PERISHABLE DURABLE GOODS

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Abstract. We examine whether the Coase conjecture [7, 14, 4, 10] is robust against slight ability of commitment of the monopolist not to sell the durable goods to consumers. We quantify the commitment ability in terms of the speed that the durable goods perish, while keeping the time between the offers small. We demonstrate that the slight commitment capability makes a substantial difference by constructing two kinds of reservation price equilibria [10] that refute the Coase conjecture.

In the first equilibrium, the monopolist can credibly delay to make an acceptable offer. All consumers are served, but only after extremely long delay. Most of gains from trading is discounted away, and the resulting outcome is extremely inefficient. In the second equilibrium, the monopolist's expected profit can be made close to the static monopoly profit, if the goods perish very slowly. By focusing on the reservation price equilibria, we rigorously eliminate any source of reputational effect. In fact, by using the first kind of reservation price equilibrium as a credible threat against the seller, we can construct many other reputational equilibria [2] to obtain the Folk theorem. Various extensions and applications are discussed.

1. Introduction

The Coase conjecture [7, 4, 14, 10] shows that the dynamic foundation of the monopolistic power lies in the monopolist's commitment ability not to sell durable goods to consumers who are willing to pay more than the marginal production cost. By commitment, we mean an action that entails an irreversible consequence. Perishability captures the irreversibility in terms of quantity while the time between the offers measure the irreversibility in terms of timing of sales which is generally considered a measure for the commitment ability of the monopolist. In order to highlight the different aspect of commitment, we focus on a model with perishable durable goods, where the time between the offers is small. Throughout this paper, we quantify the monopolist's commitment ability in terms of the speed that the durable goods perish away at the instant rate of e^{-b} for some small $b > 0$.[1]

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[1] If $b = \infty$, it is the case of complete commitment, while $b = 0$ corresponds to the classic durable good monopoly problem.

Despite an extensive literature on the dynamic monopoly problem as well as sequential bargaining models (e.g., [13, 9, 2, 5, 6, 8]), we have little understanding about how the market outcome changes with respect to the ability of commitment of the monopolist, except for the two limit cases: complete decay and no decay. In order to examine how robust the Coase conjecture is, we should examine whether the Coase conjecture continues to hold, if the monopolist can be committed not to sell a small amount of goods to consumers.

We differentiate "perish" from "depreciate," while we use "perish," "decay" and "burn off" interchangeably. Perishable goods decay before they are sold, which affects the future supply of the goods irreversibly. On the other hand, goods depreciate only after they are delivered to consumers, which generate the demand for replacement. The strategic impact of depreciation was analyzed by [3]. It is shown that with a positive rate of depreciation, the monopolist is not willing to provide the competitive level of goods. Yet, the gap between the competitive outcome and the monopolistic outcome vanishes as the time between offers and the depreciation factor converge to 0. In order to highlight the impact of slight decay, this paper assumes that the good is not depreciated after it is delivered to consumers.

In contrast to [3], we find a significant discontinuity in outcomes with respect to the rate of decay around $b = 0$ (no decay). To differentiate two cases, we call the durable good problem with $b = 0$ (no decay) *classic problem*, while the case with $b > 0$ (decay) is referred to as *perishable problem*. To highlight the impact of the slight decay, we focus on the same rule of game as the classic durable goods monopoly problem with the linear (inverse) market demand curve $p = 1 - q$ where the monopolist offer p_t in period t, which was accepted or rejected by consumers. After the offer is rejected, the monopolist has to wait for $\Delta > 0$ before offering p_{t+1}. The game continues until either all consumers are served, or all available stock is sold. All agents are risk neutral with the same discount factor $\delta = e^{-r\Delta}$ for some $r > 0$.

We focus on the case where the demand curve does not hit $p = 0$ ("gap case") in order to sharpen the comparison: $\exists q_f < 1$ such that the market demand curve is $p = 1 - q$ for $q \in [0, q_f]$. We interpret q_f as the size of the whole market. In this case, the classic problem has a unique subgame perfect equilibrium in pure strategies, where the consumer's acceptance rule can be represented as a threshold rule. We call such a subgame perfect equilibrium a reservation price equilibrium [10], for which the Coase conjecture holds: in any reservation price equilibrium, the initial offer converges to the lowest reservation value of the consumers, and all consumers are served almost immediately as $\Delta \to 0$.

We construct two reservation price equilibria, which roughly form the upper and the lower bounds of the set of all subgame perfect equilibrium payoffs of the monopolist. In the first equilibrium, the monopolist's expected profit is close to 0 if $q_f < 1$ is close to 1 and $b \to 0$. Interestingly, almost all consumers are served but the market outcome is extremely inefficient. The monopolist credibly delays to make an acceptable offer until the available stock reaches the target level. Because $b > 0$ is small, it takes exceedingly long periods for the available stock to reach the target level, and the consumer surplus is discounted away. While the monopolist generates profit slightly higher than what he could have made in the equilibrium satisfying the Coase conjecture, his profit is also very small.

In the second reservation price equilibrium, the monopolist's expected profit is close to the static monopoly profit. The slow decay opens up a strategic opportunity for the monopolist to credibly delay to make an acceptable offer for a significant time. If the consumer knows that an acceptable offer will arrive in the distant future, he is willing to accept a high price. By exploiting the impatience of the consumers, the monopolist can achieve almost the static monopolist's profit. Interestingly, this equilibrium entails a randomization strategy off the equilibrium path, in a sharp contrast to the classic problem which has a unique pure strategy subgame perfect equilibrium if the demand curve is linear and $q_f < 1$.

The first equilibrium can be served as a credible threat against a deviation by the monopolist. Following the idea of constructing reputational equilibria in [2], we can sustain any level of monopolist's profit as a subgame perfect equilibrium. Note that the second reservation price equilibrium shows that even without reputational equilibria, the monopolist can achieve almost the static monopoly profit.

The set of subgame perfect equilibria is discontinuous with respect to the perishability of the durable goods in a number of important ways. First, the subgame perfect equilibrium that satisfies the Coase conjecture is no longer an equilibrium in some nearby game to the classic problem unless the good is very plentiful in the initial round. Second, the equilibrium outcome in the nearby game is much richer than the classic problem. Thus, the equilibrium analysis does provide us a precise benchmark, against which the actual market outcome can be compared. Substantial market power does not necessarily imply substantial commitment power. Thus, the classical remedy to unravel the commitment capability of the monopolist not be as effective as the classic problem suggests.

The rest of the paper is organized as follows. Section 2 formally describes the model and the key results from the classic durable good monopoly problem. Section 3 examines a simple example where the market demand curve is a step function. Although our main result is built around a linear demand curve, we begin with this example, because we can precisely calculate the subgame perfect equilibrium to reveal the key properties of the equilibrium we will analyze. Section 4 analyzes a market with a linear demand curve. In Section 4.1, we explore an artificial game in order to highlight the mechanism that prompts the monopolist to delay to make acceptable offers. We calculate an equilibrium of the game. In Section 4.2, we construct a reservation price equilibrium, which approximates the equilibrium of the artificial game in Section 4.1. We observe that the monopolist may spend many periods without making acceptable offers, while burning off the available stock to reach the target level. The equilibrium constructed in Section 4 seems to indicate that the monopolist's profit should be small, if the monopolist has little ability to commit himself not to sell (small $\Delta > 0$ and $b > 0$). Section 5 shows the contrary by constructing an equilibrium where the monopolist can generate a large profit. Section 5.1 examines another artificial game, in which the monopolist can choose the time interval of making unacceptable offers to highlight the structure of the second equilibrium. We show that as $b \to 0$, the monopolist's equilibrium in this game converges to the static monopoly profit. In Section 5.2, we construct a reservation price equilibrium, which approximate the equilibrium constructed in Section 5.1. Section 6 concludes the paper with discussions on extensions and policy implications.

2. Preliminaries

Except for Section 3, we focus on a market where the demand curve is linear:

$$p = 1 - q \qquad q \leq q_f < 1 \tag{2.1}$$

where $q \in [0, q_f]$ is the size of the consumers who were served, and p is the delivery price. We regard each point in $[0, q_f]$ as an individual consumer. By consumer q, we mean a consumer whose reservation value is $1 - q$. We call q_f the size of the whole market.

We write a residual demand curve as $\mathsf{D}(q_0, q_f)$ after $q_0 \in [0, q_f]$ consumers are served. Following the convention of the literature, we shall treat two residual demands identical if they differ only over the null set of consumers. Let y_t be the amount of stock available at the beginning of period t. Except for Section 3, we assume that the initial stock is sufficient to meet all demand in the market: $y = y_1 > 1$.

Let q_t be the total mass of consumers who has been served by the end of period t. Thus, $q_t - q_{t-1}$ is the amount of sales in period t. Then,

$$y_{t+1} = \beta(y_t - (q_t - q_{t-1}))$$

for $\beta = e^{-\Delta b}$, $\Delta > 0$ and $b > 0$. We call $\Delta > 0$ the time interval between the offers, and $b > 0$ the instantaneous rate of decay.

Let h_t be the history at period t, that is a sequence of previous offers $(p_1, \ldots, p_{t-1})$. A strategy of the monopolist is a sequence $\sigma = (\sigma_1, \ldots, \sigma_t, \ldots)$ where $\sigma_t(h_t) = p_t \in \mathbb{R}_+$ $\forall t \geq 1$. Let Σ be the set of strategies of the monopolist. Similarly, a strategy of a consumer q is a mapping from his reservation value $1 - q$, history of offers and the present offer p to a decision to accept or reject. If he purchases the good at p, then his surplus is $(1 - q) - p$. All agents in the model have the same discount factor $\delta = e^{-r\Delta}$ for $r > 0$.

Let $\{q_0, q_1, q_2, \ldots, q_t, \ldots\}$ be a sequence of *weakly* increasing numbers, which represent the sequence of the total mass of consumers who have been served by the end of period t. Naturally, $q_0 = 0$. Let $\mathbf{Q}$ be the set of all such sequences. The monopolist's profit is

$$\sum_{t=1}^{\infty} \delta^{t-1}(q_t - q_{t-1})p_t$$

where $p_t = \sigma_t(h_t)$ where $h_t = (p_1, \ldots, p_{t-1})$.

We say that the market is cleared at t if the monopolist meets all the demand $q_t = q_f$ or sells all remaining stock $q_t - q_{t-1} = y_t > 0$ for the first time at $t < \infty$:

$$q_t = \min(q_f, y_t + q_{t-1}).$$

We know that in the classic problem, the market is cleared in a finite number of periods if $q_f < 1$ [9, 10].

Given the monopolist's strategy σ, consumer q's action is optimal if he accepts p_t in period t if

$$(1 - q) - \sigma_t(p_1, \ldots, p_{t-1}) > \sup_{k \geq 1} \delta^k((1 - q) - \sigma_{t+k}(p_1, \ldots, p_{t-1}, p_t, \ldots, p_{t+k-1}))$$

and rejects, if the inequality is reversed, where p_t is realized according to σ, $\forall t \geq 1$. By exploiting the monotonicity with respect to the reservation value, the classic problem allows us to write the optimality condition of the consumers more compactly by focusing on the *critical type* $1 - q_t$, who is indifferent between accepting the present offer and the next offer:

$$(1 - q_t) - p_t = \delta\left((1 - q_t) - p_{t+1}\right)$$

where $p_t = \sigma_t(h_t)$ and $h_t = (p_1, \ldots, p_{t-1})$. However, in the perishable problem, we have to admit the possibility that $q_t = q_{t-1}$ for some $t \geq 1$ if the monopolist makes unacceptable offers.[2] We need to write the consumer's optimality condition in a more general way:

$$(1 - q_t) - p_t = \sup_{k \geq 1} \delta^k \left((1 - q_t) - p_{t+k}\right). \tag{2.2}$$

We say p_t is unacceptable if $q_t - q_{t-1} = 0$. While any arbitrarily large offer is an unacceptable offer, it is often convenient to set an unacceptable offer as an offer that makes the highest reservation value consumer in $\mathsf{D}(q_0, q_f)$ indifferent between accepting and rejecting the present offer:

$$(1 - q_0) - p_t = \sup_{k \geq 1} \delta^k \left((1 - q_0) - p_{t+k}\right). \tag{2.3}$$

We can define a Nash equilibrium in terms of the monopolist's strategy σ that solves

$$\max_{\sigma \in \Sigma} \sum_{t=1}^{\infty} \delta^{t-1}(q_{t-1} - q_t)p_t \tag{2.4}$$

where $p_t = \sigma_t(h_t)$ where $\sigma = (\sigma_1, \sigma_2, \ldots)$ and $h_t = (p_1, \ldots, p_{t-1})$ and $\mathbf{q} = (q_1, q_2, \ldots)$ satisfies (2.2). We say that σ is a subgame perfect equilibrium if σ induces a Nash equilibrium following every history.

We shall focus on a class of subgame perfect equilibria where a consumer's strategy is characterized by a threshold rule, which is a natural state variable of the game, namely the residual demand and the available stock.

Definition 2.1. *A subgame perfect equilibrium is a reservation price equilibrium, if there exists* $\mathsf{P} : [0, q_f]^2 \times [0, y] \to \mathbb{R}$ *such that*

$$p_t = \mathsf{P}(q_t, q_f, y_t)$$

with $y_t = \beta(y_{t-1} - (q_t - q_{t-1}))$ *where* p_t *is the equilibrium price offered in period* t*,* q_t *is the total mass of consumers served by the end of period* t *and* y_t *is the available stock at the beginning of period* t*.*

The Coase conjecture holds for the classic durable good problem.

Theorem 2.2. [14, 4, 10] *Suppose that* $b = 0$*. If* $q_f < 1$*, then a (generically) unique subgame price equilibrium exists, which is a reservation price equilibrium in pure strategies. In any reservation price equilibrium, the initial offer of the monopolist converges to* $1 - q_f$ *and thus, his profit converges to* $q_f(1 - q_f)$ *as* $\Delta \to 0$*.*

Before describing and analyzing the perishable problem, let us describe the classic problem to review useful results. The optimization problem of the risk neutral monopolist for

[2] Actually, he does in an equilibrium.

the classic problem is to choose a sequence $\mathbf{q} \in \mathbf{Q}$ to maximize the discounted expected profit subject to a couple of constraints:

$$\mathcal{V}^c(q_0, q_f, y) = \max_{\mathbf{q}\in\mathbf{Q}} \sum_{t=1}^{\infty} p_t(q_t - q_{t-1})\delta^{t-1} \tag{2.5}$$

$$\text{subject to} \quad (1-q_t) - p_t = \delta((1-q_t) - p_{t+1}) \tag{2.6}$$

$$\lim_{t\to\infty}(1-q_t) - p_t = 0. \tag{2.7}$$

(2.6) is the constraint imposed by the rational expectations of the consumers, which renders p_t as a function of q_t and q_{t+1}. (2.7) implies that in order to clear the market, the "final" offer of the monopolist must be the lowest reservation value of the consumer.[3]

Let $T_f(q_0, q_f, y)$ be the total number of periods needed to make sales before serving every consumer in the market, or exhausting all available stock. If it takes infinitely many periods to serve all consumers, we let $T_f(q_0, q_f, y) = \infty$. In the classic problem, $T_f(q_0, q_f, y)$ is precisely the number of offers the monopolist makes in the game. Since the monopolist must serve a positive portion of consumers in every period,

$$q_t < q_{t+1} \qquad \forall t \geq 1$$

must hold, unless the market is closed in period t. We can understand $T_f(q_0, q_f, y)$ as the total number of periods needed to serve all consumers in the classic problem. For the later analysis, it would be more convenient to interpret $T_f(q_0, q_f, y)$ as the first period to clear the market either by serving all consumers ($q_t = q_f$) or exhausting all remaining stock ($q_t - q_{t-1} = y_t$), after making the first acceptable offer at $t_0 \geq 1$:

$$T_f(q_0, q_f, y) = \inf\{t - t_0 + 1 : q_t = \min(q_f, y_t + q_{t-1})\}.$$

In the classic problem, $T_f(q_0, q_f, y)$ is exactly the total number of periods when the monopolist keeps the market open, because $t_0 = 1$. Let us summarize the key properties of the subgame perfect equilibrium in the classic problem, which will be a key building block for constructing a reservation price equilibrium in the perishable problem. Because these properties are already proved for the analysis of the classic problem, we state them without proofs.

Lemma 2.3. *Suppose that $b = 0$.*

(1) *Fix y. If $q_f \neq q'_f$, then there is no $q \in [0, q_f]$ such that $\mathsf{P}(q, q_f, y) = \mathsf{P}(q, q'_f, y)$.*
(2) *If $q_f < 1$, $T_f(q_0, q_f, y) < \infty$.*
(3) *$\mathbf{q}(q_0, q_f, y)$ is a continuous function of (q_0, q_f, y) if $q_f < 1$.*
(4) *$T_f(q_0, q_f, y)$ is a decreasing function of q_0, but increasing function of q_f.*

The properties of Lemma 2.3 hold for general continuously downward sloping demand curve, except for the last part of the last statement. The demand is inelastic in the linear demand where q_f is close to 1, which encourages the monopolist to reduce the quantity and thus, to spend more time to sort out consumers with different reservation values. However, if the demand curve is very elastic around $q_f \simeq 1$, then the monopolist may find

[3]If $q_f = 1$ so that the lowest reservation value is 0, then the market opens indefinitely so that there is no "final" offer. Yet, the price must converge to 0 as $t \to \infty$.

it profitable to accelerate the sales. This property is used mainly to establish the existence of the reservation price equilibrium and to facilitate the construction of an equilibrium.

3. Two Types

Instead of a market with a linear demand, we first examine a market populated with a continuum of infinitesimal consumers, whose total mass is $x + \frac{1}{2}$. This example also shows the technical issues arising from analyzing a general demand curve.

Let us assume that x units of the consumers have valuation 3 and the remaining consumers have valuation 1, where $x \in (0, 0.5]$. The monopolist has y amount of perishable but durable goods. Assume that $y \geq 0.5$ so that in the initial period, the monopolist can serve every high valuation consumer. While the demand curve is not continuous, this example is sufficiently simple that we can precisely calculate a subgame perfect equilibrium to understand the structure of the equilibrium.

Let us focus on the case where

$$0 \leq x \leq y \leq x + \frac{1}{2}.$$

If $x > y$, then the monopolist can charge 3 credibly to serve every high valuation buyer and close the market. If $y \leq x + \frac{1}{2}$, the available stock is less than the total number of remaining consumers, including the low valuation buyers. We shall discuss the remaining cases after we completely analyze the most interesting cases.

3.1. Construction of an Equilibrium.

In the classic problem ($b = 0$), the optimal pricing rule is to open the market for two periods, offering $p_1 = 3 - 2\delta$ and $p_2 = 1$ unless x is too small. The initial offer will be accepted by the high valuation consumers, while the last offer serves all remaining low valuation consumers.

Suppose that the goods decay ($b > 0$). What would be the initial offer from the monopolist? The answer depends upon how quickly $y_t \leq x$, for a given level of patience of the players. For example, if $y = y_1 \leq x$, then the monopolist can credibly charge 3 from the initial period, which will be accepted by all high valuation consumers. What if y decays slowly so that $y_2 = \beta y \leq x < y$? The answer to the same question is no longer obvious. In fact, if the monopolist is sufficiently patient (small $r > 0$), he will find it optimal not to make any sales in the initial round so that the available goods can burn off as quickly as possible in order to achieve $y_2 \leq x$. In this way, he can charge 3 from period 2, which will be accepted by the high valuation consumer.

In a sharp contrast to the classic problem, making an unacceptable offer can be a part of an optimal pricing sequence. To reduce the available amount of goods, the monopolist can credibly refuse to serve any consumers. As a result, the dynamic market is exposed to two different sources of inefficiency. First, as in the classic problem, it will take more than one period to serve the consumers and this delay will be greater than in the classic problem if the monopolist chooses to burn off some of the goods. Second, if the monopolist chooses to burn off the available stocks, then some consumers may not be served. One of our objectives is to understand how the perishability factor affects the overall inefficiency of the market outcome, especially when $\Delta \to 0$.

A natural state variable is (x, y) which characterize the residual demand and the quantity of available goods. If $x \geq y$, then the monopolist can charge 3 and serve all remaining high valuation consumers, credibly excluding the low valuation consumers. The key decision is how long the monopolist has to wait before he can credibly charge 3.

Essentially, the monopolist has three options at (x, y).

- **Accelerating.** The monopolist can accelerate the sales in one of the two ways. First is to charge to serve everyone in the market. His profit will be

$$y. \tag{3.8}$$

  An alternative method is to charge $3 - 2\delta$ which is accepted by all high valuation buyers, and in the following round, charge 1 which is accepted by the remaining low valuation buyers. The average discounted profit is

$$(3 - 2\delta)x + \delta\beta(y - x). \tag{3.9}$$

  It depends upon the size of x whether (3.8) or (3.9) is optimal.
- **Delaying.** Continue to charge 3 until the high valuation consumer concludes that the monopolist will not lower the price, which will make the high valuation buyer to accept the offer. Let k be the first period that

$$\beta^{k-1} y \le x$$

  when the monopolist can charge credibly 3, which is accepted by the high valuation buyer immediately. Thus, if it takes k rounds, the expected profit is

$$3\delta^{k-1} \min(\beta^{k-1} y, x). \tag{3.10}$$

In order to delineate the optimal action of the monopolist under (x, y), let us characterize the "indifference state" between the Coase conjecture type strategy and the last one. That is state (x, y) solving

$$\max\left(y, (3 - 2\delta)x + \delta\beta(y - x)\right) = 3\delta^{k-1} \min(\beta^{k-1} y, x). \tag{3.11}$$

assuming for a moment that k can take any positive real number.

Lemma 3.1. *Suppose that k can be any non-negative real number.*

(1) $\forall (x, y)$, $\exists k \ge 0$ *such that* (3.11) *holds.*
(2) *If* (x, y) *satisfies* (3.11), *then so does* $(\lambda x, \lambda y)$ $\forall \lambda > 0$.
(3) *Define*

$$\mathcal{K} = \left\{ k : \exists (x, y) \quad \textit{such that } (3.11) \textit{ holds.} \right\}.$$

Then, $\mathcal{K}$ is a compact and connected set and therefore,

$$\sup \mathcal{K} < \infty.$$

(4) *For a fixed x, and $y' > y$. Let k and k' be the solution associated with (x, y) and (x, y') in* (3.11). *Then, $k' > k$.*

Proof. Define

$$g(k) = \max\left(y, (3 - 2\delta)x + \delta\beta(y - x)\right) - 3\delta^{k-1} \max(\beta^{k-1} y, x).$$

which is a continuous function of k. Note $g(0) < 0$ and $\lim_{k\to\infty} g(k) > 0$. Moreover, $g(k)$ is a strictly increasing function of k. Thus, there exists a unique k satisfying (3.11). Note that $g(k)$ is a linear function of (x, y) which implies the second statement. We know that the mapping $(x, y) \mapsto k$ satisfying (3.11) is continuous. Since (x, y) is contained in a compact set, $\mathcal{K}$ is compact, which implies the third statement. The last statement statement follows from the fact that the more the existing stock is, the longer it takes to reach the area where $y \le x$. □

For a fixed x, there is one-to-one correspondence between (x, y) and the solution k from (3.11). For each k, define $\alpha(k) = y/x$ where (x, y) induces k as the solution of (3.11). From Lemma 3.1,

$$\mathcal{U}(k) = \{(x, y) : k \text{ is the solution of } (3.11)\}$$

is a half line through the origin with slope $\alpha(k)$ which is a strictly decreasing function of k. The slope of $\alpha(k)$ can range from 1 to $+\infty$.

$\mathcal{U}(k)$ represents the collection of states that make the monopolist indifferent between two options of accelerating and delaying *if the delay takes k periods.* However, there is no guarantee that k is self-fulfilled. Again, let us assume for another moment that k can take any positive real number. Given (x, y), we can find a unique $k > 0$ such that

$$\beta^{k-1} y = x$$

which is the first time when the monopolist can credibly charge 3, which is accepted by the remaining high valuation buyer with probability 1. Define

$$\mathcal{V}(k) = \left\{ (x, y) : y = \frac{1}{\beta^{k-1}} x \right\}$$

as the collection of states, which takes k periods to reach the area

$$\{(x, y) : y \le x\}$$

where the monopolist's offer 3 is accepted with probability 1. Note that $\mathcal{V}(k)$ is a half-line passing through the origin. Its slope is ranging from 1 to $+\infty$, which is a strictly increasing function of k.

Therefore, there exists $k^* > 0$ such that

$$\mathcal{V}(k^*) = \mathcal{U}(k^*). \tag{3.12}$$

This k^* has a special meaning in the sense that if $(x, y) \in \mathcal{U}(k^*)$, the monopolist expects that in k^* periods, his offer 3 will be accepted with probability 1 and indeed, it takes k^* period before such event occurs.

Remark 3.2. *If k can take only a positive integer value, the same analysis proves the existence of a positive integer k^* such that*

$$\frac{1}{\beta^{k^*}} \ge \alpha(k^*) \ge \frac{1}{\beta^{k^*-1}}. \tag{3.13}$$

According to the definition of $\alpha(k^*)$ and $\mathcal{U}^*(k^*)$, if $y > \alpha(k^*)x$, then the monopolist charge 1 or $3 - 2\delta$ to 3, depending upon the size of x. Unless $x > 0$ is too small, the monopolist immediately makes an offer $3 - 2\delta$ which is accepted by all high valuation seller whose mass is x, and offer 1 to clear the market. As depicted in Figure 1, the state moves along 45 degree line passing through (x, y), because each consumer demand exactly one unit. On the other hand, if $x < y < \alpha(k^*)x$, then the monopolist refuses to make an acceptable offer. For analytic convenience, let us assume that the monopolist charges $3 + \epsilon$ for a small $\epsilon > 0$, which is rejected by all high valuation consumer. The available

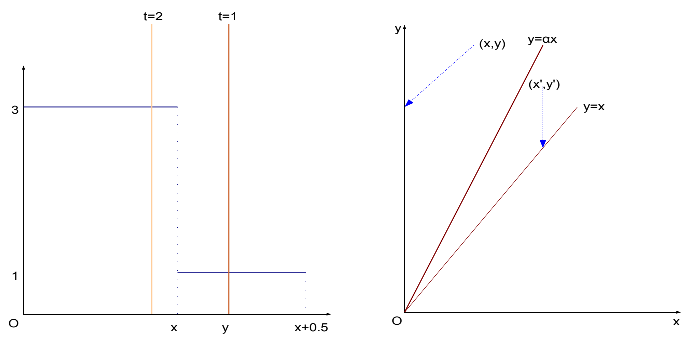

Figure 1: The left panel illustrates how the available stock decays in case that the monopolist makes no sales in the first two rounds when $\Delta > 0$ is relatively large. The right panel depicts the area of (x, y) associated with the two different strategies when $\Delta > 0$ is small. The bold straight line is $y = \alpha x$. The monopolist makes an acceptable offer immediately if $y > \alpha x$. Note that if the monopoly makes sales, x and y decreases by the same amount, and (x, y) is moving along the 45 degree line passing through (x, y). If all high valuation consumer is served, then x becomes 0. If $y' < \alpha x'$, then the state moved down vertically because no sales are made until the state hits $y = x$.

stock decays at the rate of β in each period. After k periods of rejected offers, suppose that $\beta^k y < x < \beta^{k-1} y$ holds. If the high valuation consumer rejects $3 + \epsilon$, then $\beta^k y < x$ implies that from the next period, there is excess demand among high valuation consumer and the monopolist can charge 3. Thus, all high valuation consumer is willing to accept any offer up to 3. Knowing this, the monopolist charges 3, following k unacceptable offers.

We can sustain this outcome as the subgame perfect equilibrium.

Proposition 3.3. *The above outcome path can be sustained by a subgame perfect equilibrium, which involves randomization off the equilibrium path.*

Proof. See Appendix A. □

3.2. **Properties.** The equilibrium strategy may entail a positive amount of time when the monopolist is willing to make no acceptable offers. Wasting time without making sales can never be a part of an equilibrium strategy in a classic durable good problem if the gain from trading is common knowledge, as in our example. However, because the goods are perishable, however slightly, making no sales does not mean wasting time. Rather, the monopolist can deliberately wasting some available stocks in order to manipulate the beliefs of the consumers about the monopolist's future prices.

Some consumers whose valuations are higher than the production cost may not be served. In a static monopoly problem, if the demand curve is inelastic, the monopolist finds it profitable to reduce the total sales and not to serve some consumers. The same intuition applies here. Because the monopolist may decide to waste some existing stock, which requires time, it takes more time to complete the sales.

Even if almost all consumers are served in the equilibrium, one cannot conclude that the market outcome is almost efficient. If it takes substantial time to achieve an optimal amount of stock, the realization of the gains from trading can take excessively long time. As a result, the discounted social surplus could be very small, even if almost all consumers are eventually served.

While this example is simple enough to allow us to calculate the subgame perfect equilibrium, it has a couple of rather peculiar features. Because the type space of the consumers is discrete, the gain from the reducing the available stock increase discontinuously. Combined with the fact that the initial stock is smaller than the whole market demand ($y \leq x + 0.5$), the monopolist has good reason to delay the offer, because a large return from the delay is realized fairly quickly. A natural question is whether the key properties of the equilibrium are carried over to the cases where the demand curve is continuous and the initial stock is larger than the whole market demand. To answer this question, we examine the market with a linear demand for the rest of the paper.

If we analyze a general downward sloping demand curve, we generally have to admit a randomized strategy off the equilibrium path even though only pure strategies are used along the equilibrium path, as shown in this example. By focusing on the linear demand, we can ensure that the associated optimization problem (2.4) has a unique solution as demonstrated by [9]. The equilibrium strategies off the path are essentially a properly "scaled" version of the strategies along the equilibrium path according to the size of the residual demand. The linear demand allows us to highlight the key features of the subgame perfect equilibria of the perishable problems by analyzing the equilibrium path.

4. Market with a Linear Demand

In order to highlight the key features of the equilibrium which will be constructed in this section, let us first examine a simple, but artificial, game. Then, we construct a reservation price equilibrium, which is approximated by the equilibrium of the artificial game.

4.1. **Example 2. An Artificial Game.** Let us consider an artificial game in which the monopolist in the market with a linear demand curve (2.1) has two options: make one final sale, or delay the sale. The monopolist can choose when he opens the market, say $\tau_f \geq 0$, and then, he must make an offer to serve everyone in the market or sell all the goods available at that point, if there is an excess demand.

If the monopolist charges $p_{\tau_f} = 1 - q_{\tau_f}$ after delaying τ_f time, q_{τ_f} portion of consumers will be served. Since p_{τ_f} must clear the market,

$$1 - p_{\tau_f} = \min(q_{\tau_f}, e^{-\tau_f b} y)$$

must hold. In any equilibrium, q_{τ_f} is selected in such a way that the monopolist cannot improve his profit by delaying the sale. Define

$$h(q : \tau) = e^{-\tau r} \left[e^{-\tau b} q (1 - e^{-\tau b} q) \right] - q(1 - q)$$

as the gain from delaying τ amount of real time and charging $1 - e^{-\tau b} q$ to serve everyone whose valuation is higher than $1 - q$ in the market, if the present available stock is $q \leq 1$.

It is easy to see that $h(q : 0) = 0$ and

$$\frac{\partial h(q : 0)}{\partial \tau} = -(r + b)q + (r + 2b)q^2.$$

If
$$\frac{\partial h(q:0)}{\partial \tau} \leq 0,$$
then $\forall \tau > 0$, $\frac{\partial h(q:\tau)}{\partial \tau} < 0$. If
$$\frac{\partial h(q:0)}{\partial \tau} \geq 0,$$
then $\forall q' \geq q$, $\frac{\partial h(q':0)}{\partial \tau} \geq 0$. If $\partial h(q:0)/\partial \tau > 0$, then the monopolist can be better off by delaying the sale on period. Although the total stock will be reduced to $e^{-\Delta b}q$, but he can credibly charge higher price $1 - e^{-\Delta b}q$ to generate higher profit. Similarly, if $\partial h(q:0)/\partial \tau < 0$, then he should have accelerated the sale. Thus, the optimal quantity q solves
$$\frac{\partial h(q:0)}{\partial \tau} = 0,$$
which is
$$q = \frac{r+b}{r+2b}, \tag{4.14}$$
and the discounted profit is
$$e^{-r\tau(y)} \frac{b(r+b)}{(r+2b)^2}$$
where $\tau(y)$ is defined implicitly by
$$e^{-b\tau(y)} y = \frac{r+b}{r+2b}.$$

The optimal quantity, $(r+b)/(r+2b)$, is very intuitive. For a given time preference $r > 0$, if the good is not perishable ($b = 0$), then all consumers must be served, as in the classic problem. On the other hand, if the good is perishing quickly (i.e., large $b > 0$), the quantity converges to $1/2$ which is the monopolistic profit maximizing quantity.

The ensuing analysis shows that the outcome of this artificial game approximate the outcome of the reservation price equilibrium of the dynamic monopoly problem where he can charge a series of prices over time, combined with delaying the offers. The missing step is to make it sure that the delay strategy generates a higher profit than the strategy satisfying the Coase conjecture, from which the monopolist can generate profit $q_f(1-q_f)$ almost instantaneously if $\Delta > 0$ is small. Note that
$$e^{-r\tau(y)} \frac{b(r+b)}{(r+2b)^2} > q_f(1-q_f)$$
holds as long as q_f is sufficiently close to 1 for given b, r, y.[4] Then, a substantial delay of an acceptable offer can arise in a reservation price equilibrium.

Note that for a fixed $r > 0$,
$$\lim_{b \to 0} \frac{r+b}{r+2b} = 1$$

[4]One might wonder whether we have to check the same inequality for each $\tau > 0$. From the analysis of $h(q:\tau)$, we know that if this equality holds the beginning of the game, then it continues to hold for $\tau > 0$ until the available stock reaches the optimal level.

which implies that every consumer will be served in the limit. Yet, the outcome is extremely inefficient. A simple calculation shows that

$$\tau(y) = \frac{1}{b}\left(\log y - \log \frac{r+b}{r+2b}\right).$$

If $y > 1$, as $b \to 0$, the right hand side increases indefinitely, implying that the monopolist is willing to delay the sale as long as possible in order to generate a positive profit, even if it is realized after a long delay. As a result, the market outcome becomes extremely inefficient, because the potential gains from trading is discounted away during the long delay. Even if $y = 1$, l'Hôpital's rule implies that

$$\lim_{b\to 0} \tau(y) = \frac{1}{r}$$

which implies that the delay does not vanish and can be significant if the monopolist is very patient.

4.2. **Construction of a Reservation Price Equilibrium.** We search for a reservation price equilibrium where the equilibrium path consists of two phases: the first phase where the monopolist is making unacceptable offers, and the second phase where the monopolist is making a series of acceptable offers. In the second phase, we can invoke the same insight as in the classic problem to construct the equilibrium path. In particular, we can write (2.2) in a simpler form (2.6). And, then by calculating the optimal time for delaying to make the first acceptable offer, we construct the equilibrium outcome, where the total gains from trading vanishes as $\Delta \to 0$.

We construct an equilibrium for the rest of the section, in which the total surplus from trading is arbitrarily small, despite the fact that almost every consumer is served by the monopolist. As in Section 4.1, the monopolist delays to open the market (or equivalently, making unacceptable offers) for $\tau_1 \in \{\Delta, 2\Delta, \ldots\}$ before making an acceptable offer, in order to avoid the integer problem. After the initial acceptable offer, the monopolist keeps making acceptable offers.

Given residual demand $\mathsf{D}(0, q_f)$ and the initial stock y with $q_f < 1$, the optimization problem can be written as

$$\max_{\tau_1, \mathbf{q}\in\mathbf{Q}} e^{-r\tau_1} \sum_{t=1}^{\infty} p_t(q_t - q_{t-1})\delta^{t-1} \tag{4.15}$$

$$\text{subject to} \quad (1-q_t) - p_t = \delta((1-q_t) - p_{t+1}) \tag{4.16}$$

$$p_{T_f} = 1 - q_{T_f} \tag{4.17}$$

$$\beta^{T_f}\left(e^{-r\tau_1} y - \sum_{t=1}^{T_f} \beta^{-t}(q_t - q_{t-1})\right) \geq 0 \tag{4.18}$$

$$\beta^{T_f}\left(e^{-r\tau_1} y - \sum_{t=1}^{T_f} \beta^{-t}(q_t - q_{t-1})\right)(q_{T_f} - q_f) = 0 \tag{4.19}$$

where T_f is the number of periods when a positive portion of consumers is served. τ_1 is the time during which the monopolist makes no sale, simply burning off the available stock at the rate of e^{-b}. The objective function and the first two constraints are identical to the classic problem and so is the definition of T_f.

The last two constraints warrant explanation, as they capture the key elements of the perishable problem. The first step is to observe that the trading must be completed in finite rounds, which is reminiscent to a well known result from the classic problem [9].

Lemma 4.1. *If $q_f < 1$, then in any optimal solution, $\tau_1 + \Delta T_f < \infty$.*

Proof. Given the structure of the candidate equilibrium, the proof to show $T_f < \infty$ is identical with the one in the classic problem [9, 10]. It remains to show that $\tau_1 < \infty$.

It suffices to show that $\exists \tau^* > 0$ such that if $\tau_1 > \tau^*$, then $(\tau_1, \mathbf{q})$ cannot be an optimal solution for any $\mathbf{q} \in \mathbf{Q}$.

Given demand curve $\mathsf{D}(0, q_f)$, let $q^m(0, q_f)$ be the static monopoly profit maximizing quantity. The monopolist can choose τ_1 so that $e^{-b\tau^*} y_o = q^m(0, q_f)$, and charge $1 - q^m(0, q_f)$, which will be accepted by all consumers whose valuation is at least $1 - q^m(0, q_f)$. Thus, the equilibrium payoff of the monopolist is uniformly bounded from below by

$$e^{-b\tau^*}(1 - q^m(0, q_f))q^m(0, q_f).$$

If the monopolist spends more than τ^* before making an acceptable offer, he cannot achieve this level of profit. Thus, if τ_1 is selected in an equilibrium, then $\tau_1 \leq \tau^*$. □

If $q_1 - q_0$ consumers accepts the first acceptable offer from the monopolist, then at the end of the period, $e^{-b\tau_1} y - (q_1 - q_0)$ is available, but by the beginning of period 2, only $\beta(e^{-b\tau_1} y - (q_1 - q_0))$ is available. Thus, by the time when all available goods are sold,

$$\beta\left(\cdots\left(\beta(e^{-r\tau_1} y - (q_1 - q_0)) - (q_2 - q_1)\right)\right) - (q_{T_f} - q_{T_f - 1}) \geq 0$$

must hold, because the amount of sales in period t cannot exceed the amount of stocks available in that period. The constraint can be written as

$$\beta^{T_f}\left(e^{-r\tau_1} y - \sum_{t=1}^{T_f} \beta^{-t}(q_t - q_{t-1})\right) \geq 0. \tag{4.20}$$

However, if $q_{T_f} = q_f$, then it is possible that a positive amount of goods is left over. This happens when all consumers are served. But, if $q_{T_f} < q_f$, then some consumers are not served and the final offer must be such that all remaining goods are sold. Hence, the complementary slackness condition (B.32) must hold.

We show by construction that the above optimization problem has a solution. Let $\mathcal{W}_c(\Delta)$ and $\mathcal{W}_s(\Delta)$ be the consumer and the producer surplus if the time between the offers is $\Delta > 0$.

Proposition 4.2. *Fix initial stock $y > 1$ and size of the demand $q_f < 1$. Given demand curve $\mathsf{D}(0, q_f)$ and initial stock y, there exists an optimal solution $(\tau_1, \mathbf{q})$, which can be sustained as a reservation price equilibrium.*

Proof. See Appendix B. □

Fix y, and let $q_{T_f(\Delta)}$ be the total amount that is delivered and $T_1(\Delta)$ be the first round when the monopolist is making an acceptable offer when the time between the offers is

$\Delta > 0$. Clearly, $\forall \Delta > 0$, $q_{T_f(\Delta)} \in [0, y]$ and $\Delta T_1(\Delta) \in [0, \tau^*]$. Define

$$q' = \lim_{\Delta \to 0} q_{T_f(\Delta)}$$

and

$$\tau_1(0) = \lim_{\Delta \to 0} \Delta T_1(\Delta)$$

by taking a convergent subsequence, if necessary.

Note that the sequence of acceptable offers is precisely the same as the one in the classic counter part where the demand curve is $\mathsf{D}(0, q_{T_f(\Delta)})$. Hence, the Coase conjecture implies that the profit from the perishable problem converges to

$$e^{-\tau_1(0)b} q'(1 - q').$$

Hence, the limit properties of the reservation price equilibrium can be examined through the same method as illustrated in Example 4.1.

Let $\mathcal{W}_c(\Delta)$ and $\mathcal{W}_s(\Delta)$ be the (ex ante) expected consumer surplus and the expected producer surplus from the game where the time between the offers is $\Delta > 0$. The following proposition formalizes this observation.

Proposition 4.3. *$\forall \epsilon > 0$, $\exists \overline{b} > 0$ such that $\forall b \in (0, \overline{b}]$, $\exists \overline{q}_f$ such that $\forall q_f \in (\overline{q}_f, 1)$, $\exists \overline{\Delta} > 0$ such that $\forall \Delta \in (0, \overline{\Delta})$, $\mathcal{W}_c(\Delta) < \epsilon$ and $\mathcal{W}_s(\Delta) < \epsilon$.*

The constructed equilibrium confirms our intuition that if the monopolist has little commitment power (small $b > 0$ and small $\Delta > 0$), then he can exercise little market power and entertain small profit. This observation is generally consistent with the key implication from the classic problem, and has an important policy implication. If the monopolist exercise substantial market power, then his commitment power must be substantial. Thus, by unraveling the source of the commitment power, the government can reduce the market power of the monopolist.

In the perishable problem, this conclusion does not hold in general, because we can construct another reservation price equilibrium that generate substantial profit, despite small $b > 0$ and small $\Delta > 0$. Before jumping into a formal analysis, it is useful to understand the foundation of the differences between the perishable and the classic problems.

In the classic problem with $q_f < 1$, the total quantity of sales must be q_f. Because the terminal sales amount is q_f, we can invoke the backward induction process to construct the equilibrium strategy. In particular, if $q_f < 1$, the number of periods to make acceptable offers is uniquely determined.

The equilibrium constructed in Proposition 4.3 shares some of the key features of the subgame perfect equilibrium of the classic problem. Given the terminal quantity q_{T_f}, we invoke the backward induction process to calculate the optimal pricing rule, and determine the number of acceptable offers. However, the terminal quantity q_{T_f} is generally different from the size of the whole market q_f. In fact, q_{T_f} is endogenously determined and therefore, the belief of the consumers about how the future pricing rule evolves is critical in determining q_{T_f} as well as the number of acceptable offers T_f.

We construct the beliefs off the equilibrium path so that the consumers believe that the monopolist will follow the "same kind" of pricing rule: a series of unacceptable offers followed by acceptable offers until he clears the market. The substance of Proposition 4.2

is in fact to show that this class of beliefs off the equilibrium path can sustain the optimal solution of (B.28) as a reservation price equilibrium.

But, this is not the only way to construct the equilibrium outcome. One can imagine a different pricing rule, for example, which starts with acceptable offers, followed by a series of unacceptable offers, and then resume to make acceptable offers until the monopolist clears the market. The ensuing analysis shows that the flexibility of specifying the terminal quantity q_{T_f} is the source of generating multiple equilibria with dramatically different properties.

5. Small Commitment but Large Profit

We claim that substantial market power does not imply substantial commitment power. To substantiate the claim, we need to construct a subgame perfect equilibrium in which the monopolist can generate a large profit when $b > 0$ and $\Delta > 0$ are small. The equilibrium constructed in Section 4.2 can serve as a credible threat to force the monopolist to follow a designated outcome path. Following the same idea as in [2], we can obtain the folk theorem if $\Delta \to 0$ and then $b \to 0$. In particular, we can sustain a subgame perfect equilibrium in which the monopolist generates an expected profit close to the static monopoly profit.

Proposition 5.1. *$\forall \epsilon > 0$, $\exists \bar{b} > 0$, $\exists \overline{y} > 1$, $\forall b \in (0, \bar{b})$, $\forall y \in (1, \overline{y}]$, $\exists \overline{q}_f$, $\forall q_f \in [\overline{q}_f, 1)$, $\exists \overline{\Delta} > 0$, $\forall \Delta \in (0, \overline{\Delta})$, there is a subgame perfect equilibrium in which the equilibrium payoff $\mathcal{W}_s(\Delta)$ satisfies*

$$\mathcal{W}_s^m \leq \mathcal{W}_s(\Delta) + \epsilon$$

where $\mathcal{W}_s^m$ is the static monopolist profit.

Proof. Apply [2]. □

Following [2], we differentiate two kinds of subgame perfect equilibria: *reservation price equilibria* as defined in Definition 2.1, and *reputational equilibria*, where any deviation by the monopolist triggers a punishment phase as the continuation game is played according to the equilibrium constructed in Proposition 4.3. The key idea of [2] is to use the reputational equilibria to sustain an expected payoff close to the static monopoly profit as $\Delta \to 0$. As it is stated, Proposition 5.1 does not tell us whether the reputational effect or the slight decay is the key for monopolist to generate a large profit. To crystallize the impact of a slight decay, we need to construct a reservation price equilibrium with a large profit in a perishable problem. The main goal of this section is to obtain Proposition 5.1 only with reservation price equilibria.

As in Section 4, we start with a simple artificial game to explore the key properties of the equilibrium we shall construct. Then, we construct a reservation price equilibrium, which generates expected profit close to the static monopoly profit for small $b > 0$.

5.1. **Example 3. Another Artificial Game.** The monopolist uses the delay tactic as a way to influence the consumer's belief about the future prices offered by the monopolist. Yet, the delay tactic has an obvious downside: the monopolist has to delay the realization of the profit. Because the consumers with high reservation value is willing to pay higher price, the monopolist has to balance the benefit of delaying and burning the available stock against the cost of delaying the profit, especially against the high reservation value consumers.

To explore the tension between these two strategic motivations, let us examine a slightly more elaborate version of Example 4.1 where the monopolist can only delay the beginning of the game. Instead, let us allow the monopolist to choose a time interval with length $\tau > 0$ during which he chooses to burn the stock at the instant rate of e^{-b}. Thus, the sales can occur twice, before and after the τ break. Let (q_1, q_2) represent the total amount of goods delivered after each sales, and (p_1, p_2) be the respective delivery prices. That is, at the beginning of the game, the monopolist charges p_1 to serve q_1, and then, takes break for τ time. After the break, he charges p_2 to serve additional $q_2 - q_1$ consumers. As in Example 4.1, the initial quantity of the goods is y. All other parameters of the models remain the same as in Example 4.1.

We calculate the optimal solution through backward induction. Suppose that q_1 has been served. Then, $y - q_1$ is available, and the residual demand curve is $\mathsf{D}(q_1, q_f)$. Throughout this example, we choose both $y > 1$ and $q_f < 1$ sufficiently close to 1, and $b > 0$ sufficiently small. Invoking the same logic as we did in Section 4.1., we have

$$q_2 - q_1 = (1 - q_1)\frac{r + b}{r + 2b}$$

and the monopolist has to delay the offer p_2 by τ in order to satisfy the market clearing condition:

$$e^{-b\tau}(y - q_1) = q_2 - q_1 = (1 - q_1)\frac{r + b}{r + 2b} \tag{5.21}$$

which implies that

$$p_2 = (1 - q_1)\frac{b}{r + 2b}.$$

Let $\tau(q_1)$ be the solution for (5.21). Note that

$$\tau'(q_1) > 0.$$

In order to make consumer q_1 indifferent between p_1 and p_2,

$$(1 - q_1) - p_1 = e^{-r\tau(q_1)}\left((1 - q_1) - (1 - q_1)\frac{b}{r + 2b}\right)$$

which implies that

$$p_1 = (1 - q_1)\left[1 - e^{-r\tau(q_1)}\frac{r + b}{r + 2b}\right].$$

Hence, the profit from selling q_1 in the first round can be written as

$$\begin{aligned} V(q_1) &= q_1(1 - q_1)\left[1 - e^{-r\tau(q_1)}\frac{r + b}{r + 2b}\right] + e^{-r\tau(q_1)}(1 - q_1)^2\frac{(r + b)b}{(r + 2b)^2} \\ &= (1 - q_1)\left(e^{-r\tau(q_1)}\frac{(r + b)b}{(r + 2b)^2} + q_1\left[1 - e^{-r\tau(q_1)}\frac{(r + b)(r + 3b)}{(r + 2b)^2}\right]\right). \end{aligned}$$

Note that as $b \to 0$, $V(q_1)$ converges uniformly to $(1-q_1)q_1$ over $q_1 \in [0, q_f]$. A simple calculation shows

$$\begin{aligned} V'(q_1) &= (1-2q_1)\left(1 - e^{-r\tau(q_1)}\frac{(r+b)(r+3b)}{(r+2b)^2}\right) \\ &\quad -e^{-r\tau(q_1)}\frac{(r+b)b}{(r+2b)^2} - \left[\frac{(r+b)b}{(r+2b)^2} - q_1\frac{(r+b)(r+3b)}{(r+2b)^2}\right] re^{-r\tau(q_1)}\tau'(q_1). \end{aligned}$$

As $\tau(q_1)$ is determined by (5.21), $\tau(q_1) \to \infty$ as $b \to 0$, as long as $y > 1$. Thus, the first term in the second line vanishes as $b \to 0$. To show that the second term in the second line also vanishes, recall (5.21). Thus,

$$e^{-r\tau(q_1)}\tau'(q_1) = \frac{\tilde{\omega}e^{-\frac{r\omega}{b}}}{b}$$

where

$$\omega = -\log\frac{(1-q_1)(r+b)}{(y-q_1)(r+2b)} > 0$$

and

$$\tilde{\omega} = \frac{(r+b) - e^{-b\tau(q_1)}(r+2b)}{b(1-q_1)}.$$

Thus,

$$\lim_{b\to 0} e^{-r\tau(q_1)}\tau'(q_1) = 0$$

which implies that the second line vanishes as $b \to 0$. Thus,

$$\lim_{b\to 0} V'(q_1) = 1 - 2q_1,$$

and the delivery price of q_1 converges to $p_1 = 0.5$, which generates the static monopolist's profit. The slow rate of decay combined with a negligible profit from the continuation game makes it credible for the monopolist to delay an acceptable offer for an extremely long period.

5.2. **Reservation Price Equilibrium with Large Profit.** The key feature of the equilibrium constructed in Section 4.2 is that the monopolist can credibly delay to make an acceptable offer, especially when the expected profit from accelerating the sale is small. Following the same logic as in Section 4.2, we can construct another reservation price equilibrium, which generates the monopolist the static monopolist's profit as $\Delta \to 0$ as illustrated in Section 5.1.

Imagine an equilibrium that consists of two phases, as in Section 4.2. In each phase, the monopolist is making a series of acceptable offers, denoted as $p_{1,1}$ and $\{p_{2,t}\}_{t=1}^{T_f}$ where the subscript represents the phase and the period within each phase. And, T_0 represents the number of periods during which the monopolist is making unacceptable offers. The first phase consists of a single offer, which is accepted by $\overline{q}_1$ consumers. After $\overline{q}_1$ consumers are served, the continuation game is played according to the same kind of equilibrium constructed in Section 4.2: the monopolist makes T_0 unacceptable offers and then, make a series of acceptable offers for T_f rounds to clear the market. We choose an optimal $\overline{q}_1$ that maximizes the expected discounted profit among all equilibria that have the same two phase structure as described above.

In order to formalize this idea, we need to make it sure to have a delay equilibrium in the second phase.

Lemma 5.2. *$\exists q'_f < 1, y' > 1$ such that $\forall q_f \in (q'_f, 1)$ and $\forall y \in (1, y')$, $\exists \overline{\Delta} > 0$ and $\overline{b} > 0$ such that $\forall \Delta \in (0, \overline{\Delta})$ and $\forall b \in (0, \overline{b})$, $\exists q^*$ such that if $q_0 > q^*$, then the acceleration strategy is optimal, and if $q_0 < q^*$, then the delay strategy is optimal for residual demand $\mathsf{D}(q_0, q_f)$ with available stock y.*

Proof. Since the payoff from the two strategies changes continuously with respect to $\Delta > 0$, let us consider the limit case examined in Section 4.1. Fix a residual demand $\mathsf{D}(q_0, q_f)$ and the available stock y. From the acceleration strategy, the monopolist obtains $q_f(1 - q_f)$ instantaneously. On the other hand, from the delay strategy, he obtains

$$e^{-r\tau(q_0)}(1 - q_0)^2 \frac{(r + b)b}{(r + 2b)^2}$$

where $\tau(q_0)$ is defined implicitly by (5.21) with q_1 replaced by q_0. Choose $q'_f < 1$ and $y' > 1$ sufficiently close to 1 so that

$$q_f(1 - q_f) < e^{-r\tau(q_0)}(1 - q_0)^2 \frac{(r + b)b}{(r + 2b)^2}$$

if $q_0 = 0$. We know that $\tau'(q_0) > 0$, and also, for a given q_0, $\tau(q_0)$ increases without a bound as $b \to 0$. Thus, for any sufficiently small $b > 0$, we can find a critical q_0 where the above strict inequality holds with equality. This is q^*. By the continuity of the expected payoff with respect to $\Delta > 0$, we can repeat the same reasoning for a small $\Delta > 0$ to find q^*. □

Let us consider the initial demand $\mathsf{D}(0, q_f)$ and the initial stock. After $\overline{q}_1$ consumers are served, the continuation game is played with residual demand curve $\mathsf{D}(\overline{q}_1, q_f)$ and available stock $\beta(y - \overline{q}_1)$. Since we choose $y > 1$, and $q_f < 1$, we can invoke Lemma 5.2 to identify whether the continuation game can sustain the delay strategy as a Nash equilibrium.

Corollary 5.3. *$\exists q'_f < 1, y' > 1$ such that $\forall q_f \in (q'_f, 1)$ and $\forall y \in (1, y')$, $\exists \overline{\Delta} > 0$ and $\overline{b} > 0$ such that $\forall \Delta \in (0, \overline{\Delta})$ and $\forall b \in (0, \overline{b})$, $\exists \overline{q}_1^*$ such that the continuation game after $\overline{q}_1$ consumers are served can sustain the delay strategy as a Nash equilibrium if and only if $\overline{q}_1 \leq \overline{q}_1^*$.*

Proof. Note that $y > 1$ and $q_f < 1$. It is clear that $\inf_{\overline{q}_1} \beta(y - \overline{q}_1) - (q_f - \overline{q}_1) > 0$ as long as $\Delta > 0$ is sufficiently small. Thus, $\forall \overline{q}_1 \leq q_f$, the amount of delay, $\tau(\overline{q}_1)$, defined by (5.21) increases without a bound as $b \to 0$. The conclusion follows from the same reasoning as the proof of Lemma 5.2. □

From the analysis in Section 5.1, the high expected payoff is sustained by the delay strategy in the second phase. Thus, if the initial offer $p_{1,1}$ is accepted by more than $\overline{q}_1^*$ consumers, the continuation game strategy must be an acceleration strategy, and the resulting profit is lower than otherwise.

In order to simplify the characterization of the optimal $\overline{q}_1$, let us assume for a moment that the continuation game strategy in the second phase is a delay strategy. Let $V(0, q_f, y, \overline{q}_1)$ be the expected payoff when the monopolist

By invoking the same logic as Lemma 2.3, we can show that $T_f < \infty$ and $\limsup_{\Delta\to 0} \Delta T_f < \infty$. Let us write down the optimization problem of the monopolist for a given $\overline{q}_1 = \overline{q}_1(0,\alpha)$.

$$V(0, q_f, y, \overline{q}_1) = \max_{T_0\geq 0, \mathbf{q}\in\mathbf{Q}} \overline{q}_1 p_{1,1} + \delta^{T_0} \sum_{t=1}^{\infty} p_{2,t}(q_t - q_{t-1})\delta^{t-1} \tag{5.22}$$

$$\text{such that} \quad (1 - q_t) - p_{2,t} = \delta((1 - q_t) - p_{2,t+1}) \qquad \forall 1 \leq t \leq T_f$$

$$(1 - \overline{q}_1) - p_{1,1} = \delta^{T_0}((1 - \overline{q}_1) - p_{T_0+1}) \tag{5.23}$$

$$p_{2,T_f} = 1 - q_{T_f} \tag{5.24}$$

$$\beta^{T_f-1-T_0}\left(\beta^{T_0}(y - \overline{q}_1) - \sum_{t=1}^{T_f-1-T_0} \beta^{-t}(q_t - q_{t-1})\right) \geq 0 \tag{5.25}$$

$$\beta^{T_f-1-T_0}\left(\beta^{T_0}(y - \overline{q}_1) - \sum_{t=1}^{T_f-1-T_0} \beta^{-t}(q_t - q_{t-1})\right)(q_{T_f} - q_f) = 0 \tag{5.26}$$

The optimization problem is virtually identical with (B.28). After $\overline{q}_1$ consumers are served, the continuation game is played according the equilibrium strategy constructed in Section 4.2 associated with residual demand $\mathsf{D}(\overline{q}_1, q_f)$. Then, (5.23) ensures that consumer $\overline{q}_1$ is indifferent between $p_{1,1}$ and $p_{2,t}$ after T_0 periods. Let

$$\mathcal{W}_s(\Delta) = \max_{\overline{q}_1\in[0,q_f]} V(0, q_f, y, \overline{q}_1)$$

and denote the optimal value of $\overline{q}_1$ as $\overline{q}_1^e$.

We can show that the constructed path can be sustained as a reservation price equilibrium.

Proposition 5.4. $\exists\overline{b} > 0$, $\exists\overline{y} > 1$, $\forall b \in (0, \overline{b})$, $\forall y \in (1, \overline{y}]$, $\exists\overline{q}_f$, $\forall q_f \in [\overline{q}_f, 1)$, *the constructed outcome path can be sustained as a reservation price equilibrium, which involves randomization by the monopolist following some histories off the equilibrium path.*

Proof. See Appendix C □

Recall that given demand curve $\mathsf{D}(0, q_f)$, and that

$$(1 - \overline{q}_1) - p_{1,1} = \delta^{T_0}((1 - \overline{q}_1) - p_{2,1}).$$

Let $b \to 0$. From the analysis of Section 4.1, we know that

$$\lim_{b\to 0}\lim_{\Delta\to 0} T_0 = \infty.$$

Hence,

$$1 - \overline{q}_1 - p_{1,1} \to 0.$$

In particular, if $\overline{q}_1 = 0.5$ which need not be an optimal value $\overline{q}_1^e$, the resulting expected payoff converges to the static monopoly profit. Thus, if we choose $\overline{q}_1$ optimally, the resulting profit $\mathcal{W}_s(\Delta)$ must converge to the static monopoly profit.

Proposition 5.5. $\forall\epsilon > 0$, $\exists\overline{b} > 0$, $\exists\overline{y} > 1$, $\forall b \in (0, \overline{b})$, $\forall y \in (1, \overline{y}]$, $\exists\overline{q}_f$, $\forall q_f \in [\overline{q}_f, 1)$, $\exists\overline{\Delta} > 0$, $\forall\Delta \in (0, \overline{\Delta})$, *there exists a reservation price equilibrium in which the monopolist's* *expected profit is $\mathcal{W}_s(\Delta)$ such that $\mathcal{W}_s^m \leq \mathcal{W}_s(\Delta) + \epsilon$ where $\mathcal{W}_s^m$ is the static monopolist profit.*

6. Concluding Remarks

6.1. **Delayed offer.** In order to highlight the impact of the perishability to the Coase conjecture, we literally follow the rule of the classic durable goods monopoly problem, forcing the monopoly to announce an unacceptable price in order to delay the game. Thus, the delay occurs as a positive integer multiple of $\Delta > 0$.

A more general, perhaps more natural, formulation would be to let the monopolist to delay the bargaining continuously. That is, following each history, the monopolist can choose a pair of numbers, (p, τ): p is offered but the good is delivered to the consumer τ unit of time after p is accepted. Given p, consumers decided to accept or reject. If the offer is rejected, then the monopolist has to wait $\Delta > 0$ unit of time before making another move. If p is accepted, the consumption of the good occurs in τ units of time after accepting the offer.

In the classic problem, the monopolist has no reason to delay: $\tau = 0$ following every history.[5] Thus, the Coase conjecture holds. Because the monopolist can delay the bargaining continuously, the analysis is in fact simpler and closer to the examples where we assume that the game is delayed continuously.

6.2. **Increasing Demand.** The strategic impact of the decay arises from the fact that the excess demand for the goods increases as fewer goods become available. One can apply the same logic of the perishable problem to the case where the demand is expanding. [12] investigates the dynamic sales problem with new entry of consumers in the market. Because the goods are sold to the high valuation consumers, the remaining consumers have lower reservation value and the residual demand curve becomes more elastic. As a result, the seller offers a low price in order to clear the market occasionally. We except a similar dynamics. But, we also expect that the monopolist may not serve some low valuation consumers by burning off existing stock, which results in considerably delay in offering sales price to clear the market. Formal analysis is left as a future research project.

6.3. **Endogenous Stock.** Cement is an example of perishable durable goods [11]. After it is delivered to the consumer, it generates utility to the consumers over many periods. However, while in the storage, a small portion of cement is hardened and rendered useless. Because of its weight, it is not unusual that the delivery of cement is scheduled over a certain interval. The present model provides insight about how the sales of cement can evolve *after* a fixed batch of cement is delivered, as we assume that the initial available stock is exogenous. It remains to be analyzed how the pricing rule changes, if the monopolist can control the delivery schedule and quantity as well as the pricing after the delivery.

[5] This is true because the monopolist has no private information. If the monopolist has private information, then the analysis of [1] implies that the monopolist may have incentive to delay.

Appendix A. Construction of Strategies off the Equilibrium Path

Recall that because each consumer purchases a single unit, the state moves along the 45 degree line passing through the given state (x, y), if some consumers purchase the good and $\alpha(k^*) > 1$.

If the initial state $(0.5, y)$ is above $\mathcal{U}(k^*)$, then the construction of the actions off the equilibrium path follows the same idea as the weak stationary equilibrium in [10] with minor twist. We only describe the case where the monopolist charges $3 - 2\delta$ along the equilibrium path. Let $\mathcal{U}^*$ be the half line passing through the origin along which the monopolist is indifferent between charging 1 and $3 - 2\delta$. Let α^* be the slope of $\mathcal{U}^*$. One can easily show that

$$\alpha^* > \alpha(k^*) > 1.$$

If the initial equilibrium offer is $3 - 2\delta$, then the initial state is located between $\mathcal{U}^*$ and $\mathcal{U}(k^*)$.

If $p > 3 - 2\delta^2$, then no consumer accepts the offer, expecting that in the following period, the monopolist will charge $3 - 2\delta$. If $p < 3 - 2\delta$, then every consumer purchases the good. The state moves from $(0.5, y)$ to $(0, y - 0.5)$, which implies that the monopolist has some goods for future sale, because $y > 0.5$. In the next round following such p, the monopolist charge 1 to serve all low valuation consumers.

If $3 - 2\delta < p < 3 - 2\delta^2$, we first locate a point along

$$y = \frac{\alpha^*}{\beta} x$$

that intersects with the 45 degree line passing through initial state $(0.5, y)$. Let $(0.5 - x^*, y - x^*)$ be such a point. Such p is accepted by x^* portion of high valuation consumers who expects that in the following period, the monopolist randomizes between 1 and $3 - 2\delta$ with probability λ to 1 so that

$$3 - p = \delta(3 - (\lambda + (1 - \lambda)(3 - 2\delta))).$$

In the following round, $\beta(y - x^*)$ is available and the new state $(0.5 - x^*, \beta(y - x^*))$ is located along $\mathcal{U}^*$ where the monopolist is indeed indifferent between 1 and $3 - 2\delta$.

If state (x, y) is below $\mathcal{U}(k^*)$ but $y < x$, then the high valuation consumer accepts any offer $p < 3$. Finally, suppose that state (x, y) is below $\mathcal{U}(k^*)$ but $y > x$. For simplicity, let us assume that the monopolist is indifferent between charging 3 and $3 - 2\delta$ along $\mathcal{U}(k^*)$. The other case follows from the same logic, where the monopolist is indifferent between charging 3 and 1 along $\mathcal{U}(k^*)$.

The monopolist is charging 3 in the equilibrium. If he charges $p > 3$, it is clearly optimal for the consumer to reject the offer with probability 1. If he charges $p \leq 3 - 2\delta$, then every high value consumer accepts the offer with probability 1, expecting that the monopolist will charge 1 in the following round. Indeed, after serving all high valuation consumers, the monopolist still have $\beta(y - x)$ amount for sale in the next round. He charges 1 to serve some of the low valuation consumers.

Suppose that the monopolist charges $p \in (3 - 2\delta, 3)$. Recall that $\alpha(k^*) > 1$. Find a point along

$$y = \frac{\alpha(k^*)}{\beta} x$$

that intersects with the 45 degree line passing through the given state (x, y). Let $(x - x', y - y')$ be the intersection. Given p, x' portion of consumers accepts the offer, expecting that the monopolist will randomize between 3 and $3 - 2\delta$ in the following round. Indeed, in the following round, the state is $(x - x', \beta(y - y'))$ which is located on $\mathcal{U}(\alpha^*)$, where the monopolist is indifferent between charging $3 - \delta$ and 3.

This completes the construction of the equilibrium strategy. It remains to verify that this configuration constitutes a perfect equilibrium, except for the part where the monopolist cannot benefit from accelerating the sales. In particular, given the fact that the monopolist has to charge 3 which is not accepted by any buyer for a long time, it is not obvious whether or not a slight price cut can increase the profit of the monopolist.

To complete this part of the proof, let us fix state (x, y). If $y \leq \frac{x}{\beta}$, then the equilibrium offer 3 is accepted with probability 1. Thus, it is obvious that the monopolist has no incentive to lower his price.

Fix $y \in (\frac{x}{\beta^\ell}, \frac{x}{\beta^{\ell+1}}]$ for some $\ell > 1$ but $\ell \leq k^*$. Conditioned on $p < 3$, q portion of high valuation consumer will accept the offer where

$$q = \frac{x - \beta^k y}{1 - \beta^k}$$

so that $(x - q, \beta(y - q))$ is located along $\mathcal{U}(k^*)$ where the monopolist is indifferent between the two pricing rules: charge 3 or follow the path abiding the Coase conjecture (which in this case is $3 - 2\delta$). By charging $p < 3$ in this round, the monopolist can make at most

$$3\left(q + \delta^{k^*}(x - q)\right)$$

while by following the equilibrium strategy the monopolist can make at least

$$3\delta^\ell.$$

It suffices to show that

$$3\left(q + \delta^{k^*}(x - q)\right) \geq 3\delta^\ell.$$

After substituting q, one can show that this inequality is equivalent to

$$y \geq \frac{1 - \beta^{k^*}}{\beta^{k^*}(1 - \delta^{k^*})}\left(\frac{1 - \delta^{k^*}}{1 - \beta^{k^*}} - (\delta^\ell - \delta^{k^*})\right) x.$$

If $\ell \leq k^*$ and $y \geq x/\beta^\ell$, then

$$\frac{x}{\beta^\ell} \geq \frac{1 - \beta^{k^*}}{\beta^{k^*}(1 - \delta^{k^*})}\left(\frac{1 - \delta^{k^*}}{1 - \beta^{k^*}} - (\delta^\ell - \delta^{k^*})\right) x$$

Therefore, we conclude that $y \in (\frac{x}{\beta^\ell}, \frac{x}{\beta^{\ell+1}}]$, then the monopolist's profit from deviation cannot be larger than the equilibrium payoff.

Appendix B. Proof of Proposition 4.2

We have to calculate the optimal strategy of the monopolist for all feasible configurations of $\mathsf{D}(q_0, q_f)$ and y. However, we can exploit the linearity of the demand curve to simplify the characterization substantially.

Lemma B.1. *Suppose that $\mathbf{p} = \{p_t\}$ and $\mathbf{q} = \{q_t\}$ are the optimal pricing and the quantity sequences of the constrained optimization problem* (B.28) *associated with $\mathsf{D}(0, q_f)$ and y, and it takes T_f periods to clear the market. If the demand curve is given by $\mathsf{D}(1 - \alpha, 1 - \alpha + \alpha q_f)$ and the initial stock is αy, then $\alpha\mathbf{p}$ and $\alpha\mathbf{q}$ are the solution, and the trading is completed exactly in T_f periods. This relation holds $\forall b \geq 0$ (both for the perishable and for the classic problems).*

Proof. The proof follows from the fact that the objective function and the constrains are linear functions of $q_t - q_{t-1}$. ◻

Instead of all three parameters (q_0, q_f, y), we assume without loss of generality $q_0 = 0$, and consider an arbitrary pair (q_f, y) to characterize the optimal strategy of the monopolist.

Fix $q, q_0 \in [0, q_f]$ and define

$$y_f(q_0, q) = \sup\left\{\sum_{t=1}^{T}(q_t - q_{t-1})\beta^{-t} : \exists T \geq 1, \quad \exists q_0 \leq q_1 \leq \cdots \leq q_T = q, \quad \text{satisfying} \quad \text{(B.29)}\right\}. \tag{B.27}$$

$y_f(q_0, q)$ is the minimal stock needed to serve the residual demand $\mathsf{D}(q_0, q)$ if the monopolist begins to offer an acceptable offer immediately. That is, if the monopolist begins to make an acceptable offer to meet residual demand $\mathsf{D}(q_0, q_f)$, the available stock must be $y_f(q_0, q)$.

Lemma B.2. *$y_f(q_0, q)$ is a strictly decreasing continuous function of q_0 but a strictly increasing continuous function of q.*

Proof. By the construction of $y_f(q_0, q)$, it is obvious that $y_f(q_0, q)$ is a decreasing continuous function of q_0. The continuity follows from the fact that in each period, the objective function is strictly concave, which is implied by the linearity of the demand curve [9]. To show that $y_f(q_0, q)$ is a strictly increasing function of $q < 1$, we assume without loss of generality that $q_0 = 0$ to simplify notation.

For $\alpha < 1$ which is close to 1, consider $\mathsf{D}(0, q)$ and $\mathsf{D}(1-\alpha, (1-\alpha)+\alpha q)$. By Lemma B.1, we know that the two residual demand curves generate essentially identical optimal solution, except that the solution from the second residual demand curve is obtained by multiplying α to the optimal pricing and the optimal quantity solutions of the first residual demand curves. Let $T_f(0, q)$ and $T_f(1-\alpha, 1-\alpha+\alpha q)$ be the number of periods needed serve the demand curve. We know that

$$T_f(0, q) = T_f(1-\alpha, 1-\alpha+\alpha q).$$

Given a new demand curve $\mathsf{D}(0, (1-\alpha)+\alpha q)$, the monopolist has a feasible pricing sequence that serves $1-\alpha$ in the initial round, and then follow the optimal pricing sequence induced by $\mathsf{D}(1-\alpha, 1-\alpha+\alpha q)$. Thus,

$$y_f(0, (1-\alpha)+\alpha q) \geq (1-\alpha)\beta^{-T_f(1-\alpha, 1-\alpha+\alpha q)-1} + \beta^{-1} \sum_{t=1}^{T_f(1-\alpha,(1-\alpha)+\alpha q)} (q_t - q_{t-1})\beta^{-t}$$

$$= \frac{(1-\alpha) + y_f(1-\alpha, (1-\alpha)+\alpha q)}{\beta} = \frac{(1-\alpha)+\alpha y_f(0, q)}{\beta}.$$

We can find $\alpha' < 1$ such that $\forall \alpha \in (\alpha', 1)$,

$$1-\alpha > (\beta-\alpha) y_f(0, q)$$

which implies that

$$\frac{(1-\alpha)+\alpha y_f(0, q)}{\beta} > y_f(0, q).$$

Then, $\forall \alpha \in (\alpha', 1)$,

$$y_f(0, 1-\alpha+\alpha q) > y_f(0, q).$$

Since the strict inequality holds for a small neighborhood of any $q < 1$, we conclude that $y_f(0, q)$ is a strictly increasing function of $q < 1$. Continuity follows from the maximum principle combined with the fact that the objective function is strictly concave, which is again implied by the linearity of the demand curve. □

Clearly,

$$q \leq y_f(0, q).$$

If $y_f(0, q) \geq y$, then the existing stock is too small to serve $\mathsf{D}(0, q)$. Since $y_f(q_0, q)$ is strictly decreasing in q_0, we can find $q_0 \geq 0$ such that

$$y_f(q_0, q) = y.$$

The constrained optimal pricing rule is thus an acceleration strategy defined as follows.

Definition B.3. *An acceleration strategy is an outcome path in which the monopolist serves q_0 immediately, and then follows the optimal pricing sequence associated with $\mathsf{D}(q_0, q)$. The initial offer p' is determined according to*

$$1 - q_0 - p' = \delta((1-q_0) - p_1)$$

where p_1 is the initial offer from the optimal pricing sequence associated with $\mathsf{D}(q_0, q)$.

If $y_f(0, q) \leq y$, then the existing stock is too large to credibly serve q, because the terminal condition (B.30) does not hold for y. The monopolist follows another outcome path, a delay strategy, defined as follows.

Definition B.4. *A delay strategy is an outcome path in which the monopolist makes unacceptable offers for T_1 periods, where*

$$T_1(0, q, y) = \inf\left\{T : e^{-b\Delta T} y \leq y_f(0, q)\right\}.$$

Then, the monopolist follows the acceleration strategy.

Consider the following optimization problem:

$$\max_{T_1 \geq 0, \mathbf{q} \in \mathbf{Q}} \quad \delta^{T_1} \sum_{t=1}^{T_f} p_t(q_t - q_{t-1})\delta^{t-1} \tag{B.28}$$

$$(1 - q_t) - p_t = \delta((1 - q_t) - p_{t+1}) \tag{B.29}$$

$$p_{T_f} = 1 - q_{T_f} \tag{B.30}$$

$$\beta^{T_f}\left(\beta^{T_1} y - \sum_{t=1}^{T_f} \beta^{-t}(q_t - q_{t-1}\right) \geq 0 \tag{B.31}$$

$$\beta^{T_f}\left(\beta^{T_1} y - \sum_{t=1}^{T_f} \beta^{-t}(q_t - q_{t-1})\right)(q_{T_f} - q_f) = 0. \tag{B.32}$$

A natural state variable is the residual demand $\mathsf{D}(q_0, q_f)$ and the available stock at the time when the monopolist makes the decision. By state, we mean a triple (q_0, q_f, y) representing residual demand and the available stock.

Let $\overline{q}^*(0, q_f, y)$ be the total amount of goods served in an optimal solution of (B.28) where the state is $(0, q_f, y)$. If $\overline{q}^*(0, q_f, y) = q_f$, then the associated optimal pricing sequence is precisely the optimal pricing sequence from the classic problem, because (B.31) constraint is not binding. Otherwise, (B.31) constraint is binding, and inevitably, some consumers are not served as the available goods are burned off, and therefore, the optimal solution should be a delay strategy.

Based on the analysis of the optimal strategy under state $(0, q_f, y)$, we have a "rough" characterization of optimal strategy for an arbitrary state (q_0, q_f, y) and $\overline{q}^*(q_0, q_f, y)$ which is the counter part of $\overline{q}^*(0, q_f, y)$ for state (q_0, q_f, y):

- if $\overline{q}^*(q_0, q_f, y) \geq \min(q_f, y)$, the monopolist follows the acceleration strategy, and
- if $\overline{q}^*(q_0, q_f, y) < \min(q_f, y)$, then the monopolist delays $T_1(q_0, \overline{q}^*(q_0, q_f, y), y)$ periods before making the acceptable offers. After making $T_1(q_0, \overline{q}^*(q_0, q_f, y), y)$ unacceptable offers, the monopolist follows the acceleration strategy associated with state $(q_0, q_f, e^{-b\Delta T_1(q_0, \overline{q}^*(q_0, q_f, y), y)} y)$.

It is only a rough characterization, because we have yet to identify how many consumers will accept an offer p_1' which is not an equilibrium offer. We shall focus the analysis on the deviation from the first offer in the equilibrium, because the general case follows from the same logic.

We need to consider two separate cases depending upon whether the initial offer is acceptable (i.e., the monopolist follows an acceleration strategy), or the initial offer is unacceptable (i.e., the monopolist follows a delay strategy).

B.1. p_1 **is an acceptable offer.** Fix $p_1' \neq p_1$. We only examine the case where $p_1' < p_1$, because the other case follows from the symmetric logic. If the acceptable strategy does not bind (B.31), then the complementary slackness condition implies that

$$q_f = q^*(0, q_f, y).$$

Since we are considering an acceleration strategy, it is precisely the total number of periods when the market is open. In this case, the equilibrium strategy off the equilibrium path is identical with that in the classic problem. Because the unique subgame perfect equilibrium in the classic problem is a reservation price equilibrium, the acceleration strategy can be sustained by a reservation price equilibrium. By the nature of the reservation price equilibrium, a lower than an equilibrium offer increases the sales in that period. As a result, (B.31) condition is not binding in any continuation game. That is why we can use the same reservation price equilibrium strategy of the classic problem, as if the good does not perish.

On the other hand, if the acceptable strategy binds (B.31) so that

$$q_f > q^*(0, q_f, y),$$

then the equilibrium strategy off the equilibrium path differs from that from the classic problem. Yet, we can show that an offer lower than an equilibrium price always increases the sales in that period.

Because (B.31) holds with an equality, the market must be cleared in the sense that the monopolist sells all available stocks, even though some consumers are not served. Since p_1 is an acceptable offer, all ensuing offers from the monopolist are also acceptable for some consumers. Thus, after q_1 consumers are served,

$$y_f(q_1, \overline{q}^*(q_1, q_f, y)) = \beta(y - q_1)$$

must hold, where the left hand side is the amount of goods needed to serve the remaining consumers in the continuation game following p_1, while the right hand side is the good available at the beginning of the second round. We can re-write the same equality as

$$q_1 + \frac{1}{\beta} y_f(q_1, \overline{q}^*(q_1, q_f, y)) = y. \tag{B.33}$$

Recall the definition of $y_f(q', q'')$. It is clear that

$$\frac{\partial y_f(q', q'')}{\partial q''} \geq 1 \quad \text{and} \quad \frac{\partial y_f(q', q'')}{\partial q'} \leq -1.$$

Fix $p_1' < p_1$, and let q_1' and q_1 be the mass of consumers who accept p_1' and p_1, respectively. We claim that

$$q_1' \geq q_1.$$

To prove our claim by way of contradiction, suppose that

$$q_1' < q_1.$$

Even though p_1 is an acceptable offer, we have yet to prove that p_1' is also an acceptable offer.

Lemma B.5. *If p_1 is an acceptable offer, then $\forall p_1' < p_1$ is an acceptable offer.*

Proof. Suppose that p_1' is not an acceptable offer ($q_1' = 0$). By the definition of a delay strategy, p_1' is a part of a delay strategy. By the definition of a delay strategy, the continuation strategy involves T_1' periods of delay, followed by a sequence of acceptable offers. The sequence of acceptable offers is identical to the optimal pricing sequence associated some residual demand $\mathsf{D}(0, q')$ where $q' > \overline{q}^*(0, q_f, y)$. Thus,

$$\frac{1}{\beta^{T_1}} y_f(0, q') \leq y$$

which is impossible, because

$$\frac{1}{\beta^{T_1}} y_f(0, q') \geq \frac{1}{\beta} y_f(0, q') > q_1 + \frac{1}{\beta} y_f(q_1, \overline{q}^*(0, q_f, y)) = y.$$

□

Now, we know $p_1' < p_1$ is an acceptable offer. By the construction, every offer following p_1' is an acceptable offer. Thus, all offers following p_1' is identical to an optimal solution from the classic problem associated with demand $\mathsf{D}(q_1', q'')$ for some $q'' \leq 1$. By Lemma 2.3, we know that no two reservation price functions associated with two demand curves with different lowest reservation value consumers intersect with each other. In particular, if $p_1' < p_1$ and $q_1' < q_1$, then the reservation price function associated with (p_1', q_1') is located "below" the reservation price function associated with (p_1, q_1). Since the demand curve is downward sloping,

$$q'' > \overline{q}^*(0, q_f, y).$$

Since (B.31) holds following p_1', we have

$$q_1' + \frac{1}{\beta} y_f(q_1', q'') \leq y.$$

From (B.33),

$$q_1' + \frac{1}{\beta} y_f(q_1', q'') \leq y = q_1 + \frac{1}{\beta} y_f(q_1, \overline{q}^*(0, q_f, y))$$

which is impossible, because $q_1' < q_1$ and $q'' > \overline{q}^*(0, q_f, y)$ imply that

$$q_1' + \frac{1}{\beta} y_f(q_1', q'') > q_1 + \frac{1}{\beta} y_f(q_1, \overline{q}^*(0, q_f, y)).$$

B.2. p_1 **is not an acceptable offer.** The construction follows almost the same idea. While there are many unacceptable offers, let us streamline the construction by focusing on a series of "lowest" unacceptable offers. Suppose that the monopolist makes T_1 unacceptable offers, before making the first acceptable offer p_{T_1+1}. Define p_1 $(t \leq T_1)$ implicitly as

$$1 - q_0 - p_t = \delta^{T_1-t+1}(1 - q_0 - p_{T_1+1})$$

or equivalently as

$$p_t = (1 - \delta^{T_1-t+1})(1 - q_0) + \delta^{T_1-t+1} p_{T_1+1}.$$

In particular, if $q_0 = 0$,

$$p_1 = (1 - \delta^{T_1})(1 - q_0) + \delta^{T_1} p_{T_1+1}.$$

Fix $p_1' \neq p_1$. As in the previous case, let us focus on the case where $p_1' < p_1$. We need to find an optimal strategy with an additional constraint that the initial offer is p_1'. By Lemma 2.3, we know that the initial acceptable offer is a continuous function of the terminal offer. Since the initial unacceptable offer is a continuous function of the first acceptable offer, it is also a continuous function of the terminal offer. Recall that $\overline{q}^*(0, q_f, y)$ is the equilibrium quantity delivered to the consumers. For each $\overline{q} \geq \overline{q}^*(0, q_f, y)$, Constructing an optimal pricing rule with the terminal condition that $p_{T_f} = 1 - \overline{q}$. Let $p_1^*(\overline{q})$ be the first offer (which may be unacceptable) in the optimal pricing rule that terminates with $p_{T_f} = 1 - \overline{q}$.

Since $p_1^*(\overline{q})$ is a continuous function of $\overline{q}$, $\forall p_1' < p_1$, there exists $\overline{q} > \overline{q}^*(0, q_f, y)$ such that $p_1' = p_1^*(\overline{q})$. If $p_1' = 1 - q_f$, then the consumer must accept the offer, because the monopolist will never charge a price lower than $1 - q_f$. Thus, there exists $\overline{q}'$ such that $p_1' = p^*(\overline{q}') \geq 1 - q_f$ is an acceptable offer. By the construction of the strategy off the equilibrium path from the acceptable offer, we know that the mass of consumers who accepts $p_1'' \leq p_1'$ does not decrease.

Appendix C. Proof of Proposition 5.4

Choose the parameters according to Lemma 5.2 so that $q^* \in [0, q_f]$ exists. Fix the outcome path associated the optimal value $\overline{q}_1^e$. We need to construct the strategy off the equilibrium path. We focus on the initial offer, because we already know that the second phase can be sustained by a reservation price equilibrium, if $\overline{q}_1^e \leq q^*$. If we choose the parameters according to Lemma 5.2, then $\overline{q}_1^e \leq q^*$ for any sufficiently small $\Delta > 0$.

Fix $p_1' \neq p_{1,1}$. We focus on the case $p_1' < p_{1,1}$, because the other case follows from the symmetric reasoning. Since $p_{1,1}$ must satisfy (5.23), $\forall p_1'$ there exists $\overline{q}_1'$ and a delay strategy in $\mathsf{D}(\overline{q}_1', q_f)$ with the available stock $\beta(y - \overline{q}_1')$ such that

$$1 - \overline{q}_1' - p_1' = \delta^{T_0'}(1 - \overline{q}_1' - p_{2,1}')$$

where T_0' and $p_{2,1}'$ are the number of unacceptable offers and the initial offer in the delay strategy associated with state $(\overline{q}_1', q_f, \beta(y-\overline{q}_1'))$. Since the initial offer of the delay strategy associated with state $(\overline{q}_1', q_f, \beta(y-\overline{q}_1'))$ is continuous function of $\overline{q}_1'$, we can choose

$$\overline{q}_1(p_1') = \sup\{q_1' : \text{there is a delay strategy associated with } (\overline{q}_1', q_f, \beta(y - q_1')) \text{ satisfying (5.23)}\}. \tag{C.34}$$

Remark C.1. *In the limit as $\Delta \to 0$ and $b \to 0$, the initial offer to serve $\overline{q}_1$ becomes $1 - \overline{q}_1$. Thus, for each $p_1' < p_{1,1}$, we can associate $\overline{q}_1' > \overline{q}_1^e$, which implies that a lower price is always accepted by more consumers. However, for a positive $\Delta > 0$, this sort of monotonicity may fail. Note that the amount of delay in the continuation game is determined by the time needed to achieve the desired target level of the stock. We know that as more consumers are served in the first phase, it will take more time to achieve the desired target level. Let $\overline{q}_1, p_{1,1}, p_{2,1}, T_0$ be the size of consumers served by $p_{1,1}$, the initial offer, the first acceptable offer of the second phase and the number of unacceptable offers. Recall that $\forall \Delta > 0$, $\forall b > 0$,*

$$1 - \overline{q}_1 - p_{1,1} = \delta^{T_0}(1 - \overline{q}_1 - p_{2,1}) > 0.$$

Since T_0 is increasing as $\overline{q}_1$ increases, it is possible that $1 - \overline{q}_1 - p_{1,1}$ is decreasing as $\overline{q}_1$ is increasing and $p_{1,1}$ is decreasing. Because of this possible failure of monotonicity, (C.34) *may not generate a decreasing function for $\Delta > 0$ and $b > 0$, although it does in the limit. To address this issue, we need to do some additional work.*

Note that $\overline{q}_1^e$ has to converge to 0.5, which generates the maximum profit in the limit. We can choose the parameters according to Lemma 5.2 so that $\overline{q}_1^e < q^*$, and the initial offer associated with the original demand $\mathsf{D}(0, q_f)$ is within ϵ neighborhood of $1 - \overline{q}_1^e$, while the initial offer associated with the residual demand $\mathsf{D}(q^*, q_f)$ is also within ϵ neighborhood of $1 - q^*$.

Thus, the initial offer $p_{1,1}$ changes from the neighborhood of $1 - \overline{q}_1^e$ to the neighborhood of $1 - q^*$, which is smaller than $1 - \overline{q}_1^e$. We know the mapping $\overline{q}_1 \mapsto p_{1,1}$ may not be strictly decreasing over $[\overline{q}_1^e, q^*]$ but the value around $\overline{q}_1^e$ is strictly larger than the value around q^*, if Δ is sufficiently small. Therefore, (C.34) is a strictly decreasing function.

Define

$$p_1^* = \sup\{p_{1,1} \geq 1 - q_f : \overline{q}_1(p_1') \geq q^*\}.$$

The right hand side is not empty, because if the monopolist offers $1 - q_f$, all consumers must accept immediately. In fact, if $q_1^* = 1 - q_f$, then the proof is completed, as we have already shown that whenever the monopolist deviates to a lower price, more consumers accept the offer.

Suppose that $p_1^* > 1 - q_f$. Then, $\forall p_1' < p_1^*$, the continuation game cannot sustain the delay strategy as a Nash equilibrium outcome. As a result, we need a randomization by the monopolist to smooth out the transition from the delay strategy to the acceleration strategy.

Consider a continuation game after q^* consumers are served. The residual demand is $\mathsf{D}(q^*, q_f)$ and the available stock is $\beta(y - q^*)$. By the definition of q^*, both the acceleration and the delay strategies are optimal. Let $p_{2,1}^a$ be the initial offer of the acceleration strategy, and $p_{2,1}^d$ be the initial offer of the delay strategy, which is offered after T_0 periods. Since $p_1' < p_1^*$,

$$1 - q^* - p_1' > \delta^{T_0}(1 - q^* - p_{2,1}^d).$$

If $p_1' > p_{2,1}^a$, then

$$1 - q^* - p_{2,1}^a > 1 - q^* - p_1'.$$

Choose $\alpha \in (0, 1)$ so that

$$1 - q^* - p_1' = \alpha\delta^{T_0}(1 - q^* - p_{2,1}^d) + (1 - \alpha)(1 - q^* - p_{2,1}^a).$$

That is, the consumers expect that the monopolist randomize over two strategies so that q^* consumer is indifferent between accepting and rejecting p_1'.

If $p_1' \leq p_{2,1}^a$, the consumers expect that the monopolist follow an acceleration strategy. The continuation strategy is identical to the continuation game following a deviation from an acceptable offer (which in this case is $p_{2,1}^a$.

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