The Dynamics of Knowledge Diversity and Economic Growth*

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Abstract

How is long run economic growth related to the diversity of knowledge? We formulate and study a microeconomic model of knowledge creation, through the interactions among a group of R & D workers, embedded in a growth model to address this question. Income to these workers accrues as patent income, whereas transmission of newly created knowledge to all such workers occurs due to public transmission of patent information. Our model incorporates two key aspects of the cooperative process of knowledge creation: (i) heterogeneity of people in their state of knowledge is essential for successful cooperation in the joint creation of new ideas, while (ii) the very process of cooperative knowledge creation affects the heterogeneity of people through the accumulation of knowledge in common. The model features myopic R & D workers in a pure externality model of interaction. Surprisingly, in the general case for a large set of initial conditions we find that the equilibrium process of knowledge creation converges to the most productive state, where the population splits into smaller groups of optimal size; close interaction takes place within each group only. Equilibrium paths are found analytically. Long run economic growth is positively related to both the effectiveness of pairwise R & D worker interaction and to the effectiveness of public knowledge transmission. JEL Classification Numbers: D83, O31, D90 Keywords: knowledge creation, knowledge externalities, microfoundations of endogenous growth, knowledge diversity and growth

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1 Introduction

How is economic growth related to the diversity of knowledge? How does knowledge diversity change as an economy grows? Can more effective public knowledge transmission, via the patenting process or the internet, cause the knowledge base to become too homogeneous and slow growth? Given spillovers in the creation of new knowledge, is the equilibrium knowledge production path efficient?

To address these questions, we attempt to provide microfoundations for aggregate models of knowledge creation and transfer. The basic framework that employs knowledge creation as a black box driving economic growth is usually called the endogenous growth model. Here we make a modest attempt to open that black box. The literature using this black box includes Shell (1966), Romer (1986, 1990), Lucas (1988), Jones and Manuelli (1990), and many papers building on these contributions.[1]

In particular, the model proposed below is closely related to the endogenous growth model developed by Romer (1990) in which R & D firms invest resources to develop new products. In Romer's model, the productivity of each R & D firm rises in proportion to the stock of general knowledge capital; the latter is assumed to be equal to the cumulative number of products invented in the R & D sector in the past. In addition, all workers in the R & D sector are assumed to be homogeneous. Hence, in Romer's model, when labor is the unique input in the R & D sector, the number of new products developed per unit of time is also proportional to the number of R & D workers at that time.

While maintaining the assumption of monopolistic competition in the sector that produces horizontally differentiated consumption goods, this paper introduces several innovative features into the details of the R & D sector. First, at any given time, all knowledge workers (K-workers) engaged in R & D are heterogeneous in the sense that for any pair of K-workers, each has knowledge distinct from the other as well as a stock of knowledge in common. Such heterogeneity in K-workers provides them with an opportunity to cooperate in R & D work. Second, the heterogeneity is endogenous to the model. At each moment of time, each K-worker will want to conduct research with their best partner (or partners); the new knowledge jointly created becomes shared

[1] We note that differentiation of agents in terms of *quality* (or vertical characteristics) of knowledge is studied in Jovanovic and Rob (1989) in the context of a search model. In contrast, our model examines (endogenous) horizontal heterogeneity of agents and its effect on knowledge creation and consumption.

knowledge, thus dynamically building up knowledge in common. When people are not meeting, their knowledge bases grow more different. Thus, the history of meetings and their content is important. Moreover, some of the new knowledge created by any K-worker, either alone or in partnership with others, is revealed in the form of patent registration and thus learned by all K-workers, yielding additional knowledge in common. In this way, the heterogeneity or diversity of all K-workers changes endogenously over time. Third, the effectiveness of cooperation between K-workers can change over time, and this change is endogenous. If two K-workers have too much knowledge in common, little synergy can be expected from their joint work, since neither brings originality to the partnership. Analogously, if two K-workers have very different knowledge bases, they have little common ground for communication, so their partnership will not be very productive. Thus, a partnership in knowledge creation is most productive when common and differential knowledge are in balance. Then, since the heterogeneity among K-workers changes endogenously over time, the effectiveness of cooperation among K-workers also changes endogenously.

We model endogenous agent heterogeneity, or horizontal agent differentiation, in order to look at the *permanent* effects of knowledge creation on growth.[2] For simplicity, we assume that it is not possible for more than two knowledge creators to meet or work at one time, though more than one couple can work simultaneously. When agents meet, they create new, shared knowledge, thus building up knowledge in common. When agents are not meeting with each other, their knowledge bases grow more different. The fastest rate of knowledge creation occurs when common and differential knowledge are in balance. The knowledge creation workers can work alone or with a partner. The suitability of partners depends on the stock of knowledge they have in common and their respective stocks of exclusive knowledge at a given time.

In order to provide microfoundations for behavior in the R & D sector, the model of knowledge production detailed in Berliant and Fujita (forthcoming) is embedded, with an extension to allow public knowledge transmission via the patenting process, in a growth model. A manufacturing sector produces consumption goods for both their workers and the knowledge workers, using a Dixit and Stiglitz (1977) monopolistic competition framework. To produce a consumption commodity under constant returns to manufacturing labor input,

[2] We employ a deterministic framework. It seems possible to add stochastic elements to the model, but at the cost of complexity. It should also be possible to apply the law of large numbers to a more basic stochastic framework to obtain equivalent results.

a patent must be purchased from the R & D sector. Manufacturing workers, firms, and consumers in the R & D sector are all farsighted, in the sense that they have rational expectations about prices. The knowledge workers themselves can either be farsighted or myopic in their choices concerning R & D partnerships; surprisingly, the equilibrium is the same.

For simplicity, we deal exclusively with the case when the agents are symmetric. Our model is analytically tractable, so we do not have to resort to simulations; we find each equilibrium path explicitly.

Our results are summarized as follows. When the initial state features relative homogeneity of knowledge between knowledge workers, the sink will be the most productive state, where the population splits into smaller groups of optimal size; close interaction takes place within each group only.[3] This optimal size is larger as the heterogeneity of knowledge is more important in the knowledge production process and as the transmission of public knowledge becomes more effective. The efficiency result is the most surprising to us, as we posit a model with myopic knowledge workers and with only externalities in interactions between knowledge workers, so one would not expect efficient outcomes.

Long run economic growth is positively related to both the effectiveness of pairwise knowledge worker interaction and, more importantly, to the effectiveness of public knowledge transmission. The latter is due, in part, to the endogenous adjustment of group size to a better public knowledge transmission technology. Finally, if we define efficiency constrained by the monopolistic competition environment for consumption goods, our equilibrium paths are constrained efficient.

The model is also at an intermediate level of aggregation. That is, although it is at a more micro level than large aggregate models such as those found in the endogenous growth literature, we do not work out completely its microfoundations. That is left to future research.

Section 2 gives the model and notation, Section 3 analyzes the equilibrium path of dynamics in the knowledge production sector, Section 4 analyzes the equilibrium growth path for the entire economy, whereas Section 5 explores the efficiency properties of the equilibrium path. Section 6 gives our conclusions and suggestions for future knowledge workers. Two appendices provide the proofs of key results.

[3] It would be reasonable to call these groups R & D teams.

2 The Model

In this section, we introduce the basic model. There are three types of activity in the economy. There are consumers of physical goods, producers of physical goods, and the R & D sector. The activities in the economy representing physical commodity production and consumption are standard models of product variety with monopolistic competition. The major difference between our model and others is the level of detail in the R & D sector, that generates patents sold to the producers of physical, differentiated products. We shall describe first the consumer side of the economy, namely a market for differentiated products. In the following subsection, we describe the production side of this market. Finally, we describe the R & D sector, the focus of our work.

To begin, there are two types of workers: knowledge workers (K-workers) engaged in R & D, and manufacturing workers (M-workers) producing differentiated products. For simplicity, we assume that the type of each worker is exogenously given, so workers cannot change sectors. Let N denote the number of K-workers, and let L denote the number of M-workers.

Before getting into the details of the model, it is useful to discuss the rationality assumptions we make regarding the agents. For the producers and the manufacturing workers, we assume that they all have perfect foresight, including knowledge of future prices. When knowledge workers consume, they also have perfect foresight.

The important assumption concerns knowledge workers when they make decisions about knowledge production, in particular which partner to work with at any given time or whether to work alone. In our previous work, we have used a myopic core solution concept. That is, workers in the R & D sector make decisions about their research teams in a cooperative manner but without looking ahead at the long term consequences. Such a concept will be used below. But we also show in Section 5 (under some restrictions) that the same solution arises if the workers have perfect foresight and use a core concept or utilitarian social welfare function. Thus, the solution path we propose for a large set of parameters and initial conditions is at the intersection of many solution concepts, and is efficient for the R & D sector (constrained efficient in the entire economy).

In our view, this result is strongest when considering the myopic core solution concept, since in that case we have postulated a model with externalities and R & D agents who are myopic, but attain a constrained efficient outcome in spite of this.

2.1 Consumers

First, we describe consumers' preferences (the time argument is suppressed when no confusion arises). All workers have the same instantaneous utility function given by

$$u = \left[\int_0^M q(h)^\rho \, dh \right]^{1/\rho} \qquad 0 < \rho < 1 \tag{1}$$

In this expression, M is the total mass of varieties available in the economy at a given time, whereas $q(h)$ represents the consumption of variety $h \in [0, M]$.

If E denotes the expenditure of a consumer at a given time while $p(h)$ is the price of variety h, then the demand function is as follows:

$$q(h) = Ep(h)^{-\sigma} P^{\sigma-1} \qquad h \in [0, M] \tag{2}$$

where P is the price index of varieties given by

$$P \equiv \left[\int_0^M p(h)^{-(\sigma-1)} dh \right]^{-1/(\sigma-1)} \tag{3}$$

Introducing (3) and (2) into (1) yields the indirect utility function

$$v = E/P$$

We now describe the behavior of an arbitrary consumer i, who is either a K-worker or an M-worker. If this consumer chooses an expenditure path, $E_i(t)$ for $t \in [0, \infty)$ such that $E_i(t) \geq 0$, then his indirect utility at time t is given by

$$v_i(t) = E_i(t)/P(t) \tag{4}$$

where $P(t)$ is the price index of the manufactured goods at time t.

The lifetime utility of consumer i at time 0 is then defined by

$$U_i(0) \equiv \int_0^\infty e^{-\gamma t} \ln[v_i(t)] dt \tag{5}$$

where $\gamma > 0$ is the subjective discount rate common to all consumers.

The intertemporal allocation of resources is governed by an interest rate equal to $v(t)$ at time t. We must now specify consumer i's intertemporal budget constraint, that is, the present value of expenditure equals wealth. Let $y_i(t)$ be the income that this consumer receives at time t. For any M-worker, their income at time t will be their wage at that time, whereas for any K-worker, their income at time t will be the value of the patents they create at that time. Then, the present value of income is given by

$$W_i(0) = \int_0^{\infty} e^{-\bar{\nu}(t)t} y_i(t) dt \tag{6}$$

where $\overline{\nu}(t) \equiv (1/t) \int_0^t \nu(\tau) d\tau$ is the average interest rate between 0 and t; in (6), the term $\exp[-\overline{\nu}(t)]$ converts one unit of income at time t to an equivalent unit at time 0. Using the budget flow constraint, Barro and Sala-i-Martin (1995, p. 66) show that the consumer's intertemporal budget constraint may be written as follows:

$$\int_0^{\infty} E_i(t) e^{-\overline{\nu}(t)t} dt = \omega_i + W_i(0) \tag{7}$$

where ω_i is the value of the consumer's *initial assets*, specified as follows:

$$\omega_i = 0 \text{ for } M\text{-worker } i$$

and

$$\omega_i = \frac{\Pi(0) \cdot M(0)}{N} \text{ for K-worker } i \tag{8}$$

So each K-worker owns the same number $\frac{M(0)}{N}$ of patents at time 0, where the price of patents at time 0 is $\Pi(0)$.

Then, if $E_i(\cdot)$ stands for an expenditure path that maximizes (5) subject to (7), the first order condition implies that

$$\dot{E}_i(t)/E_i(t) = \nu(t) - \gamma \qquad t \geq 0 \tag{9}$$

where $\dot{E}_i(t) \equiv dE_i(t)/dt$. Since (9) must hold for every consumer, it is clear that the following relation must hold

$$\dot{E}(t)/E(t) = \nu(t) - \gamma \qquad t \geq 0 \tag{10}$$

where $E(t)$ stands for the total expenditure in the economy at time t.

2.2 Producers

We now turn to the production side of the economy. We normalize the wage rate of manufacturing workers to 1:

$$w^M = 1 \qquad t \geq 0 \tag{11}$$

The production of any variety, say h, requires the use of the patent specific to this variety, which has been developed in the R & D sector. Once a firm has acquired the patent at the market price (which corresponds to this firm's fixed cost), it can produce one unit of this variety by using one unit of M-labor. When the manufacturer of variety h produces $q(h)$ units, the profit is

$$\pi(h) = [p(h) - 1]q(h)$$

which yields the equilibrium price common to all varieties produced:

$$p^* = 1/\rho \tag{12}$$

Then, if M denotes the number of varieties produced at the time in question, substituting (12) into (3) yields

$$P = (1/\rho)(M)^{-1/(\sigma-1)} \tag{13}$$

Furthermore, substituting (12) and (13), we obtain the equilibrium output of any variety produced in the economy:

$$q^* = \rho E/M \tag{14}$$

whereas the equilibrium profit is given by

$$\pi^* = q^*/(\sigma - 1) \tag{15}$$

since

$$\frac{1}{\rho} - 1 = \frac{1}{\sigma - 1}$$

We now study the labor market clearing conditions for the M-workers. In equilibrium, labor demand is equal to labor supply, so

$$L = Mq^* \tag{16}$$

and, by (14),

$$L = \rho E \tag{17}$$

so that in equilibrium, the total expenditure

$$E^* = L/\rho \tag{18}$$

is independent of time since L is constant. Therefore, we may conclude from (10) that *the equilibrium interest rate is equal to the subjective discount rate over time*

$$\nu^*(t) = \gamma \qquad \text{for all } t \geq 0 \tag{19}$$

As a result, using (9), the expenditure of any specific consumer i is also a constant, which is readily obtained from (7) and (19):

$$E_i^* = \gamma[\omega_i + W_i(0)] \tag{20}$$

Substituting (13) into (4) and setting $E_i(t) = E_i^*$ yields

$$v_i(t) = \rho \cdot E_i^* \cdot M(t)^{\frac{1}{\sigma-1}} \tag{21}$$

Finally, using (5) and (21), we obtain the lifetime utility of consumer i as

$$U_i(0) = \frac{E_i^*}{\sigma} \int_0^\infty e^{-\gamma t} \ln(M(t))dt \tag{22}$$

2.3 R & D Sector

Production of a new manufactured commodity requires the purchase of a patent. These patents are produced by the R & D sector, consisting of N workers, and they are the only output of this sector. Each new patent embodies a new idea. Not all new ideas result in patents. New ideas are produced by K-workers using their prior stock of knowledge. The scheme for producing new ideas is described as a knowledge production process. The basic layout of this sector is as follows, and is similar to Berliant and Fujita (forthcoming). At any given time, each K-worker has a stock of knowledge that has some commonalities with other K-workers but some knowledge distinct from other workers. Since workers possess knowledge exclusive of others, they may wish to cooperate with each other in the knowledge production process. Heterogeneity of knowledge in a partnership brings more originality, but knowledge in common is important for communication. Thus, K-worker heterogeneity is an essential feature of the model and of the knowledge production process. The K-workers choose to work alone or with a partner, maximizing their myopic payoff, namely the value of patents produced at that time. The solution concept used is the myopic core. If they work alone, new ideas are produced as a function of the total number of ideas known by a K-worker. If a pair of workers produces new ideas together, their knowledge production is a function of their knowledge in common on the one hand and the knowledge they have that is distinct from their partner on the other. Knowledge that is produced by an agent at a given time becomes part of the stock of knowledge for that agent in the future. In addition, some of these ideas become patented and sold to the manufacturing sector. The ideas embodied in the patents become public, and will be learned by all the agents in the R & D sector.

The basic unit of knowledge is called an idea.[4] The number of potential ideas is infinite. In this paper, we will treat ideas symmetrically.[5] In describing the process of knowledge production, that is either accomplished alone or in cooperation with another K-worker, the sufficient statistics about the state of knowledge of a K-worker i at a given time can be described as follows. We shall focus on K-worker i and her potential partner K-worker j. First, $n_i(t)$ represents the total stock of i's ideas at time t. Second, $n_{ij}^c(t)$ represents the total stock of ideas that i has in common with K-worker j at time t. Third, $n_{ij}^d(t)$ represents the stock of ideas that i knows but j doesn't know at time t. Finally, $n_{ji}^d(t)$ represents the stock of ideas that j knows but i doesn't know at time t.

By definition, $n_{ij}^c(t) = n_{ji}^c(t)$.[6] It also holds by definition that

$$n_i(t) = n_{ij}^c(t) + n_{ij}^d(t) \tag{23}$$

Knowledge is a set of ideas that are possessed by a person at a particular time. However, knowledge is not a static concept. New knowledge can be produced either individually or jointly, and ideas can be shared with others. But all of this activity takes time.

Now we describe the components of the rest of the model. To keep the description as simple as possible, we focus on just two agents, i and j. At each time, each agent faces a decision about whether or not to meet with others. If two agents want to meet at a particular time, a meeting will occur. If an agent decides not to meet with anyone at a given time, then the agent produces separately and also creates new knowledge separately, away from everyone

[4] In principle, all of these time-dependent quantities are positive integers. However, for simplicity we take them to be continuous (in $\mathbb{R}_+$) throughout the paper. One interpretation is that the creation of an idea occurs at a stochastic time, and the real numbers are taken to be the expected number of jumps (ideas learned) in a Poisson process. The use of an integer instead of a real number adds little but complication to the analysis.

[5] Extensions to idea hierarchies and knowledge structures will be discussed in the conclusions.

[6] In general, however, it is not necessary that $n_{ij}^d(t) = n_{ji}^d(t)$.

else. If two persons do decide to meet at a given time, then they collaborate to create new knowledge together.

At each moment of time, there are two mutually exclusive ways to produce new knowledge. The first way is to work alone, away from others. We denote the event that K-worker i does research alone at time t by $\delta_{ii}(t) = 1$, indicating that i works with herself. Otherwise, $\delta_{ii}(t) = 0$. Alternatively, K-worker i can choose to work with a partner, say K-worker j. We denote the event that K-worker i wishes to work with j at time t by $\delta_{ij}(t) = 1$. Otherwise, $\delta_{ij}(t) = 0$. In equilibrium, this partnership is realized at time t if $\delta_{ij}(t) = \delta_{ji}(t) = 1$.

Consider first the case where K-worker i works alone. In this case, idea production is simply a function of the stock of i's ideas at that time. Let $a_{ii}(t)$ be the rate of production of new ideas created by person i in isolation at time t. Then we assume that the creation of new knowledge during isolation is governed by the following equation:

$$a_{ii}(t) = \alpha \cdot n_i(t) \text{ when } \delta_{ii}(t) = 1. \tag{24}$$

If a meeting occurs between i and j at time t ($\delta_{ij}(t) = \delta_{ji}(t) = 1$), then joint knowledge creation occurs, and it is governed by the following dynamics:[7]

$$a_{ij}(t) = 2\beta \cdot (n_{ij}^c)^\theta \cdot (n_{ij}^d \cdot n_{ji}^d)^{\frac{1-\theta}{2}} \text{when } \delta_{ij}(t) = \delta_{ji}(t) = 1 \text{ for } j \neq i \tag{25}$$

where $0 < \theta < 1$, $\beta > 0$. So when two people meet, joint knowledge creation occurs at a rate proportional to the normalized product of their knowledge in common, the differential knowledge of i from j, and the differential knowledge of j from i. The rate of creation of new knowledge is high when the proportions of ideas in common, ideas exclusive to person i, and ideas exclusive to person j are in balance. The parameter θ represents the weight on knowledge in common as opposed to differential knowledge in the production of new ideas. Ideas in common are necessary for communication, while ideas exclusive to one person or the other imply more heterogeneity or originality in the collaboration. If one person in the collaboration does not have exclusive ideas, there is no reason for the other person to meet and collaborate. The multiplicative nature of the function in equation (25) drives the relationship between knowledge

[7] We may generalize equation (25) as follows:

$$a_{ij}(t) = \max\left\{2(\alpha - \varepsilon)\, n_i(t), 2(\alpha - \varepsilon)\, n_j(t), 2\beta \cdot (n_{ij}^c)^\theta \cdot (n_{ij}^d \cdot n_{ji}^d)^{\frac{1-\theta}{2}}\right\}$$

where $\varepsilon > 0$ represents the costs from the lack of concentration. This generalization, however, does not change the results presented in this paper in any essential way.

creation and the relative proportions of ideas in common and ideas exclusive to one or the other agent. Under these circumstances, no knowledge creation in isolation occurs.

Income for the research sector derives from selling patents. But not all ideas are patentable. For every collection of ideas created, we assume that η proportion are patentable as blueprints of new products. Thus, they are sold to the manufacturing sector. The residual ideas, namely $1-\eta$ proportion of new ideas, becomes tacit knowledge that is only known to the creator or creators of these ideas. They are useful for future creation of yet further ideas.

Let $y_i(t)$ to be the income of K-worker i at time t, and let $\Pi(t)$ be the price of patents at time t. Then, suppressing t for notational simplicity:

$$y_i = \Pi \cdot \eta \cdot (\delta_{ii} \cdot a_{ii} + \sum_{j \neq i} \delta_{ij} \cdot a_{ij}/2) \tag{26}$$

The formula implies that the revenue from new patents is split evenly if two K-workers are producing new ideas together.

Concerning the rule used by an agent to choose their best partner, to keep the model tractable in this first analysis, we assume a myopic rule. At each moment of time t, person i would like a meeting with person j when her income while meeting with j is highest among all potential partners, including herself. Maximizing income at a given time amounts to choosing $\{\delta_{ij}\}_{j=1}^{N}$ so that the right hand side of (26) is highest, meaning that a selection is made only among the most productive partners. In other words, as we are attempting to model close interactions within groups, we assume that at each time, the myopic persons interacting choose a core configuration. That is, we restrict attention to configurations such that at any point in time, no coalition of persons can get together and make themselves better off *in that time period.* In essence, our solution concept at a point in time is the myopic core.

In the case of a tie between income generated by several possible partnerships, agent i chooses an option that maximizes the derivative of income, $\dot{y}_i$. Furthermore, when the derivative of income is still the same among best options, agent i chooses an option that maximizes the second derivative of income, $\ddot{y}_i$and so on.

All agents take prices, in this case Π, as given, implying:

$$\max_{\{\delta_{ij}\}_{j=1}^{N}} (\delta_{ii} \cdot a_{ii} + \sum_{j \neq i} \delta_{ij} \cdot a_{ij}/2) \tag{27}$$

subject to the obvious constraints:

$$\sum_{j=1}^{N} \delta_{ij} = 1 \text{ for } i = 1, ..., N \tag{28}$$

Since n_i is a stock variable, this is equivalent to

$$\max_{\{\delta_{ij}\}_{j=1}^{N}} \left(\frac{\delta_{ii} \cdot a_{ii} + \sum_{j\neq i} \delta_{ij} \cdot a_{ij}/2}{n_i}\right) \tag{29}$$

In order to rewrite this problem in a convenient form, we first define the total number of ideas possessed by i and j:

$$n^{ij} = n_{ij}^{d} + n_{ji}^{d} + n_{ij}^{c} \tag{30}$$

and define new variables

$$\begin{aligned} m_{ij}^{c} &\equiv m_{ji}^{c} = \frac{n_{ij}^{c}}{n^{ij}} = \frac{n_{ji}^{c}}{n^{ij}} \\ m_{ij}^{d} &= \frac{n_{ij}^{d}}{n^{ij}}, \; m_{ji}^{d} = \frac{n_{ji}^{d}}{n^{ij}} \end{aligned}$$

By definition, m_{ij}^{d} represents the proportion of ideas exclusive to person i among all the ideas known by person i or person j. Similarly, m_{ij}^{c} represents the proportion of ideas known in common by persons i and j among all the ideas known by the pair. From (30), we obtain

$$1 = m_{ij}^{d} + m_{ji}^{d} + m_{ij}^{c} \tag{31}$$

whereas (30) and (23) yield

$$n_i = (1 - m_{ji}^{d}) \cdot n^{ij} \tag{32}$$

Using these identities and new variables, while recalling the knowledge production function (25) and (24), we obtain (see Technical Appendix a for details)

$$\frac{a_{ij}/2}{n_i} = G(m_{ij}^{d}, m_{ji}^{d}) \tag{33}$$

where

$$G(m_{ij}^{d}, m_{ji}^{d}) \equiv \frac{\beta \left(1 - m_{ij}^{d} - m_{ji}^{d}\right)^{\theta} \cdot (m_{ij}^{d} \cdot m_{ji}^{d})^{\frac{1-\theta}{2}}}{1 - m_{ji}^{d}} \tag{34}$$

Using (24) and (33), we can rewrite the income function (26) as

$$y_i = \Pi \cdot \eta \cdot n_i \cdot (\delta_{ii} \cdot \alpha + \sum_{j\neq i} \delta_{ij} \cdot G(m_{ij}^{d}, m_{ji}^{d})) \tag{35}$$

and the optimization problem (29) as follows:

$$\max_{\{\delta_{ij}\}_{j=1}^{N}} (\delta_{ii} \cdot \alpha + \sum_{j \neq i} \delta_{ij} \cdot G(m_{ij}^d, m_{ji}^d)) \tag{36}$$

We now describe the dynamics of the knowledge system, dropping the time argument. To describe these dynamics, in the end we require only an expression relating $\dot{m}_{ij}^d$ $(= dm_{ij}^d/dt)$ to m_{ij}^d. There are two ways to acquire new knowledge for a K-worker: internal production of new ideas and information from public sources. The first way has the feature that ideas produced alone are attributed to that worker, whereas ideas produced in pairs are attributed to both K-workers who produce them. In either case, the new ideas are learned by exactly the people who produce them. The second source of knowledge acquisition derives from the new ideas that are patented. The patented ideas become public information. A certain proportion of patented ideas, $\mu(N)$, are learned by all of the K-workers. In general, $\mu(N)$ will be a decreasing function of N. Limited time and energy determine how many of these new, public ideas can be learned. Due to these limitations, the amount of information a K-worker can learn from patents at a given time is, roughly, proportional to the number of new ideas she can create in that time. The number of new ideas and thus patents is proportional to the number of K-workers, so $\mu(N)$ will be inversely proportional to N.[8] Thus, these ideas become knowledge in common for all agents in the research sector. The net result is an increase in n_{ij}^c for all i and j of $\mu(N) \cdot \eta$ proportion of new ideas created in the economy. The workers in the K-sector see this flow of new ideas from patents, and account for it in when they choose actions at each moment of time. To obtain an expression relating $\dot{m}_{ij}^d$ to m_{ij}^d, we must first examine the knowledge dynamics in terms of the original variables, n_i, n_{ij}^c, and n_{ij}^d.

Let us focus on agent i, as the expressions for the other agents are analogous. Let A be the total number of ideas created at a given moment:

$$\begin{aligned} A &= \sum_{k=1}^{N} \delta_{kk} \cdot a_{kk} + (\sum_{k=1}^{N} \sum_{l \neq k} \delta_{kl} \cdot a_{kl})/2 && (37) \\ &= \sum_{k=1}^{N} \delta_{kk} \cdot \alpha n_k + \sum_{k=1}^{N} \sum_{l \neq k} \delta_{kl} \cdot n_k \cdot G(m_{kl}^d, m_{lk}^d) && (38) \end{aligned}$$

[8] In theory, it might be possible to accumulate a stock of ideas patented in past periods to learn in the future. The problem with this is that such information perpetually accumulates, and thus due to time constraints there is never an opportunity to learn the content of older patented ideas.

The dynamics of the knowledge system are based on the assumption that once learned, ideas are not forgotten. Using the argument above, we obtain knowledge system dynamics:

$$\begin{aligned}
\dot{n}_i &= \sum_{l=1}^{N} \delta_{il} \cdot a_{il} + \mu(N) \cdot \eta \cdot (A - \sum_{l=1}^{N} \delta_{il} \cdot a_{il}) && (39)\\
\dot{n}_{ij}^c &= \delta_{ij} \cdot a_{ij} + \mu(N) \cdot \eta \cdot (A - \delta_{ij} \cdot a_{ij}) \text{ for all } j \neq i && (40)\\
\dot{n}_{ij}^d &= (1 - \mu(N) \cdot \eta) \cdot \sum_{k \neq j} \delta_{ik} \cdot a_{ik} \text{ for all } j \neq i && (41)
\end{aligned}$$

Thus, equation (39) says that the increase in the knowledge of person i is the sum of: the knowledge created in isolation, the knowledge created jointly with someone else, and the transfer of new knowledge from new patents. Equation (40) means that the increase in the knowledge in common for persons i and j equals the new knowledge created jointly by them plus the transfer of knowledge from new patents. Finally, equation (41) means that all the knowledge created by person i either in isolation or joint with persons other than person j becomes a part of the differential knowledge of person i from person j, *except* for patented ideas that are learned by all K-workers.

Using (24) and (33), equation (39) can be rewritten as

$$\dot{n}_i = (1 - \mu(N) \cdot \eta) \cdot n_i \cdot (\delta_{ii}\alpha + \sum_{l \neq i}^{N} \delta_{il} \cdot G(m_{il}^d, m_{li}^d)) + \mu(N) \cdot \eta \cdot A \qquad (42)$$

where A is given by (38). Furthermore, using (24), (25), and (33), we have (see Theorem A2 of Technical Appendix a)

$$\begin{aligned}
\dot{m}_{ij}^d &= (1 - \mu \cdot \eta)(1 - m_{ij}^d)(1 - m_{ji}^d) \left\{ \delta_{ii} \cdot \alpha + \sum_{k \neq i,j} \delta_{ik} \cdot 2G(m_{ik}^d, m_{ki}^d) \right\} \qquad (43)\\
&\quad - m_{ij}^d [\mu\eta\alpha(1 - m_{ji}^d) \cdot \sum_{k=1}^{N} \delta_{kk} \cdot \frac{n_k}{n_i} + (1 - \mu \cdot \eta) \cdot \delta_{ij} \cdot \left(1 - m_{ji}^d\right) \cdot 2G(m_{ij}^d, m_{ji}^d)\\
&\quad + \mu \cdot \eta \cdot \left(1 - m_{ji}^d\right) \sum_{k=1}^{N} \sum_{l \neq k} \delta_{kl} \cdot \frac{n_k}{n_i} \cdot G(m_{kl}^d, m_{lk}^d)]\\
&\quad - m_{ij}^d \cdot (1 - \mu \cdot \eta) \cdot \left(1 - m_{ij}^d\right) \cdot \left\{ \delta_{jj} \cdot \alpha + \sum_{k \neq i,j} \delta_{jk} \cdot 2G(m_{jk}^d, m_{kj}^d) \right\}
\end{aligned}$$

for $i, j = 1, 2, \cdots, N,\ i \neq j$. Thus, using (42) and (43), the knowledge dynamics are described in terms of n_i and m_{ij}^d $(i, j = 1, ..., N)$ only.

3 Knowledge Dynamics

3.1 The Model

Since we are concerned with the macro behavior of the economy and the big picture in terms of growth, we make a number of simplifying assumptions. We impose the assumption that the initial state of knowledge for all K-workers is pairwise symmetric in terms of heterogeneity. The initial state of knowledge is given by

$$n_{ij}^c(0) = n^c(0) \text{ for all } i \neq j \tag{44}$$
$$n_{ij}^d(0) = n^d(0) \text{ for all } i \neq j \tag{45}$$

At the initial state, each pair of K-workers has the same number of ideas, $n^c(0)$, in common. Moreover, for any pair of K-workers, the number of ideas that one K-worker knows but the other does not know is the same and equal to $n^d(0)$. Given that the initial state of knowledge is symmetric among the K-workers, it turns out that the equilibrium configuration at any time also maintains the basic pairwise symmetry among K-workers.

Suppose that at some given time, all K-workers are pairwise symmetric to each other. Namely, when

$$m_{ij}^d = m_{ji}^d \quad \text{for all } i \neq j \tag{46}$$

(36) is simplified as

$$\max_{\{\delta_{ij}\}_{j=1}^N} (\delta_{ii} \cdot \alpha + \sum_{j \neq i} \delta_{ij} \cdot g(m_{ij}^d)) \tag{47}$$

where the function g is defined as

$$g(m) \equiv G(m, m) \equiv \beta \frac{(1-2m)^\theta \, m^{(1-\theta)}}{1-m} \tag{48}$$

Since $n^{ij} = n^{ji}$ by definition, we can readily see, by using (32), that condition (46) is equivalent to

$$n_i = n_j \text{ for all } i \text{ and } j \tag{49}$$

Furthermore, since $a_{ij} = a_{ji}$ by definition, substituting (46) into (33) yields

$$\frac{a_{ij}/2}{n_i} = \frac{a_{ji}/2}{n_j} = g(m_{ij}^d) \tag{50}$$

Thus, when two K-workers i and j cooperate in knowledge production and their knowledge states are symmetric, $g(m_{ij}^d)$ represents the creation of new ideas per capita (normalized by the size of individual knowledge input, n_i). In this context, condition (47) means that each K-worker wishes to engage in knowledge production in a partnership with a person (possibly including herself) leading to the highest K-productivity.

Figure 1 illustrates the graph of the function $g(m)$ as a bold curve for parameter values $\beta = 1$ and $\theta = 1/3$.

FIGURE 1 GOES HERE

Differentiating $g(m)$ yields

$$g'(m) = g(m) \cdot \frac{(1-\theta) - (2-\theta)\cdot m}{(1-2m)\cdot m \cdot (1-m)}$$

implying that

$$g'(m) \gtrless 0 \text{ as } m \lessgtr \frac{1-\theta}{2-\theta} \quad \text{for } m \in (0, \frac{1}{2}) \tag{51}$$

Thus, $g(m)$ is strictly quasi-concave on $[0, 1/2]$, achieving its maximal value at

$$m^B = \frac{1-\theta}{2-\theta} \tag{52}$$

which we call the "Bliss Point." It is the point where knowledge productivity is highest for each person. In the remainder of the paper, our main concern is whether or not the dynamics of knowledge interaction will, starting at the initial state given by (44) and (45), lead the system of K-workers to this bliss point.

When condition (46) holds, using (48) and (49), the dynamics can be written as

$$\begin{aligned}
\frac{\dot{m}_{ij}^d}{1-m_{ij}^d} \;&=\; (1-\mu\cdot\eta)\cdot(1-m_{ij}^d)\cdot\left\{\delta_{ii}\cdot\alpha + \sum_{k\neq i,j}\delta_{ik}\cdot 2g(m_{ik}^d)\right\} \\
&\quad -m_{ij}^d\left\{\mu\cdot\eta\cdot\alpha\cdot\sum_{k=1}^{N}\delta_{kk} + (1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2g(m_{ij}^d) + \mu\cdot\eta\cdot\sum_{k=1}^{N}\sum_{l\neq k}\delta_{kl}\cdot g(m_{kl}^d)\right\} \\
&\quad -m_{ij}^d(1-\mu\cdot\eta)\cdot\left\{\delta_{jj}\cdot\alpha + \sum_{k\neq i,j}\delta_{jk}\cdot 2g(m_{kj}^d)\right\} \qquad (53) \\
\dot{n}_i \;&=\; (1-\mu(N)\cdot\eta)\cdot n_i\cdot\left(\delta_{ii}\cdot\alpha + \sum_{k\neq i,j}\delta_{ik}\cdot g(m_{il}^d)\right) + \mu(N)\cdot\eta\cdot A \qquad (54)
\end{aligned}$$

for $i, j = 1, 2, ..., N$, where A is given by

$$A = \sum_{k=1}^{N} \delta_{kk} \cdot \alpha n_k + \sum_{k=1}^{N} \sum_{l \neq k} \delta_{kl} \cdot n_k \cdot g(m_{kl}^d) \tag{55}$$

We observe that the basic rules, (47), (53), and (42), that govern the knowledge dynamics are described in terms of m_{ij}^d and n_i $(i, j = 1, 2, ...N)$ only. Notice that no market variable is used. This enables us first to solve for the equilibrium path of knowledge dynamics independent of commodity and capital markets.

Next, taking the case of $N = 4$, we illustrate the possible equilibrium configurations of partnerships in knowledge creation, noting that the equilibrium configuration can vary with time. Figure 2 gives the possibilities at any fixed time for $N = 4$. Given that the initial state of knowledge is symmetric among the four K-workers, as noted above, the equilibrium configuration at any time also maintains the basic symmetry among K-workers.

FIGURE 2 GOES HERE

Panel (a) in Figure 2 represents the case in which each of the four K-workers is working alone, creating new ideas in isolation. Panels (b-1) to (b-3) represent the three possible configurations of partnerships, in which two couples each work separately but simultaneously. In panel (b-1), for example, 1 and 2 work together. At the same time, 3 and 4 work together.

Although panels (a) to (b-3) represent the basic forms of knowledge creation with four persons, it turns out that the equilibrium path often requires a mixture of these basic forms. That is, on the equilibrium path, people wish to change partners as frequently as possible. The purpose is to balance the number of different and common ideas with partners as best as can be achieved. This suggests a work pattern with rapidly changing partners on the equilibrium path.

Please refer to panels (c-1) to (c-3) in Figure 2. Each of these panels represents a work pattern where a worker rotates through two fixed partners as fast as possible in order to maximize the instantaneous increase in their income. In panel (c-1), for example, worker 1 chooses K-workers 2 and 3 as partners, and rotates between the two partners under equilibrium values of δ_{12} and δ_{13} such that $\delta_{12} + \delta_{13} = 1$. K-workers 2, 3 and 4 behave analogously. In order for this type of work pattern to take place, of course, all four persons must agree to follow this pattern. Finally, panel (d) depicts a work pattern in which each worker rotates though all three possible partners as fast as possible. That is, for all $i \neq j$, $\delta_{ij} \in (0,1)$, and for all i, $\delta_{ii} = 0$ and $\sum_{j \neq i} \delta_{ij} = 1$.

At this point, it is useful to remind the reader that we are using a myopic core concept to determine equilibrium at each point in time. In fact, it is necessary to sharpen that concept in the model with N persons. *When there is more than one vector of strategies that is in the myopic core at a particular time, namely more than one vector of joint strategies implies the same, highest income for all persons, the one with the highest first derivative of income is selected, and so on.* The justification for this assumption is that at each point in time, people are attempting to maximize the flow of income. The formal definition of the myopic core and proof that it is nonempty can be found in Berliant and Fujita (forthcoming, Appendix 0). Although the theorem is general, in the remainder of this paper we shall focus on the symmetric case.

3.2 Equilibrium Path of Knowledge Dynamics

Now we are ready to investigate the actual equilibrium path, depending on the given initial composition of knowledge,

$$m_{ij}^d(0) = m^d(0) = \frac{n^d(0)}{n^c(0) + 2n^d(0)}$$

which is common for all pairs i and j $(i \neq j)$. In order to sharpen the results that follow, we introduce a specific form of the parametric function $\mu(N)$, representing the proportion of the public information on new patents that is actually learned by K-workers as knowledge in common. Assuming that the flow of knowledge that each K-worker can acquire from public information on new patents is proportional to the flow of new knowledge she can produce, we use the following relation in the analysis below (see Appendix 1 for a justification):

$$\mu(N) = \frac{C}{\eta N}$$

or

$$\mu(N) \cdot \eta = \frac{C}{N} \tag{56}$$

where C is a positive constant representing the learning capacity (l-capacity) of each K-worker.

In the remainder of this paper, we assume that

$$\alpha < g(m^B) \tag{57}$$

so as to avoid the trivial case of all agents always working in isolation.

In Figure 1, let m^J and m^I be defined on the horizontal axis at the left intersection and the right intersection between the $g(m)$ curve and the horizontal line at height α, respectively.

In the following analyses, the various cases are determined by the initial heterogeneity of the K-workers. For each case, the associated form of work in equilibrium is illustrated using the diagrams in Figure 2, that provides detail for the case $N = 4$. To be precise:

Proposition 1: *The equilibrium path of K-worker interactions and the sink point of the knowledge creation process depend discontinuously on the initial condition, $m^d(0)$. Assuming that the number of K-workers N is large, the pattern of interaction between K-workers and the sink point as a function of the initial condition are as follows.*

(i) For $m^J < m^d(0) \leq m^B$, we define two subcases. Let $\widetilde{C} \equiv \frac{2\theta}{1-\theta}$.

(a) $C < \widetilde{C}$. The equilibrium path consists of an initial time interval in which each K-worker is always paired with another but trade partners as rapidly as possible (with $\delta_{ij} = 1/(N-1)$ for all i and for all $j \neq i$). When the bliss point, $\widetilde{m}^B = \frac{1-\theta}{2-\theta}$, is attained, the agents split into groups of $\widetilde{N}^B = 1 + \frac{1}{\theta - \frac{(1-\theta)C}{2}}$, and they remain at the bliss point.

(b) $C > \widetilde{C}$. The equilibrium path has all K-workers paired with another but trading partners as rapidly as possible (with $\delta_{ij} = 1/(N-1)$ for all i and for all $j \neq i$). This continues forever. The equilibrium path remains to the left of the bliss point, so the bliss point is never attained. The sink point is $\widetilde{m}^{d*} = \frac{1}{2+\frac{C}{2}}$.

(ii) $m^d(0) < m^J < m^B$. Once again, there are two subcases. If C is large, then all K-workers are in isolation producing new ideas alone forever. The sink point is $\widetilde{m}^{d**} = \frac{1}{2+C}$. If C is not large, then the equilibrium path consists of a first phase in which all K-workers are in isolation producing new ideas. Once the system reaches m^J, the equilibrium path follows that given in case (i).

(iii) $m^B < m^d(0)$ The equilibrium path consists of many phases. First, the N K-workers are paired arbitrarily and work with their partners for a nonempty interval of time. Second, they switch to new partners and work with their new partners for a nonempty interval of time. Finally, each K-worker pairs alternately with the two partners with whom they worked in the first two phases, but not with a K-worker with whom they have not worked previously. This process continues, possibly adding more partners.

We wish to alert the reader that the focus of the remainder of

the paper, in particular our analysis of economic growth, will be on case (i). Thus, we shall not discuss the other cases in great detail.

3.2.1 Case (i): $m^J < m^d(0) \leq m^B$

First suppose that the initial state is such that

$$m^J < m^d(0) < m^B$$

Then, since $g(m^d_{ij}(0)) = g\left(m^d(0)\right) > \alpha$ for any possible work pairs consisting of i and j, no person wishes to work alone at the start. However, since the value of $g(m^d_{ij}(0))$ is the same for all possible pairs, all forms of (b-1) to (d) in Figure 2 are possible equilibrium work configurations at the start. To determine which one of them will actually take place on the equilibrium path, we must consider the first derivative of income for all persons.

In general, consider any time at which all persons have the same composition of knowledge:

$$m^d_{ij} = m^d \text{ for all } i \neq j \tag{58}$$

where

$$g(m^d) > \alpha$$

Focus on person i; the equations for other persons are analogous. Since person i does not wish to work alone, it follows that

$$\delta_{ii} = 0 \text{ and } \sum_{j \neq i} \delta_{ij} = 1 \tag{59}$$

Substituting (58) and (59) into (26) and using (50) yields

$$y_i = \Pi \cdot \eta \cdot n_i \cdot g(m^d) \tag{60}$$

Likewise, substituting (56), (58) and (59) into (53) and arranging terms gives

$$\dot{m}^d_{ij} = \dot{m}^d = 2\left(1 - m^d\right) \cdot g(m^d) \cdot \left\{(1 - \frac{C}{N}) \cdot (1 - 2m^d) - (1 - \frac{C}{N}) \cdot (1 - m^d) \cdot \delta_{ij} - \frac{C}{2} \cdot m^d\right\} \tag{61}$$

for $i \neq j$.

Since the income function (60) is independent of the values of δ_{ij} $(j \neq i)$, in order to examine what values of $\delta_{ij} (j \neq i)$ person i wishes to choose, we must consider the time derivative of y_i. In doing so, however, we cannot use equation (60) because the original variables have been replaced. Instead, we must go back to the original equation (35). Then, using equations (58) to (61) and setting $\delta_{ij} = \delta_{ji}$ (which must hold for any feasible meeting), we obtain the following (see Berliant and Fujita, 2006, Technical Appendix b for proof):

$$\begin{aligned} \dot{y}_i &= \dot{\Pi} \cdot \eta \cdot n_i \cdot g(m^d) + \Pi \cdot \eta \cdot \dot{n}_i \cdot g(m^d) \\ &\quad + \Pi \cdot \eta \cdot n_i \cdot \sum_{j \neq i} \delta_{ij} \cdot g'(m^d) \cdot \dot{m}_{ij}^d \end{aligned} \tag{62}$$

where

$$\dot{n}_i = g(m^d) \cdot n_i \cdot (2 + \frac{N-2}{N} \cdot C) \tag{63}$$

and $\dot{m}_{ij}^d$ is given by (61). Substituting (61) into (62) and setting $\sum_{j \neq i} \delta_{ij} = 1$ yields

$$\begin{aligned} \dot{y}_i &= \dot{\Pi} \cdot \eta \cdot n_i \cdot g(m^d) + \Pi \cdot \eta \cdot \dot{n}_i \cdot g(m^d) \\ &\quad + \Pi \cdot \eta \cdot n_i \cdot 2\left(1 - m^d\right) \cdot g(m^d) \cdot g'(m^d) \cdot \\ &\quad \left\{ (1 - \frac{C}{N}) \cdot (1 - 2m^d) - (1 - \frac{C}{N}) \cdot (1 - m^d) \cdot \sum_{j \neq i} \delta_{ij}^2 - \frac{C}{2} \cdot m^d \right\} \end{aligned} \tag{64}$$

All K-workers take Π and $\dot{\Pi}$ as given, whereas n_i is a state variable. Furthermore, the value of $\dot{n}_i$ given above is independent of the values of δ_{ij} for $j \neq i$. Thus, choosing the values of δ_{ij} for $j \neq i$ is equivalent to choosing the values that maximize the last term in (64).

Now, suppose that

$$m^d < m^B$$

and hence $g'(m^d) > 0$. Then, assuming that $\frac{C}{N} < 1$, in order to maximize the time derivative of the income, person i must solve the following quadratic minimization problem:

$$\min \sum_{j \neq i} \delta_{ij}^2 \quad \text{subject to} \quad \sum_{j \neq i} \delta_{ij} = 1 \tag{65}$$

which yields the solution for person i:

$$\delta_{ij} = \frac{1}{N-1} \quad \text{for all } j \neq i \tag{66}$$

Although we have focused on person i, the vector of optimal strategies is the same for all persons. Thus, all persons agree to a square work in which each person rotates through all $N-1$ possible partners while sharing the time equally.

The intuition behind this result is as follows. The condition $m^d < m^B$ means that the K-workers have relatively too many ideas in common, and thus they wish to acquire ideas that are different from those of each possible partner as fast as possible. That is, when $m^J < m_{ij}^d = m^d < m^B$ in Figure 1, each K-worker wishes to move the knowledge composition m_{ij}^d to the right as quickly as possible, thus increasing the K-productivity $g(m_{ij}^d)$ as fast as possible.

Concerning the general case with $N \geq 4$, when $m^J < m^d(0) = m_{ji}^d(0) < m^B$ for all $i \neq j$, on the equilibrium path, each K-worker i spends the same amount of time $\delta_{ij} = 1/(N-1)$ for all $j \neq i$ with every other K-worker at the start. Then, since the symmetric condition (58) holds from the start onward, the same work pattern will continue as long as $m^J < m^d < m^B$. The dynamics of this work pattern are as follows. The creation of new ideas always takes place in pairs. Pairs are cycling rapidly with $\delta_{ij} = 1/(N-1)$ for all $j \neq i$. K-worker 1, for example, spends $1/(N-1)$ of each period with K-worker 2, for example, and $(N-2)/(N-1)$ of the time working with other partners. Setting $m_{ij}^d = m^d$ and $\delta_{ij} = 1/(N-1)$ in (61), we obtain

$$\dot{m}^d = 2(1-m^d) \cdot g(m^d) \cdot \frac{1-\frac{C}{N}}{N-1} \left\{ (N-2) - m^d \left[(2N-3) + \frac{C}{2} \cdot \frac{N-1}{1-\frac{C}{N}} \right] \right\} \tag{67}$$

Setting $\dot{m}^d = 0$ and considering that $m^d < 1$, we obtain the sink point

$$m^{d*} = \frac{N-2}{(2N-3) + \frac{C}{2} \cdot \frac{N-1}{1-\frac{C}{N}}} \tag{68}$$

As N increases, the value of m^{d*}increases monotonically (provided $N > C$) eventually reaching the limit

$$\widetilde{m}^{d*} = \frac{1}{2 + \frac{C}{2}} \tag{69}$$

In the upper half of Figure 3, the K-productivity curve $g(m)$ is transferred from Figure 2. In the bottom half of Figure 3, the bold curve depicts the limiting sink, $\widetilde{m}^{d*}$, as a function of the l-capacity parameter C. When N is sufficiently large, the actual sink curve, m^{d*}, is close to this limiting curve.

FIGURE 3 GOES HERE

In the context of Figure 3, we can identify two different possibilities. Suppose that

$$m^B < m^{d*} \tag{70}$$

That is, the sink point of the dynamics given in (67) is on the right side of the bliss point. In this case, beginning at any point $m^J < m^d(0) < m^B$, the system reaches the bliss point in finite time. In terms of the original parameters, using (52) and (68), condition (70) can be rewritten as

$$C < \frac{\left\{\frac{2-\theta}{1-\theta} - \frac{2N-3}{N-2}\right\} \cdot (N-1)}{\frac{2-\theta}{1-\theta} - \frac{2N-3}{N-2} + \frac{N}{2}} \tag{71}$$

Since $m^{d*} \to \widetilde{m}^{d*}$, when N is sufficiently large, condition (71) can be expressed as

$$C < \widetilde{C} \equiv \frac{2\theta}{1-\theta} \tag{72}$$

In Figure 3, C_1 provides an example of this case. The associated sink point is given by m_1^{d*}.

In contrast, suppose that

$$m^{d*} < m^B \tag{73}$$

This occurs exactly when the inequality in (71) is reversed. Assuming that N is sufficiently large, it occurs when the inequality in (72) is reversed. In Figure 3, C_2 represents an example of such a value of C, whereas the associated sink point is given by m_2^{d*}. In this case, starting with any initial point $m^J < m^d(0) < m^B$, the system moves automatically toward $m^{d*} < m^B$, but never reaches the bliss point.

On the downward vertical axis of Figure 3, $\widetilde{C}$ gives the value of the parameter C at the boundary of the two cases. The case (70) occurs exactly when the value of the l-capacity C is relatively small, whereas the case (73) occurs when C is relatively large. In what follows, under the assumption that N is large, we examine the actual dynamics in each of the two cases.

Case (a): $m^J < m^d(0) \leq m^B$ and $C < \widetilde{C}$ When condition (70) holds, starting with any initial point $m^J < m^d(0) \leq m^B$, the system following the dynamics (61) reaches the bliss point m^B in finite time. When the bliss point is reached, we have

$$m_{ij}^d = m^d = m^B \text{ for } i \neq j \tag{74}$$

and $g'(m^d) = g'(m^B) = 0$. Thus, (64) becomes

$$\dot{y}_i = \dot{\Pi} \cdot \eta \cdot n_i \cdot g(m^d) + \Pi \cdot \eta \cdot \dot{n}_i \cdot g(m^d) \tag{75}$$

that is, again, independent of the values of δ_{ij} $(j \neq i)$. Thus, we consider the second order condition for income maximization. Replace $g(m^d)$ with

$G(m_{ij}^d, m_{ji}^d)$ in (62) and take the time derivative of the resulting equation. Using (74) and the fact that $g'(m^B) = 0$, by following the logic in Berliant and Fujita (2006, Technical Appendix b) we obtain

$$\begin{aligned}
\ddot{y}_i &= \ddot{\Pi} \cdot \eta \cdot n_i \cdot g(m^B) + 2\dot{\Pi} \cdot \eta \cdot \dot{n}_i \cdot g(m^B) + \Pi \cdot \eta \cdot \ddot{n}_i \cdot g(m^d) \\
&\quad + \Pi \cdot \eta \cdot n_i \cdot (1 - m^B)^2 \cdot 4(m^B)^2 g''(m^B) \cdot \\
&\quad \left\{ \sum_{i \neq j} \delta_{ij} \cdot \left[(1 - \frac{C}{N}) \cdot (1 - 2m^B) - (1 - \frac{C}{N}) \cdot (1 - m^B) \cdot \delta_{ij} - \frac{C}{2} m^B \right] \right\}^2
\end{aligned} \tag{76}$$

where, using (63),

$$\begin{aligned}
\dot{n}_i &= g(m^B) \cdot n_i \cdot (2 + \frac{N-2}{N} C) \\
\ddot{n}_i &= g(m^B) \cdot \dot{n}_i \cdot (2 + \frac{N-2}{N} C)
\end{aligned}$$

Since the first three terms on the right hand side of (76) are independent of the values of δ_{ij} $(j \neq i)$ whereas $g'' < 0$, choosing the values of δ_{ij} $(j \neq i)$ to maximize $\ddot{y}_i$ is equivalent to the following optimization problem:

$$\min_{\{\delta_{ij}\}} \left\{ \sum_{i \neq j} \delta_{ij} \cdot \left[(1 - \frac{C}{N}) \cdot (1 - 2m^B) - (1 - \frac{C}{N}) \cdot (1 - m^B) \cdot \delta_{ij} - \frac{C}{2} m^B \right] \right\}^2 \tag{77}$$

$$\text{subject to } \sum_{j \neq i} \delta_{ij} = 1$$

This problem can be solved by using the rule that whenever $\delta_{ij} > 0$, the value of the terms inside the square brackets in expression (77) must be zero, or

$$\delta_{ij} > 0 \implies \delta_{ij} = \frac{(1 - \frac{C}{N}) \cdot (1 - 2m^B) - \frac{C}{2} m^B}{(1 - \frac{C}{N}) \cdot (1 - m^B)} \equiv \delta^B \tag{78}$$

whereas the number of partners for K-worker i must be chosen to satisfy the constraint $\sum_{j \neq i} \delta_{ij} = 1$. This applies to all K-workers.

This equilibrium configuration of partnerships at the bliss point m^B can be achieved as follows: When the system reaches m^B, the population splits into smaller groups of equal size,[9]

$$N^B \equiv 1 + \frac{1}{\delta^B} \tag{79}$$

[9] The configuration of workers necessary to maintain the bliss point is not unique. Each K-worker must have $N^B - 1$ links to other K-workers, communicating with each for an equal share of time. For example, when $N^B = 4$, groups of 4 may form, where each worker within a group communicates equally with every other worker in that group. However, with

so each person works with $N^B - 1$ other persons in their group for the same proportion of time, δ^B. Recalling (61), rule (78) is equivalent to

$$\delta_{ij} > 0 \implies \dot{m}^d_{ij} = 0 \text{ at } m^d_{ij} = m^B$$

That is, when all K-workers reach the bliss point, they stay there by splitting into smaller groups of the same size, N^B, so direct interactions take place only within each group. In this way, each K-worker maintains the highest K-productivity while enjoying the knowledge externalities derived from public information on new patents. Figure 4 depicts an example of an equilibrium configuration of K-worker interactions in which four groups of K-workers form at the bliss point. The dotted arrows represent indirect interactions through the public revelation of patent information.

FIGURE 4 GOES HERE

Substituting (52) into (78), using (79) and arranging terms, the optimal group size N^B is given by

$$N^B = 1 + \frac{1}{\theta - \frac{(1-\theta)\cdot C}{2} \cdot \frac{N}{N-C}} \tag{80}$$

As N becomes large, the optimal group size approaches

$$\widetilde{N}^B = 1 + \frac{1}{\theta - \frac{(1-\theta)\cdot C}{2}} \tag{81}$$

The optimal group size for large population $\widetilde{N}^B$ (as well as the optimal group size for finite population N^B) increases monotonically with the l-capacity, C; as C increases, the transmission of public knowledge in common increases, so it is necessary to have a larger group in order to maintain heterogeneity among agents within the group. Recalling that $\widetilde{C}$ was defined in (72), the group size becomes infinitely large as C approaches $\widetilde{C}$ from the left. Recalling that θ is the weight given to knowledge in common in the K-production function, as

$N^B = 4$ it is also possible to have, say, groups of six forming. With such groups, each K-worker has communication links to only three other K-workers within their group. So not all possible links within a group are actually active. If groups at the bliss point are larger, then their communication structure must become more sparse to maintain the bliss point. The minimal size of groups that coalesce at the bliss point is clearly N^B. Nevertheless, all of the calculations apply independent of the size of groups that form at the bliss point. The same remarks apply to the various cases detailed below, except when K-workers are in isolation.

the value of θ increases, $\widetilde{N}^B$ decreases, which is not surprising. In Figure 5, for each fixed value of the parameter C, the optimal group size $\widetilde{N}^B$ is graphed as a function of θ.

FIGURE 5 GOES HERE

Substituting $\delta^B = 1/(N^B - 1)$ for δ_{ij} in equation (61), then by construction, m^B is the sink point of the dynamics

$$\dot{m}_{ij}^d = 2\left(1 - m^d\right) \cdot g(m^d) \cdot \left\{(1 - \frac{C}{N}) \cdot (1 - 2m^d) - (1 - \frac{C}{N}) \cdot (1 - m^d) \cdot \frac{1}{N^B - 1} - \frac{C}{2} \cdot m^d\right\} \tag{82}$$

for $i \neq j$ but i and j in the same group. Thus, starting with any initial point $m_{ij}^d(0) = m^d(0) \in (0, 1/2)$, if each person participates in a group of N^B persons, and if they maintain the same group structure where each person works with each of the $N^B - 1$ other people in their group for the same proportion of time δ^B, then the system monotonically approaches the bliss point m^B. However, when $m_{ij}^d(0) = m^d(0) < m^B$, if all N persons form a single group while setting $\delta_{ij} = 1/(N - 1)$, the system can reach the bliss point m^B fastest.[10]

When the system reaches the bliss point, the workers break into groups and the system becomes asymmetric, in the following sense. If K-worker i belongs to the same group as K-worker k, then their differential knowledge remains at the bliss point m^B, maintaining the highest K-productivity $g(m^B)$. If K-worker j belongs to a different group, then the differential knowledge between i and j on the other diverges, namely it moves away from m^B, thus reducing $g(m_{ij}^d)$. So once the population splits into groups, K-workers i and j will not want to collaborate again.

Formally, setting $\delta_{ij} = 0$ in equation (61), the dynamics of differential knowledge for K-workers i and j in different groups is given by

$$\dot{m}_{ij}^d = \dot{m}^d = 2\left(1 - m^d\right) \cdot g(m^d) \cdot \left\{(1 - \frac{C}{N}) \cdot (1 - 2m^d) - \frac{C}{2} \cdot m^d\right\} \tag{83}$$

that yields a sink point

$$m^{d*} = \frac{1 - \frac{C}{N}}{2 + \frac{C}{2}}$$

As $N \to \infty$, the sink point becomes

$$m^{d*} = \frac{1}{2 + \frac{C}{2}} = \widetilde{m}^{d*} \tag{84}$$

[10] With a starting point $m_{ij}^d(0) = m^d(0) < m^B$, if the population forms groups of size less than N but larger than N^B, then the system will still reach the bliss point, but at a slower speed than if the group size were N.

Notice that this is the same as expression (69). As the number of K-workers becomes large, the difference between pairs of workers who interact at intensity $1/(N-1)$ and pairs of workers in different groups who don't interact is close to zero, so they tend to the same sink point.

To sum up, for partnerships of K-workers within the same group, their productivity is $g(m^B)$. For potential partnerships of K-workers in different groups, their potential productivity is $g(\widetilde{m}^{d*}) < g(m^B)$. So these potential partnerships are never formed.

The implication is that we have endogenous formation of cohesive groups. One interpretation of this phenomenon is that the groups represent firms, so we have endogenous formation of firm boundaries.

Case (b): $m^J < m^d(0) \leq m^B$ **and** $C > \widetilde{C}$ As explained previously, in this case the dynamics imply that only one large group forms, so each agent works with everyone else an equal amount of time. Heterogeneity m^d changes, approaching the sink point m^{d*} given by (68) to the left of the bliss point, so the bliss point is never reached. In this case

$$m^J < m^{d*} < m^B$$

and one large group is maintained forever, without achieving the highest possible productivity. Intuitively, this is due to the large externality from public knowledge, so it is impossible to attain sufficient heterogeneity.

3.2.2 Case (ii): $m^d(0) < m^J < m^B$

Under this set of parameters, $g(m^d(0)) < \alpha$. In other words, at time 0 it is best for everyone to work in isolation rather than in pairs. Substituting $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ into (82), and using (56), we obtain dynamics for work in isolation:

$$\dot{m}^d_{ij} = \dot{m}^d = \left(1 - m^d\right) \cdot \alpha \cdot \left\{ (1 - \frac{C}{N}) \cdot (1 - 2m^d) - C \cdot m^d \right\} \tag{85}$$

that yields the sink point

$$m^{d**} = \frac{1}{2 + \frac{C}{1-\frac{C}{N}}}$$

As $N \to \infty$, the sink point approaches

$$\widetilde{m}^{d**} = \frac{1}{2 + C}$$

Evidently, $\widetilde{m}^{d**} < \widetilde{m}^{d*}$. When N is sufficiently large, it follows that

$$m^{d**} < m^{d*}$$

Focusing on this case, there are two possibilities, namely $m^J < m^{d**}$ and $m^J > m^{d**}$.[11] Assuming C is not too large, we concentrate on the first possibility,

$$m^J < m^{d**} \tag{86}$$

The equilibrium path has every K-worker in isolation to begin, creating new knowledge on their own and moving to the right until they all reach the point m^J. Then one large group forms and all K-workers create new knowledge working in pairs where each spends equal time with every other. From here, the equilibrium path is exactly the same as in case (i).

3.2.3 Case (iii): $m^B < m^d(0)$

Next, let us consider the dynamics of the system when it begins to the right of m^B. First we consider the situation where $m^B < m^d(0) < m^I$, where m^I was introduced in Figure 1. In other words, the initial state reflects a higher degree of heterogeneity than the bliss point, but $g(m^d(0)) > \alpha$. Since the initial state reflects a higher degree of heterogeneity than the bliss point, the K-workers want to increase the knowledge they have in common as fast as possible, leading to fidelity and *pairwise knowledge creation.*

To be precise, since $m^d_{ij}(0) = m^d(0)$ for all $i \neq j$ and $g(m^d(0)) > \alpha$, the situation at time 0 is the same as that in Case (i) except that we now have $m^d(0) > m^B$. Hence, focusing on person i as before, the time derivative of income y_i at time 0 is given by (64). However, since $g'(m^d) = g'(m^d(0)) < 0$ at time 0, in order to maximize the right hand side of equation (64), person i now must solve now the following quadratic maximization problem:

$$\max \sum_{j \neq i} \delta_{ij}^2 \;\; \text{subject to} \;\; \sum_{j \neq i} \delta_{ij} = 1 \tag{87}$$

Thus, person i wishes to choose any partner, say k, and set $\delta_{ik} = 1$, whereas $\delta_{ij} = 0$ for all $j \neq k$. The situation is the same for all K-workers. Hence, without loss of generality, we can assume that N persons agree at time 0 to form the following combination of partnerships:

$$P_1 \equiv \{\{1,2\}, \{3,4\}\{5,6\}, \cdots, \{N-1, N\}\} \tag{88}$$

[11] Under the second possibility, $m^{d**} < m^J$, each K-worker creates knowledge in isolation forever, approaching the sink point m^{d**}.

and initiate pairwise dancing such that[12]

$$\delta_{ij} = \delta_{ji} = 1 \text{ for } \{i,j\} \in P_1,\ \delta_{ij} = \delta_{ji} = 0 \text{ for } \{i,j\} \notin P_1 \tag{89}$$

Similar to Berliant and Fujita (2006, case (ii)), the equilibrium path can be described as follows. The equilibrium path consists of several phases. First, in order to increase income and K-productivity as fast as possible, they want to develop knowledge in common with their partner as fast as possible. Therefore, the N persons are paired arbitrarily and work with their partners for a nonempty interval of time. This implies fast movement to the left, because there is both shared knowledge creation and public knowledge transfer. If potential partners are not actually meeting, their differential knowledge will converge to the sink point of the process where no persons meet, given by (84) and illustrated by m_1^{d*} in Figure 3. This process moves to the left beyond the bliss point because they cannot switch to any new partner that will allow them to maintain the bliss point. The actual partners move quickly to the left of the bliss point and their K-productivity decreases rapidly. When their productivity matches that of a potential partner with whom they have not worked, they switch to new partners and work with their new partners for a nonempty interval of time. Once again, the two actual partners increase their knowledge in common quickly, past the bliss point, and their productivity decreases rapidly, while the differential knowledge with their potential partners moves slowly toward m^{d*}, until the productivity of their current partnership and their previous partnership are the same. Next, each person works alternately with the two partners with whom they worked in the first two phases, but not with a person with whom they have not worked previously. This process continues, but the productivity of each K-worker remains near $g(m_1^{d*})$. The equilibrium path in this case crosses the bliss point, but this is not a sink of the process, due to the myopic behavior of the K-workers.

4 Growth

Next we assemble the various pieces of our general equilibrium model. Our focus is on case (i.a) of the knowledge dynamics, where the initial state of knowledge heterogeneity is to the left of the bliss point: $m^J < m^d(0) < m^B$.

Proposition 2: *Assuming that the number of K-workers N is large, long run economic growth as a function of the initial condition are as follows.*

[12] Here we adopt the convention that $\{i,j\} \in P_1$ means either $\{i,j\} \in P_1$ or $\{j,i\} \in P_1$, whereas $\{i,j\} \notin P_1$ means neither $\{i,j\} \in P_1$ nor $\{j,i\} \in P_1$.

(i) For $m^J < m^d(0) \leq m^B$*, we define two subcases. Let* $\widetilde{C} \equiv \frac{2\theta}{1-\theta}$*.*

(a) $C < \widetilde{C}$*. Let* t^B *be the time that all* K*-workers reach the bliss point* m^B*. Then* $\frac{\dot{n}(t)}{n(t)} = g(m^B)(2 + \frac{N-2}{N}C)$ for $t \geq t^B$*. Moreover,* $\lim_{t\to\infty} \frac{\dot{M}(t)}{M(t)} = \beta\theta^\theta(1-\theta)^{1-\theta} \cdot (2 + \frac{N-2}{N}C)$ *and* $\lim_{t\to\infty} \frac{\dot{v}_i(t)}{v_i(t)} = \frac{\beta\theta^\theta(1-\theta)^{1-\theta}\cdot(2+\frac{N-2}{N}C)}{\sigma-1}$*.*

(b) $C > \widetilde{C}$*. Then* $\lim_{t\to\infty} \frac{\dot{n}(t)}{n(t)} = \lim_{t\to\infty} \frac{\dot{M}(t)}{M(t)} = 2\beta(\frac{C}{2})^\theta$ *whereas* $\lim_{t\to\infty} \frac{\dot{v}_i(t)}{v_i(t)} = \frac{2\beta(\frac{C}{2})^\theta}{\sigma-1}$*.*

Therefore long run economic growth is positively related to both β*, the parameter reflecting* K*-productivity of work in pairs, and* C*, the speed of public knowledge transmission.*

4.1 Case (a): $m^J < m^d(0) \leq m^B$ and $C < \widetilde{C}$

Recall that from equation (66) that the initial pattern of knowledge creation has each K-worker interacting with every other K-worker for an equal share of time, so the dynamics are symmetric and given by (67). The associated sink point is given by (68). Summarizing, the assumption of case (i.a) is that the sink point, when each K-worker is interacting with every other K-worker with the same intensity, is to the right of the bliss point:

$$m^J < m^d(0) < m^B < m^{d*}$$

Let t^B be the unique finite time such that the dynamics reach the bliss point, so that:

$$m^d_{ij}(t) = m^d(t) = m^B \text{ for } t \geq t^B.$$

Due to the symmetry of the path in case (i.a), for all t and for all $j \neq i$,

$$\begin{aligned} g(m^d_{ij}(t)) &= g(m^d(t)) \\ n_i(t) &= n(t) \\ a_{ij}(t) &= a(t) = n(t) \cdot 2g(m^d(t)) \end{aligned}$$

In particular

$$\begin{aligned} g(m^d(t)) &= g(m^B) \text{ for } t \geq t^B \\ a(t) &= n(t) \cdot 2g(m^B) \text{ for } t \geq t^B \end{aligned}$$

Setting $m^d = m^B$ in equation (63), we have

$$\dot{n}(t) = n(t) \cdot \phi^B \text{ for } t \geq t^B$$

where

$$\phi^B \equiv g(m^B)(2 + \frac{N-2}{N}C)$$

so

$$\frac{\dot{n}(t)}{n(t)} = \phi^B \text{ for } t \geq t^B \tag{90}$$

Thus, once the system reaches the bliss point, the size of each K-worker's knowledge expands at an exponential rate of ϕ^B, and we have:

$$n(t) = n(t^B) \cdot e^{\phi^B(t-t^B)} \text{ for } t \geq t^B$$

Recall that the number of varieties of manufactured goods at time t, that is equal to the number of patents present at time t, is given by $M(t)$. Since $A(t)$ is the total number of ideas created at time t, whereas the the proportion of new ideas that are patented is given by η, the rate of increase in patents at time t is given by

$$\dot{M}(t) = \eta \cdot A(t)$$

In the present context of case (i.a), using equation (37),

$$A(t) = \frac{N \cdot a(t)}{2} = N \cdot n(t) \cdot g(m^d(t))$$

and hence

$$\dot{M}(t) = \eta \cdot N \cdot n(t) \cdot g(m^d(t))$$

In particular,

$$\dot{M}(t) = \eta \cdot N \cdot g(m^B) \cdot n(t^B) \cdot e^{\phi^B(t-t^B)} \text{ for } t \geq t^B \tag{91}$$

With this in hand, we can proceed to the calculation of the asymptotic rate of growth of patents. First, for $t \geq t^B$:

$$M(t) = M(t^B) + \int_{t^B}^{t} \dot{M}(\tau)d\tau$$

Using (91),

$$\int_{t^B}^{t} \dot{M}(\tau)d\tau = \eta \cdot N \cdot g(m^B) \cdot n(t^B) \cdot \frac{e^{\phi^B(t-t^B)} - 1}{\phi^B}$$

Hence

$$\frac{\dot{M}(t)}{M(t)} = \frac{\eta \cdot N \cdot g(m^B) \cdot n(t^B) \cdot e^{\phi^B(t-t^B)}}{M(t^B) + \eta \cdot N \cdot g(m^B) \cdot n(t^B) \cdot \frac{e^{\phi^B(t-t^B)}-1}{\phi^B}}$$

implying

$$\lim_{t\to\infty}\frac{\dot M(t)}{M(t)} = \phi^B \equiv g(m^B)(2+\frac{N-2}{N}C) \tag{92}$$
$$= \beta\theta^\theta(1-\theta)^{1-\theta}\cdot(2+\frac{N-2}{N}C)$$

Notice that the asymptotic growth rate of M is the same as the asymptotic growth rate of n.

Next we calculate the rate of growth of indirect utility of consumers. Using (4) and (20), for any consumer i, we have:

$$v_i(t) = E_i \cdot \rho M(t)^{1/(\sigma-1)}$$

Since E_i is constant, this leads to

$$\frac{\dot v_i(t)}{v_i(t)} = \frac{1}{\sigma-1}\cdot\frac{\dot M(t)}{M(t)}$$

Thus, using (92)

$$\lim_{t\to\infty}\frac{\dot v_i(t)}{v_i(t)} = \frac{\phi^B}{\sigma-1} = \frac{g(m^B)(2+\frac{N-2}{N}C)}{\sigma-1} \tag{93}$$
$$= \frac{\beta\theta^\theta(1-\theta)^{1-\theta}\cdot(2+\frac{N-2}{N}C)}{\sigma-1}$$

Therefore the growth rate of indirect utility approaches a constant.

In summary, the growth rate of the individual stock of knowledge (90), the growth rate of patents (92) and the growth rate of indirect utility (93) approach constants as t tends to infinity. These constants are positively related to both C and β.

It is not surprising that β, the coefficient on the joint knowledge production function, is positively related to the growth of the economy. In contrast, it is surprising that C is positively related to economic growth. On the face of it, when C is higher, agents become relatively homogeneous quicker, since the public transmission of patent knowledge is faster. In theory, it could be the case that the result is lower K-productivity and thus lower economic growth because the higher homogeneity reduces knowledge productivity. This was our initial conjecture. However, in the model, as indicated by (80) or (81), the group size at the bliss point adjusts optimally for the speed of public transmission of knowledge. Group size increases to offset the higher speed of public knowledge transmission and the resulting increase in group homogeneity. The effect of larger groups at the bliss point is to create more heterogeneity within groups, thus maintaining higher economic growth. This is, in essence, a general equilibrium effect that allows the economy to take advantage of a higher speed of public information transmission.[13]

4.2 Case (b): $m^J < m^d(0) \leq m^B$ and $C > \widetilde{C}$

The assumption that applies for this case implies that $m^{d*} < m^B$. This case is very similar to the previous one. The only change in the calculations is that $g(m^B)$ is replaced with $g(\widetilde{m}^{d*})$, where $\widetilde{m}^{d*}$ is given by equation (69).[14] The system tends to m^{d*} as $t \to \infty$. Defining

$$
\begin{aligned}
\phi^* &\equiv g(\widetilde{m}^{d*}) \cdot (2 + C) \\
&= 2\beta(\frac{C}{2})^{\theta}
\end{aligned}
$$

Analogous calculations yield

$$
\lim_{t\to\infty} \frac{\dot{n}(t)}{n(t)} = \phi^* \tag{94}
$$

$$
\lim_{t\to\infty} \frac{\dot{M}(t)}{M(t)} = \phi^* \equiv 2\beta(\frac{C}{2})^{\theta}
$$

$$
\lim_{t\to\infty} \frac{\dot{v}_i(t)}{v_i(t)} = \frac{\phi^*}{\sigma - 1} = \frac{2\beta(\frac{C}{2})^{\theta}}{\sigma - 1} \tag{95}
$$

Again, the asymptotic growth rate of individual knowledge stock, patents, and indirect utility are constants (different from case (i.a)), and depend positively on β and C. The surprising result here is that even though the system does not achieve the bliss point, a higher rate of public knowledge transmission results in higher economic growth. Even though m^{d*} decreases as C increases, and thus the productivity of partnerships $g(m^{d*})$ declines, notice that $g(m^{d*})$ represents the normalized productivity of partnerships. In fact, the total productivity of partnerships is $n \cdot g(m^{d*})$. In the end, the positive effect of increasing n due to public knowledge spillovers more than offsets the negative effect of a decline in $g(m^{d*})$.

[13] Based on macro equilibrium conditions of the economy, we have derived the relation (20), meaning that the total expenditure per unit of time is a constant independent of time. In Technical Appendix b, using individual budget constraints, we show that the relation (20) indeed holds along the equilibrium path.

[14] For the sake of simplicity, we assume that N is large, so we can use $\widetilde{m}^{d*}$ given in (69) instead of m^{d*} given by (68).

5 Efficiency

Next we consider the welfare properties of the equilibrium path. Clearly, it is first necessary to introduce a concept of constrained efficiency that accounts for the nature of the monopolistic competition environment in the market for consumption commodities. There is a market failure associated with this feature of the model in itself. However, that is not the focus of our work. Therefore, we employ a notion of constrained efficiency that allows a planner to search for Pareto improvements by using only the choice of the time path of partnerships in the R & D sector, with perfect foresight of the consequences for the other sectors of the model; in particular the consumption good market features monopolistic competition, once the time path in the R & D sector is chosen.

Here we discuss efficiency in the context of an intertemporal utilitarian social welfare function. We consider the following planner's problem, where the planner chooses $\{\delta_{ij}(\cdot)\}_{i,j=1}^{N}$ in order to maximize the sum of M-workers' and K-workers' utility:

$$\max W = \frac{L}{\sigma - 1}\int_0^\infty e^{-\gamma t} \cdot \ln(M(t))dt$$

subject to

$$\begin{aligned}
\dot{M} &= \eta \cdot A \\
&= \eta \cdot \sum_{k=1}^{N} n_k \left(\delta_{kk} \cdot \alpha + \sum_{l \neq k} \delta_{kl} \cdot G(m_{kl}^d, m_{lk}^d) \right)
\end{aligned}$$

$$\begin{aligned}
\dot{n}_i &= (1 - \mu\eta) \cdot n_i \cdot \left\{ \delta_{ii} \cdot \alpha + \sum_{j \neq i} \delta_{ij} \cdot G(m_{ij}^d, m_{ji}^d) \right\} \\
&\quad + \mu\eta \cdot \sum_{k=1}^{N} n_k \left(\delta_{kk} \cdot \alpha + \sum_{l \neq k} \delta_{kl} \cdot G(m_{kl}^d, m_{lk}^d) \right)
\end{aligned}$$

and

$$
\begin{aligned}
\dot{m}_{ij}^{d} &= (1-\mu\cdot\eta)(1-m_{ij}^{d})(1-m_{ji}^{d})\left\{\delta_{ii}\cdot\alpha+\sum_{k\neq i,j}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})\right\} \\
&\quad -m_{ij}^{d}[\mu\eta\alpha(1-m_{ji}^{d})\cdot\sum_{k=1}^{N}\delta_{kk}\cdot\frac{n_k}{n_i}+(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot\left(1-m_{ji}^{d}\right)\cdot 2G(m_{ij}^{d},m_{ji}^{d}) \\
&\quad +\mu\cdot\eta\cdot\left(1-m_{ji}^{d}\right)\sum_{k=1}^{N}\sum_{l\neq k}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G(m_{kl}^{d},m_{lk}^{d})] \\
&\quad -m_{ij}^{d}\cdot(1-\mu\cdot\eta)\cdot\left(1-m_{ij}^{d}\right)\cdot\left\{\delta_{jj}\cdot\alpha+\sum_{k\neq i,j}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})\right\}
\end{aligned}
\tag{96}
$$

given $n_i(0) > 0$ and $m_{ij}^{d}(0) > 0$, for $i,j = 1,...,N$. The equality in the objective function follows from (22) and (106). We must also account for the obvious constraints:

$$
\begin{aligned}
\sum_{j=1}^{N}\delta_{ij} &= 1 \text{ for each } i=1,...,N \\
\delta_{ij} &= \delta_{ji} \text{ for each } i,j=1,...,N \\
\delta_{ij} &\geq 0 \text{ for each } i,j=1,...,N
\end{aligned}
$$

We assume that the discount rate is sufficiently large, $\gamma > g(m^B)$, in order to ensure that the objective is finite. Optimality requires that at each moment of time, the following Hamiltonian is maximized by choosing $\{\delta_{ij}\}_{i,j=1}^{N}$ and taking into account the obvious constraints:

$$
H=\frac{L}{\sigma-1}e^{-\gamma t}\cdot\ln(M(t))+\lambda\cdot\dot{M}+\sum_{i=1}^{N}\xi_i\cdot\dot{n}_i+\sum_{i=1}^{N}\sum_{j\neq i}\chi_{ij}\cdot\dot{m}_{ij}^{d}
$$

where the multipliers follow the dynamics:

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial H}{\partial M}=-\frac{L}{\sigma-1}e^{-\gamma t}\cdot\frac{1}{M} \\
\dot{\xi}_i &= -\frac{\partial H}{\partial n_i} \text{ for } i=1,...,N \\
\dot{\chi}_{ij} &= -\frac{\partial H}{\partial m_{ij}^{d}} \text{ for } i,j=1,...,N,\ i\neq j
\end{aligned}
\tag{97}
$$

and satisfy the following transversality condition:[15]

$$
\lim_{t\to\infty}H(t)=0
$$

[15] This transversality condition comes from Léonard and Van Long (1992), Theorem 9.6.1, p. 299.

Equation (97) implies

$$\begin{aligned}
\lambda(t) &= \frac{L}{\sigma-1}\int_t^\infty e^{-\gamma t}\cdot\frac{1}{M(\tau)}d\tau \\
&= \frac{L}{\sigma-1}e^{-\gamma t}\int_t^\infty e^{-\gamma(\tau-t)}\cdot\frac{1}{M(\tau)}d\tau \\
&= e^{-\gamma t}\cdot\frac{L}{\sigma-1}\int_t^\infty e^{-\gamma(\tau-t)}\cdot\frac{1}{M(\tau)}d\tau
\end{aligned}$$

From equation (101), it follows that

$$\lambda(t) = e^{-\gamma t}\cdot\Pi(t)$$

Moreover, from equation (103),

$$\lim_{t\to\infty} e^{-\gamma t}\cdot M(t)\cdot\Pi(t) = 0$$

So

$$\lim_{t\to\infty} M(t)\cdot\lambda(t) = 0$$

Suppose that the following symmetric initial conditions for case (i) are satisfied:

$$\begin{gathered}
n_i(0) = n(0) > 0 \text{ for } i = 1, ..., N \\
m^J < m_{ij}^d(0) = m^d(0) < m^B \text{ for } i, j = 1, ..., N,\ i \neq j \\
\text{and } g(m^B) > \alpha
\end{gathered}$$

Recall that the myopic equilibrium path for case (i) when $m^J < m_{ij}^d(0)$ is:

$$\begin{aligned}
\delta_{ij}(t) &= \frac{1}{N-1} \text{ for } t^J \leq t < t^B \text{ for } i, j = 1, ..., N,\ i \neq j \qquad (98)\\
\delta_{ij}(t) &= \frac{1}{N^B-1} \text{ for } t > t^B \text{ when } i \text{ and } j \text{ belong to the same group}
\end{aligned}$$

where t^B is the first time t such that $m(t) = m^d(t) = m^B$, the bliss point m^B is given by (52) and the group size N^B is given by (80).

Under these initial conditions, it can be verified that if N is sufficiently large, then there exists a set of multipliers such that the myopic equilibrium path detailed in (98) for case (i) satisfies the necessary conditions for optimality.

When $m^d(t) < m^B$ and therefore $t < t^B$, then knowledge productivity is higher and m_{ij}^d moves almost as fast to the right as working in isolation if each person works with every other person with equal intensity. The intuition for this result follows from a combination of two reasons. Productivity is higher when working with others as opposed to working alone on this part of the path. When N is sufficiently large, working with others is very close to working in isolation when the accumulation of differential knowledge is considered, so cooperation with others will be better on net. Once the bliss point is attained, the system reaches the highest productivity possible, and remains there.

This intuition indicates that, when $m^d(t) < m^B$, working with a smaller group than the other $N-1$ dancers, then movement to the right is slower than working with everyone but oneself. So coalitions cannot block this path. Furthermore, once the bliss point is achieved, this is the highest productivity possible, so coalitions cannot block this part of the path either. Thus, the path chosen by myopic agents, that coincides with the utilitarian welfare optimal path, is in the core with rational expectations.

6 Conjectures and Conclusions

We have considered a model of knowledge creation and economic growth that is based on individual behavior, allowing knowledge workers to decide whether joint or individual production is best for them at any given time. We have allowed them to choose their best partner or to work in isolation. One would not expect that equilibria would be efficient for two reasons: there are externalities in R & D (both from pairwise interactions and from public knowledge transmission through patents), and the markets for consumption goods are characterized by monopolistic competition. The emphasis of our model is on endogenous agent heterogeneity, whereas we examine the permanent effects of knowledge creation and accumulation on growth.

With N persons, assuming that N is large enough, we find that, surprisingly, for a range of initial conditions that imply a large degree of homogeneity among agents, the sink is the most productive state in the R & D sector. The population breaks into optimal size groups when it reaches the most productive state. The size of these groups is inversely related to the weight given to homogeneity in knowledge production.

Our equilibrium is efficient, subject only to the constraint that the market for consumption goods features monopolistic competition.

Long run economic growth is positively related to both the effectiveness of pairwise knowledge worker interaction and, more importantly, to the effectiveness of public knowledge transmission. The latter is due, in part, to the endogenous adjustment of R & D group size to a better public knowledge transmission technology.

In applying our results to real life issues, we must be very careful about interpreting the meaning of the comparative dynamics that we have derived. According to equation (81), for example, the optimal group size $\widetilde{N}^B$ increases as C (the speed of public knowledge transmission) increases or θ (the weight given to knowledge in common in K-production) *decreases*. In real life, however, once the optimal group size is reached under a fixed set of parameters, group size does not easily adjust to a new optimal size under a new set of parameters. This is because of the *lock-in effect* of the optimal group size that was explained at the end of Case (a) in Section 3.2.1 Case (i). In particular, the knowledge of a K-worker in one group will drift apart from the knowledge of K-workers in other groups. Thus, once the most productive state is achieved under one set of parameters, realigning the K-workers into larger groups when parameters change will not result in optimal knowledge production, since the K-workers initially in different groups have differentiated themselves too much from each other. This lock-in effect inherent in an R & D system may partly explain, for example, why the Japanese economy has been suffering from a prolonged recession and slow growth since the early 1990s. Specifically, the so called IT revolution has significantly increased the value of C, whereas new industries displaying rapid growth, such as computer software and advanced service industries (including global finance), tend to have a lower value of θ (i.e., a higher weight on knowledge diversity in K-production) than traditional manufacturing industries (based mainly on incremental improvements in Japan). Due to the lock-in effect, R & D group size and composition were inherited from past economic circumstances. Our model implies low mobility of Japanese workers and researchers beyond existing institutions, through no fault of their own. But the Japanese R & D system has not adapted adequately to the new situation. Our analysis implies that research groups in the new industries should be made more diverse and larger. Such a change would imply short term reductions in R & D productivity in exchange for long term gains.

Many extensions of our work come to mind. It is important and interesting to add direct pairwise knowledge transfer between knowledge workers on a team, as opposed to public knowledge that is learned by everyone, to the model. Then we can study comparative statics with respect to speeds of knowledge transfer and knowledge creation on the equilibrium outcome and on its efficiency. Markets for ideas would also be a nice feature. One set of extensions would allow agents to decide, in addition to the people they choose with whom to work, the intensity of knowledge creation and exchange.

Another set of extensions would be to add stochastic elements to the model, so the knowledge creation and transfer process is not deterministic. Probably our framework can be developed from a more primitive stochastic model, where the law of large numbers is applied to obtain our framework as a reduced form.[16]

An important application of our work would be to the literature on intellectual property, where the idea production process is often modeled as a black box; see Scotchmer (2004) and Boldrin and Levine (2005) for interesting and provocative treatments.

Location seems to be an important feature of knowledge creation and transfer, so regions and migration are important, along with urban economic concepts more generally; for example, see Duranton and Puga (2001) and Helsley and Strange (2004). A natural extension of our model would have knowledge workers in regions, allowing only those in the same region to interact, but making migration of knowledge workers between regions feasible.

It would be very useful to extend the model to more general functional forms. It would be interesting to proceed in the opposite direction by putting more structure on our concept of knowledge, allowing asymmetry or introducing notions of distance, such as a metric, on the set of ideas[17] or on the space of knowledge. Finally, it would be useful to add vertical differentiation of knowledge, as in Jovanovic and Rob (1989), to our model of horizontally differentiated knowledge.

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[16] We confess that our first attempts to formulate our model of knowledge creation were stochastic in nature, using Markov processes, but we found that they quickly became intractable.

[17] See Berliant *et al* (2006).

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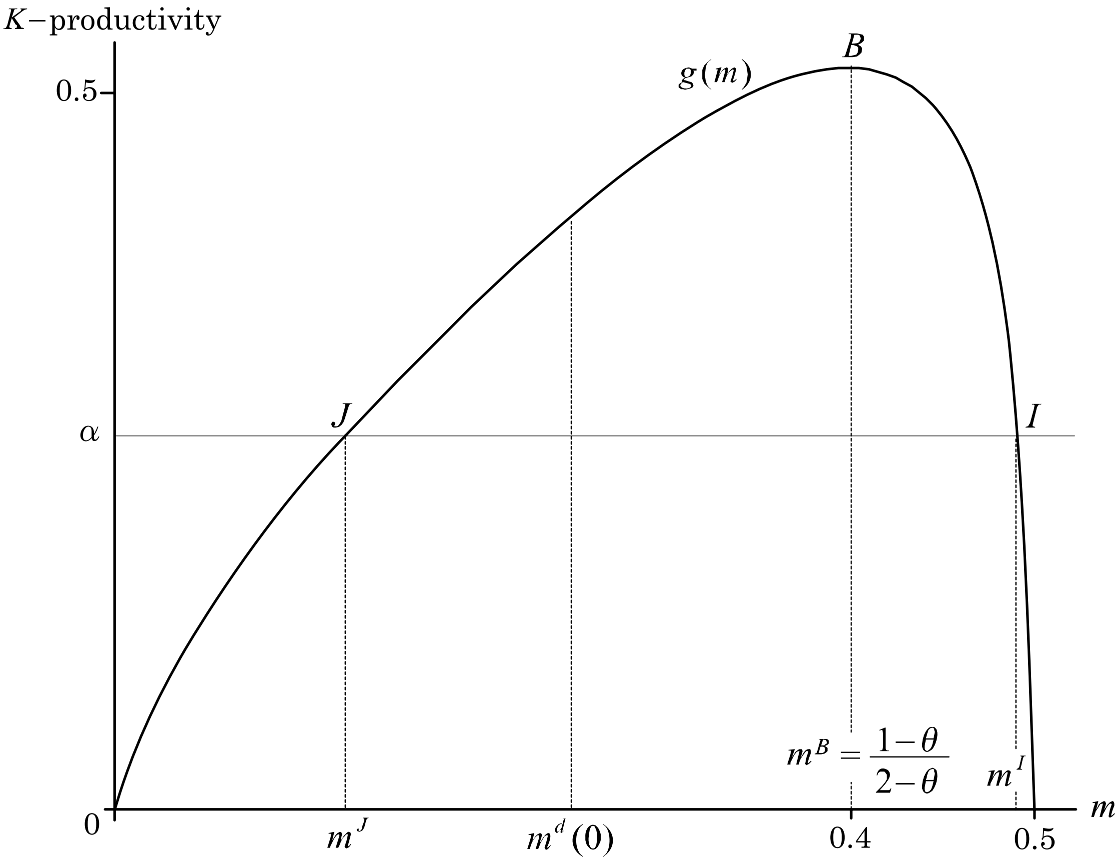

Figure 1: The $g(m)$ curve and the bliss point when $\beta = 1$ and $\theta = 1/3$.

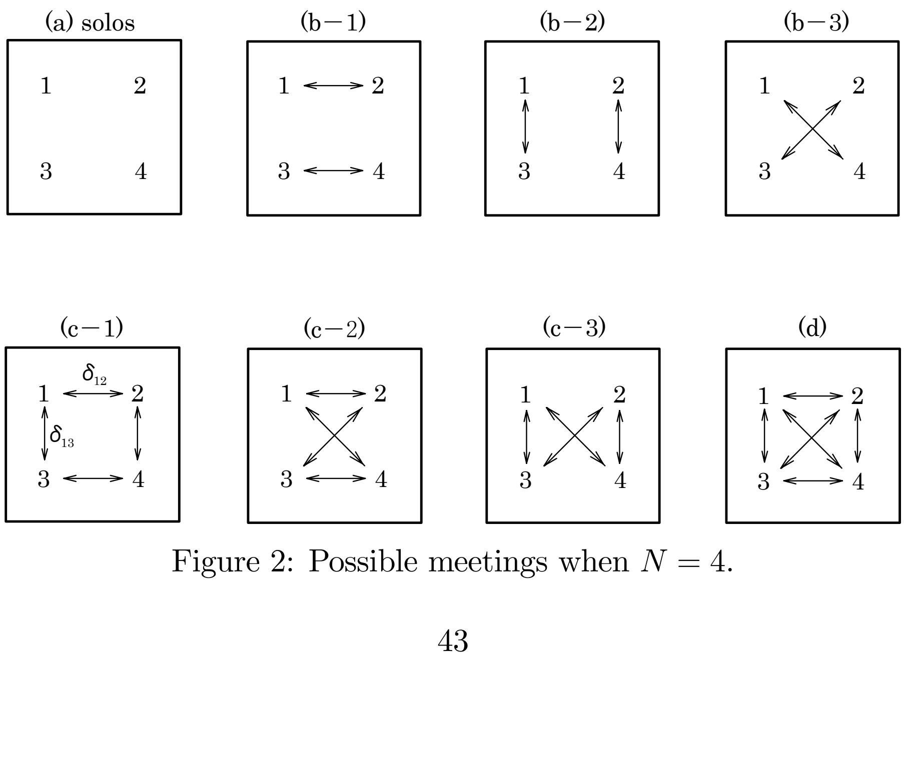

Figure 2: Possible meetings when $N = 4$.

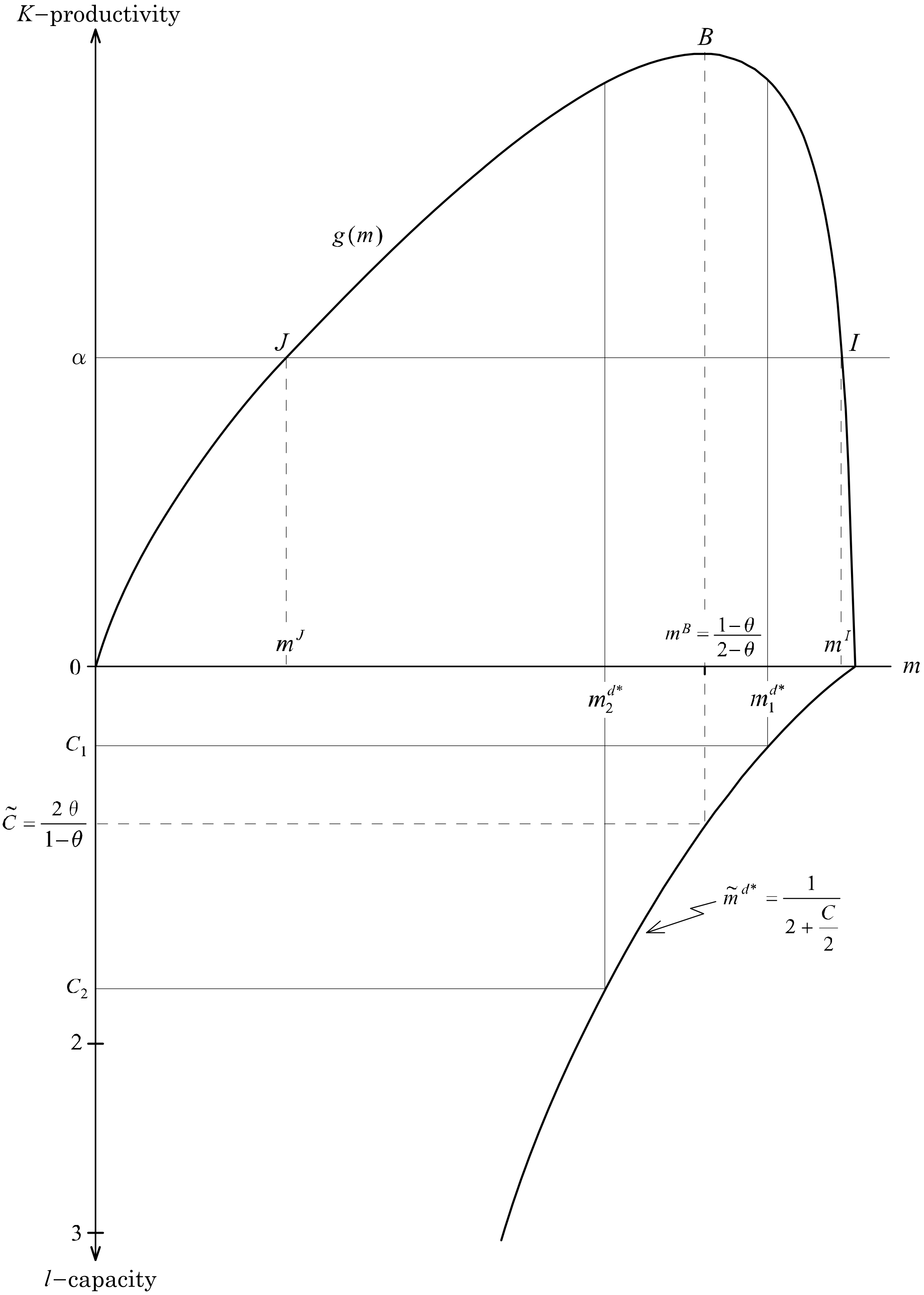

Figure 3: The K-productivity curve $g(m)$ and the limiting sink curve $\widetilde{m}^{d*}$.

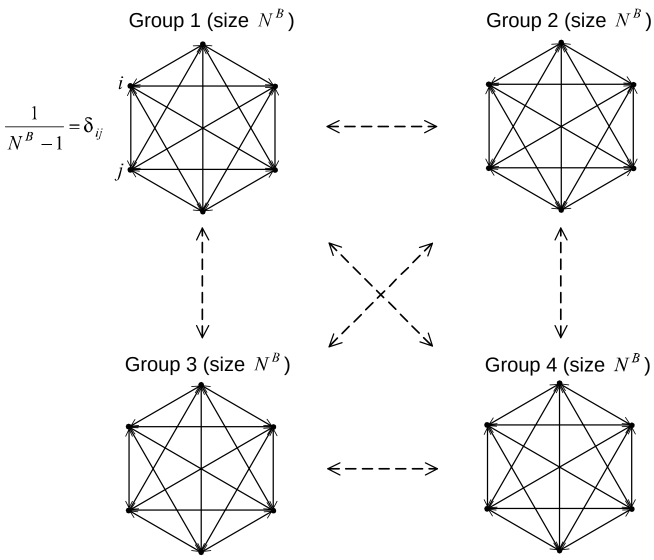

Figure 4: An example of K-interactions at the bliss point.

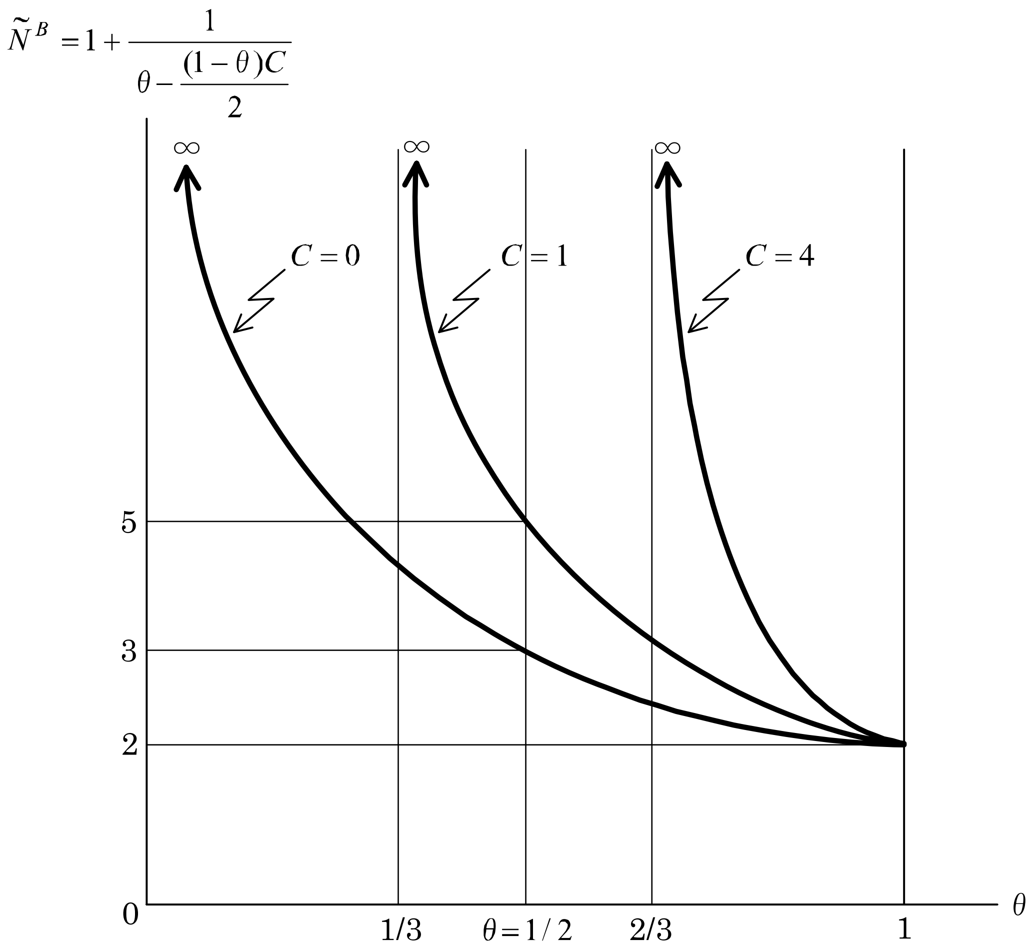

Figure 5: The optimal group size $\widetilde{N}^B$.

7 Appendix 1: Justification of Knowledge Absorption Function

Consider the following statement of the capacity constraint on knowledge absorption from public information on K-worker i:

$$C \cdot \left(\delta_{ii} \cdot a_{ii} + \sum_{j \neq i} \delta_{ij} \cdot (a_{ij}/2) \right) = \mu\eta \left(A - \sum_{j=1}^{N} \delta_{ij} \cdot a_{ij} \right) \tag{99}$$

We shall explain the content of this equation piece by piece. On the right hand side of the equation, the term in brackets $A - \sum_{j=1}^{N} \delta_{ij} \cdot a_{ij}$ represents the new knowledge produced in the economy that does not involve partnerships including K-worker i. Recall that η gives the rate at which new ideas are patented, whereas μ gives the rate at which publicly revealed ideas can be absorbed by a K-worker. Therefore the right hand side of the equation represents the public knowledge revealed by patents that is absorbed by K-worker i. The term in brackets on the left hand side represents new knowledge created by K-worker i at an instant. In total, the equation means that the new public knowledge that can be absorbed by K-worker i is proportional to their capacity to produce new ideas. In essence, this is due to the constraint on their time and the productivity of their effort both to absorb new ideas and to produce them.

Equation (99) implies:

$$\mu = \frac{C}{\eta} \cdot \frac{\delta_{ii} \cdot a_{ii} + \sum_{j \neq i} \delta_{ij} \cdot (a_{ij}/2)}{A - \sum_{j=1}^{N} \delta_{ij} \cdot a_{ij}}$$

Next we consider two special cases, where we assume pairwise symmetry: $n_k = n_i \equiv n$ for all i and k. First, when each agent in the knowledge sector is working alone, namely $\delta_{ii} = 1$ for all i and $\delta_{ij} = 0$ for all $i \neq j$, then

$$A = \sum_{k=1}^{N} \alpha \cdot n_k = \alpha \cdot n \cdot N$$

and

$$\mu = \frac{C}{\eta} \cdot \frac{\alpha \cdot n}{\alpha \cdot n \cdot (N-1)} = \frac{C}{\eta} \cdot \frac{1}{N-1}$$

The second special case is given by $m_{ij}^d = m^d$ for all $i \neq j$ and $g(m^d) > \alpha$. Thus, $\delta_{ii} = 0$ for all i and $a_{ij} = a$ for all $i \neq j$. In this special case, we have

$$\mu = \frac{C}{\eta} \cdot \frac{a/2}{\frac{Na}{2} - a} = \frac{C}{\eta} \cdot \frac{1}{N-2}$$

Assuming N is sufficiently large, we employ the following specification.

$$\mu(N) \approx \frac{C}{\eta N}$$

8 Appendix 2: Technical Appendix

8.1 Appendix a

Theorem A1: The following identity holds:

$$\frac{a_{ij}/2}{n_i} = G(m_{ij}^d, m_{ji}^d)$$

where $G$is defined in (34).
Proof: Using (25) and (32),

$$\begin{aligned}
& \frac{a_{ij}/2}{n_i} \\
= & \frac{n^{ij}}{n_i} \cdot \frac{a_{ij}/2}{n^{ij}} \\
= & \frac{1}{1-m_{ji}^d} \cdot \beta(m_{ij}^c)^{\theta} \cdot (m_{ij}^d \cdot m_{ji}^d)^{\frac{1-\theta}{2}} \\
= & \frac{\beta\left(1-m_{ij}^d-m_{ji}^d\right)^{\theta} \cdot (m_{ij}^d \cdot m_{ji}^d)^{\frac{1-\theta}{2}}}{1-m_{ji}^d} \\
= & G(m_{ij}^d, m_{ji}^d)
\end{aligned}$$

which leads to (33).
Theorem A2: Knowledge dynamics evolve according to the system:

$$\begin{aligned}
\dot{m}_{ij}^d = & (1-\mu\cdot\eta)(1-m_{ij}^d)(1-m_{ji}^d)\left\{\delta_{ii}\cdot\alpha + \sum_{k\neq i,j}\delta_{ik}\cdot 2G(m_{ik}^d, m_{ki}^d)\right\} \\
& -m_{ij}^d[\mu\cdot\eta\cdot\alpha\cdot\sum_{i=1}^{N}\delta_{ii}(1-m_{ji}^d) + (1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2\beta\cdot\left(1-m_{ji}^d\right)\cdot 2G(m_{ij}^d, m_{ji}^d) \\
& +\mu\cdot\eta\cdot\sum_{i=1}^{N}\sum_{l\neq k}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot\beta\cdot\left(1-m_{ji}^d\right)\cdot G(m_{kl}^d, m_{lk}^d)] \\
& -m_{ij}^d[(1-\mu\cdot\eta)\cdot\left(1-m_{ij}^d\right)\cdot\left\{\delta_{jj}\cdot\alpha + \sum_{k\neq i,j}\delta_{jk}\cdot 2G(m_{jk}^d, m_{kj}^d)\right\}]
\end{aligned}$$

for $i, j = 1, 2, \cdots, N$, $i \neq j$

Proof: By definition,

$$\begin{aligned}
\dot{m}_{ij}^d &= \frac{d\left(n_{ij}^d/n^{ij}\right)}{dt} \\
&= \frac{\dot{n}_{ij}^d}{n^{ij}} - \frac{n_{ij}^d}{n^{ij}} \cdot \frac{\dot{n}^{ij}}{n^{ij}} \\
&= \frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d \cdot \frac{\dot{n}^{ij}}{n^{ij}} \\
&= \frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d \cdot \left(\frac{\dot{n}_{ij}^c}{n^{ij}} + \frac{\dot{n}_{ij}^d}{n^{ij}} + \frac{\dot{n}_{ji}^d}{n^{ij}}\right) \\
&= \left(1 - m_{ij}^d\right) \cdot \frac{\dot{n}_{ij}^d}{n^{ij}} - m_{ij}^d \cdot \left(\frac{\dot{n}_{ij}^c}{n^{ij}} + \frac{\dot{n}_{ji}^d}{n^{ij}}\right)
\end{aligned}$$

Setting $\mu = \mu(N)$, and using (41) and (32), we have

$$\begin{aligned}
\frac{\dot{n}_{ij}^d}{n^{ij}} &= \frac{(1-\mu\cdot\eta)\cdot\sum\limits_{k\neq j}\delta_{ik}\cdot a_{ik}}{n^{ij}} \\
&= (1-\mu\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\neq i,j}\delta_{ik}\cdot\frac{a_{ik}}{n^{ij}}] \\
&= (1-\mu\cdot\eta)\cdot[\frac{\delta_{ii}\cdot\alpha\cdot n_i}{n^{ij}} + \sum_{k\neq i,j}\delta_{ik}\cdot\frac{n_i}{n^{ij}}\cdot\frac{n^{ik}}{n_i}\cdot\frac{a_{ik}}{n^{ik}}] \\
&= (1-\mu\cdot\eta)\cdot\frac{n_i}{n^{ij}}\cdot\left\{\delta_{ii}\cdot\alpha + \sum_{k\neq i,j}\delta_{ik}\cdot\frac{n^{ik}}{n_i}\cdot\frac{a_{ik}}{n^{ik}}\right\} \\
&= (1-\mu\cdot\eta)\cdot\left(1-m_{ji}^d\right)\cdot\left\{\delta_{ii}\cdot\alpha + \sum_{k\neq i,j}\delta_{ik}\cdot\frac{1}{1-m_{ki}^d}\cdot 2\beta\left(1-m_{ik}^d-m_{ki}^d\right)^{\theta}\cdot\left(m_{ik}^d\cdot m_{ki}^d\right)^{\frac{1-\theta}{2}}\right. \\
&= (1-\mu\cdot\eta)\cdot\left(1-m_{ji}^d\right)\cdot\left\{\delta_{ii}\cdot\alpha + \sum_{k\neq i,j}\delta_{ik}\cdot 2G(m_{ik}^d, m_{ki}^d)\right\}
\end{aligned}$$

Similarly,

$$\frac{\dot{n}_{ji}^d}{n^{ij}} = (1-\mu\cdot\eta)\cdot\left(1-m_{ij}^d\right)\cdot\left\{\delta_{jj}\cdot\alpha + \sum_{k\neq i,j}\delta_{jk}\cdot 2G(m_{jk}^d, m_{kj}^d)\right\}$$

while using (40) yields

$$\begin{aligned}
\frac{\dot{n}_{ij}^c}{n^{ij}} &= \frac{(1-\mu\cdot\eta)\delta_{ij}\cdot a_{ij} + \mu\cdot\eta A}{n^{ij}} \\
&= \frac{(1-\mu\cdot\eta)\delta_{ij}\cdot a_{ij} + \mu\cdot\eta\cdot\sum_{k=1}^N\delta_{kk}\cdot a_{kk} + \mu\cdot\eta\cdot\sum_{k=1}^N\sum_{l\neq k}\delta_{kl}\cdot(a_{kl}/2)}{n^{ij}} \\
&= \frac{\mu\cdot\eta\cdot\sum_{k=1}^N\delta_{kk}\cdot a_{kk}}{n^{ij}} + \frac{(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot a_{ij}}{n^{ij}} + \frac{\mu\cdot\eta\cdot\sum_{k=1}^N\sum_{l\neq k}\delta_{kl}\cdot(a_{kl}/2)}{n^{ij}}
\end{aligned}$$

Using equations (24), (25), (33), and (32), we have:

$$
\begin{aligned}
\frac{\dot{n}_{ij}^{c}}{n^{ij}} &= \mu\cdot\eta\cdot\alpha\cdot(1-m_{ji}^{d})\cdot\sum_{k=1}^{N}\delta_{kk}\cdot\frac{n_k}{n_i}+(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot 2\beta\cdot\left(1-m_{ij}^{d}-m_{ji}^{d}\right)^{\theta}\cdot(m_{ij}^{d}\cdot m_{ji}^{d})^{\frac{1-\theta}{2}} \\
&\quad +\mu\cdot\eta\cdot\sum_{k=1}^{N}\sum_{l\neq k}\delta_{kl}\cdot\frac{n^{kl}}{n^{ij}}\cdot\beta\cdot\left(1-m_{kl}^{d}-m_{lk}^{d}\right)^{\theta}\cdot(m_{kl}^{d}\cdot m_{lk}^{d})^{\frac{1-\theta}{2}} \\
&= \mu\cdot\eta\cdot\alpha\cdot(1-m_{ji}^{d})\cdot\sum_{k=1}^{N}\delta_{kk}\cdot\frac{n_k}{n_i}+(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot(1-m_{ji}^{d})\cdot 2G(m_{ij}^{d},m_{ji}^{d}) \\
&\quad +\mu\cdot\eta\cdot\sum_{k=1}^{N}\sum_{l\neq k}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot(1-m_{ji}^{d})\cdot G\left(m_{kl}^{d},m_{lk}^{d}\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\dot{m}_{ij}^{d} &= (1-\mu\cdot\eta)(1-m_{ij}^{d})(1-m_{ji}^{d})\left\{\delta_{ii}\cdot\alpha+\sum_{k\neq i,j}\delta_{ik}\cdot 2G(m_{ik}^{d},m_{ki}^{d})\right\} \\
&\quad -m_{ij}^{d}[\mu\eta\alpha(1-m_{ji}^{d})\cdot\sum_{k=1}^{N}\delta_{kk}\cdot\frac{n_k}{n_i}+(1-\mu\cdot\eta)\cdot\delta_{ij}\cdot\left(1-m_{ji}^{d}\right)\cdot 2G(m_{ij}^{d},m_{ji}^{d}) \\
&\quad +\mu\cdot\eta\cdot\left(1-m_{ji}^{d}\right)\sum_{k=1}^{N}\sum_{l\neq k}\delta_{kl}\cdot\frac{n_k}{n_i}\cdot G(m_{kl}^{d},m_{lk}^{d})] \\
&\quad -m_{ij}^{d}\cdot(1-\mu\cdot\eta)\cdot\left(1-m_{ij}^{d}\right)\cdot\left\{\delta_{jj}\cdot\alpha+\sum_{k\neq i,j}\delta_{jk}\cdot 2G(m_{jk}^{d},m_{kj}^{d})\right\}
\end{aligned}
$$

8.2 Appendix b

Here we confirm that when the expenditure of any specific consumer i is constant over time and given by (20), the total equilibrium expenditure in the economy is indeed given by equation (18).

For case (a), along the equilibrium path, the present value of income at time 0 for K-worker i is given by

$$
\begin{aligned}
W_i(0) &= \int_0^{\infty} e^{-\gamma t}y_i(t)dt \\
&= \int_0^{\infty} e^{-\gamma t}\cdot[\frac{\eta\cdot a(t)}{2}\cdot\Pi(t)]dt
\end{aligned}
$$

Furthermore, as explained in case (a) of the Growth section,

$$\dot{M}=\eta\cdot A(t)=\frac{\eta\cdot N\cdot a(t)}{2}$$

Thus,

$$W_i(0) = \frac{1}{N}\int_0^\infty e^{-\gamma t}\cdot \dot{M}(t)\cdot \Pi(t)dt \tag{100}$$

where the patent price $\Pi(t)$ is obtained as follows, by using (15) and (16):

$$\begin{aligned}\Pi(t) &= \int_t^\infty e^{-\gamma(\tau-t)}\cdot \pi^*(\tau)d\tau \\ &= \int_t^\infty e^{-\gamma(\tau-t)}\cdot \frac{q^*(\tau)}{\sigma-1}d\tau \\ &= \frac{L}{\sigma-1}\cdot \int_t^\infty e^{-\gamma(\tau-t)}\cdot \frac{1}{M(\tau)}d\tau \end{aligned}\tag{101}$$

that yields

$$\dot{\Pi}(t) = \gamma\Pi(t) - \frac{L}{\sigma-1}\cdot\frac{1}{M(t)} \tag{102}$$

Next, integrating (100) by parts and using (102), we obtain:

$$\begin{aligned}W_i(0) &= \frac{1}{N}\int_0^\infty \dot{M}(t)\cdot(e^{-\gamma t}\cdot\Pi(t))dt \\ &= \frac{1}{N}\left\{\left[e^{-\gamma t}\cdot M(t)\cdot\Pi(t)\right]|_0^\infty - \int_0^\infty M(t)\cdot\frac{d(e^{-\gamma t}\cdot\Pi(t))}{dt}dt\right\} \\ &= \frac{1}{N}\left\{\left[e^{-\gamma t}\cdot M(t)\cdot\Pi(t)\right]|_0^\infty - \int_0^\infty M(t)\cdot(-\gamma e^{-\gamma t}\cdot\Pi(t)+e^{-\gamma t}\cdot\dot{\Pi}(t))dt\right\} \\ &= \frac{1}{N}\left\{\left[e^{-\gamma t}\cdot M(t)\cdot\Pi(t)\right]|_0^\infty +\gamma\int_0^\infty M(t)\cdot e^{-\gamma t}\cdot\Pi(t)dt - \int_0^\infty e^{-\gamma t}\cdot M(t)\cdot\dot{\Pi}(t))dt\right\} \\ &= \frac{1}{N}\{\left[e^{-\gamma t}\cdot M(t)\cdot\Pi(t)\right]|_0^\infty +\gamma\int_0^\infty e^{-\gamma t}\cdot M(t)\cdot\Pi(t)dt \\ &\quad -\int_0^\infty e^{-\gamma t}\cdot(M(t)\cdot\gamma\Pi(t)-\frac{L}{\sigma-1})dt\} \\ &= \frac{1}{N}\left\{\left[e^{-\gamma t}\cdot M(t)\cdot\Pi(t)\right]|_0^\infty + \int_0^\infty e^{-\gamma t}\cdot\frac{L}{\sigma-1}dt\right\}\end{aligned}$$

that leads to

$$W_i(0) = \frac{1}{N}\left\{\lim_{t\to\infty} e^{-\gamma t}\cdot M(t)\cdot\Pi(t) - M(0)\cdot\Pi(0) + \frac{L}{\sigma-1}\cdot\frac{1}{\gamma}\right\}$$

Thus, using (20) and (8), for any specific K-worker i, expenditure is:

$$\begin{aligned}E_i &= \gamma\left(\frac{\Pi(0)\cdot M(0)}{N} + W_i(0)\right) \\ &= \frac{1}{N}\left\{\gamma\cdot\lim_{t\to\infty} e^{-\gamma t}\cdot M(t)\cdot\Pi(t) + \frac{L}{\sigma-1}\right\}\end{aligned}$$

In order to evaluate the first term in this expression, observe that by (101),

$$\begin{aligned} M(t) \cdot \Pi(t) &= \frac{L}{\sigma-1} \cdot \int_t^\infty e^{-\gamma(\tau-t)} \cdot \frac{M(t)}{M(\tau)} d\tau \\ &< \frac{L}{\sigma-1} \cdot \int_t^\infty e^{-\gamma(\tau-t)} d\tau = \frac{L}{\sigma-1} \cdot \frac{1}{\gamma} \end{aligned}$$

The second line follows since for $\tau > t$, $\frac{M(t)}{M(\tau)} < 1$. So

$$\lim_{t \to \infty} e^{-\gamma t} \cdot M(t) \cdot \Pi(t) = 0 \tag{103}$$

and

$$E_i = \frac{1}{N} \cdot \frac{L}{\sigma - 1}$$

Therefore, the total expenditure of all K-workers together is:

$$N \cdot E_i = \frac{L}{\sigma - 1} \tag{104}$$

For any specific M-worker i, we have $y_i(t) = w^M \equiv 1$ for every time t. Thus,

$$W_i(0) = \int_0^\infty e^{-\gamma t} \cdot y_i(t) dt = \frac{1}{\gamma}$$

Noting that $\omega_i = 0$ by assumption for any M-worker, equation (20) yields:

$$E_i = 1$$

So the total expenditure of all M-workers together is:

$$L \cdot E_i = L \tag{105}$$

Summing (104) and (105) yields the total expenditure of consumers in the economy:

$$E^* = \frac{L}{\sigma - 1} + L = \frac{\sigma L}{\sigma - 1} = \frac{L}{\rho} \tag{106}$$

Therefore relation (18) is verified for the equilibrium path.