64 and 70 (minutes). It is assumed that people commute to Tokyo station (office area) from their apartments. The first raw designates the sizes of apartments, and the three intervals are represented by the medians 15, 35 and 55 (m^2) when we plug them into a utility function.[9] Thus, the apartments are classified into $T = 6 \times 3 = 18$ categories.

We assume that the households have the common *basic utility function* as follows:

$$U^0(t, s, m_i) = -2.2t + 4.0s + 100\sqrt{m_i}, \tag{2.1}$$

where t takes possible values 18, 23, 31, 52, 64, 70 and s takes values, 15, 35, 55. A pair (t, s) determines a category. By calculating the first part $-2.2t + 4.0s$ of $U(t, s, c)$, we obtain h_k for the corresponding cell of Table 2.1. These h_k's give the ordering over the 18 categories: For example, $-2.2t + 4.0s$ takes the largest value at $(t, s) = (18, 55)$; we label $k = 1$ to the category of $(t, s) = (18, 55)$. Similarly, $-2.2t + 4.0s$ takes the 7-th

[9] The first interval is taken from 5 to 25. In the Japanese standard, one-room apartments are categolized into this class.

value at $(t,s) = (64,55)$, and thus $k = 7$. Those labels are denoted by the first numbers k of those cells. Thus, we have the correspondence λ between the characteristics (t,s) and category k.

Based on this correspondence λ, we can define the utility function $u : X = \{\mathbf{e}^0, \mathbf{e}^1, ..., \mathbf{e}^{18}\} \times \mathbf{R}_+ \to \mathbf{R}$ as follows:

$$u(\mathbf{e}^k, m_i) = h_k + 100\sqrt{m_i}, \tag{2.2}$$

where $\lambda(t,s) = k$ and $h_k = -2.2t + 4.0s$ for $k \geq 1$ and h_0 is chosen so that $h_0 + 100\sqrt{I_m} > h_1$.

Table 2.1: | k | h_k | w_k |

time (min) \ size (m^2)	< 25			$25-45$			$45-65$		
18:Nakano	11	20.4	1176	5	100.4	761	1	180.4	269
23:Ogikubo	12	9.4	1153	6	89.4	739	2	169.4	367
31:Mitaka	14	-8.2	716	8	71.8	571	3	151.8	267
52:Tachikawa	16	-54.4	460	10	25.6	283	4	105.6	260
64:Hachio-ji	17	-80.8	1095	13	-0.8	346	7	79.2	184
70:Takao	18	-94.0	103	15	-14.0	105	9	66.0	102

The derived utility function in (2.2) satisfies Assumption A. The concavity of $100\sqrt{m_i}$ expresses the law of diminishing marginal utility of consumption, which will be discussed in Section 5.

The third entry w_k of category k in Table 2.1 is the number of units listed for sale in the housing magazine; particularly, *the Yahoo Real Estate* (15, June 2005). The largest number of supplied units is $w_{11} = 1176$ for the smallest apartments in the Nakano area, and the smallest number is $w_9 = 102$ for the largest apartments in the Takao area. The total number of apartment units on the market is $\sum_{k=1}^{18} w_k = 8957$. These large numbers will be important for statistical treatments in later sections.

We *assume* that the same number of households are coming to the market to look for apartments and they rent all the units in this week. Thus, the number of households m is also the same as 8957.

Suppose that only those units appear in the market and no units are newly built. For the purpose of determination of a competitive equilibrium, we separate the cost functions for $k = 1, ..., T-1$ and $k = T$. For $k = 1, ..., T-1$, we formulate the cost function $C_k(y_k)$ as:

$$C_k(y_k) = \begin{cases} c_k y_k & \text{if } y_k \leq w_k \\ \text{“large”} & \text{if } y_k > w_k, \end{cases} \tag{2.3}$$

where $c_k > 0$ for $k = 1, ..., T-1$ and "large" is a number greater than I_1. Thus, only the supplied units are in the scope of cost functions. In the case of $k = T$, we assume that more units are waiting for the market. Let w_T^0 be an integer with $w_T^0 > w_T$. We define $C_T(y_T)$ by:

$$C_T(y_T) = \begin{cases} c_T y_T & \text{if } y_T \leq w_T^0 \\ \text{"large"} & \text{if } y_T > w_T^0, \end{cases} \tag{2.4}$$

where $c_T > 0$. Hence, the market price for an apartment in category T must be c_T.

Landlord k has the reservation price c_k for all units he provides, but the cost to build a new unit is too large relative to this housing market. This satisfies Assumption B. For the calculation of the maximal competitive rent vector, let $c_{18} = 48.0$ and $c_1, ..., c_{17}$ are "small" in the sense that all the $w_k^0 = w_k$ units are supplied at the competitive prices for $k = 1, ..., 17$.

The remaining element of the housing market model $\mathbb{E} = (M, u, I; N, C)$ is the incomes for the households. We assume that the (monthly) income distribution over $M = \{1, ..., m\} = \{1, ..., 8957\}$ is uniform from 100,000 yen to 850,000 yen. Hence, $I_{8957} = 100,000$ and $I_1 = 850,000$. In fact, this uniform distribution is just for the purpose of calculation, and can be changed into other distributions.[10]

Under the above specification of the housing market model $\mathbb{E} = (M, u, I; N, C)$, we can calculate the maximal competitive rent vector $p = (p_1, ..., p_T)$, which is given in Table 2.2. The average rents $\overline{p} = (\overline{p}_1, ..., \overline{p}_{18})$ as well as the standard deviation $(s_1, ..., s_T)$ from the data of *the Yahoo Real Estate* are given. Figure 2.2 depicts $p = (p_1, ..., p_T)$ as well as $\overline{p} = (\overline{p}_1, ..., \overline{p}_{18})$.

Table 2.2: | k | p_k | $\overline{p}_k$ | s_k | (1,000yen)

time (min) \ **size** (m^2)	< 25				$25 - 45$				$45 - 65$			
18:Nakano	11	78.5	74.4	12.7	5	113.9	112.5	23.8	1	154.8	162.7	26.7
23:Ogikubo	12	74.3	75.8	13.6	6	108.6	107.0	23.1	2	149.0	146.2	20.9
31:Mitaka	14	68.7	68.9	9.8	8	110.6	102.1	21.2	3	140.0	143.1	21.6
52:Tachikawa	16	56.4	59.8	11.0	10	80.7	78.1	12.5	4	116.6	116.0	16.5
64:Hachio-ji	17	50.0	51.5	7.5	13	71.0	73.3	11.3	7	104.0	103.5	17.9
70:Takao	18	48.0	46.4	5.9	15	67.2	65.1	9.6	9	98.1	86.1	11.3

To consider how much the calculated price vector $p = (p_1, ..., p_T)$ fits with the data from the housing magazine, we will define the discrepancy measure in Section 4.1. In

[10] At this stage, the result is not sensitive with the uniform distribution assumption, i.e., if we change it to a truncated normal destitution, the calculated rents are not much changed. However, in the later calculation in Section 5, a change of this assumption seems to affect the result ($\eta = 1.025$).

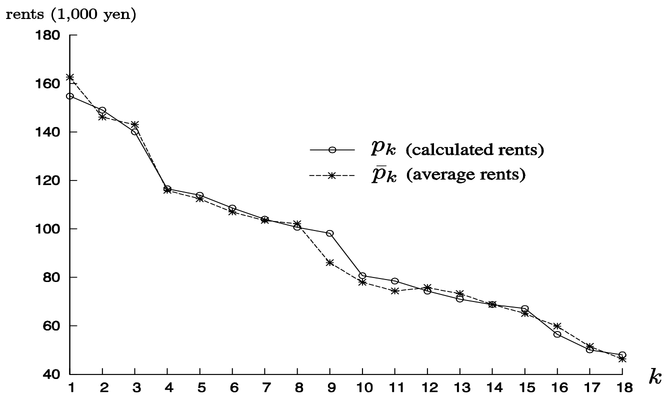

Figure 2.2: The Computed Prices and Average Prices

the present data set from the *Yahoo Real Estate*, the measure is calculated as follows:

$$\frac{\sum_{k=1}^{18}\sum_{d\in D_k}(P_{kd}(\omega)-p_k)^2}{\sum_{k=1}^{18}\sum_{d\in D_k}(P_{kd}(\omega)-\overline{p}_k)^2}=\frac{2466003.250}{2389029.280}\doteqdot 1.032. \tag{2.5}$$

where $D_1, ..., D_{18}$ are the sets of apartment units for categories $k = 1, ..., 18$, e.g., D_1 is the set of largest apartment units in the Nakano area, and $P_{kd}(\omega)$ is the listed price of unit d in category k in the data. Here, ω is the primitive event determining the (random) error terms (see Section 4). That is, the sum of square errors from the predicted prices is divided by the sum of square errors from the average prices. We will argue in Section 4 that this ratio (2.5) has some specific meaning. Here, we emphasize that this ratio is already close to 1.

3. Rental Housing Markets with the Housing Magazine

In the competitive equilibrium in $\mathbb{E} = (M, u, I; N, C)$, all the apartment units in each category are uniformly priced. In reality, however, the prices for apartments in a category are not uniform. As mentioned in Section 1, this non-uniformness may be interpreted as the effects of non-systematic factors, which households are confronting. Here, we modify a housing market model by taking non-systematic factors into account. Then we show that the market model $\mathbb{E}$ can be regarded as an idealization and be used as an analytic tool for the rental housing markets with non-uniform rents.

3.1. Time Structure of the Rental Housing Market

First, we note that our approach is a snap-shot equilibrium theory. Thus, the time index would be unnecessary. However, it would be easier first to describe the economy with the time structure for the consideration of decision making and trades with the use of the housing magazine. Only for this explanation, we will use the time index.

The market situation is recurrent and is described using the "week" due to Hick [6] in Fig.3.1:

$$\begin{array}{ccccccc} \longrightarrow & | & \mathbb{E}^{t-1}(\boldsymbol{\epsilon}^{t-1}) & | & \mathbb{E}^{t}(\boldsymbol{\epsilon}^{t}) & | & \longrightarrow \\ \text{week } t-2 & & \text{week } t-1 & & \text{week } t & & \text{week } t+1 \end{array}$$

Fig.3.1

In week t, market $\mathbb{E}^t(\boldsymbol{\epsilon}^t) = (M^t, u^t, I^t; N^t, C^t; \boldsymbol{\epsilon}^t)$ has a perturbation term $\boldsymbol{\epsilon}^t = (\boldsymbol{\epsilon}^t_1, ..., \boldsymbol{\epsilon}^t_T)$ as the summary of non-systematic components to the systematic part already specified in $\mathbb{E}^t = (M^t, u^t, I^t; N^t, C^t)$.

In reality, interactions between information by the housing magazine and decision making by households and landlords have a complex temporal structure. But for logical clarity, we simplify the story in the following manner: Households M^t and landlords N^t go to the market of week t, and there they trade apartment units and disappear from the market. Before going to the market, households M^t look at the weekly housing magazine of week $t-1$ and decide which category they go to. Landlords N^t, also looking at the housing magazine of week $t-1$, decide to supply apartment units. We assume this delay for logical simplicity.

In $\mathbb{E}^t(\boldsymbol{\epsilon}^t) = (M^t, u^t, I^t; N^t, C^t; \boldsymbol{\epsilon}^t)$ of week t, the rental prices are realized with error term $\boldsymbol{\epsilon}^t$. The term $\boldsymbol{\epsilon}^t$ is a T-vector of independent random variables which perturb the market rents p^t_k for apartments in category $k = 1, ..., T$ to $p^t_k + \boldsymbol{\epsilon}^t_k$.

We will have various random variables and their realizations. To distinguish between random variables and their realizations, we prepare the underlying probability space $(\Omega, \mathcal{F}, \mu)$ which all the random variables in this paper follow.

In week $t-1$, apartment units of categories $1, ..., T$ are brought to the housing market. Let $D^{t-1}_1, ..., D^{t-1}_T$ are the (finite nonempty) sets of those apartment units. Each unit d in D^{t-1}_k and its rental price are listed in the housing magazine as $p^{t-1}_k + \boldsymbol{\epsilon}^{t-1}_{kd}(\omega^{t-1})$. The entire *housing magazine of week* $t-1$ is described as

$$\{p^{t-1}_1 + \boldsymbol{\epsilon}^{t-1}_{1d}(\omega^{t-1}) : d \in D^{t-1}_1\}, \quad ..., \quad \{p^{t-1}_T + \boldsymbol{\epsilon}^{t-1}_{Td}(\omega^{t-1}) : d \in D^{t-1}_T\}. \tag{3.1}$$

Here, we assume that $\{\boldsymbol{\epsilon}^{t-1}_{kd} : d \in D^{t-1}_k\}$ consists of independent random variables identical to $\boldsymbol{\epsilon}^{t-1}_k$. Thus, the rental price of each unit d is independently affected by $\boldsymbol{\epsilon}^{t-1}_{kd}$.

Term $\boldsymbol{\epsilon}^{t-1}_{kd}$ (or $\boldsymbol{\epsilon}^{t-1}_k$) represents the effects of non-systematic components such as the local environment of apartment unit d. The market mechanism with real estate agents determine the price for each unit taking its local environment. From the econometric point of view, we focus only on the systematic components mainly explaining the market functioning, but we cannot ignore the non-systematic components in the real data.

We do not assume that the households and landlords know the error terms $\boldsymbol{\epsilon}^{t-1} = (\boldsymbol{\epsilon}_1^{t-1}, ..., \boldsymbol{\epsilon}_T^{t-1})$. Instead, we assume that each $i \in M^t$ looks at the housing magazine (3.1) listing the supplied units and prices (3.1) at week $t-1$, and then forms an estimator of the price distribution:

$$P_k^{i,t} = p_k^{t-1} + \boldsymbol{\epsilon}_k^{i,t} \text{ for each } k = 1, ..., T. \tag{3.2}$$

We allow $\boldsymbol{\epsilon}_k^{i,t}$ to be a random or nonrandom (degenerated) variable. In general, $P_k^{i,t}$ is a random variable for each k. Household i makes a choice of a category by looking at his price estimators in (3.2). That is, he maximizes the expected utility (subject to the budget constraint) relative to this price expectation.

Each landlord k ($k = 1, ..., T$) supplies only apartment units only in category k. Landlord j is assumed to make his estimator $p_k^{t-1} + \boldsymbol{\epsilon}_k^{k,t}$ of only apartments k.

3.2. Equilibrium with Subjective Estimations

We do not explicitly consider a dynamic structure of the housing market by assuming that the market is stationary and $P_k^{i,t}$ is independent of week t. Thus, we drop the superscript t from $\mathbb{E}^t(\boldsymbol{\epsilon}^t)$ and $P_k^{i,t}$. Hence, our snapshot model is described as $\mathbb{E}(\boldsymbol{\epsilon}) = (M, u, I; N, C; \boldsymbol{\epsilon})$.

Each household $i \in M$ has his own subjective estimation of a price distribution $P_k^i = p_k + \boldsymbol{\epsilon}_k^i$ in each $k = 1, ..., T$. Note that he himself does not know p_k itself, but he has the distribution P_k^i. Each landlord $k \in N$ has the price expectation $P_k^k = p_k + \boldsymbol{\epsilon}_k^k$. We assume that these price expectations do not take negative values:

$$P_k^i(\omega) \geq 0 \text{ and } P_k^k(\omega) \geq 0 \text{ for all } \omega \in \Omega. \tag{3.3}$$

We give two examples for such subjective price expectation.

Example 3.1.(Average Prices): Looking at the housing magazine (3.1), household i (landlord j) takes some samples of prices from category k. Let L_i is the samples taken. Then, he takes the average P_k^i of these samples:

$$P_k^i(\omega) = \sum_{d \in L_k} (p_k + \boldsymbol{\epsilon}_{kd}(\omega)) / |L_k| \, . \tag{3.4}$$

The primitive event ω refers to the one occurring in week $t-1$. Since households and landlords are not statistical analyzers, the number of samples $|L_k|$ is small such as $10 \sim 25$. In this case, the variance of $P_k^i(\cdot)$ becomes $1/10 \sim 1/25$. Accordingly, the corresponding standard deviations in Table 2.2 should become $1/3 \sim 1/5$.

We have still two possible interpretations here. One possibility is that he adopts this particular (non-stochastic) value $P^i_k(\omega)$ as his expectation for the price in category k. The other possible interpretation is that he knows from the previous history of the market that his expectation is given as the random variable $P^i_k(\cdot)$. Of course, actual D_k may differ from week by week. But here, we assume that the households take these as constant and only $P^i_k(\cdot)$ as a random variable.

The second example is to assume that households (landlords) have the ability of the outside objective observer.

Example 3.2.(True Price Distribution): Suppose that household i (landlord j) carefully scrutinizes the housing magazine by drawing a histogram. Since the number of units listed in the magazine is quite large, it might be a possible idealization that household i's price expectation is the true one $P^i_k = p_k + \boldsymbol{\epsilon}^i_k = p_k + \boldsymbol{\epsilon}_k$. In the case of landlord j, also, his price expectation is given as $P^i_k = p_k + \boldsymbol{\epsilon}^j_k = p_k + \boldsymbol{\epsilon}_k$.

In this case, the condition $E(\boldsymbol{\epsilon}_k) = 0$ has a clear-cut implication for a landlord: If landlord j has a *risk-nuetral utility function*, then his expected profit maximization problem becomes simply a profit maximization problem, that is,

$$E(y_j P^j_k(\omega) - C_j(y_j)) = E(y_j(p_k + \boldsymbol{\epsilon}^j_k(\omega)) - C_j(y_j)) = y_j p_k - C_j(y_j). \tag{3.5}$$

However, the magazine is quite large and not well-organized. It is very costly to extract the distribution $p_k + \boldsymbol{\epsilon}_k(\cdot)$. Instead, often, the information publicly announced is the average price of samples; Example 3.1 is a better fitting to reality.

In the rental housing market $\mathbb{E}(\boldsymbol{\epsilon}) = (M, u, I; N, C; \boldsymbol{\epsilon})$, the concept of a competitive equilibrium should be modified since each economic agent forms a price estimation of prices and his decision making is based on his estimation. We, first, take this estimation into account in utility maximization for each household, and then we formulate a landlord's profit maximization.

Household i chooses one category based on his price estimation $P^i = (P^i_1, ..., P^i_T)$. A choice for household i should satisfy the budget constraint. Taking his budget constraint into account, we define the following utility: for $x_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$,

$$U_i(x_i, I_i - P^i(\omega) \cdot x_i) = \begin{cases} u_i(x_i, I_i - P^i(\omega) \cdot x_i) & \text{if } 0 \leq I_i - P^i(\omega) \cdot x_i \\ \\ u_i(\mathbf{0}, I_i) & \text{otherwise.} \end{cases} \tag{3.6}$$

In the second case of (3.6), his budget is not met and no trade occurs. In general, this utility function $U_i(x_i, I_i - P^i(\cdot) \cdot x_i)$ is a random variable.

Using the above notation, we define the expected utility before going to a category:

$$EU_i(x_i, I_i - P^i \cdot x_i) = \int_{\omega \in \Omega} U_i(x_i, I_i - P^i(\omega) \cdot x_i) d\mu(\omega). \tag{3.7}$$

His category choice is made by maximizing this expected utility function over $\{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$.

We assume that each landlord k has a *risk-neutral utility function.* Then his expected utility is calculated as the expected payoff:

$$E(y_k P_k^k - C_k(y_k)) = y_k E(P_k^k) - C_k(y_k). \tag{3.8}$$

If $E(\boldsymbol{\epsilon}_k^k) = 0$, i.e., $E(P_k^k) = p_k$, (3.8) becomes simply profits $y_k p_k - C_k(y_k)$. However, we treat landlords in the same way as households in that he may construct his price expectation P_k^j without assuming $E(\boldsymbol{\epsilon}_k^j) = 0$.

Combining (3.7) and (3.8), we have now the definition of a competitive equilibrium in the housing market $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N}) = (M, u, I;\ N, C; \boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N})$, where $\boldsymbol{\epsilon}^{M \cup N} = (\{\boldsymbol{\epsilon}^i\}_{i \in M}, \{\boldsymbol{\epsilon}^k\}_{k \in N})$. A competitive equilibrium is simply defined by replacing the utility functions and profit functions in UM and PM of Section 2 with the objective functions (3.7) and (3.8). The balance of demand and supply is the same. Thus, it is given as a triple $(p, x, y) = ((p_1, ..., p_T),\ (x_1, ..., x_m), (y_1, ..., y_n))$, where $p \in \mathbf{R}_+^T, x \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}^m$ and $y \in \mathbf{Z}_+^n$. Hence, we can make a direct comparison of a competitive equilibrium in the present sense with that defined in Section 2.

Since we allow perturbed price expectations for households and landlords, the exact forms of UM and PM in Section 2 may not be converted. However, they can approximately. Here, we need to define the two notions: an ε-competitive equilibrium and a convergent sequence of price expectations.

Let ε be a nonnegative real number. We call (p, x, y) is an *ε-competitive equilibrium* $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N})$ when the following two conditions and BDS hold:

(ε-Expected Utility Maximization): for all household $i \in M$,

$$EU_i(x_i, I_i - P^i \cdot x_i) + \varepsilon \geq EU_i(x_i', I - P^i \cdot x_i') \text{ for all } x_i' \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}.$$

(ε-Expected Profit Maximization): for all landlord $k = 1, ..., T$,

$$E(P_k^k y_k - C_k(y_k)) + \varepsilon \geq E(P_k^k y_k' - C_k(y_k')) \text{ for all } y_k' \in \mathbf{Z}_+^*.$$

Both are obtained by modifying UM and PM with expected utility theory and ε-maximization.

We will convert a competitive equilibrium (p, x, y) in $\mathbb{E}$ into $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N})$. Our concern is the preservation of the equilibrium property of (p, x, y) in $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N})$. Exactly speaking, this preservation does not necessarily hold, but if we take the perturbations $\boldsymbol{\epsilon}^{M \cup N}$ to be small enough, we would succeed in the conversion of a competitive equilibrium (p, x, y) into $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N})$.

To express "small perturbations", we introduce the convergence of the vectors of estimators $\boldsymbol{\epsilon}^{M \cup N}$. We say that an error sequence $\{\boldsymbol{\epsilon}^{M \cup N, \nu} : \nu = 1, ...\} = \{(\{\boldsymbol{\epsilon}^{i, \nu}\}_{i \in M}, \{\boldsymbol{\epsilon}^{k, \nu}\}_{k \in N}) : \nu = 1, ...\}$ is *convergent in probability* iff for any $\delta > 0$,

$$\mu(\{\omega : \max_{j \in M \cup N} \left\| \boldsymbol{\epsilon}^{j, \nu}(\omega) \right\| < \delta\}) \to 1 \text{ as } \nu \to +\infty, \tag{3.9}$$

where $\|\cdot\|$ is the max-norm $\|(y_1, ..., y_T)\| = \max_{1 \le t \le T} |y_t|$. This mean that when ν is large enough, the estimation $\boldsymbol{\epsilon}^{j,\nu}(\omega)$ is distributed closely enough to $\mathbf{0}$.

We have the following theorem. The proof will be given in the Appendix.

Theorem 3.1 (Correspondence to $\mathbb{E}$): Suppose that the sequence of estimation errors $\{\boldsymbol{\epsilon}^{M \cup N, \nu} : \nu = 1, ...\}$ is converging to 0 in probability. Then,

(1): If (p, x, y) be a competitive equilibrium in $\mathbb{E}$, then for any $\varepsilon > 0$, there is a ν_o such that for any $\nu \ge v_o$, (p, x, y) is an ε-competitive equilibrium in $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N, \nu})$.

(2): Suppose that a triple (p, x, y) satisfies $px^i < I_i$ for all $i \in M$. Then, the converse of (1) holds.

The theorem is represented in terms of an error sequence $\{\boldsymbol{\epsilon}^{M \cup N, \nu}\}$, but the point is to express that the subjective price expectation of each household (landlord) has a small variance. That is, when each household i (landlord j) has his price expectation $\boldsymbol{\epsilon}^i$ (or $\boldsymbol{\epsilon}^j$) with a small variance, his utility maximization (or profit maximization) in the idealized market $\mathbb{E}$ is preserved approximately in the market $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N, \nu})$ for large ν, and *vice versa*.

In Example 3.1, if a household takes 10 samples from category k, then the variance of his estimator becomes the 1/10 of the variant of the objective $\boldsymbol{\epsilon}^k$. Hence, perhaps, our theorem is applied, and we can use the market model $\mathbb{E}$ as an for an ideal approximation of $\mathbb{E}(\boldsymbol{\epsilon})$.

On the other hand, if a household draws a histogram from the data like in Example 3.2, then his price estimation is, more or less, independent of the numbers of units listed. In this case, the above theorem is not applied. Recall our basic assumption that households and landlords do not thoroughly investigate the housing magazine, but, instead, they use some summary statistics such as the average. In this case, drawing a histogram from the data is not a good description of the formation of a subjective estimate for an economic agent.

Theorem 3.1 can be simplified under some conditions. When the price expectation for landlord $k \in N$ satisfies $E(\boldsymbol{\epsilon}^k) = 0$, then his expected profit is simply given as the profit function, and so we do not need to consider the convergent sequence for k. Here, the ε-modification is not required for him either.

Also, if a competitive equilibrium (p, x, y) is strict in the sense that a household (landlord) maximizes his utility at a unique choice, then we do not need the ε-modification for the household (landlord).

4. Statistical Analysis of Rental Housing Markets

Now, our aim is to estimate the structures of the rental housing market from the data given in the housing magazine. In Section 4.1, we develop various concepts to connect the data with possible market models and to evaluate such a connection. In Section 4.2, we specify a class of market models for our estimation.

4.1. Estimation of the Market Structure

Let $\mathbb{E}^o(\boldsymbol{\epsilon}^o) = (M^o, u^o, I^o; N^o, C^o; \boldsymbol{\epsilon}^o)$ be the true market. We call $\mathbb{E}^o = (M^o, u^o, I^o; N^o, C^o)$ the *latent* true market structure. We assume that this $\mathbb{E}^o$ satisfies Assumptions A and B of Section 2, i.e., $\mathbb{E}^o \in \Gamma_0$. The maximal competitive rent vector $p^o = (p_1^o, ..., p_T^o)$ of $\mathbb{E}^o$ is called the *latent market price vector*. Let D_k^o be a nonempty set of apartment units listed in category $k = 1, ..., T$. Once the perturbation term $\boldsymbol{\epsilon}^o$ from the non-systematic factors is given, we have the housing magazine $\{P_{1d}^o(\omega^o) : d \in D_1^o\}, ..., \{P_{Td}^o(\omega^o) : d \in D_T^o\}$. The listed price for a unit d ($d \in D_k^o$ and $k = 1, ..., T$) is given as:

$$P_{kd}^o(\omega^o) = p_k^o + \boldsymbol{\epsilon}_{kd}^o(\omega^o),$$

where ω^o is the primitive event of the specific date. We will estimate some components of $\mathbb{E}^o$ from the housing magazine, specifically, the utility functions of households.

Again, let $p = (p_1, ..., p_T)$ be an arbitrary price vector in $\mathbb{R}^T$. We assume that $P_{kd}^o(\omega)$ is defined for an arbitrary $\omega \in \Omega$. Then, the *total sum of square residuals* $T_R(P_D^o(\omega), p)$ is given as

$$T_R(P_D^o(\omega), p) = \sum_{k=1}^{T} \sum_{d \in D_k^o} (P_{kd}^o(\omega) - p_k)^2. \tag{4.1}$$

We adopt this as a measure to express the difference between the data and the estimated price vector.

Our estimation problem is to choose $\mathbb{E} = (M, u, I; N, C)$ in order to minimize $T_R(P_D^o(\omega), \psi(\mathbb{E}))$ in a subclass Γ of Γ_0, where $\psi(\mathbb{E})$ is the maximal competitive price vector in $\mathbb{E}$. The choice of Γ is essential for this problem. We write our estimation problem explicitly:

Definition 4.1 (Γ-MSE Problem): Let Γ be a subset of Γ_0. Then, we call the following minimization problem the Γ-*market strucutre estimation problem*: Choose a model $\mathbb{E} = (M, u, I; N, C)$ from the set Γ to minimize $T_R(P_D^o(\omega), p)$ subject to the condition:

($*$): (p, x, y) is a maximal competitive equilibrium in $\mathbb{E}$ for some (x, y)
with $y_k = |D_k^o|$ for $k = 1, ..., T$.

The additional condition $y_k = |D_k^o|$ for $k = 1, ..., T$ means that each maximal competitive equilibrium (p, x, y) is compatible with the number of apartment units listed in the housing magazine.

If the latent true structure $\mathbb{E}^o$ belongs to Γ, it would be a candidate for the solution of the Γ-MSE problem. In the first place, however, we do not know whether or not $\mathbb{E}^o$ belongs to Γ. A simple idea is to choose a large class for Γ to guarantee that $\mathbb{E}^o$ could be in Γ. In fact, this idea does not work well: In Section 6.1, we discuss the negative result for this idea, and argue rather that we should look at a narrow class for Γ.

Before going to a concrete treatment of the Γ-MSE problem, we consider the benchmark case and also a certain comparison measure from the benchmark case with a result of the Γ-MSE problem.

The benchmark case is the *average rent estimator*. Given a sample $P_D^o = \{P_{kd}^o : d \in D_k^o \text{ and } k = 1, ..., T\}$, we define $\overline{P}^o = (\overline{P}_1^o, ..., \overline{P}_T^o)$ by

$$\overline{P}_k^o(\omega) = \frac{\sum_{d \in D_k^o} P_{kd}^o(\omega)}{|D_k^o|} \text{ for each } \omega \in \Omega \text{ and } k = 1, ..., T. \tag{4.2}$$

This is defined as a function of $\omega \in \Omega$. We can regard this as the best estimator of the latent market price vector $p^o = (p_1^o, ..., p_T^o)$. Each realization $\overline{P}^o(\omega)$ is the unique minimizer of $T_R(P_D^o(\omega), p)$ with no constraints. When $E(\boldsymbol{\epsilon}_k^o) = 0$, $\overline{P}_k^o$ is an unbiased estimator of p_k^o. Both are simply proved but will be used a lot.

Lemma 4.1.(1): for all $\omega \in \Omega$, $T_R(P_D^o(\omega), \overline{P}^o(\omega)) \leq T_R(P_D^o(\omega), p)$ for any $p = (p_1, ..., p_T) \in \mathbf{R}^T$.

(2): When $E(\boldsymbol{\epsilon}_k^o) = 0$, $\overline{P}_k^o$ is an unbiased estimator of p_k^o, i.e., $E(\overline{P}_k^o) = p_k^o$.

Proof.(1): Let $\omega \in \Omega$ be fixed. Since $T_R(P_D^o(\omega), p)$ is a strictly convex function of $p = (p_1, ..., p_T) \in \mathbf{R}^T$, the necessary and sufficient condition for p to be a minimizer of $T_R(P_D^o(\omega), p)$ is given as $\partial T_E(P_D(\omega), p)/\partial p_k = 0$ for all $k = 1, ..., T$. Only the average $\overline{P}^o(\omega) = (\overline{P}_1^o(\omega), ..., \overline{P}_T^o(\omega))$ satisfies this condition.

(2): Since $\boldsymbol{\epsilon}_{kd}^o$ is identical to $\boldsymbol{\epsilon}_k^o$ for all $d \in D_k^o$ and $E(\boldsymbol{\epsilon}_k^o) = 0$, we have $E(\boldsymbol{\epsilon}_{kd}^o) = 0$ for all $d \in D_k^o$. Hence $E(\overline{P}_k^o) = \sum_{d \in D_k^o} E(P_{kd}^o)/|D_k^o| = \sum_{d \in D_k^o}(p_k^o + E(\boldsymbol{\epsilon}_{kd}^o))/|D_k^o| = p_k^o$. ■

As far as $E(\boldsymbol{\epsilon}_k^o) = 0$ for $k = 1, ..., T$, $\overline{P}^o$ is an unbiased estimator of the latent market price vector p^o, though for each ω, $\overline{P}^o(\omega)$ may not yet coincide with p^o. This estimator $\overline{P}^o$ enjoys various desired properties such as consistency (i.e., convergence to the latent market price vector p^o in probability as the number $\max_k |D_k|$ tends to infinity) and efficiency in the sense of Cramer-Rao. For these, see van der Vaart [24]. Also, it will be obtained also by the maximum likelihood method, which will be discussed in Section 7.

However, we are interested in estimating the structure of $\mathbb{E}^o$ rather than the price vector p^o. For this purpose, we can use the average price vector $\overline{P}^o$ as the benchmark case, and Lemma 4.1.(1) plays a more important role.

We have the decomposition of the total sum of square residulas, which corresponds to the well-known decomposition property of the total sum of square residuals in the regression model (cf. Wooldridge [23]). However, this will be essential for our further analysis, so we give a proof for completeness of the paper.

Lemma 4.2 (Decomposition): For each $\omega \in \Omega$,

$$T_R(P_D^o(\omega), p) = T_R(P_D^o(\omega), \overline{P}^o(\omega)) + \sum_{k=1}^{T} |D_k^o| \, (\overline{P}_k^o(\omega) - p_k)^2. \tag{4.3}$$

Proof. Consider the component of $T_R(P_D^o(\omega), p)$ for each k :

$$\sum_{d \in D_k^o} (P_{kd}^o(\omega) - p_k)^2 = \sum_{d \in D_k^o} (P_{kd}^o(\omega) - \overline{P}_k^o(\omega) + \overline{P}_k^o(\omega) - p_k)^2 =$$

$$\sum_{d \in D_k^o} (P_{kd}^o(\omega) - \overline{P}_k^o(\omega))^2 + \sum_{d \in D_k^o} 2(P_{kd}^o(\omega) - \overline{P}_k^o(\omega)) \cdot (\overline{P}_k^o(\omega) - p_k) + \sum_{d \in D_k^o} (\overline{P}_k^o(\omega) - p_k)^2.$$

The second term of the last expression vanishes by (4.2). The third is written as $|D_k^o| \, (\overline{P}_k^o(\omega) - p_k)^2$. We have (4.3) by summing these over $k = 1, ..., T$. ■

The second term of (4.3) is the total sum of the differences between the average $\overline{P}^o(\omega)$ and given p. Each square difference is counted by the number of occurrences of trades. We call this the *theoretical discrepancy.*

We call the ratio

$$\eta(p)(\omega) = \frac{T_R(P_D^o(\omega), p)}{T_R(P_D^o(\omega), \overline{P}(\omega))} = 1 + \frac{\sum_{k=1}^{T} |D_k^o| \, (\overline{P}_k^o((\omega)) - p_k)^2}{T_R(P_D^o(\omega), \overline{P}^o(\omega))} \tag{4.4}$$

the *discrepancy measure* of p from of $\overline{P}^o(\omega)$. The portion exceeding 1 is the theoretical discrepancy, relative to the smallest total sum of residuals. In the example in Section 2.2, this discrepancy is given in (2.5) as about 1.032. That is, the theoretical discrepancy is only 3.2%. In Section 5, we will see more accurate estimations.

The *coefficient of determination* in our context may help us also. It indicates how much the systematic components explain the observed rental prices. For this, consider the following total variation $T_V(P_D^o, \xi)$ relative to a given reference point $\xi \in \mathbf{R}$: for any fixed $\omega \in \Omega$,

$$T_V(P_D^o(\omega), \xi) = \sum_{k=1}^{T} \sum_{d \in D_k^o} (P_{kd}^o(\omega) - \xi)^2. \tag{4.5}$$

This is simply the total square variations of the data P_D measured from the reference point ξ. Then we have the following lemma, which can be proved in a similar manner to Lemmas 4.1.(1) and 4.2.

Lemma 4.3.(1). $T_V(P_D^o(\omega), \overline{\overline{P}}(\omega)) \leq T_V(P_D^o(\omega), \xi)$ for any $\xi \in \mathbf{R}$, where $\overline{\overline{P}}(\omega)$ is the entire average of P_D^o, i.e., $\overline{\overline{P}}(\omega) = \sum_k \sum_{d \in D_k} P_{kd}^o(\omega) / \sum_k |D_k^o|$.

(2): It holds that

$$T_V(P_D^o(\omega), \overline{\overline{P}}(\omega)) = \sum_{k=1}^{T} \sum_{d \in D_k^o} (P_{kd}^o(\omega) - \overline{P}_k(\omega))^2 + \sum_{k=1}^{T} \sum_{d \in D_k^o} (\overline{P}_k(\omega) - \overline{\overline{P}}(\omega))^2.$$

Hence, we can use the entire average $\overline{\overline{P}}(\omega)$ for the reference point to measure the total variation. Then we have the decomposition of the total variation as in the standard regression analysis. Thus, we define the *coefficient of determination* as:

$$\delta(P_D^o(\omega)) = \frac{\sum_{k=1}^{T} |D_k| \, (\overline{P}_k(\omega) - \overline{\overline{P}}(\omega))^2}{T_V(P_D^o(\omega), \overline{\overline{P}}(\omega))}. \tag{4.6}$$

When this coefficient $\delta(P_D^o(\omega))$ is large, the systematic components of the market is dominant. In the example of Section 2.2, $\delta(P_D^o(\omega^o)) = 0.757$.

4.2. Subclass Γ_{sep} of Γ_0

Among the components in $\mathbb{E}^o = (M^o, u^o, I^o; N^o, C^o)$, some are more observable and some others are less. The most unobservable components are the individual utility functions, since they are internal in the households' minds. For the other components, some information is available from the different sources. In this paper, we focus on the estimation of individual utility functions. That is, we target to estimate $u^o = (u_1^o, ..., u_m^o)$ among the components $(M^o, u^o, I^o; N^o, C^o)$. The other components are simply assumed, based on the data from the housing magazine or some other sources. For example, the set of households M is taken be a set of cardinality of the data set D^o.

The set of market models Γ_{sep} consists of $\mathbb{E} = (M, u, I; N, C)$ satisfying the following three conditions:

S1: The incomes of households are ordered as $I_1 \geq ... \geq I_m > 0$.

S2: Every household in M has the same utility function $u_1 = ... = u_m$ expressed as

$$u_i(\mathbf{e}^k, m_i) = h_k + g(m_i) \text{ for all } (\mathbf{e}^k, m_i) \in X, \tag{4.7}$$

where $h_0, h_1, ..., h_T$ are given real numbers with $h_k > h_0$ for $k = 1, ..., T$ and $g : \mathbf{R}_+ \to \mathbf{R}$ is a monotonically increasing and continuous concave function with $g(m_i) \to +\infty$ as $m_i \to +\infty$ and $h_0 + g(I_i) > h_k + g(0)$ for $k = 1, ..., T$.

S3: Each landlord $k = 1, ..., T$ has a cost function of the form (2.3) and (2.4).

First of all, Γ_{sep} is a subset of Γ_0. In S1, the households are ordered by their incomes. Condition S2 has two parts: First, every household has the same utility function; and second, the utility function is expressed in the separable form. The first is restrictive in the respect that the households have the same location of their offices. The second part itself is less restrictive; and it should be remarked that this is not quasi-linearity. With respect to consumption, the law of diminishing marginal utility could strictly hold. Condition S3 is for simplification: Our theory emphasizes on the households' side.

The reader who is familiar to mathematical economics or game theory may think that the set Γ_{sep} is very narrow in that the households have the same utility functions of the separable form of (4.7) and the landlords' cost functions are also very specific. This opinion is totally incorrect from our research perspective with the results we will present in the subsequent sections. That is, the above class is too large from the view point of the Γ-MSE problem. It will be shown in Section 6.1 that any given price vector can be rationalized as the optimum of the Γ_{sep}-MSE problem. In this case, however, the estimated model has no prediction power, and it is an *ex post* rationalization.

Our estimation problem requires us to calculate a maximal competitive equilibrium (p, x, y) in $\mathbb{E} = (M, u, I; N, C)$. The method of calculation was given in Kaneko [9] and Kaneko *et al* [12]. This method is used to implement our econometrics and also is used to prove two theorems in Section 6. Here, we describe this method without a proof.

Consider any price vector $p = (p_1, ..., p_T)$ with $p_1 \geq ... \geq p_T > 0$. The cases of different orders are changed into this case by renaming $1, ..., T$. Then, we will regard the apartment units in category 1 as the best, and will suppose that the richest households $1, ..., |D_1^o|$ rent them. Similarly, the apartments in category 2 are the second best and the second richest households $|D_1^o| + 1, ..., |D_1^o| + |D_2^o|$ rent them. In general, we define

$$G(k) = \sum_{t=1}^{k} |D_t^o| \quad \text{for all } k = 1, ..., T. \tag{4.8}$$

Then we suppose that the households $G(k-1)+1, ..., G(k)$ rent apartments in category k.

Then, we focus the boundary households $G(1), G(2), ..., G(T-1)$ with their incomes $I_{G(1)}, I_{G(2)}, ..., I_{G(T-1)}$, respectively.

We have the following lemma due to Kaneko [9] and Kaneko, *et al* [12]. Our econometric calculation is based on this lemma.

Lemma 4.4. Consider a vector $(p_1, ..., p_T)$ with $p_1 \geq ... \geq p_T > 0$. Let $\mathbb{E} = (M, u, I; N, C) \in \Gamma_{sep}$ satisfying
(1): $p_k \leq I_{G(k)}$ for all $k = 1, ..., T-1$;
(2): $c_k \leq p_k$ and $w_k = |D_k^o|$ for all $k = 1, ..., T$.
Recall that c_k is the marginal cost given in (2.3) and (2.4). Suppose also that $(p_1, ..., p_T)$ satisfies

$$\begin{array}{rcl} h_{G(T-1)} + g(I_{G(T-1)} - p_{T-1}) & = & h_{G(T-1)} + g(I_{G(T-1)} - p_T) \\ h_{G(T-2)} + g(I_{G(T-2)} - p_{T-2}) & = & h_{G(T-2)} + g(I_{G(T-2)} - p_{T-1}) \\ & & \cdot \quad \cdot \quad \cdot \\ h_{G(1)} + g(I_{G(1)} - p_1) & = & h_{G(1)} + g(I_{G(1)} - p_1). \end{array} \tag{4.9}$$

Then, there is an allocation (x, y) such that (p, x, y) is a maximal competitive equilibrium in $\mathbb{E}$ with $y_k = |D_k^o|$ for all $k = 1, ..., T$.

Thus, boundary household $G(T-1)$ compares his utility from staying in an apartment in category $T-1$ with that from category T. Also, boundary household $G(T-2)$ makes a parallel comparison, and so on. The logic of this argument is essentially the same as that for Ricardo's [18] differential rents (in the second category). The rent in the worst category T is regarded as the land rent-cost of farm lands, which corresponds to Ricardo's absolute rent (or in the first category).

5. Application of Our Theory to the Market in Tokyo

In this section, we apply the econometric method developed in Section 4 to the rental housing market in Tokyo described in Section 2.2. First, we give a simple heuristic discussion on our application, and then give a more systematic study of it. The prediction of our Γ-MSE problem with an appropriate choice of Γ is satisfactory from various points of view. Nevertheless, some reader may think from the viewpoint of mathematical economics that the set Γ is very restrictive. This will be discussed in Section 6.

5.1. Heuristic Discussion

For an econometric study of a specific target, we need to consider a more concrete class for Γ than the class Γ_{sep} given in Section 4.2. In Section 2.2, we used a specific form of the basic utility function $U^0(t,s,m_i) = -2.2t + 4.0s + 100\sqrt{m_i}$ of (2.1) and obtained the resulting value of the discrepancy measure, $\eta = 1.032$. Perhaps, we need to explain how we have found it and how good it is relative to others.

First, let us compare several other basic utility functions with (2.1):

(1): $U^1(t,s,m_i) = -t + s + 100\sqrt{m_i}$; $\eta^1 = 3.259$;

(2): $U^2(t,s,m_i) = -2t + 255\sqrt{s+1000} + 100\sqrt{m_i}$; $\eta^2 = 1.036$;

(3): $U^3(t,s,m_i) = -74t + 165s + 100m_i$; $\eta^3 = 1.124$.

The first means that if we adopt the specific basic utility function of (1), then the discrepancy measure η takes large value $\eta^1 = 3.259$. Thus, the total sum of squred residuals from the predicted prices is more than the three-times of that from the average prices. In the case (2), it is already almost as small as $\eta = 1.032$ given by of (2.1). In (3), it is larger than this value, but it should be noticed that the basic utility function of (3) is entirely linear.

The utility function U^1 of (1) is simply adopted so as to show that if coefficients are arbitrarily chosen, the discrepancy would be large. On the other hand, the utility function U^0 of (2.1) is chosen by minimizing the discrepancy measure η by changing the coefficients of t and s. More explicitly, let $\mathcal{U}(1,1,\frac{1}{2})$ be the class of basic utility functions:

$$\{U(t,s,m_i) = -\alpha_1 t + \alpha_2 s + 100\sqrt{m_i} : \alpha_1, \alpha_2 \in R\}, \tag{5.1}$$

where the coefficient 100 of the third term is simply chosen to make the values of α_1, α_2 more clearly visible. Then, $U^0(t, s, m_i)$ is obtained by minimizing η (equivalently, the total sum of square residuals from the predicted prices) in this class. This is not the exact solution but is calculated using a method of grid-search with a computer.

Let us explain our procedure of computation more concretely. Now, suppose that $U \in \mathcal{U}(1, 1, \frac{1}{2})$ is given. Then, for each $(t, s) \in \{18, 23, 31, 52, 64, 70\} \times \{15, 35, 55\}$, we have the value $-\alpha_1 t + \alpha_2 s$, which gives the ranking, $1, ..., 18$ over $\{18, 23, 31, 52, 64, 70\} \times \{15, 35, 55\}$. Then, the k-th category has $h_k = \alpha_1 t + \alpha_2 s$ and $\lambda(k) = (t, s)$. This method is the same as in Section 2.2. Hence, U determines

$$u(\mathbf{e}^k, m_i) = h_k + 100\sqrt{m_i} \text{ for } k = 0, 1, ..., T. \tag{5.2}$$

Thus, each $U \in \mathcal{U}(1, 1, \frac{1}{2})$ determines u.

Now, we consider the subclass $\Gamma(1, 1, \frac{1}{2})$ of Γ_{sep} defined by:

$$\{(M, u, I; N, C) \in \Gamma_{sep} : u \text{ is determined by some } U \in \mathcal{U}(1, 1, \frac{1}{2})\}. \tag{5.3}$$

Then, we apply the $\Gamma(1, 1, \frac{1}{2})$-MSE problem to the data discussed in Section 2.2. So far, we have no method of finding an exact solution for the $\Gamma(1, 1, \frac{1}{2})$-MSE problem. The objective of the present research is not to implement to construct such a method, but it would be more important to know what shape a solution has. That is, we should be satisfied by finding an approximate solution (α_1, α_2) for the $\Gamma(1, 1, \frac{1}{2})$-MSE problem. If the values of the discrepancy measure η are close enough, then we should not care about which point the exact solution is.

An approximate solution will be obtained by the following process.

Step 1: We assume that each of α_1 and α_2 takes a (integer) value from some intervals, say, $[1, 100]$. Then, we have $100^2 = 10^4$ combinations of (α_1, α_2).

Step 2: For each combination (α_1, α_2), we find a maximal competitive price vector p compatible with the data set $P^o_D(\omega^o)$ and we have the value $\eta(p)$ of discrepancy measure. The algorithm to find a maximal competitive rent vector given by Lemma 4.4 is used to find the price vector.

Step 3: Then, we find a combination (α_1, α_2) with the minimum value of η among 10^4 combinations of (α_1, α_2).

If a solution is on the boundary, we calibrate the intervals, and if it is not on the boundary, we repeat these steps by choosing a smaller intervals with finer grids. Hence, the computation to obtain the minimum value of η is not exact: It may be a local optimum as well as an approximation.

By the above simulation method, we have found the utility function $U^0(t, s, m_i)$ of (2.1) in the class $\Gamma(1, 1, \frac{1}{2})$ with $\eta = 1.032$.

The basic utility function U^2 of (2) is obtained by minimization in the class $\mathcal{U}(1, \frac{1}{2} \circledast \beta_2, \frac{1}{2})$ with the fixed $\beta_2 = 1000$:

$$\{U(t, s, m_i) = -\alpha_1 t + \alpha_1 \sqrt{s + \beta_2} + 100\sqrt{m_i} : \alpha, \beta_1 \in R\}. \tag{5.4}$$

In fact, when β_2 is increased, the optimal value of η is decreasing (we calculated η up to 400,000) but it does not reach $\eta^0 = 1.032$. Since β_2 is getting large, the second term is getting closer to the linear function. Therefore, we interpret this result as meaning that the basic utility function $U^0(t, s, m_i) = -2.2t + 4.0s + 100\sqrt{m_i}$ of (2.1) would be the limit function.

The utility function $U^3(t, s, m_i)$ of (3) is obtained by minimizing the value η in the class $\mathcal{U}(1, 1, 1)$:

$$\{U(t, s, m_i) = -\alpha_1 t + \alpha_2 s + 100 m_i : \alpha_1, \alpha_2 \in R\}. \tag{5.5}$$

That is, the utility functions are entirely linear. The estimation in this class is only interested in seeing the relationship between our Γ-MSE problem and the standard linear regression. This will be discussed in Section 6.2.

5.2. Law of Diminishing Marginal Utility

By the estimations in the above classes of basic utility functions, $U^0(t, s, m_i)$ of (2.1) gave the best value to the discrepancy measure with our computations. The law of diminishing marginal utility holds strictly only for the consumption term m_i, but not for the other variables, the commuting time-distance and size of an apartment. One possible test of this observation is to broaden the class of basic utility functions. In this subsection, we will give this test.

Instead of the classes of basic utility functions discussed above, we consider the following class $\mathcal{U}(\pi_1 \circledast \beta_1, \pi_2 \circledast \beta_2, \pi_3 \circledast \beta_3)$:

$$U(t, s, m_i) = \alpha_1(\beta_1 - t)^{\pi_1} + \alpha_2(s + \beta_2)^{\pi_2} + 100(m_i + \beta_3)^{\pi_3}, \tag{5.6}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ and π_1, π_2, π_3 are all real numbers. Since we would like to have the above classes as subsets of $\mathcal{U}(\pi_1 \circledast \beta_1, \pi_2 \circledast \beta_2, \pi_3 \circledast \beta_3)$, this class allows us to have additional parameters $\beta_1, \beta_2, \beta_3$ and π_1, π_2, π_3. The introduction of β_1 is natural, since the commuting time-distance must have a limit. The parameters β_2 and β_3 will be interpreted after stating the calculation result. The parameters π_1, π_2, π_3 are related to the law of diminishing marginal utility. When they are close to 1, the law is regarded as not holding in the strict sense, and when they are small and far away from 1, the law is regarded as valid.

Keeping the remark in mind that the given minimum value of η may be a local optimum as well as is an approximation, we give our computation result: First, the resulting basic utility function is given as

$$U^{MU}(t,s,m_i) = 3.53(140-t)^{0.75} + 2.68(s+200)^{0.91} + 100(m_i-25)^{0.40}, \qquad (5.7)$$

and the incomes are uniformly distributed from $I_{8957} = 94$ to $I_1 = 1120$. Finally, the discrepancy measure is calculated as

$$\eta = 1.025.$$

Obtaining this value, we adjusted parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \pi_1, \pi_2, \pi_3$, and the lowest price p_{18}. In each combination, we calculated the value of η through solving the equation (4.9). Also, we have adjusted the lowest income I_{8957} and the highest income I_1.[11]

Consider the implications of the above computation result. First, strictly speaking, the law of diminishing marginal utility holds for each variable. However, the degree is quite different: the utility is quite close to a linear function with respect to the size. The degree of diminishing marginal utility is higher with the commuting time-distance, and is the highest with consummation other than housing.

First, the observation that the degree of diminishing marginal utility is least with respect to the size may be caused by our restriction on apartments up to $65m^2$. In Tokyo, we find apartments up to $85m^2$ (and may find a quite small number of apartments larger than $85m^2$). We omitted these "large" apartments, since the number of supply for them is much small than the types we have treated and their prices behave quite differently. This may be the reason for almost constant marginal utility.

Second, the degree of diminishing marginal utility for the commuting time-distance is higher than that for the apartment size. This suggests, perhaps, that the time-distance 70 minutes to Takao station is already quite large. Our computation result is affected if we change $\beta_1 = 140$ slightly either up or down. Therefore, this constant has a specific meaning.

Finally, the degree of diminishing marginal utility for consumption other than housing is quite large. This means that the choice by a household renting an apartment crucially depends upon its income level. The dependence of willingness-to-pay for an apartment upon income is quite strong: A poor people do not want to pay for a market rent (of course, they cannot) for a good apartment, but if they get rich, they would

[11] One possible amendment of our estimation is to change the assumption on the income distribution. We have assumed that the incomes are distributed from the lowest I_{8957} to the highest I_1. When we started this study, it seemed that this assumption was complemented by the other variables. However, the above computation result seems to be quite sensitive by changing these lowest and highest income levels. Hence, it could give a better result if we replace the assumption of a uniform distribution by the data available from the other source. This will be a future problem.

change their the amount of willingness-to-pay. Hence, this is rather opposite case to the quasi-linear utility function case.

Since the basic utility function given by (2.1) has similar to that of (5.7), it can be regarded as a simple version of (5.7). In the case of simple demonstration or computation for an illustration purpose, this utility function will be useful.

Our result is in strong contrast with the estimation result of a utility fuction in Kanemoto-Nakamura [14] in the hedonic price approach. Their result states that the degreee of diminishing marginal utility is very low, for example, consumption term is $x^{0.978}$. Since their model and the data are quite different, we cannnot make direct comparisons. Nevertheless, since the treatments are richer than ours in several respects that the size variable takes finer values and it has more explanatory variables e.g., such as time distance to the nearest statation. Perhaps, we should take more explanatory variables in the criteria listed in the housing magazine so as to make comparions with their estimation results, which will be a future work.

6. Some Classes of Market Models

The classes of market models discussed so far look very restrictive relative to the standard of mathematical economics. For example, the classes Γ_{sep} and $\Gamma(1, 1, \frac{1}{2})$ could be regarded as extremely restrictive, while $\Gamma(1, 1, \frac{1}{2})$ has strong explanatory power as shown in Section 5. Then, what status does the class Γ_{sep} have in the Γ-MSE problem? First in this section, we show that Γ_{sep} is too large as a set of candidate models. Second, we ask the relationship of our Γ-MSE problem to linear regression. We show that if we choose the class of models with linear utility functions, our Γ-MSE problem turns out to be equivalent to linear regression.

6.1. The Γ_{sep}-Market Structure Estimation: *Ex Post* Rationalization

The class Γ_{sep} of market models is too large to have a meaningful estimation. That is, in the class Γ_{sep}, we can always find a model $\mathbb{E}$ to "fully explain" a given set of realized rental prices, but it is an *ex post* explanation in the sense to be explained presently. This theorem is given to think where we should go with our estimation problem. A proof will be given in the end of this section.

Theorem 6.1 (*Ex Post* Rationalization): Let $P_D(\omega) = \{P_{kd}(\omega) : d \in D_k$ and $k = 1, ..., T\}$ be any data set. Suppose that each D_k is nonempty and the average prices $\overline{P}(\omega) = (\overline{P}_1(\omega), ..., \overline{P}_T(\omega))$ are positive. Then, there exists a market model $\mathbb{E} = (M, u, I; N, C)$ in the class Γ_{sep} such that for some (x, y), $(\overline{P}(\omega), x, y)$ is a maximal competitive equilibrium in $\mathbb{E}$ with $y_k = |D_k| > 0$ for $k = 1, ..., T$.

This existence assertion holds for any given $g : R_+ \to R$ of S2.

It claims that if we choose the class Γ_{sep}, then we can "fully explain" any data set from the housing magazine. The discrepancy measure η can take the value 1 in the class Γ_{sep}. Should we be pleased by finding a class to guarantee to always "fully explain" a given data set? Or, even, should we interpret this theorem as meaning that the true market $\mathbb{E}^0$ is included in the class Γ_{sep}? Contrary to these interpretations, we should regard this theorem as a negative result. Another negative part is the remark appended with the theorem that the "full explanation" can be done with any *a priori* given function g satisfying condition S2: With the Γ_{sep}-MSE problem, we are incapable to talk about the choice of g.

Theorem 6.1 can be compared with the fact that in linear regression, if we extend the set of linear functions to that of piecewise linear functions, then we could draw a piecewise linear curve to fit fully any data; but this explains nothing about the data. Like a geocentric theory of the universe, if the structure of explanatory variables is too rich, we can explain anything; but it is an *ex post* rationalization of the observed fact. Theorem 6.1 is of this sort[12].

In our case, the number of explained variables $\overline{P}(\omega) = (\overline{P}_1(\omega), ..., \overline{P}_T(\omega))$ is T, and the number of explaining variables $(h_1, ..., h_T)$ in utility function $u(e^k, m_i) = h_k + g(m_i)$ is also T. Thus, we have a perfect match: For different $\overline{P}(\omega) = (\overline{P}_1(\omega), ..., \overline{P}_T(\omega))$, the theorem gives different $(h_1, ..., h_T)$, which can be known after the observation and gives no predictions of new prices. Therefore, the theorem asserts that the class Γ_{sep} is too large to have a meaningful result from the Γ_{sep}-MSE problem. On the other hand, the $\Gamma(1, 1, \frac{1}{2})$-MSE problem in Section 5 has a clear-cut contrast with Theorem 6.1: There 18 average rental prices are explained by the choice of parameters by changing essentially 3 parameter values.

Proof of Theorem 6.1: Let us denote $(\overline{P}_1(\omega), ..., \overline{P}_T(\omega))$ by $(p_1, ..., p_T)$. Also, let $G(k) = \sum_{t=1}^{k} |D_t|$. We assume without loss of generality that $p_1 \geq ... \geq p_T$. First, we let $g : \mathbf{R}_+ \to \mathbf{R}$ be any monotone, strictly concave and continuous function with $\lim_{m_i \to +\infty} g(m_i) = +\infty$.

Also, let $h_0 = 0$. Then choose $I_{G(T)} = I_n, I_{G(T-1)}, ..., I_{G(1)}$ and define $h_T, h_{T-1}, ..., h_1$ inductively as follows: The base case is as follows:

(T-0): Choose an income level $I_{G(T)} = I_m$ so that $I_m > p_T > 0$, and then define $h_T := h_0 + g(I_m) - g(I_n - p_T)$.

This choices of I_n and h_T are possible by the monotonicity of g. In this case, $h_T > h_0 = 0$.

[12] The reader may recall the Debrue-Mandel-Sonnenshein Theorem in general equilibrium theory (see Mas-Colell, *et al* [16]) stating that any demand function with a certain required condition is derived from some economic model. While Theorem 6.1 is a negative result as argued, the Debrue-Mandel-Sonnenshein theorem is not in that it describes the equivalence between the set of demand curves and the set of economic models. In this sense, the similarity is rather superficial.

Let k be an arbitrary number with $1 \leq k \leq T$. The inductive hypothesis is that $I_{G(k)}$ and h_k are already defined. First, we choose $I_{G(k-1)}$ so that

(k-1) : $I_{G(k-1)} > p_{k-1}$ and $I_{G(k-1)} \geq I_{G(k)}$.

This choice is simply possible. Then we define h_{k-1} by

(k-2) : $h_{k-1} = h_k + g(I_{G(k-1)} - p_k) - g(I_{G(k-1)} - p_{k-1})$.

Since $g(I_{G(k-1)} - p_{k-1}) \leq g(I_{G(k-1)} - p_k)$, we have $h_{k-1} \geq h_k$.

By the above induction definition, we have $I_n, I_{G(T-1)}, ..., I_{G(1)}$ and $h_T, h_{T-1}, ..., h_1$. We also choose other I_i's ($i \neq n$ and $i \neq G(k)$ for $k = 1, ..., T-1$) so that $I_n \leq I_{n-1} \leq ... \leq I_1$.

Thus, we have the utility function $u(\mathbf{e}^k, m_i) = h_k + g(m_i)$ for $(\mathbf{e}^k, m_i) \in X$. By the above inductive definition, $(p_1, ..., p_T)$ satisfies the recursive equation (4.9).

We define the cost function $C_j(\cdot)$ for the landlord j with $\{j\} = N_k$. We assume $0 < c_k \leq p_k$ for all $k = 1, ..., T$. Then, by Lemma 4.4, $(p_1, ..., p_T)$ is the maximal competitive price vector of $\mathbb{E}$ with $y_k = |D_k|$ for $k = 1, ..., T$. ■

6.2. Linear Regression and Linear Utility Functions

Now, a reader may be curious about comparisons between our equilibrium-econometric analysis and the standard econometrics approach. In this section, we compare our approach only with linear regression.

We assume that there are L attributes for the basic utility function U for each household, and the domain of U is expressed as $Y = \mathbf{R}_+^L \times \mathbf{R}_+$. In the example of Section 2.2, there are only two attributes, the commuting time and the size of an apartment. A linear basic utility function over Y is expressed as

$$U(a_1, ..., a_L, m_i) = \sum_{l=1}^{L} \alpha_l a_l + m_i \text{ for all } (a_1, ..., a_L, m_i) \in Y. \tag{6.1}$$

Here, a_l represents the degree of the l-th attribute of an apartment and α_l is its coefficient. We denote the set of all basic utility function of the form (6.1) by $\mathcal{U}_{lin}$.

An *attribute vector* $\tau^k = (\tau_1^k, ..., \tau_L^k)$ in $\mathbf{R}^L$ is given for each $k = 0, 1, ..., T$. That is, the choice $\mathbf{e}^k$ gives the attribute vector τ^k. It means that an apartment in category k has the magnitudes $\tau_1^k, ..., \tau_L^k$ of attributes $1, ..., L$. For $k = 0$, τ^0 is interpreted as the attributes of the outside option. In Example of Section 2.2, category $k = 5$ (Nakano, size:$25 - 45$) has the attribute vector $\tau^5 = (18\,\text{min}, 35m^2)$. Then, each U in $\mathcal{U}_{lin}$ determines

$$u(\mathbf{e}^k, m_i) = U(\tau^k, m_i) = \sum_{l=1}^{L} \alpha_l \tau_l^k + m_i \text{ for all } k = 0, 1, ..., T. \tag{6.2}$$

Now, we define the subclass Γ_{lin} of Γ_{sep} by

$$\{\mathbb{E} \in \Gamma_{sep} : u \text{ is determined by } U \in \mathcal{U}_{lin} \text{ with some } \tau^0, ..., \tau^T\}. \tag{6.3}$$

The boundary condition $u(0, I_i) > u(\mathbf{e}^k, 0)$ for all $k = 1, ..., T$ should hold for $(M, u, I; N, C) \in \Gamma_{lin}$, because it belongs to Γ_{sep} by (6.3). Once this set is defined, we have the Γ_{lin}-MSE problem. Our present target is to show that this problem is equivalent to linear regression.

Consider the following the *linear regression model*:

$$p_k = \sum_{l=1}^{L} \alpha_l \tau_l^k + \beta + \boldsymbol{\epsilon}_k \text{ for } k = 1, ..., T. \tag{6.4}$$

That is, rental prices, $p_1, ..., p_T$, are assumed to be linear functions of attributes $1, ..., L$, which are explanatory variables. Coefficients $\alpha_1, ..., \alpha_L$ and constant β will be estimated. Given the housing magazine $P_D^o(\omega)$ as data, we estimate them by minimizing the sum of square residuals, i.e., *the method of least squares*. It is formulated by the following minimization problem:

$$\min_{\alpha,\beta} \sum_k \sum_d (P_{kd}^o(\omega) - p_k)^2 = \min_{\alpha,\beta} \sum_k \sum_d \left(P_{kd}^o(\omega) - (\sum_{l=1}^{L} \alpha_l \tau_l^k + \beta) \right)^2. \tag{6.5}$$

This is a no-constraint minimization problem and has a solution $(\hat{\alpha}, \hat{\beta})$.

Now, we compare (6.5) with the Γ_{lin}-MSE problem. First, we should observe a certain difference between these two approaches. That is, the minimization problem (6.5) makes sense for any given data set $P_D^o(\omega)$: Even if the data $P_D^o(\omega)$ consisted of all negative elements, (6.5) would still provide some solution $\hat{\alpha} = (\hat{\alpha}_1, ..., \hat{\alpha}_L)$ and $\hat{\beta}$. On the other hand, the Γ_{lin}-market estimation problem does not make sense: It cannot predict negative prices since if the estimated rental price for category k is negative (nonpositive), landlord k provides no apartments, i.e., condition $y_k = |D_k^o|$ is violated. Thus, we need certain conditions on the data $P_D^o(\omega)$ to guarantee that such a case does not happen. The following lemma gives such conditions.

Lemma 6.2. If $p = (p_1, ..., p_T)$ is a maximal competitive price vector in any market model $\mathbb{E} = (M, u, I; N, C)$ in Γ_{lin}, then there is some β such that

$$\beta < -\sum_{l=1}^{L} \alpha_l \tau_l^0; \tag{6.6}$$

and $p_1, ..., p_T$ are described as

$$p_k = \sum_{l=1}^{L} \alpha_l \tau_l^k + \beta > 0 \text{ for all } k = 1, ..., T, \tag{6.7}$$

where $(\alpha_1, ..., \alpha_L)$ is the coefficients of the utility function u in $\mathbb{E}$.

Proof. Let (p, x, y) be any competitive equilibrium in $\mathbb{E} = (M, u, I; N, C)$ in Γ_{lin} with $|D_k^o| = y_k > 0$ for all $k = 1, ..., T$. Without loss of generality, we assume that $p_k \geq p_T$ for $k = 1, ..., T-1$. First, we show

$$p_k - p_T = \sum_l \alpha_l \tau_l^k - \sum_l \alpha_l \tau_l^T \text{ for all } k = 1, ..., T. \tag{6.8}$$

Suppose that this is shown. Now, let $\beta = p_T - \sum_l \alpha_l \tau_l^T$. We have, by (6.8), $p_k = \sum_l \alpha_l \tau_l^k + \beta$ for $k = 1, ..., T$. Since $|D_k^o| = y_k > 0$ and the cost c_k for one apartment unit in category k is positive for all $k = 1, ..., T$, the market price is equal to or greater than c_k. Hence $p_k > 0$ for all $k = 1, ..., T$, which is (6.7). The constant β must satisfy $\beta < -\sum_l \alpha_l \tau_l^0$, since any household i in D_T^o chooses the T-th apartment rather than $(0, I_i)$, i.e., $U(e^T, I_i - p_T) = \sum_l \alpha_l \tau_l^T + I_i -$ $(\sum_l \alpha_l \tau_l^T + \beta) = I_i - \beta > u(0, I_i) = h_0 + I_i = \sum_l \alpha_l \tau_l^0 + I_i$.

Now let us prove (6.8). Since $|D_k^o| > 1$ for $k = 1, ..., T$, we can take a household i with $x_i = e^k$, i.e., he chooses $x_i = e^k$ as a utility maximization point under $p = (p_1, ..., p_T)$. Hence,

$$\sum_l \alpha_l \tau_l(k) + I_i - p_k \geq \sum_l \alpha_l \tau_l(T) + I_i - p_T.$$

By the same argument for a household i' with $x_{i'} = e^T$, we have

$$\sum_l \alpha_l \tau_l(t) + I_{i'} - p_k \leq \sum_l \alpha_l \tau_l(T) + I_{i'} - p_T.$$

Equation (6.8) follows from these two inequalities. ■

Lemma 6.3 (Sustainability): If $p = (p_1, ..., p_T)$ is expressed by some $\alpha = (\alpha_1, ..., \alpha_L)$ and $\beta < -\sum_l \alpha_l \tau_l^0$ as (6.7), then p is sustained by a market model $\mathbb{E}$ in Γ_{lin}.

Proof: Without loss of generality, we assume $p_1 \geq ... \geq p_T$.

First, we define the basic utility function by $U(a_1, ..., a_L, m_i) = \sum_l \alpha_l a_l + m_i$. Let $I_1, ..., I_n$ be incomes with $I_1 > ... > I_n > p_1$. We define cost functions $C_1, ..., C_{T-1}$ by (2.3) with $w_k = |D_k^o|$ and $c_k < p_k$ for $k = 1, ..., T-1$. Define C_T by (2.4) with $w_T^0 > |D_T^o|$ and $c_T = p_T$. In this case, for each $k = 1, ..., T$, $y_k = |D_k^o|$ maximizes landlord k's profits.

The prices given by (6.7) satisfies the rent equation (4.9). Also, since $\beta < -\sum_l \alpha_l \tau_l^0$, each household i has the utility:

$$u(e^k, I_i - p_k) = I_i - \beta > I_i + \sum_l \alpha_l \tau_l^0 = u(\mathbf{0}, I_i).$$

Hence, his choice of an apartment is better than choosing no apartments. ■

Conditions (6.6) and (6.7) are on the data in the housing magazine P_D^o. The second condition states that the estimated prices are positive, and should be satisfied by any real data. The first means that a household having a linear utility function prefers renting an apartment to not renting in this market. This condition is needed to take into account the outside opportunity "**0**" from the domain of a utility function $\{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\} \times \mathbf{R}_+$.

We have the following equivalent theorem between the Γ_{lin}-market estimation problem and linear regression.

Theorem 6.4 (Linear Regression): We assume that any optimal solution of (6.5) satisfies (6.6) and (6.7). Now, let (α, β) be any $(L+1)$-vector. Then, a vector (α, β) is a solution of the minimization problem (6.5) if and only if there is a solution model $\mathbb{E} = (M, u, I; N, C)$ of the Γ_{lin}-MSE problem such that u of $\mathbb{E}$ is determined by U of (6.1) with (α, β) and its resulting price vector is given as

$$p_l = \sum_{l=1}^{L} \alpha_l \tau_l^k + \beta \quad \text{for all } k = 1, ..., T. \tag{6.9}$$

Proof. Suppose that $p = (p_1, ..., p_T)$ is given by a solution of the Γ_{lin}-market estimation problem. Then it is a feasible solution of the problem (6.5) with $\beta < -\sum_l \alpha_l \tau_l^0$ by Lemma 6.2. Any solution of (6.5) with $\beta < -\sum_l \alpha_l \tau_l^0$ should be supported by some model by Lemma 6.3. Hence, $p = (p_1, ..., p_T)$ is an optimal solution of the problem (6.5) with $\beta < -\sum_l \alpha_l \tau_l^0$. The converse can be shown by tracing this argument backward. ■.

In the example of Section 2.2, $(\alpha, \beta) = ((\alpha_1, \alpha_2), \beta)$ is given as $((-0.74, 1.65), 41.3)$, i.e.,

$$U(t, s, m_i) = -0.74t + 1.65s + m_i \tag{6.10}$$

$$p_{k(t,s)} = -0.74t + 1.65s + 41.3$$

Since (6.7) and $\beta < -\sum_l \alpha_l \tau_l^0$ hold, we have a linear market model $\mathbb{E}$. Incidentally, the discrepancy measure $\eta(p)(\omega^0) = 1.123$, which is larger than the corresponding value $\eta(p)(\omega^0)$ given in Section 2.2 or Section 5 except (1).

7. Likelihood Ratio Test (Imcomplete)

We have developed the econometric method through market equilibrium theory, and have obtained the best prediction value $\eta(p)(\omega^o) = 1.025$ of the discrepancy measure in Section 5. Nevertheless, this value solely does not state how good the prediction is. After all, it would be the only way to compare our result with the result in terms of some other criteria. In this section, we consider the likelihood ratio test for our framework. First, we need the maximum likelihood estimators for the mean and variant for the price of each category. Then, we will consider the likelihood ratio test for our estimation. Theoretically, it could be possible to do hypothesis testing based on the discrepancy measure $\eta(p)(\omega)$ and if this is possible, this would give a convenient way to think about the value such as $\eta(p)(\omega^o) = 1.028$. However, at present moment, the computer implementation of probability needs too much steps, and we should be satisfied by a more standard approach, *the likelihood ratio test*, which is not directly related to the discrepancy measure. Still, we need to develop one further step, since our problem is multi-dimensional.

7.1. Maximum Likelihood Estimators for the Means and Variants of the Rental Prices

For In this section, we assume that each $\boldsymbol{\epsilon}_k^0$ of $\boldsymbol{\epsilon}_1^0, ..., \boldsymbol{\epsilon}_T^0$ follows normal distribution $N(0, \sigma_k^2)$ and they are independent. Then, $P_k^o = p_k^o + \boldsymbol{\epsilon}_k^0$ follows normal distribution $N(p_k^o, \sigma_k^2)$. Here, both p_k^o and σ_k^2 are unknown. First, the maximum likelihood estimation of p_k^o and σ_k^2 is obtained by the following maximization problem from the same set $P_D^o(\cdot)$: for each $\omega \in \Omega$,

$$\max_{p_k, \sigma_k} \log \prod_{k=1}^{T} (\frac{1}{\sqrt{2\pi}\sigma_k})^{|D_k^o|} \exp - \sum_{d \in D_k^o} \frac{(P_{kd}^o(\omega) - p_k)^2}{2(\sigma_k)^2} \tag{7.1}$$

By the standard argument (cf. Rohatgi [19], p.678), the maximum likelihood estimator of p_k is given as the average estimator $\overline{P}_k^o$ of (4.2), and the maximum likelihood estimator for σ_k is given by

$$\overline{\sigma}_k(\omega)^2 = \frac{\sum_{d \in D_k^o} (P_{kd}^o(\omega) - \overline{P}_k(\omega))^2}{|D_k^o|} \quad \text{for all } \omega \in \Omega. \tag{7.2}$$

We will use those estimators $\overline{P}^o = (\overline{P}_1^o, ..., \overline{P}_T^o)$ and $\overline{\sigma}^2 = ((\overline{\sigma}_1)^2, ..., (\overline{\sigma}_T)^2)$ as the benchmark.

Now, suppose that a predicted rent vector is given as $r = (r_1, ..., r_T)$ such as the price vector given in Section 5. Giving one constraint $p = r$ on the maximization problem (7.1), we have the maximum likelihood estimator of $\sigma(r) = (\sigma_1(r), ..., \sigma_T(r))$ is given as: for $k = 1, ..., T$,

$$\sigma_k(r)^2 = \frac{\sum_{d \in D_k^o} (P_{kd}^o(\omega) - r_k)^2}{|D_k^o|} \quad \text{for } \omega \in \Omega. \tag{7.3}$$

Thus, each $\sigma_k(r)$ depends upon r_k. Now, we have two sets of estimators $(\overline{P}^o, \overline{\sigma}^2)$ and $(r, \sigma(r))$. Hence, when $r = (r_1, ..., r_T)$ is given by our equilibrium theory, the variant $\sigma_k(r)^2$ for category k are obtained simply as the total sum, divided by $|D_k^o|$, of squared errors around each r_k.

7.2. Likelihood Ratio Test

We assume that the true market prices given as $p^o = (p^o_1, ..., p^o_T)$ and variants $\sigma^2 = (\sigma^2_1, ..., \sigma^2_T)$ are unknown and only the data set P^o_D is available. According to the above consideration of maximum likelihood estimators, we represent them by $\overline{P}^o = (\overline{P}^o_1, ..., \overline{P}^o_T)$ and $\overline{\sigma}^2 = ((\overline{\sigma}_1)^2, ..., (\overline{\sigma}_T)^2)$ based on the data set P^o_D. On the other hand, our theory has a prediction $r = (r_1, ..., r_T)$, the best of which was given in Section 5.2. The hypothesis testing takes the form that under the hypothesis that $r = (r_1, ..., r_T)$ is the true market prices, how probable the data set P^o_D happens. If this probability is small, say 0.05, the hypothesis would be reject. Otherwise, $r = (r_1, ..., r_T)$ and its underlying basic utility function given (5.7) would remain as candidates for the true price vector and market structure.

Now, we formulate the above argument as the testing hypothesis:

H_0(*null hypothesis*): $p^o = r$.

In particular, we follow the idea of the likelihood ratio test. However, it is a higher dimensional likelihood ratio test. First, we need to reconsider the theory slightly.

First, we compare the maximal likelihood given by $(\overline{P}^o, \overline{\sigma}^2)$ with the likelihood given $(r, \sigma(r)^2)$ by taking the ratio:

$$\rho = \frac{\max_{p,\sigma} \prod_k (\frac{1}{\sqrt{2\pi}\sigma_k})^{|D_k|} \exp - \sum_{D_k} \frac{(P_{kd}-p_k)^2}{2(\sigma_k)^2}}{\max_{\sigma} \prod_k (\frac{1}{\sqrt{2\pi}\sigma_k})^{|D_k|} \exp - \sum_{D_k} \frac{(P_{kd}-r_k)^2}{2(\sigma_k)^2}},$$

where this ratio ρ depends upon a primitive event ω, but ω is not written for notational simplicity. The numerator is the likelihood given by $(\overline{P}^o, \overline{\sigma}^2)$, and the denominator is given by $(r, \sigma(r)^2)$. This ratio ρ is a random variable over Ω.

Plugging the formulae of $(\overline{P}^o, \overline{\sigma}^2)$ and $(r, \sigma(r))$, the likelihood ratio becomes

$$\rho = \frac{\prod_k (\frac{1}{\sqrt{2\pi}\overline{\sigma}_k})^{|D_k|} \exp[-\frac{|D_k|}{2}].}{\prod_k (\frac{1}{\sqrt{2\pi}\sigma_k(r_k)})^{|D_k|} \exp[-\frac{|D_k|}{2}]} = \prod_k \left(\frac{\sum_{D_k}(P_{kd} - r_k)^2}{\sum_{D_k}(P_{kd} - \overline{P}_k)^2} \right)^{|D_k|/2}.$$

Since $\sum_{D_k}(P_{kd} - r_k)^2 = \sum_{D_k}(P_{kd} - \overline{P}_k)^2 + |D_k| (\overline{P}_k - r_k)^2$ by the null hypothesis H_0, we have

$$\rho = \prod_k \left(1 + \frac{|D_k| (\overline{P}_k - r_k)^2}{\sum_{D_k}(P_{kd} - \overline{P}_k)^2} \right)^{|D_k|/2} = \prod_k \left(1 + \frac{(\overline{P}_k - r_k)^2}{\sum_{D_k}(P_{kd} - \overline{P}_k)^2 / |D_k|} \right)^{|D_k|/2}.$$

Now, letting

$$T_k = \frac{\overline{P}_k - r_k}{\sqrt{V_k / |D_k|}} \text{ and } V_k = \sum_{D_k}(P_{kd} - \overline{P}_k)^2/(|D_k| - 1) \text{ for } k = 1, ..., T,$$

we have

$$\rho = \prod_{k=1}^{T} (1 + \frac{(T_k)^2}{|D_k| - 1})^{|D_k|/2}. \tag{7.4}$$

Since $E(P_{kd}) = r_k$ for all $d \in D_k$ by H_0,, each T_k follows the t-distribution of freedom $|D_k| - 1$.

In the unidimensional case, i.e., $T = 1$, we can use the table of the 1-dimensional t-distribution. However, our target is the multidimensional case, i.e., $T = 18$ in the Tokyo rental housing market example. Hence, first, we make the histogram of $\log \rho(\omega)$ by computer simulation, which is simply based on the random number generator based on a t-distribution. Figure 7.1 gives the histogram of $\log \rho(\omega)$.

Finally, we plug the rent vector $r = (r_1, ..., r_T)$ given in Section 5.2 and the data set P_D^o already used in Section 2.2. Then, we find that the value $\log \rho(\omega_0)$ is in the interval before the critical point c_0 for the probability 0.05. Hence, we do not reject the null hypothesis H_0.

8. Conclusions

We have developed the equilibrium-econometric analysis of rental housing markets. Our analysis has the three salient features as stated in the beginning of Section 1. Let us look at these features.

The first salient feature, i.e., the development of an econometric analysis directly through a market equilibrium theory, is shown in the entire paper. It provides a bridge between a market equilibrium theory and an econometric analysis. This bridge is built by focussing on the housing magazine as serving information about rental houses to economic agents (households, landlords) and as the source of data for the econometric analysis. This explains the source of the error terms in our econometric analysis. This is the former half of the second feature mentioned in Section 1.

This second feature forces us to modify our market equilibrium theory so that market prices are perturbed by some error terms. Nevertheless, we have shown that we can ignore the error terms, which is the correspondence theorem (Theorem 3.1). This is the latter half of the second salient feature.

Then the third salient feature was discussed in Sections 4 and 5. We introduced the discrepancy measure as the ratio of the total sum of square residuals from the predicted rental prices over the that from the average prices. In the best estimation we obtained in Section 5, the measure takes about the value 1.025. Also, this estimation result has some implications on the law of diminishing marginal utility. It does not hold in a strict form for the commuting time-distance to the office area from apartments, but it holds strictly for the apartment-sizes and consumption other than housing. The degree for consumption is quite large, which implies that income effects on the housing quality is quite large, i.e., there is a strong tendency to rent a better apartment if a household has a larger income.

In spite of the above seemingly very accurate prediction with respect to our discrepancy measure, it was shown that the best predicted rental prices are still rejected in our likelihood ratio test. The likelihood ratio test compares the predicted rental prices with the "true" rental prices. Since our data set of apartments listed is very large, it would be rejected by the test unless the predicted prices are very close to the "true prices". Our best estimation result given in Section 5 is not yet close enough to the "true" one.

Scrutinizing closely on the calculations for the likelihood ratio test, we would find some reason for the deviation of our prediction. For example, only some category show some significant deviations. This fact may imply that these categories have some different reasons not following our predictions, for examples, the average ages of apartment units are older than the others. Or, maybe, we may find some other sociological reasons such as high crime rates. These will be subjects of the future paper.

We have many untouched problems. They are divided into three classes: (1) theoretical problems; (2) applications to housing markets along different railway lines and in different cities (in Japan and other countries); and (3) applications to panel data. They include a lot of problems, but here, we discuss some problems in each class.

(1): *Theoretical problems*: One is the formation of individual subjective estimation of the price distribution from the housing magazine. In this paper, we simply assumed that each economic agent forms such an estimation, and showed the correspondence theorem, which is interpreted as meaning that if such estimations have small variances, then we could ignore the disturbances in the econometric study. The formation of an estimation of a price distribution may be a theoretical problem of interests from the viewpoint of inductive game theory (Kaneko-Matusi [11] and Kaneko-Kline [10]): The question is whether an agent with a limited analytical ability can derive a meaningful estimation or what he can derive. This should be studied not only theoretical but also empirically using the data.

(2): *Applications to housing markets along different railway lines and in different cities*: In this paper, we discussed only a submarket along the Chuo railway line of JR. The authors have been applying our theory to various railway lines in the Tokyo area, but those have been done as pilot studies. A more systematic study of those rental housing markets in different places and in different time will be an important future problem. Then, for example, the law of diminishing marginal utility can be tested with different housing markets.

One important feature in the Tokyo area (as well as in Japan) is that we find almost no clear-cut segregations. On the other hand, in many American cities we find segregations with different income groups and/or ethnic groups in housing markets. Our approach can be applied to those cities with no conceptual difficulty, but we have some technical difficulties: The assignment market has not been developed to be capable to treat such segregation problems or widespread externalities. For example, calculation of a competitive equilibrium has been done by using a certain result (Lemma 4.4 in this paper). Taking income or ethnic differences and segregation, we may need to return to a more general procedure to calculate a competitive equilibrium.

An application to such cases will make our theory more fruitful. Also, it may be related to the problem of discrimination and prejudices, which was also discussed in inductive game theory. In this sense, an extended study of our approach will make some contributions to inductive game theory.

(3): *Applications to panel data*: This is related to (2). The housing magazine is issued daily or weekly. Accumulating the housing magazines, we can regard them as panel data. Then, we study the temporary changes of the housing market. One small problem is to check the comparative statics results obtained in Kaneko *et al* [12] and Ito [7] with those railway lines. In doing so, we may have better understanding of the structure of the housing market.

The above three problems are simply listed and are interrelated to each other. Also, each takes some time to be checked. Nevertheless, a steady study of them gives a better understanding of our socio-economic behavior and the social-economic structure of society.

9. Appendix

Proof of Theorem 3.1.(1): Since the condition BDS in $\mathbb{E}$ are preserved to $\mathbb{E}(\boldsymbol{\epsilon}\backslash\boldsymbol{\epsilon}^{M\cup N,\nu})$, we should show that the ε-modifications of UM and PM hold for $\mathbb{E}(\boldsymbol{\epsilon}\backslash\boldsymbol{\epsilon}^{M\cup N,\nu})$ for all $\nu \geq$ some ν_0. However, we show it only for a household $i \in M$. It is similar to prove it for $j \in N$. The additional assumption that the domain of the profit function is finite is used for it.

Now, let ε be an arbitrary positive number, and let $P^{i,\nu} = p + \boldsymbol{\epsilon}^{i,\nu}$ for $\nu = 1, ...$ Consider an arbitrary household i. Let $z^i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$ with $I_i - pz^i \geq 0$. Then, by UM,

$$u_i(x^i, I_i - px^i) \geq u_i(z^i, I_i - pz^i). \tag{9.1}$$

We should consider two cases: $x^i = \mathbf{e}^t$ $(t \neq 0)$ and $x^i = \mathbf{0}$. Consider the case of $x^i = \mathbf{e}^t$.

As $\delta \to 0$, the utility value $u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t))$ converges to $u_i(x^i, I_i - px^i) = u_i(\mathbf{e}^t, I_i - p_t)$ by continuity of u_i in the composite commodity. Since $\{\boldsymbol{\epsilon}^{i,\nu}\}$ converges to $\mathbf{0}$ in probability, for any $\delta > 0$, there is a $\nu(\delta)$ such that for any $\nu \geq \nu(\delta)$,

$$\mu(\{\omega : \left\|\boldsymbol{\epsilon}^{i,\nu}(\omega)\right\| < \delta\}) < 1 - \frac{\delta}{2}. \tag{9.2}$$

Since u_i is monotonic in the composite commodity, it holds that for all $\nu \geq \nu(\delta)$,

$$EU_i(\mathbf{e}^t, I_i - P^{i,\nu} \cdot \mathbf{e}^t) \geq (1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i). \tag{9.3}$$

Since the right-hand side of (9.3) converges to $u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t)$ as $\delta \to 0$, there is some δ_1 such that for all $\delta \geq \delta_1$,

$$(1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i) \geq u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) - \frac{\varepsilon}{2}. \tag{9.4}$$

Since δ in (9.3) is arbitrary, we can take the above δ_1 for δ. Hence, from (9.3) for δ_1 and (9.4), for any $\nu \geq \nu(\delta_1)$, we have

$$EU_i(\mathbf{e}^t, I_i - P^{i,\nu}\mathbf{e}^t) \geq u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) - \frac{\varepsilon}{2}. \tag{9.5}$$

Now, let z^i be an arbitrary element in $\{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$. Again, since u_i is monotonic in the composite commodity, we have, using (9.2), for all $\nu \geq \nu(\delta)$,

$$(1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_{t'} - \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i) \tag{9.6}$$

$$\geq \quad EU_i(\mathbf{e}^t, I_i - P^{i,\nu}\mathbf{e}^t) \geq EU_i(z^i, I_i - P^{i,\nu}z^{it})$$

The first term converge to $u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t)$ as $\delta \to 0$. Hence, there is some δ_2 such that for any $\delta \geq \delta_2$,

$$u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) + \frac{\varepsilon}{2} \geq (1 - \frac{\delta}{2})u_i(\mathbf{e}^t, I_i - (p_t + \delta)\mathbf{e}^t)) + \frac{\delta}{2}u_i(\mathbf{e}^t, I_i). \tag{9.7}$$

Hence, from (9.6) and (9.7), it holds that for any $\nu \geq \nu(\delta_2)$,

$$u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) + \frac{\varepsilon}{2} \geq EU_i(z^i, I_i - P^{i,\nu}z^i) \tag{9.8}$$

Now, we take $\delta_3 = \min(\delta_1, \delta_2)$. Then, it follows from (9.5) and (9.8) that for all $\nu \geq \delta_3$,

$$EU_i(\mathbf{e}^t, I_i - P^{i,\nu}e^t) + \frac{\varepsilon}{2} \geq u_i(\mathbf{e}^t, I_i - p_t\mathbf{e}^t) \geq EU_i(z^i, I_i - P^{i,\nu}z^i) - \frac{\varepsilon}{2}.$$

Connecting the first term with the last term, we have the final target: $EU_i(\mathbf{e}^t, I_i - P^{i,\nu}\mathbf{e}^t) + \varepsilon \geq EU_i(z^i, I_i - P^{i,\nu}z^i)$.

In the case $x^i = \mathbf{0}$, the first half of the above proof should be modified.

(2): Suppose the *if clause* of the assertion. Now, let $\{\varepsilon_\lambda\}$ a decreasing positive and converging sequence to 0. Then, for each ε^{λ}, we can find a ν_{λ} such that for all $\nu \geq \nu_{\lambda}$, (p, x, y) is an ε_{λ}-competitive equilibrium in $\mathbb{E}(\boldsymbol{\epsilon} \backslash \boldsymbol{\epsilon}^{M \cup N, \nu})$. It suffices to show that the utility maximization and profit maximization hold under price vector p.

Consider utility maximization for x_i. Then, we have, for all λ,

$$EU_i(x_i, I_i - P^{i,\nu_\lambda} x_i) + \varepsilon_\lambda \geq EU_i(z^i, I_i - P^{i,\nu_\lambda} z_i) \text{ for all } z_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}. \quad (9.9)$$

Now, let $z_i \in \{\mathbf{0}, \mathbf{e}^1, ..., \mathbf{e}^T\}$ be fixed. Suppose $I_i - pz_i > 0$. Then, both $EU_i(x_i, I_i - P^{i,\nu_\lambda} x_i)$ and $EU_i(z^i, I_i - P^{i,\nu_\lambda} z_i)$ converge to $u_i(x_i, I_i - px_i)$ and $u_i(z_i, I_i - pz_i)$, which together with (9.9) imply

$$u_i(x_i, I_i - px_i) \geq u_i(z^i, I_i - pz_i).$$

Now, suppose $I_i - pz_i = 0$. Since $u_i(z_i, 0) < u_i(0, I_i)$ by Assumption A, there is a λ_0 such that for all $\lambda \geq \lambda_0$,

$$EU_i(z^i, I_i - P^{i,\nu_\lambda} z_i) > u_i(z_i, 0).$$

Hence, by (9.9), we have $u_i(x_i, I_i - px_i) \geq u_i(z_i, 0) = u_i(z^i, I_i - pz_i)$.

The profit maximization for y_j can be proved even in a simpler manner. ■

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