EXPECTATIONS FORMATION AND VOLATILITY

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(in progress, comments welcome)

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Abstract. I show in this paper that the possibility of expectations-driven volatility of prices and trades as an equilibrium phenomenon has been downplayed by the use of the rational expectations hypothesis. Starting from the fact that the rational expectations hypothesis remains silent about how the agents may have ended up holding the expectations they hold, I consider the consequences of imposing rationality assumptions on the way the agents form their expectations. More specifically, I assume that each agent holds *rationally formed expectations* in the sense that any other expectations consistent with his choices that he might have held would imply a smaller likelihood for the history he observes. Then I establish that in simple overlapping generations economies without intrinsic uncertainty there are expectations-driven fluctuations of prices and trades that are rationally formed expectations equilibrium outcomes, but would never be the outcome of a rational expectations equilibrium. This result suggests that the strong requirements underlying the rational expectations hypothesis may have led to a serious underestimation of the role of expectations in the generation of volatility of equilibrium prices and allocations.

1. Introduction

In this paper I argue, in an overlapping generations framework, that the use of the rational expectations hypothesis underestimates the possibility of expectations-driven volatility of prices and trades as an equilibrium outcome. In effect, whenever one modifies the equilibrium notion in order to add a rationality condition *on the formation of expectations*, new instances of expectations-driven fluctuations of prices and trades turn out to be equilibrium outcomes. The need to add some rationality condition on the formation of expectations could already be felt after realizing that the rational expectations hypothesis imposes just a consistency condition that remains silent on the issue of how do the agents form their expectations.

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But this need becomes an urge once it is noticed that in a rational expectations equilibrium the expectations held by the agents need not be those that account in the best way for the history observed by the agent *at the time of making his decision*, even among the expectations that are consistent with the agents' decisions. In addition to this, whenever a rational expectations equilibria is stationary the agents' expectations are history independent, which is clearly counterfactual. From a positive viewpoint, expectations *do* result from past experiences, and the dependence is clearly non-trivial. Since any sensible theory about the formation of the agents' expectations needs to make them follow from the information available to agents at the time they make their decisions, then an alternative assumption about the expectations at equilibrium is needed.

Accordingly, I adopt in this paper the view that, whenever an agent's decision is consistent with several expectations, to assume that the expectations the agent holds do not provide among these the best explanation of the observed history amounts to assume that he did not formed his expectations rationally. One could say that, if the rational expectations hypothesis "is nothing else than the extension of the rationality hypothesis to expectations" (Guesnerie, AER(1992)), the hypothesis above is nothing else than the extension of the rationality hypothesis to the *formation* of expectations. Thus, I will consider, in an overlapping generations economy without intrinsic uncertainty, that agents hold instead *rationally formed expectations*, in the sense that any other expectations, consistent with their choices, that they may hold would imply a smaller likelihood for the history they observe, and therefore it would be irrational for the agents to hold them. I show then to be a consequence of this assumption that there exist in this economy rationally formed expectations equilibria exhibiting fluctuations that follow finite-state first order Markov chains, that no rational expectations (sunspot) equilibrium could generate.[1] The existence of such equilibria suggests that the role of expectations in the generation of excess volatility in the economy may be much more important than what the widespread use of the rational expectations hypothesis may have led to think.

In order to see this result let us stress first that the currently standard approach to modelling expectations —i.e. making the rational expectations hypothesis— consists only of imposing a consistency requirement on the agents' expectations without actually addressing the question of where do the agents' expectations come from. Alternatively, one can still require the expectations to be consistent while adding specific rationality requirements for the way in which the agents form their expectations. More specifically, the equilibrium of an economy is usually defined in terms of feasible allocations and prices, i.e. in terms of compatible consumption decisions (and possibly production decisions also) and actual prices at which trades take place. Each consumption decision is supposed to result from a utility maximization given the prices. Nevertheless, more accordingly to our intuition the equilibrium of an economy should rather be defined in terms of *perceived prices*, consumption decisions, and actual prices, satisfying all some conditions. The actual prices are supposed to be determined by the consumption decisions (i.e. by the "law of supply and demand"), which are determined themselves by the perceived prices. This is a sequence of causalities that one may be tempted to close

[1] See Cass and Shell (1983) for the seminal paper on the notion of sunspot equilibrium, and Chiappori and Guesnerie (1991) for a survey on these equilibria in dynamic economies.

into a loop, and the rational expectations hypothesis does just that in the most straightforward way, i.e. requiring that the perceived prices and the actual prices coincide. Of course, this distinction between the perceived prices and actual prices is void of content whenever there is no room for uncertainty about prices —as in a one-shot, static set-up for instance— although there remains the usual problem of simultaneous causation (of the kind "which came first, the chicken or the egg") that haunts any notion of equilibrium in multiple-agents simultaneous decision making problems. In that case perceived prices and actual prices can only but coincide at equilibrium if agents are to pick choices within their actual budget sets.

The distinction becomes nonetheless meaningful when there is room for uncertainty about prices, as for instance in dynamic setups where, although current perceived prices can still be required to coincide with the current actual prices for the same reasons as in the static case, there may be room now for the perceived distribution of future prices (or price expectations) to depart from whatever the actual distribution of future prices may be, as long as the consumers' equilibrium behavior is consistent with both. Still, the rational expectations hypothesis wipes out the distinction quite drastically by requiring the outright exact coincidence between the distributions of expected prices and actual prices. This certainly simplifies things, even though at the cost of making an admittedly extremely strong assumption. But more importantly, it also certainly overlooks the issue of where do price expectations come from, since although one can have no problem to admit that perceived current prices come from the direct observation of actual current prices on catalogues or shop windows, nothing of the kind (barring crystal balls) exists for future prices.

More reasonably, an equilibrium should be defined instead as, for each agent and every history of past and current prices he may observe, (1) *expectations on future prices* (or, equivalently, a belief that the prices follow a particular stochastic process), and (2) consumption decisions, such that, for any history of prices that may realize, (i) the resulting allocation of resources is feasible, (ii) the agents' consumption choices maximize their utilities given the price process they believe they face,[2] and (iii) *the agents' beliefs about the price process are formed rationally*, i.e. their beliefs constitute their optimal estimates of the price process among those consistent with their choices, given the information they have. More specifically, the agents' beliefs should be such that no other beliefs consistent with their choices imply that the observed history is a likelier outcome.

Note that no specific condition other than leading to compatible consumption decisions by the agents is required from the history of prices that may actually realize. Thus there is no room for agents to mistake a price process they face, since there is no such thing as an independent actual price process *prior* to their decisions. This intends to capture the idea that nothing else other than the confrontation of demand and supply in the marketplace determines the actual prices at which trades take place. The consistency of price expectations with actual prices is required to hold backwards and in terms of their ability to account in the best possible way —among the expectations consistent with their choices— for the observed history

[2] For the overlapping generations economies considered below it will turn out to be the case that in doing so they will be maximizing their utilities subject to the prices, resulting from their decisions, that they *actually* face as well.

of prices, instead of forwards and in terms its ability to predict in the best possible way future realizations of a given process supposedly driving the realizations of prices, as in the rational expectations assumption.

It will be established below that, in a rationally formed expectations equilibrium of an overlapping generations economy, different agents of any given generation (let alone of different generations) typically hold different expectations, so that it cannot be that for all agents their believed processes coincide with a "true" process supposed to drive the prices. This diversity of beliefs is only a natural consequence of the fact that different generations observe histories of prices and consumptions of different length and therefore form their beliefs optimally using different information sets. Note also that the need for each agents' expectations to be consistent with their different choices may lead them to hold different expectations even within any given generation.

In considering explicitly rationality conditions on the way the agents form their beliefs, as part of the definition of the equilibrium, I depart thus from the rational expectations equilibrium concept insofar the latter actually remains silent about how the agents may have arrived to hold such expectations to begin with. The approach followed in this paper thus adds an element missing in the rational expectations equilibrium concept. This will show itself clearly in the definition of a rationally formed expectations equilibrium below, which will embed that of a rational expectations equilibrium as a rationally formed expectations equilibrium constrained to satisfy the strong and, more importantly, counterfactual requirement of having the agents to hold expectations that are independent of the histories they observe.

The rationality condition imposed on the formation of expectations is reminiscent of the one underlying the rational beliefs equilibrium concept of Kurz (1994). Nonetheless, rationally formed expectations differ essentially from Kurz's rational beliefs. Although the notion of rational belief equilibrium shares with the rationally formed expectations equilibrium concept that I consider in this paper the idea that rationality of expectations or "beliefs should be defined relative to what is learnable from the data" (Kurz (1994), p.879), the consistency condition in Kurz (1994) requires that the price process that each agent believes is driving the prices must imply the same *long term* behavior of prices as the true process. In order to infer such long term behavior, Kurz assumes the agents have access to infinitely long histories of past prices, a formidable feat that the rationally formed expectations equilibrium does not require. It could be said that Kurz's criterion stresses the long term consistency of beliefs with data, while the rationally formed expectations considered in this paper stress the short term consistency. Arguably the latter may seem particularly more meaningful in overlapping generations economies considered in this paper in which agents do not care about prices beyond their life spans.

The approach considered here is also distinct from the *adaptive learning* approach, as in for instance Woodford (1990), insofar in that paper the agents learn some information about the "support" (specifically the optimal labor supply for each value of the sunspot) while I focus on how the agents infer the probabilities of transition between states that have been historically observed. Moreover, in Woodford (1990) agent t's preferences suffer from an additive shock $\varepsilon_t s_t$, linear in the agent savings (labor supply in Woodford's interpretation), where ε_t is an independently and identically distributed process with $E_t(\varepsilon_t) = 0$ and variance $\sigma^2 > 0$ whose distribution the agents learn from past realizations. It is from the accidental sample correlation between the small fluctuations in the relative price $\frac{p_t}{p_{t+1}}$ generated by these shocks and the sunspot realizations that agents' beliefs in the sunspot theory can get reinforced and thus convergence to a stationary sunspot equilibrium obtains. As Woodford points out, the presence of these shocks to the fundamentals is thus essential for the learning to occur. In contrast, in the rationally formed expectations equilibria considered in this paper there is no room for an intrinsic uncertainty to play any role whatsoever.

More generally, the adaptive learning literature (see for instance Evans and Honkapohja (2001) for a recent account) looks for conditions under which a rational expectations equilibrium is a fixed point of the process of updating an expectations formation rule (usually a linear statistical one) depending on past realizations. Therefore, the main goal of this literature is to explain how a *rational expectations* equilibrium might be reached. Similarly, according to the other main approach to the problem of expectations formation, that of *eductive learning* initiated by Guesnerie (1992), the agents may coordinate on a rational expectations equilibrium if they can deduce from the problem that there is no other equilibrium, at least locally. Thus as opposed to the learning happening in real time in the adaptive learning models, as the agents update their beliefs, eductive learning is supposed to happen in no time, i.e. in agents' minds out of the immediate understanding of the problem they face. Nonetheless, the goal of the explanation is still to give an account of how a *rational expectations* equilibrium might be realized. At any rate, both the adaptive and the eductive learning approaches look essentially for ways to discern whether some rational expectations equilibria are more "reasonable" than others, i.e. following a logic of refinement of the equilibrium concept. Here instead I address rather the issue of whether the rational expectations equilibrium concept itself has actually led to overlooking some phenomena as equilibrium phenomena.

In order to make the point as simply and clearly as possible I focus in this paper on a specific type of rationally formed expectations equilibrium, namely those that are stationary and markovian over a finite support in simple overlapping generations economies without intrinsic uncertainty. The remainder of the paper is organized as follows: Section 2 conveys the main ideas by means of a simple example of an overlapping generations economy that allows to "visualize" what is driving the result. Section 3 produces in this example rationally formed expectations equilibria exhibiting fluctuations distinct from those of any rational expectations equilibrium. Section 4 generalizes the setup, provides a precise definition of the notion of rationally formed expectations equilibrium for the overlapping generations economies, and establishes the existence of rationally formed expectations equilibria exhibiting fluctuations that no rational expectations equilibrium could generate. The constructive argument used to establish this result reveals a high level of degrees of freedom to produce rationally formed expectations equilibria. Therefore, in the next Section ? it is proved that still not anything can be supported as a rationally formed expectations equilibrium. Finally Section 5 concludes with a discussion of the equilibrium concept.

2. The leading example

In order to fix ideas, and for the sake of simplicity, consider a 2-period lived representative agent overlapping generations economy. Thus the representative agent born in period t lives for two periods and has to make a decision about how much to save from (real) income y when young (date t) in order to be able to keep consuming when old (date $t+1$; assume for instance that the income when young comes from working, which he cannot do or is prevented legally from doing when old). His decision depends on the purchasing power he expects his savings to have when old. If the current level of prices is p_t and he *expects* the level of prices when old to take any of, say, three possible values p^j_{t+1}, for $j = 1, 2, 3$, with probabilities m^j respectively, then his chosen level s^t of savings would be determined by the solution to

$$\begin{aligned} \max_{s^t, c^t_t, c^{tj}_{t+1} \geq 0} \sum_{i=1}^{3} m^j u(c^t_t, c^{tj}_{t+1}) \\ c^t_t + s^t = y \\ c^{tj}_{t+1} = \frac{p_t}{p^j_{t+1}} s^t \end{aligned} \tag{1}$$

where c^t_t is his consumption when young and c^{tj}_{t+1} is his consumption when old if the level of prices then is p^j_{t+1}. Under standard assumptions guaranteeing differentiability and the interiority of the solution, the necessary and sufficient first-order condition characterizing the solution, of the problem above, along with its budget constraints, is

$$\sum_{j=1}^{3} m^j \Big(-u_1(y - s^t, \frac{p_t}{p^j_{t+1}} s^t) + u_2(y - s^t, \frac{p_t}{p^j_{t+1}} s^t) \frac{p_t}{p^j_{t+1}} \Big) = 0. \tag{2}$$

At equilibrium every agent chooses his consumption rationally according to his expectations about future prices, and individual consumption decisions are compatible. Consider, for instance, an equilibrium of this economy in which the level of prices takes at any period one of three possible values p^1, p^2, p^3 (so that we can drop the time index from the prices) and the representative agent *expects* the probability of the price being p^j when old to depend only on the price p^i he faces when young (conditions for the existence of such an equilibrium are well known and reminded below). Accordingly, let us denote m^{ij} this probability. The representative agent's savings decision depends thus only on the level of prices p^i he faces when young, in such a way that we can denote s^i the solution to the necessary and sufficient first-order condition of this agent's problem,

$$\sum_{j=1}^{3} m^{ij} \Big(-u_1(y - s^i, \frac{p^i}{p^j} s^i) + u_2(y - s^i, \frac{p^i}{p^j} s^i) \frac{p^i}{p^j} \Big) = 0. \tag{3}$$

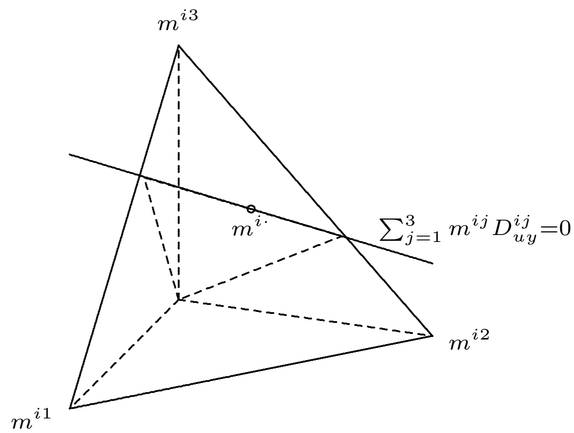
If moreover the contingent savings s^i and prices p^i, for all $i = 1, 2, 3$, satisfy

$$s^i = \frac{p^j}{p^i} s^j \tag{4}$$

then, not only all the agents are behaving rationally according to their expectations, but markets clear as well (in effect, in any state i the desired saving s^i by the young agent equals then the desired consumption $\frac{p^j}{p^i}s^j$ by the contemporary old agent born in any state j). Conditions for the existence of prices p^i, savings s^i, and probabilities m^{ij}, for $i, j = 1, 2, 3$, such that (3) and (4) hold for all $i = 1, 2, 3$ are well known (see Azariadis (1981), Azariadis and Guesnerie (1986), Chiappori and Guesnerie (1988, 1989), Guesnerie (1989)) and such an equilibrium is known as a 3-state markovian stationary sunspot equilibrium.

Note however that in equation (3) above the same savings decision s^i can follow from different probabilities of transition m^{i1}, m^{i2}, and m^{i3}. In effect, at any such equilibrium each vector (m^{i1}, m^{i2}, m^{i3}) of probabilities of transition from each state $i = 1, 2, 3$, must satisfy the two linear equations consisting of (i) being in the unit simplex in $\mathbb{R}^3$ and (ii) satisfying equation (2), so that there remains one degree of freedom for each row (m^{i1}, m^{i2}, m^{i3}) of the Markov matrix $(m^{ij})_{i,j=1}^k$, as illustrated in Figure 1 below, where $m^{i\cdot} \equiv (m^{i1}, m^{i2}, m^{i3})$ and $D_{uy}^{ij} \equiv -u_1(y - s^i, \frac{p^i}{p^j}s^i) + u_2(y - s^i, \frac{p^i}{p^j}s^i)\frac{p^i}{p^j}$.

Figure 1

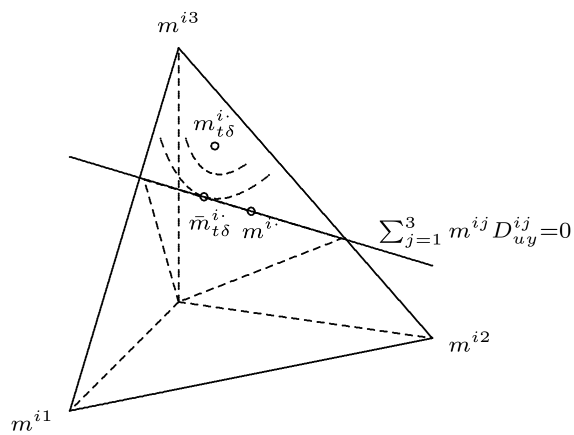


Thus, if no further requirement is made on the agents' beliefs, then equations (3) and (4) may hold true —i.e., (i) everyone behaves rationally *given his believes* and (ii) markets clear— while agents across generations *hold different beliefs* about the probabilities of transition m^{ij}. This possibility is excluded if the agents are supposed to hold rational expectations, since in that case all the agents must share the same "true" m^{ij}'s. Note that until now we have not mentioned the existence of a "true" objective process, but only the existence of agents' expectations about future prices. That is because in the description above of the 3-state stationary sunspot equilibrium, and in the related literature, the probability m^{ij} with which the agent *expects* the transition from a state i to a state j to happen is implicitly assumed to be the actual probability with which such transition does happen.

As a matter of fact, plenty of price process beliefs are compatible with any given equilibrium behavior. The rational expectations hypothesis imposes the additional condition requiring *every* agent's expectations to coincide with a particular objective process. Nevertheless, alternative consistency conditions at equilibrium other than the rational expectations hypothesis can be imposed on the agents' expectations. This is the more so given that, anyway, the rational expectations hypothesis does not explain why the agents would hold the expectations they hold or, in other words, how do they form their expectations. The rational expectations hypothesis is just a consistency requirement that turns out to be both extremely simple and extremely demanding. But it is far from obvious that it is the most natural one, mainly because of its deficiency with with respect to the issue of the expectations formation.

Regarding the origins of the agents' expectations, they can only come from the optimal inference the agents can make from the information available to them at the time of making their decisions. Specifically, for any given agent, his estimate of the probabilities of transition from any given price p^i to any other price level p^j must, on the one hand, be consistent with the agent's own saving decision, so that it must satisfy the first-order condition (3). But on the other hand, it would be irrational for the agent to hold expectations, among all the expectations that are consistent with his decision, that follow from beliefs that do not make the likelihood of the history he observes of transitions from that price to every other price as big as possible. Thus, the agent's rationally formed estimate of these probabilities of transition is the point $\bar{m}^{i\cdot}_{t\delta}$ (where t stands for the date up to which the generation t can observe a history δ of actual prices extending infinitely into the future) attaining the highest likelihood level curve on the unit simplex, *among those consistent with the first-order condition* necessarily satisfied by the agent's saving decision (represented by the plane intersecting the unit simplex in Figure 2 below), Note that the empirical frequencies of transitions starting from i (the number of observed transitions from state i to each state j over the number of times i has been attained, depicted as $m^{i\cdot}_{t\delta}$ in Figure 2) would be the unconstrained maximum likelihood estimator of $m^{i\cdot}$, but typically such expectations will be inconsistent with the agents behavior.

Figure 2



Thus positive prices $\bar{p}^i$, savings $\bar{s}^i$, for all $i = 1, 2, 3$, and history-dependent beliefs about the price process $(\bar{m}^{ij}_{t\delta})^k_{i,j=1}$, *for every history δ of prices and up to every date t*, such that

$$\sum_{j=1}^{3} \bar{m}^{ij}_{t\delta}\Big(- u_1(y - \bar{s}^i, \frac{\bar{p}^i}{\bar{p}^j}\bar{s}^i) + u_2(y - \bar{s}^i, \frac{\bar{p}^i}{\bar{p}^j}\bar{s}^i)\frac{\bar{p}^i}{\bar{p}^j}\Big) = 0 \tag{5}$$

and

$$\bar{s}^j = \frac{\bar{p}^i}{\bar{p}^j}\bar{s}^i,\ \forall j = 1, 2, 3 \tag{6}$$

holds for all $i = 1, 2, 3$, and every history δ up to every period t, constitute a 3-state markovian stationary rationally formed expectations equilibrium if, and only if, any other $m^{i\cdot}$ satisfying (5) implies a lower likelihood than $\bar{m}^{i\cdot}_{t\delta}$ of the realization of history δ up to period t, for all $i = 1, \ldots, 3$ and *for every history* δ *and up to every period* t.

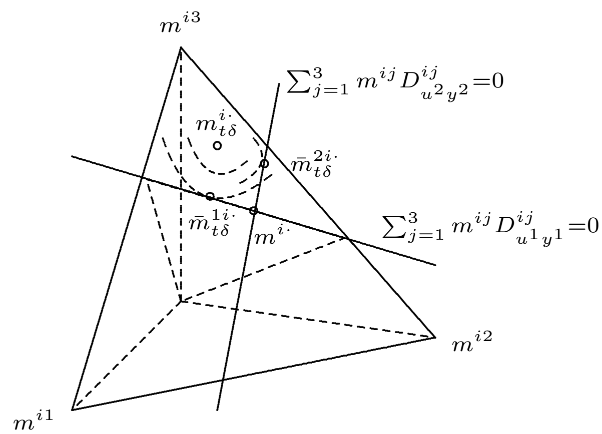
Intuitively, as this example with a representative agent allows to see, at equilibrium the constrained maximum likelihood estimates $\bar{m}^{i\cdot}_{t\delta}$ will typically be different for different generations since they will have access to histories of different length, and hence their empirical frequencies of transitions $m^{i\cdot}_{t\delta}$ will typically be different for different t's even for a given history δ. The example above allows to see also that there may be stationary markovian sunspot equilibria that are observationally equivalent to the rationally formed expectations equilibrium described by Figure 2. In such an equilibrium it would be impossible to discern empirically whether the agents hold rational expectations or rationally formed expectations. Nonetheless, assuming they hold rational expectations amounts to assume they hold expectations that do not account for the empirical evidence as well as other expectations consistent with their choices do. At any rate, and more importantly, there needs not be any rational expectations equilibrium observationally equivalent to any given rationally formed expectations equilibrium, as Section 3 below shows.

Nevertheless, in the representative agent overlapping generations economy example above there does exist a continuum of rational expectations equilibria that are observationally equivalent to a rationally formed expectations equilibrium. But not even this needs to be the typical case. In effect, consider now a similar equilibrium of an overlapping generations economy in which there is two (different) agents in each generation with utilities u^h and real incomes y^h, for $h = 1, 2$. This equilibrium is characterized by the feasibility conditions and the first-order conditions for the two agents $h = 1, 2$ of each generation

$$\sum_{j=1}^{3} \bar{m}^{hij}_{t\delta} \Big(-u^h_1(y^h - \bar{s}^{hi}, \frac{\bar{p}^i}{\bar{p}^j}\bar{s}^{hi}) + u^h_2(y^h - \bar{s}^{hi}, \frac{\bar{p}^i}{\bar{p}^j}\bar{s}^{hi})\frac{\bar{p}^i}{\bar{p}^j} \Big) = 0 \qquad (2)$$

for all $i = 1, 2, 3$, all δ, and all t, where the believed probabilities $\bar{m}^{hij}_{t\delta}$ are now indexed also by the agent h who holds them. Therefore, each agent's estimate of the probabilities of transition from any state i given any observed history must satisfy now, on top of the condition of being in the unit simplex, the linear constraint corresponding to his own first-order condition. As a consequence, the agents' beliefs about these probabilities will typically not coincide (see Figure 3, where $D^{ij}_{u^h y^h} \equiv -u^h_1(y^h - \bar{s}^{h,i}, \frac{\bar{p}^i}{\bar{p}^j}\bar{s}^{h,i}) + u^h_2(y^h - \bar{s}^{h,i}, \frac{\bar{p}^i}{\bar{p}^j}\bar{s}^{h,i})\frac{\bar{p}^i}{\bar{p}^j}$, for all $h = 1, 2$).

Figure 3



Note, however, that in Figure 3 there is only one process (the Markov matrix given by the rows $m^{i\cdot}$, for all $i = 1, 2, 3$) that is compatible with this equilibrium and that could drive the equilibrium prices and allocation as a rational expectations sunspot equilibrium. At any rate, the existence of an observationally equivalent rational expectations sunspot equilibrium is nonetheless neither needed for the existence of rationally formed expectations equilibria, nor guaranteed, as shown in the next section.

3. Rationally formed expectations equilibria distinct from rational expectations equilibria

From the previous example one could be tempted to think that every rationally formed expectations equilibrium of this overlapping generations economy without intrinsic uncertainty is observationally equivalent to some rational expectations sunspot equilibrium and that, therefore, the point made in this paper is void of real content. In order to show that this is not the case, I will establish now constructively the existence of rationally formed expectations equilibria that follow first-order Markov chains fluctuating between three states in economies that do not have 3-state markovian stationary sunspot equilibrium fluctuating between those states.

Consider first a 3-state markovian stationary sunspot equilibrium of a simple overlapping generations economy with a representative agent with utility function u and endowments $e = (e_1, e_2)$. That is to say, consider, for all $i = 1, 2, 3$, prices p^i, first and second period consumptions c_1^i and c_2^i and a Markov matrix of probabilities of transition $(m^{ij})_{i,j=1}^3$ such that, for all $i = 1, 2, 3$,

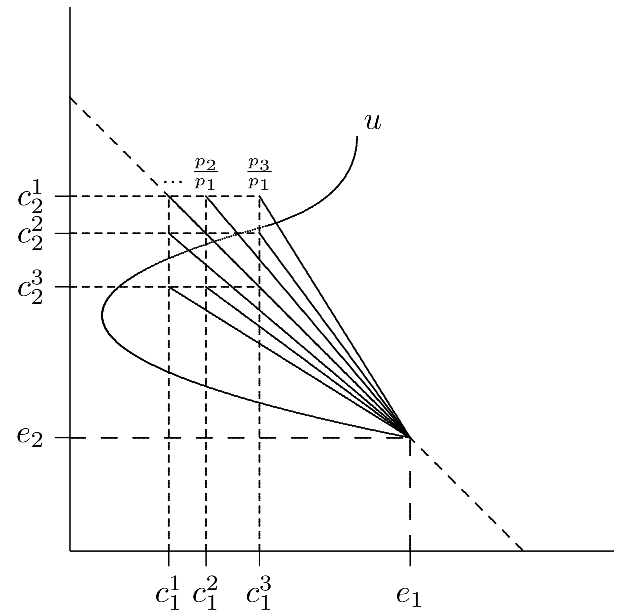
$$c_1^i + c_2^i = e_1 + e_2 \tag{8}$$

and

$$(c_1^i, (c_2^j)_{j=1}^3) \in \underset{p^i(\tilde{c}_1^i - e_1) + p^j(\tilde{c}_2^j - e_2) = 0,\, \forall j}{\arg\max} \sum_{j=1}^{3} m^{ij} u(\tilde{c}_1^i, \tilde{c}_2^j) \tag{9}$$

Figure 4 shows the offer curve of the representative agent, the contingent consumption bundles and the budget lines corresponding to the seven distinct relative prices of such an equilibrium.

Figure 4

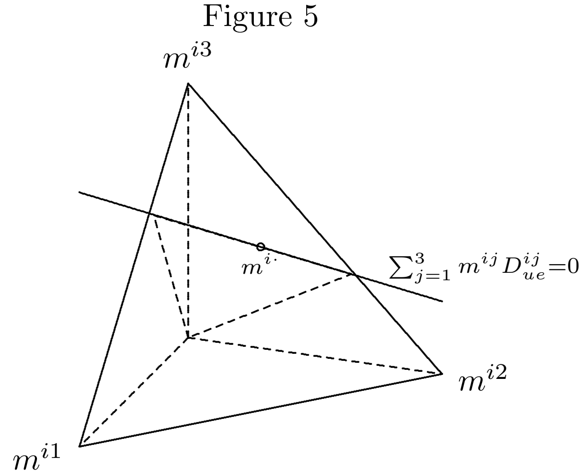


Then necessarily the contingent consumptions c_1^i, c_2^i satisfy the equations, for all $i = 1, 2, 3$

$$\sum_{j=1}^{3} m^{ij} D_{ue}^{ij} = 0 \tag{10}$$

where $D_{ue}^{ij} \equiv u_1(c_1^i, c_2^j)(c_1^i - e_1) + u_2(c_1^i, c_2^j)(c_2^i - e_2)$. Figure 5 below shows the linear constraint on the simplex that the equilibrium equations imposes on the probabilities of transition from any state i.

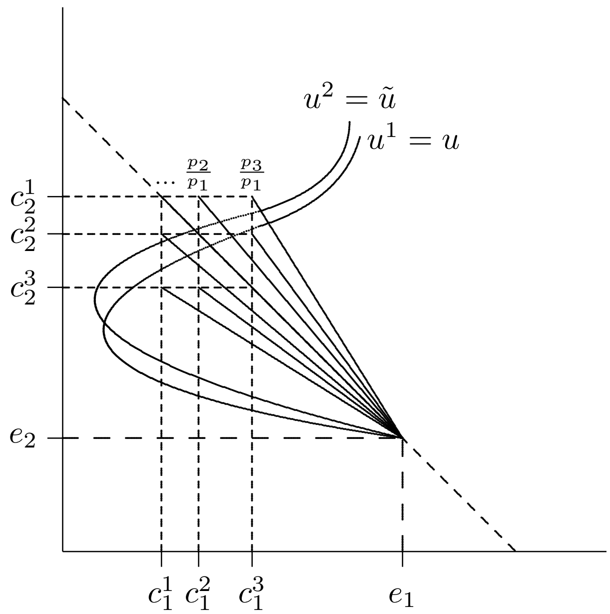
Figure 5



Now imagine this was in fact an economy of two *identical* agents 1 and 2 per generation, so that u and e are the utility and preferences u^h and e^h of both agents $h = 1, 2$ and, for all $i, j = 1, 2, 3$, c_1^i and c_2^j are the equilibrium contingent consumptions c_1^{hi} and c_2^{hj} of both $h = 1, 2$. Consider instead a nearby economy in which agent 2 has a different utility function $\tilde{u}$ (while u^1 continues to be u). Since

u^2 is now different from but close enough to u, then (i) agent 2's offer curve will continue to separate top and bottom points of the grid of contingent consumptions (see Figure 6 below),

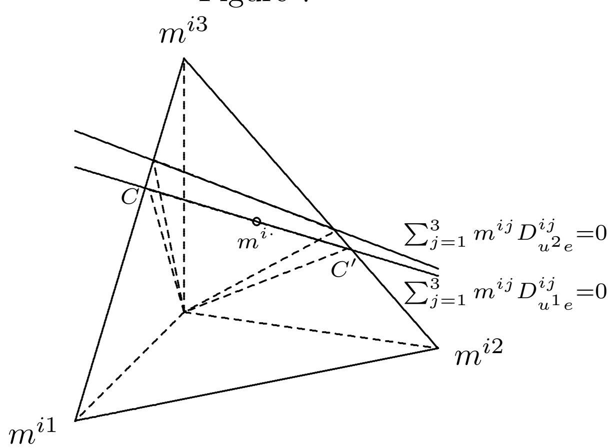
Figure 6



and (ii) the first order conditions of agent 2 will impose on the probabilities of transition a different constraint.

For some robust perturbations, the new linear constraint on the probabilities of transition has no intersection with the old one on the unit simplex, as shown in Figure 7 below.

Figure 7



This implies that for the resulting economies there is no Markov matrix that makes *both* agents 1 and 2 choose the contingent consumptions c_1^i and c_2^j whenever facing the prices p^i, p^j, for all $i, j = 1, 2, 3$. In effect, as long as the perturbation makes the normal vector to the plane defined by the first-order condition (10) for agent 2, i.e. $(D^{i1}_{u^2 e}, D^{i2}_{u^2 e}, D^{i3}_{u^2 e})$, to be spanned by the corresponding vector for agent 1, $(D^{i1}_{u^1 e}, D^{i2}_{u^1 e}, D^{i3}_{u^1 e})$, and the normal vector to the unit simplex $(1, \ldots, 1)$, but distinct from the former, then the system

$$\sum_{j=1}^{3} m^{ij} D^{ij}_{u^1 e} = 0$$
$$\sum_{j=1}^{3} m^{ij} D^{ij}_{u^2 e} = 0 \tag{12}$$

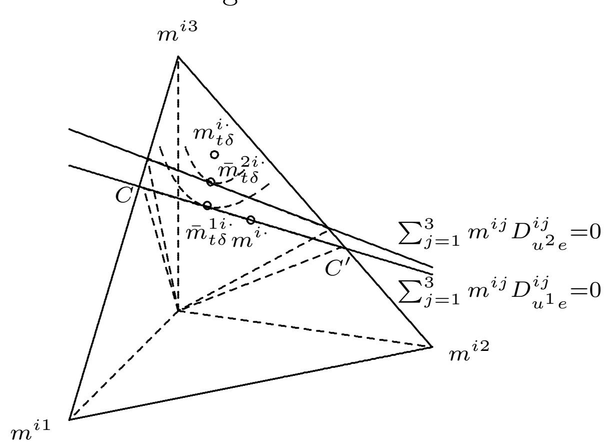
has no solution within the unit simplex. Note that there is a 1-dimensional manifold (after normalization) of possible vectors $(D^{i1}_{u^2e}, D^{i2}_{u^2e}, D^{i3}_{u^2e})$ satisfying this condition. Of course, any other perturbation close enough to anyone on this manifold would still be such that no 3-state markovian stationary sunspot equilibrium exists with this support for the corresponding 2-agent overlapping generations economy.

Notwithstanding, there do exist rationally formed expectations equilibria of any of the 2-agent economies resulting from such robust perturbations, that follow a Markov chain over the given support. In effect, since the offer curves of agents 1 and 2 still separate top and bottom points of the grid of contingent consumptions in Figure 6, then the unit simplex has always a nonempty intersection with the linear subspaces corresponding to the agents' first-order conditions, and hence there exist probabilities $(m^{hij}_{t\delta})^3_{i,j=1}$, for all h, δ, and t, that maximize the likelihood $\prod_{i,j=1}^3 (m^{ij})^{\sum_{\tau=1}^t \delta^i_{\tau-1}\delta^j_\tau}$ of observing the history δ and up to period t, among the probabilities of transition in the unit simplex that are consistent with the agent's first-order conditions

$$\sum_{j=1}^{3} m^{ij} D^{ij}_{u^h e} = 0$$

for all $i = 1, 2, 3$ (the existence, illustrated in Figure 8 below, is guaranteed by the continuity of the likelihood function and the compactness of the constrained domain).

Figure 8



4. The general case

In what follows entire histories $\{p_t\}_{t\in\mathbb{N}}$ of the price level, taking at any period any of k possible values $\bar{p}^1, \ldots, \bar{p}^k$, are going to be formalized by means of the function δ^i_t indicating whether the price i has been realized at period t or not. Thus $\delta^i_t = 1$

whenever $p_t = \bar{p}^i$ and is 0 otherwise. Since only one price can prevail at any period t, it must hold that $\sum_{i=1}^k \delta_t^i = 1$ for all $t \in \mathbb{N}$. Therefore, a history of realizations is a sequence $\delta = \{\delta_t\}_{t\in\mathbb{N}}$ in $\{0,1\}^k$ such that, for all $t \in \mathbb{N}$, $\sum_{i=1}^k \delta_t^i = 1$. Let Δ denote the set of such sequences.

Next I provide the formal definition of a k-state markovian stationary rationally formed expectations equilibrium of a stationary overlapping generations with a 2-period lived representative generation with H agents with utility function u^h and endowments $e^h = (e_1^h, e_2^h)$, for all $h = 1, \ldots, H$.

Definition. *A k-state markovian stationary rationally formed expectations equilibrium of the stationary overlapping generations economy with representative generation $(u^h, e^h)_{h=1}^H$ consists of*

(1) *a positive price for consumption in each state, i.e. for all $i = 1, \ldots, k$, some $p^i > 0$,*

(2) *allocations of nonnegative first-period consumptions and contingent plans of second-period consumptions for each agent at each possible state when young, i.e. for all $h = 1, \ldots, H$ and all $i = 1, \ldots, k$, some $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$, and*

(3) *beliefs about the probabilities of transition between states for each agent and any history of states up to his date of birth, i.e. for all $h = 1, \ldots, H$, all $t \in \mathbb{N}$, and all realization $\delta \in \Delta$, a Markov matrix $(m_{t\delta}^{hij})_{i,j=1}^k$,*[3]

such that

(1) *the allocation is feasible at every state, i.e. for all $i = 1, \ldots, k$*

$$\sum_{h=1}^H (c_1^{hi} + c_2^{hi}) = \sum_{h=1}^H (e_1^h + e_2^h)$$

(2) *for every agent h and any history δ up to the date t in which he is born, his first-period consumption and contingent plan of second-period consumptions $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ are optimal whenever at t the price is p^i, i.e. for all $h = 1, \ldots, H$, all $\delta \in \Delta$, all $t \in \mathbb{N}$, and all $i = 1, \ldots, k$, it holds*

$$\left(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k\right) = \arg\max \sum_{j=1}^k m_{t\delta}^{hij} u^h(c_1^i, c_2^j)$$
$$\text{s.t. } p^i(c_1^i - e_1^h) + p^j(c_2^j - e_2^h) = 0, \quad \forall j$$

and

(3) *each agent's beliefs maximize the likelihood of the history he observes among those for which his first-period consumption and contingent plan of second-period consumptions $(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k)$ are optimal whenever at t the price is p^i, i.e. for all $h = 1, \ldots, H$, all $t \in \mathbb{N}$, all $\delta \in \Delta$, and all $i = 1, \ldots, k$, if $m^{i\cdot} \in S^{k-1}$ is such that*

$$\left(c_1^{hi}, \{c_2^{hj}\}_{j=1}^k\right) = \arg\max \sum_{j=1}^k m^{ij} u^h(c_1^i, c_2^j)$$
$$\text{s.t. } p^i(c_1^i - e_1^h) + p^j(c_2^j - e_2^h) = 0, \quad \forall j$$

[3] See the remark below after this definition regarding redundant beliefs.

then

$$\prod_{j=1}^{k}(m^{ij})^{\sum_{\tau=1}^{t}\delta_{\tau-1}^{i}\delta_{\tau}^{j}} \leq \prod_{j=1}^{k}(m_{t\delta}^{hij})^{\sum_{\tau=1}^{t}\delta_{\tau-1}^{i}\delta_{\tau}^{j}}$$

Note first that although with the chosen notation every agent is supposed to hold beliefs about the price process after *every* partial history of realizations (even those beyond his life-span), in fact only the finitely many possible histories of realizations up to the date of his decision making are relevant for the equilibrium condition (2) above. Thus every member of every generation holds actually only finitely many beliefs.

Note also that if, in the definition above, the expectations are constrained to be history and agent independent and the last condition (3) is dropped, then it becomes the definition of a stationary rational expectations (sunspot) equilibrium following a k-state Markov chain. As a consequence, it is worth noticing that, typically, in such a rational expectations equilibrium there exist, for every agent, expectations consistent with his consumption choice that make the history he may observe likelier than the equilibrium expectations do. This fact should be seen as a shortcoming of the rational expectations equilibrium notion in this context, since according to it in the rational expectations equilibria of this kind the agents hold expectations that, among all the expectations consistent with their choices, do not give the best account of whatever they observe. Of course the discrepancy of the agents' expectations with the likelihood maximizing expectations after each possible history vanishes in the limit if the prices are supposed to follow a given Markov chain. But this assumption seems to be at odds with the fundamental idea that market prices are determined by the agents' choices. Moreover, in the case one wants the equilibrium concept to at least aspire to have some positive content, the counterfactual assumption that expectations are history independent points to another weakness of the rational expectations equilibrium concept.

The next proposition establishes that any stationary overlapping generations with sunspot equilibria can be perturbed robustly in order to produce rationally formed expectations equilibria that no sunspot equilibrium can match. Therefore the use of the rational expectations hypothesis discards all the expectations-driven fluctuations generated by these equilibria and, in this sense, downplays the role that expectations have in the excess volatility of the economy.

Proposition. *Arbitrarily close (in the topology of C^1-convergence over compacta in the space of utility functions) to every stationary overlapping generations economy with a finite-state markovian stationary sunspot equilibrium there exists an economy with finite-state markovian stationary rationally formed expectations equilibria distinct from any sunspot equilibrium.*

Proof. Let $(u^h, e^h)_{h=1}^{H}$ be the representative generation of a stationary overlapping generations economy, and let $\{p^i, c_1^{hi}, c_2^{hi}\}_{i=1,h=1}^{i=k,h=H}$ be the contingent prices and consumptions of a k-state markovian stationary sunspot equilibrium of the economy driven by a Markov chain with matrix of probabilities of transition $(m^{ij})_{i,j=1}^{k}$.

Assume, without loss of generality, that the allocation corresponding in this equilibrium to the agents of type H only is feasible with their only resources, i.e.[4]

$$c_1^{Hi} + c_2^{Hi} = e_1^H + e_2^H.$$

Consider a new economy with a representative generation $(u^h, e^h)_{h=1}^{H+1}$ consisting of adding an agent $H+1$ with the same endowments and consumptions as agent H (the new allocation of the new economy is feasible because of the assumption above), and a utility function u^{H+1} with gradients at the consumption bundles $(c_1^{Hi}, c_2^{Hj})_{i,j=1}^k$ such that, for some $i = 1, \ldots, k$, the vector $(D^{i1}_{u^{H+1}}, \ldots, D^{ik}_{u^{H+1}})$, where $D^{ij}_{u^h} \equiv u_1^h(c_1^{hi}, c_2^{hj})(c_1^{hi} - e_1^h) + u_2^h(c_1^{hi}, c_2^{hj})(c_2^{hj} - e_2^h)$ for $h = H+1$, is both not collinear to $(D^{i1}_{u^H}, \ldots, D^{ik}_{u^H})$ and in the span of this vector and $(1, \ldots, 1)$, i.e.

$$\begin{pmatrix} D^{i1}_{u^{H+1}} \\ \vdots \\ D^{ik}_{u^{H+1}} \end{pmatrix} = \alpha \begin{pmatrix} D^{i1}_{u^H} \\ \vdots \\ D^{ik}_{u^H} \end{pmatrix} + \beta \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{*}$$

for some α and β, with $\beta \neq 0$ (note first that since $\sum_{j=1}^k m^{ij} D^{ij}_{u^H} = 0$, the vector $(D^{i1}_{u^H}, \ldots, D^{ik}_{u^H})$ cannot be collinear to $(1, \ldots, 1)$; moreover there is a 1-dimensional manifold of *directions* that the vector $(D^{i1}_{u^{H+1}}, \ldots, D^{ik}_{u^{H+1}})$ can take while satisfying this conditions). Then the system of equations in the probabilities m^{ij}

$$\begin{aligned} m^{i1} D^{i1}_{u^H} + \cdots + m^{ik} D^{ik}_{u^H} &= 0 \\ m^{i1} D^{i1}_{u^{H+1}} + \cdots + m^{ik} D^{ik}_{u^{H+1}} &= 0. \end{aligned}$$

has no solution. In effect, using the equation (*) above, the second equation can be written equivalently as

$$\alpha\big(m^{i1} D^{i1}_{u^H} + \cdots + m^{ik} D^{ik}_{u^H}\big) + \beta(m^{i1} + \cdots + m^{ik}) = 0$$

but since $\sum_{j=1}^k m^{ij} D^{ij}_{u^H} = 0$ and $\beta \neq 0$, then one would have to have that

$$m^{i1} + \cdots + m^{ik} = 0!!$$

This establishes that the prices and consumptions $\big\{p^i, c_1^{hi}, c_2^{hi}\big\}_{i=1,h=1}^{i=k,h=H+1}$, with $(c_1^{H+1i}, c_2^{H+1j})_{i,j=1}^k = (c_1^{Hi}, c_2^{Hj})_{i,j=1}^k$, are not those of a sunspot equilibrium of the economy with representative generation $(u^h, e^h)_{h=1}^{H+1}$ (in effect, otherwise the system above would have a solution). They nevertheless are the allocation and prices of a rationally formed expectations equilibrium of this economy.

In effect, if u^{H+1} is close enough to u^H in the topology of C^1-convergence over compacta, then for all $h = 1, \ldots, H+1$, all $t \in \mathbb{N}$, and all $\delta \in \Delta$, there exists

[4] There is always a subset of types of agents for which this is true (note that this subset needs not be proper). In general, the replication and perturbation argument would be done on all the types of agents of the subset.

$\left(m_{t\delta}^{hij}\right)_{i,j=1}^{k}$ satisfying

$$\left(m_{t\delta}^{hij}\right)_{i,j=1}^{k} = \arg\max_{m^{ij}} \prod_{i,j=1}^{k} (m^{ij})^{\sum_{\tau=1}^{t} \delta_{\tau-1}^{i}\delta_{\tau}^{j}}$$
$$\text{s.t. } \forall i,\ \ m^{i\cdot}\in S^{k-1}$$
$$\forall i,\ \left(c_1^{hi},\{c_2^{hj}\}_{j=1}^{k}\right)=\arg\max \textstyle\sum_{j=1}^{k} m^{ij} u^{h}(c_1^{i},c_2^{j})$$
$$\text{s.t. } p^{i}(c_1^{i}-e_1^{h})+p^{j}(c_2^{j}-e_2^{h})=0,\quad \forall j$$

since the likelihood function that is being maximized is continuous in every case, and the constrained set consisting of the probabilities vectors $(m^{i1},\ldots,m^{ik})$ in the unit simplex satisfying the linear first-order conditions of each agent maximizing his utility at $(\bar{c}_1^{hi},\{\bar{c}_2^{hj}\}_{j=1}^{k})$ whenever the first-period price is p^i, is compact.

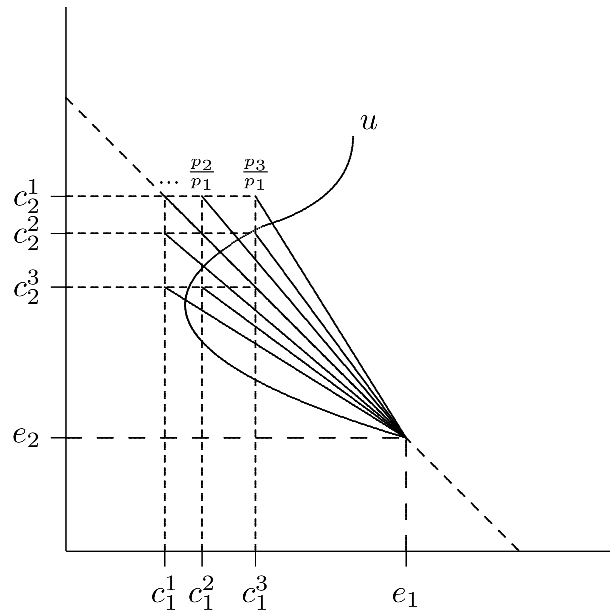
The same is true for any robust and small enough perturbation $\tilde{u}^{H+1}$ of u^{H+1}, therefore not necessarily putting $(D_{\tilde{u}^{H+1}}^{i1},\ldots,D_{\tilde{u}^{H+1}}^{ik})$ in the span of $(D_{u^H}^{i1},\ldots,D_{u^H}^{ik})$ and $(1,\ldots,1)$. Q.E.D.

4. Fluctuations not supported by rationally formed expectations equilibria

Given that the notion of rationally formed expectations equilibrium is able to account for more fluctuations in the allocation and prices as equilibrium phenomena than the rational expectations equilibrium concept, one would like to have an idea of where do the limits of this expansion lay. Or at least whether the proposed equilibrium notion does not go too far as to be able to rationalize *any* fluctuations as an equilibrium phenomenon.

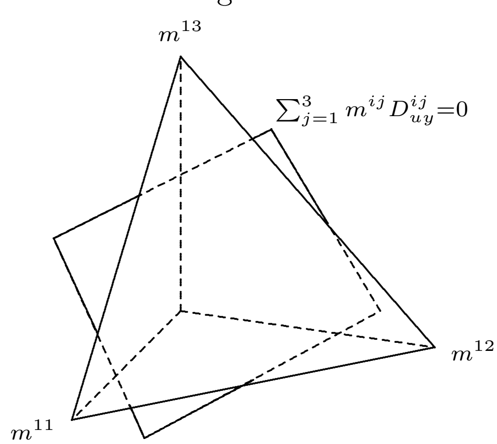
In order to see that not anything can be made into a rationally formed expectations equilibrium, consider a feasible allocation of consumptions c_1^i, c_2^i, and prices p^i, for all $i,j=1,\ldots,k$, such that for some agent h, not all the grid's top and bottom nods are separated by the agent's offer curve (see Figure 9).

Figure 9



Then the set of expectations consistent with some agent h's choice (in particular when when the price is p^1 in Figure 9) is empty (see Figure 10 below). As a consequence, no fluctuations between the consumption levels shown in Figure 9 can result from a rationally formed expectations equilibrium.

Figure 10



5. Discussion

At least two points are worth to be discussed in some detail here. The first is whether one should include as a constraint of the likelihood maximization problem of any given agent the consistency of the believed probabilities of transition not only with the agent's own behavior, but also with that of every other agent. Note that if the consistency of the believed probabilities of transition, i.e. the fact that they have to be such that each agents choice has to be optimal according to them, was generalized to hold with respect to every other agent as well, then any given agent's choice would have to be optimal according to every other agent's beliefs. This may not be a problem if one imposes the rational expectations assumption, since in that case all the agents share the same beliefs. Nonetheless, whenever a consistency requirement for the expectations other than the rational expectations hypothesis is made (like that of being rationally formed), there seems to be no compelling reason for an agent to make decisions that are optimal according to beliefs other than his own.

A second point of interest arises when one considers what happens if one insists on assuming that there is such thing as a "true" process driving the prices. If that is the case, then the empirical frequencies of transition will certainly converge to the true ones eventually. While rationally formed expectations equilibria in which agents hold beliefs distinct from those following from the true process still exist, one could rightfully wonder why would any agent insist on holding beliefs that are distinct from empirical frequencies that have barely changed for, say, the last thousand years. Clearly to reply that this can be still an equilibrium phenomenon is not satisfactory enough.

Two other answers could be given to this question. One is that the stationary equilibria may not be the sensible ones to focus on, albeit their welcomed simplicity for the sake of the analysis. It may well be that the right choice is to allow for a non-stationary behavior that converges to the rational expectations equilibrium driven by the true process. The problem is that this only makes sense whenever there exists such a rational expectations equilibrium, which in the setup considered in this paper could only be a sunspot equilibrium. But in this case considering rationally formed expectations equilibria would not disclose any new equilibrium phenomenon that the rational expectations equilibrium concept would have not revealed before, as for example the expectations-driven volatility not supported by sunspot equilibria shown in this paper. Rather rationally formed expectations equilibria would just show a rational way in which beliefs about prices converge to the true price process.

A second more honest answer, in my view, would be that the right question is instead why to assume that, in the considered overlapping generations setup without intrinsic uncertainty, there is such a thing as a "true" process for the prices to begin with. The idea of the existence of a true process is in this case only a consequence of adopting the rational expectations hypothesis, in such a way that unless one stays in that framework the very idea makes no sense. But the approach followed in this paper explicitly avoids using the rational expectations approach. It actually tries to make the point that the very use of the rational expectations hypothesis has been preventing us from seeing as equilibrium phenomena some economic phenomena.

More precisely, while in the case in which the economy is subject to intrinsic uncertainty there is a rationale for the objective process driving the shocks on the fundamentals to help conforming the process followed by the prices, in the absence of intrinsic uncertainty this dependence of prices on an objective stochastic process does not exist, for the simple reason that no objective process is there anymore. Nor is it needed to assume the existence of an objective sunspot signal that the agents use to coordinate their expectations. According to their beliefs, the observation of the current and past prices is enough to form their possibly different expectations about next period price. To apply mechanically the rational expectations hypothesis to this case amounts, in my view, to substitute a normative causality by which current decisions are determined by perfectly (if stochastically) *foreseen* future events, to the positive causality by which current decisions are determined by the expectations about future events based on *seen* current and past events.

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