Contributing or Free-Riding? A Theory of Endogenous Lobby Formation*

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Abstract

We consider a two-stage public good provision game: In the first stage, players simultaneously decide if they join a contribution group or not. In the second stage, players in the contribution group simultaneously offer contribution schemes in order to influence a third party agent's policy choice (say, the government chooses a level of public good provision). We use a communication-based self-enforcing equilibrium concept in a noncooperative two stage game, *perfectly coalition-proof Nash equilibrium* (Bernheim, Peleg and Whinston, 1987 JET). We show that, in public good economy, the outcome set of this equilibrium concept is equivalent to an "intuitive" hybrid solution concept *free-riding-proof core*, which always exists but does not necessarily achieve global efficiency. It is not necessarily true that the formed lobby group is the highest willingness-to-pay players, nor is a consecutive group with respect to their willingnesses-to-pay. We also check how the set of equilibrium outcomes would exhibit completely different characteristics when "public good" assumption is dropped.

Preliminary

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1 Introduction

In their seminar paper, Grossman and Helpman (1994) consider an endogenous trade policy formation problem when industries can influence the government's trade policy through lobbying activities by applying common agency game defined by Bernheim and Whinston (1986). A common agency game is a menu auction game in which there are multiple players/principals and an agent who can choose an action that affects all players' payoffs. Each player offers a contribution scheme to the agent promising how much money she will pay for each action. Observing contribution schemes, the agent chooses her action in order to maximize the total benefit she can obtain. Bernheim and Whinston (1986) analyze a communication-based equilibrium concept, coalition-proof Nash equilibrium (CPNE), in order to analyze common agency games. In Grossman and Helpman (1994), players/principals are lobbies who represents industries, and an agent is the government. The government cares about social welfare, while it also cares about flexible contribution money provided by lobby groups. Each lobby contributes money to the government in order to influence the government's trade policy for its favor. Each lobby represents one industry, and it prefers a high price for a commodity that is produced by the industry, while prefers low prices for all other commodities.[1] That is, in Grossman and Helpman (1994), there are conflicts of interests among lobbies. One of their main results is that in equilibrium lobby powers are cancelled out and that the government chooses a free trade (no tariff) policy, it can collect a big amount of contributions from conflicting industries.

Although the free trade outcome is an interesting result, it is based on special assumptions.[2] They assume that industry lobbies are preorganized, and that each lobby act as a single player. This implies that if there are multiple firms in an industry, each industry lobby has power to allocate contribution shares efficiently and forcefully among the member firms. However, in the real world, it is not necessarily the case that all firms in the same industry participate in a lobbying group. Since trade policies affect all firms

[1] This is because lobbies representing industries are ultimately consumers.

[2] Actually, their clean result is crucially based on their assumption that all lobbies are ultimately consumers who have identical utility functions. Bernheim and Whinston (1986) show that in equilibrium the agent chooses an action that maximizes the total surplus of the game. The result by Grossman and Helpman (1994) is a direct corollary of this under the representative consumer assumption.

in an industry in the same way, there are free-riding motives for firms.

Motivated by this, we consider a common agency game with players' endogenous participation decisions. In the first stage, players choose if they participate in lobbying or not.[3] In the second stage, among players who chose to participate in lobbying activities, a common agency game is played: contribution schemes are offered simultaneously, and the agent chooses an action. We use a dynamic extension of coalition-proof Nash equilibrium (CPNE), perfectly coalition-proof Nash equilibrium (PCPNE), as the solution concept. This equilibrium concept has a solid theoretical ground in a certain sense,[4] but characteristics of equilibria are not immediately clear. We use "guess and verify" method in order to characterize PCPNE: we define intuitive hybrid solution concepts for special classes of common agency games, and verify them with the PCPNE.[5]

We ask who participate in lobbying (and who free-ride others) in two different environments.[6] One is an environment without conflict of interests, in which all players have comonotonic preferences.[7] This environment mimics an import competing industry case in which many firms decide lobbying or free-riding (Bombardini, 2005, and Paltseva, 2006), and pure public good provision problem. Although Bombardini (2005) provides some empirical evidence of free-rider firms in industries. Assuming symmetric firms and focusing on symmetric outcome among lobby participants in a common agency game, Paltseva (2006) consider Nash equilibrium to analyze free-riding incentives. The other is an environment in which there are two groups with

[3]This is called open membership game (Yi, 1996).

[4]If only nested coalitional deviations are allowed, CPNE (and credible core in Ray, 1989) is a consistent solution concept in the sense that the original strategy profile and strategy profiles that are generated from deviations are treated in the same manner. However, it does not mean that CPNE is the only satisfactory solution concept. There can be many formulation of describing coalition formation process in noncooperative games. For example, Konishi and Ray (2003) and Gomes and Jehiel (2005) allow non-nested future deviations in defining consistent solutions.

[5]In various games, it is sometimes possible to show the equivalence between CPNE and intuitive solution concepts. See Bernheim and Whinston (1986), Thoron (1999), Conley and Konishi (2002) and Konishi and Ünver (2006).

[6]It is hard to characterize payoff structure of CPNEs in common agency games under general setup (see Laussel and Le Breton, 2001). Characterization of PCPNE is even harder. In fact, PCPNE may not exist in common agency games with endogenous participation decision.

[7]Preferences are **comonotonic** if for all pair of players i and j, and all pair of actions a and a', if i prefers a to a', then j also prefers a to a'.

pure conflicts of interests. This environment mimics the situation in which there are firms in import competing industries and exporting firms, and the government is deciding if it signs a free trade agreement with a foreign country.[8]

In the former environment, there is no rent for the government in PCPNE, while free-riding incentive is strong (equilibrium lobby participation is small). The equilibrium outcome is highly nonconvex, and the equilibrium lobby group may not be consecutive: i.e., weak firms may join the lobby together with strong firms, yet some medium firms may not. In contrast, if conflicts are present (export lobby and import lobby), the government gets a big rent, while lobby participation is strong.

This paper is organized as follows. In the next subsection, some related literature is discussed briefly. In Section 2, the common agency game is reviewed, then our game and the equilibrium concept, PCPNE, is introduced. In Section 3, we consider the environment without conflict of interests. We define an intuitive hybrid solution concept, free-riding-proof core, and prove the equivalence between PCPNE and the free-riding-proof core (Theorem 1). In Section 4, we consider the environment with pure conflicts of interests. We define the maximal pivotal lobby allocations, and prove the equivalence between PCPNE and the maximal pivotal lobby allocations. Section 5 concludes.

1.1 Related Literature

Le Breton and Salaniè (2003) analyze a common agency problem with asymmetric information on agent's preferences. They show that equilibria can be inefficient even in the case that there is only one player in each interest group.[9] If there are multiple players in each interest group, then the failure in internalizing the benefits of contributions within the group makes contributions even less. In this sense, Le Breton and Salaniè (2003) generate free-riding incentives under compulsory lobby participation. In contrast, we generate "free-riding" is an obvious way by introducing participation decisions.

[8] A free trade agreement abandons trade barriers of the two countries. Exporting firms prefer a free trade agreement, while import competing firms prefer a protection policy.

[9] Laussel and Le Breton (1998) analyze public good case when the agent must sign a contract of participation when all contribution schemes are proposed before knowing her cost type (then Nature plays and the agent chooses an agenda). They show that all equilibria are efficient, and there is no free-riding incentive.

The environment without conflict of interests can be regarded as a public good provision problem. Groves and Ledyard (1977), Hurwicz (1979), and Walker (1981) showed that efficient public good provision can be achieved despite of Samuelson's pessimism (1954). However, they all assume that players must participate in the game. Saijo and Yamato (1999) considered voluntary participation game of public good provision by constructing a two stage game (participation, and public good provision). Negative results for efficiency due to free-riding incentives. Shinohara (2003) considers coalition-proof Nash equilibrium in the voluntary participation game by Saijo and Yamato (1999) with the Lindahl mechanism in the second stage. He shows that there can be multiple coalition-proof Nash equilibria with different sets of players participating in the mechanism in heterogenous player case. One of our results exhibits the same result but with a common agency game in the second stage (thus, payoffs are not fixed unlike in Shinohara, 2003). Such a voluntary public good provision problem can idealize the case of no conflict of interests. Maruta and Okada (2005) analyze a similar sort of heterogeneous agent binary public good provision game with evolutionary stability (see also Palfrey and Rosenthal, 1984).[10]

2 A Noncooperative Game

We will consider a two stage game: in the first stage, each player decides if she join the lobby (contribution) group, or she stay outside (free-riding).[11]

[10] In contrast, Nishimura and Shinohara (2007) consider a multi-stage voluntary participation game in a discrete multi-unit public good problem, and show that Pareto-efficient allocations in subgame perfect Nash equilibrium through a mechanism that determines public good provision unit-by-unit. Their efficiency result crucially depends on the following assumption: a player who did not participate in the mechanism in early stages can participate in public good provision later on. This forgiving attitude allows a player not contributing at all until the time that all other players are no longer interested in contributing without her participation, and then contribute money just to bring one more unit of public good. Thus, we may say that their mechanism achieves Pareto efficiency by accommodating players' free-riding incentives.

[11] Note that in our game, there can be only one lobby that contributes to provision of public good. This does not seem a bad assumption given the nature of common agency game played in the second period. In contrast, Ray and Vohra (2001) assume that the second stage is a voluntary contribution game, thus it makes sense to assume that many groups can be formed in the first stage and they all provide public good simultaneously (see Ray and Vohra, 2001).

In the second stage, among the lobby group members, a common agency game by Bernheim and Whinston (1986). If a player choose to free-ride in the first stage, she cannot participate in the contribution game. Without free-riding incentive ($S = N$), Laussel and Le Breton (1998, 2001) extensively studied the equilibrium payoff structures of common agency games on general versions of this public good problem, and obtained many interesting and useful results. Our analysis will be built on theirs, but we consider possible free-riding: our focus is the conflict between contributing and free-riding.

We will focus on players' lobbying activities over government policies. We will consider a two stage game. In stage 1, players decide if they join a lobbying process or not (lobby formation stage).[12] In stage 2, the lobbying group lobby over government policies. In the next section, we analyze the second stage game.

2.1 Common Agency Game (the Second Stage)

There is a set of players, $N = \{1, ..., n\}$ and the government G. Suppose that $S \subseteq N$ is the contribution group, and $N \backslash S$ are passive free-riders. The government G can choose an agenda a from the set of agendas A. Each player i has utility function $v_i : A \to \mathbb{R}_+$, and similarly the government has utility function $v_G : A \to \mathbb{R}_+$. In public good provision problem, $v_G(a) = -C(a)$. Each player i offers a contribution scheme $\tau_i : A \to \mathbb{R}_+$. If the government chooses $a \in A$, then the government gets the payoff

$$u_G(a; (\tau_i(a))_{i \in S}) = \sum_{i \in S} \tau_i(a) + v_G(a),$$

and player i gets payoff

$$u_i(a; \tau_i(a)) = v_i(a) - \tau_i(a).$$

The government chooses a policy $a \in A$ that maximizes u_G:

$$a^*(S, \tau_S) \in \arg\max_{a \in A} u_G(a; (\tau_i(a))_{i \in S}).$$

[12]This is called open membership game (no exclusion is possible). See d'Aspremont et al. (1983), Yi (1996) and Thoron (1998). For excludable coalitions, see Hart and Kurz (1983), Yi (1996), and Ray and Vohra (2001). Bloch (1997) has a nice survey on the rules of coalition formation games.

In the game, the government is not a player: it is just a machine that maximizes its payoff given the contribution schemes.[13] A second-stage **common agency game** Γ is a list $\Gamma = (S, (\mathcal{T}_i, u_i)_{i\in S})$, where $\mathcal{T}_i$ is collection of all contribution schemes for i. Note that $N\backslash S$ are simply free-riders, and they do not affect game Γ. Thus, $N\backslash S$ can be regarded as irrelevant players in game Γ.

First consider joint payoff that can be achieved by each subgroup $T \subseteq S$. For each $T \subseteq S$, let

$$W_\Gamma(T) \equiv \max_{a\in A}\left[\sum_{i\in T} v_i(a) + v_G(a)\right],$$

and

$$W_\Gamma(\emptyset) \equiv \max_{a\in A} v_G(a),$$

The efficient public good provision for S (and G) is

$$a^*(S) \in \arg\max_{a\in A}\left(\sum_{i\in N} v_i(a) + v_G(a)\right).$$

Let

$$Z_\Gamma \equiv \left\{ \begin{array}{c} u \in \mathbb{R}^N_+ : \sum_{i\in T} u_i \leq W_\Gamma(S) - W_\Gamma(S\backslash T) \text{ for all } T \subset S \\ \text{and } u_j = v_j(a^*(S)) \text{ for all } j \notin S \end{array} \right\}.$$

The inequality that Z_Γ satisfies can be interpreted as what T can get since the complement set $S\backslash T$ can achieve total payoff $W_\Gamma(S\backslash T)$ by themselves (T cannot ask more than $W_\Gamma(S) - W_\Gamma(S\backslash T)$). We take the Pareto-frontier of Z_Γ:[14]

$$\bar{Z}_\Gamma \equiv \{u \in Z_\Gamma : \nexists u' \in Z_\Gamma \text{ such that } u' > u\}.$$

Now, let us state the results in the literature. Bernheim and Whinston (1986) introduced a concept of truthful strategies, where τ_i is **truthful relative to** $\bar{a}$ if and only if for all $a \in A$ either $v_i(a) - \tau_i(a) = v_i(\bar{a}) - \tau_i(\bar{a})$, or

[13] Strictly speaking, since the government may have multiple optimal policy, we need to introduce a tie-breaking rule. However, it is easy to check the set of truthful equilibria (see below) would not depend on the choice of tie-breaking rules.

[14] We follow the standard notational convention: $u' > u$ means (i) for all $i \in N$, $u'_i \geq u_i$, and (ii) $u' \neq u$.

$v_i(a) - \tau_i(a) < v_i(\bar{a}) - \tau_i(\bar{a})$ and $\tau_i(a) = 0$. A **truthful Nash equilibrium** (τ_S^*, a^*) is a Nash equilibrium such that τ_i^* is truthful relative to $a^* \in A$ for all $i \in S$. The first result by Bernheim and Whinston (1986) is the following:

Fact 1. (Bernheim and Whinston, 1986) Consider a common agency game Γ. In all truthful Nash equilibria G chooses an efficient action $a^*(S)$, and the vector of players' payoffs belongs to the Pareto frontier $\bar{Z}_\Gamma$. Moreover, every vector $u \in \bar{Z}_\Gamma$ can be supported by a truthful Nash equilibrium.

Bernheim and Whinston (1986) defined (strictly) coalition-proof Nash equilibrium. First define a reduced game. A **reduced game** of Γ is $\Gamma(T, \tau_{-T})$ that is a game with players in T by letting players in $S \backslash T$ passive players in Γ, who always play τ_{-T}. A **(strictly) coalition-proof Nash equilibrium (CPNE)** of common agency game Γ is defined as follows (Bernheim and Whinston, 1986; Bernheim, Peleg and Whinston, 1987):[15]

1. In a single player game Γ, (τ_1^*, a^*) is a CPNE of reduced game $\Gamma(\{i\}, \mathcal{T}_i, \bar{\tau}_{-i})$ if and only if it is a Nash equilibrium.
2. Let n be the number of players of the game. In a game $\Gamma(S, \mathcal{T}_S, \bar{\tau}_{-S})$ where $|S| = n$, $(\tau_S^*, a^*) = ((\tau_i^*)_{i \in S}, a^*)$ is a **(strictly) self-enforcing** strategy profile if for all $T \subsetneqq S$, $(\tau_i^*)_{i \in T}$ is a CPNE of the reduced game $\Gamma(T, \mathcal{T}_T, \tau_{S \backslash T}^*, \bar{\tau}_{-S})$.
3. Let n be the number of players of the game. In a game $\Gamma(S, \mathcal{T}_S, \bar{\tau}_{-S})$ where $|S| = n$, $(\tau_S^*, a^*) = ((\tau_i^*)_{i \in S}, a^*)$ is a **CPNE** if it is self-enforcing and there is no other self-enforcing strategy profile τ_S' that yields at least as high a payoff to each player and a strictly higher payoff to at least one player in S.

The second result in Bernheim and Whinston (1986) is as follows:

Fact 2. (Bernheim and Whinston, 1986) Consider a common agency game Γ. In all CPNEs G chooses an efficient action $a^*(S)$, and the vector of players' payoffs belongs to the Pareto frontier $\bar{Z}_\Gamma$. Moreover, every truthful Nash

[15] The definitions of CPNE in Bernheim and Whinston (1986) and Bernheim, Peleg and Whinston (1987) are different in defining coalitional deviations. The former uses weakly improving deviations, while the latter uses strictly improving deviations. On this issue, see Konishi, Le Breton and Weber (1999).

equilibrium is coalition-proof, thus every vector $u \in \bar{Z}_\Gamma$ can be supported by a CPNE.

That is, there is essentially a one-to-one relationship between truthful Nash equilibria and CPNEs. Note that a truthful Nash equilibrium is a CPNE (this property will be used in the proof of Theorem 1). One of many results in Laussel and Le Breton (2001) provided a useful property, convex-game property, which applies to an interesting class of games. Consider a characteristic function $(W_\Gamma(T))_{T\subseteq S}$ generated from a common agency game Γ. We say that Γ has **convex-game property** if for all $T \subset T' \subset S$ with $i \in S\backslash T'$, $W_\Gamma(T\cup\{i\}) - W_\Gamma(T) \leq W_\Gamma(T'\cup\{i\}) - W_\Gamma(T')$ holds. Laussel and Le Breton (2001) shows the following:

Fact 3. (Laussel and Le Breton, 2001) Consider a common agency game Γ with convex-game property. Then, in all CPNEs G obtains $u_G = W_\Gamma(\emptyset)$ (no rent property), and the set of CPNE payoff vectors is equivalent to the core of characteristic function game $(W_\Gamma(T))_{T\subseteq S}$.[16]

This fact will be useful in analyzing public good case below, since public good economy satisfies the convex game property.[17]

2.2 Lobby Formation Game

In this section, we analyze an equilibrium lobby group and its allocation. Note that we are not only talking about coalition-proof Nash equilibrium allocation in the menu auction stage. We also require that the lobby group formation itself is coalition-proof as well. In order to do so, we first need to define the first stage lobby formation game in an appropriate manner, assuming that the outcome of each possible lobby S is a coalition-proof Nash equilibrium of a common agency game played by S. As an extension of CPNE in strategic form games to extensive form games, Bernheim, Peleg and

[16]Here, we normalize $W_\Gamma(\emptyset) = 0$ in order to make $(W_\Gamma(T))_{T\subseteq S}$ a characteristic function game. A payoff vector $u_S = (u_i)_{i\in S}$ is in *the core* iff $\sum_{i\in S} u_i = W_\Gamma(S)$, and $\sum_{i\in T} u_i \geq W_\Gamma(T)$ for all $T \subset S$.

[17]Actually, with the no-rent property (in all CPNE, $u_G = W_\Gamma(\emptyset)$ holds), the set of CPNEs is eqivalent to the set of strong equilibria (see Aumann, 1959, for the definition) in common agency games (see Konishi, Le Breton and Weber, 1999). Thus, with the convex-game property, the set of CPNE, the set of strong equilibria, and the core of $(W_\Gamma(T))_{T\subseteq S}$ are all equivalent. However, with participation stage, strong equilibrium tends to be empty.

Whinston (1987) provide a definition of coalition-proof Nash equilibrium for multi-stage games, *perfectly coalition-proof Nash equilibrium (PCPNE)*. The first stage **lobby formation game** is such that N is the set of players, and player i's action set is a list $A_i = \{0, 1\}$: i.e., player i announces if she wants to participate in the lobby. Once action profile $\sigma^1 = (\sigma_1^1, ..., \sigma_n^1) \in A = \Pi_{j \in N} A_j$ is determined, then in the second stage, lobbying game takes place with the set of active players $S(\sigma^1) = \{i \in N : \sigma_i^1 = 1\}$.

Next we extend the definition of CPNE for multi-stage games by following the definition by Bernheim, Peleg and Whinston (1987). In our game, there are only two stages $t = 1, 2$. Player i's strategy $\sigma_i = (\sigma_i^1, \sigma_i^2) \in \Sigma_i = \Sigma_i^1 \times \Sigma_i^2$ is such that $\sigma_i^1 \in \Sigma_i^1$ denotes i's lobby participation choice, and $\sigma_i^2 \in \Sigma_i^2$ is a function $\sigma_i^2 : \mathcal{S}(i) \to \mathcal{T}_i$ if $\sigma_i^1 = 1$, where $\mathcal{S}(i) = \{S \in 2^N : i \in S\}$ and $\mathcal{T}_i$ is the space of bid functions in the common agency game (if $\sigma_i^1 = 0$, then σ_i^2 is a trivial strategy).[18] When $\sigma_i^1 = 0$ (no participation in lobbying), the second stage strategy σ_i^2 is irrelevant. Each player's payoff function is $u_i : \Sigma \to \mathbb{R}$ that is the same payoff function of lobbying game when lobby group S is determined by $S(\sigma^1)$. For $T \subseteq N$, consider a **reduced game** $\Gamma(T, \sigma_{-T})$ that is a game with players in T by letting players in $N \backslash T$ passive players in Γ, who always play σ_{-T}. We also consider **subgames** for all $\sigma^1 \in \Sigma^1$, and **reduced subgames** $\Gamma(T, \sigma^1, \sigma_{-T}^2)$ in similar ways. A **perfectly coalition-proof Nash equilibrium** (**PCPNE**) $(\sigma^*, a^*) = ((\sigma_i^{1*}, \sigma_i^{2*})_{i \in N}, a^*)$ is recursively defined as follows:[19]

(a) In a single player, single stage subgame $\Gamma(\{i\}, \Sigma_i^2, \sigma^1, \sigma_{-\{i\}}^2)$, strategy $\sigma_i^{2*} \in \Sigma_i^2$ and the agenda chosen by the agent a^* is a **PCPNE** if σ_i^{2*} maximizes u_i via a^*.

(b-1) Let $(n, 2)$ be the numbers of players and stages of games. Pick any positive pair of integers $(m, r) \leq (n, 2)$ with $(m, r) \neq (n, 2)$.[20] For any $T \subseteq N$ with $|T| \leq m$, assume that PCPNE has been defined for all

[18] Thus, $\sigma_i^2(S) \in \mathcal{T}_i$ is $T_i : A \to \mathbb{R}_+$ in the last section.

[19] Note that in Bernheim, Peleg and Whinston (1987), the definition of PCPNE is based on strictly improving coalitional deviations. However, we adopt a definition based on weakly improving coalitional deviations, since the theorem on menu auction in Bernheim and Whinston (1986) uses CPNE based on weakly improving deviation. For details on these two definitions, see Konishi, Le Breton and Weber (1999).

[20] The numbers n and t represent the numbers of players and stages of a reduced (sub) game, respectively.

reduced games $\Gamma(T, \sigma_{-T})$ and their subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$ (if $r = 1$, then only for all reduced subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$). Then,

(i) for all reduced games $\Gamma(S, \sigma_{-S})$ and their subgames $\Gamma(S, \sigma^1, \sigma^2_{-S})$ with $|S| = n$, $(\sigma^*, a^*) \in \Sigma \times A$ is **perfectly self-enforcing** if for all $T \subset S$ we have (σ^*_T, a^*) is a PCPNE of reduced game $\Gamma(T, \sigma^*_{S\setminus T}, \sigma_{-S})$, and σ^{2*}_T is a PCPNE of reduced subgame $\Gamma(T, \sigma^1, \sigma^{2*}_{S\setminus T}, \sigma^2_{-S})$, and

(ii) for all $S \subseteq N$ with $|S| = n$, (σ^*_S, a^*) is a **PCPNE** of reduced game $\Gamma(S, \sigma_{-S})$ if (σ^*_S, a^*) is perfectly self-enforcing in reduced game $\Gamma(S, \sigma_{-S})$, and there is no other perfectly self-enforcing σ'_S such that $u_i(\sigma'_S, \sigma_{-S}) \geq u_i(\sigma^*_S, \sigma_{-S})$ for every $i \in S$ with at least one strict inequality.

(b-2) Let $(n, 1)$ be the numbers of players and stages of games. Pick any positive integer $m < n$. For any $T \subseteq N$ with $|T| \leq m$, assume that PCPNE has been defined for all reduced subgames $\Gamma(T, \sigma^1, \sigma^2_{-T})$. Then,

(i) for all reduced subgame $\Gamma(S, \sigma^1, \sigma^2_{-S})$ with $|S| = n$, $(\sigma^*, a^*) \in \Sigma \times A$ is **perfectly self-enforcing** if for all $T \subset S$ we have (σ^{2*}_T, a^*) is a PCPNE of reduced subgame $\Gamma(T, \sigma^1, \sigma^{2*}_{S\setminus T}, \sigma^2_{-S})$, and

(ii) for all $S \subseteq N$ with $|S| = n$, (σ^{2*}_S, a^*) is a **PCPNE** of reduced game $\Gamma(S, \sigma^1, \sigma_{-S})$ if (σ^{2*}_S, a^*) is perfectly self-enforcing in reduced subgame $\Gamma(S, \sigma^1, \sigma_{-S})$, and there is no other perfectly self-enforcing $\sigma^{2\prime}_S$ such that $u_i\left(\sigma^1, \sigma^{2\prime}_S, \sigma^2_{-S}\right) \geq u_i\left(\sigma^1, \sigma^{2*}_S, \sigma^2_{-S}\right)$ for every $i \in S$ with at least one strict inequality.

For any $T \subseteq N$ and any strategy profile σ, let $PCPNE(\Gamma(T, \sigma_{-T}))$ denote the set of PCPNE strategy profiles on T for the game $\Gamma(T, \sigma_{-T})$. For any strategy profile (σ, a), a strategic coalitional deviation (T, σ'_T, a') from (σ, a) is **credible** if $(\sigma'_T, a') \in PCPNE(\Gamma(T, \sigma_{-T}))$. A PCPNE is *a strategy profile that is immune to any credible coalitional deviation.*

First note that PCPNE coincides with CPNE in the second stage. Thus, a CPNE needs to be assigned to each subgame. Second, if a coalition T wants to deviate in the first stage, within the reduced game $\Gamma(T, \sigma_{-T})$, it can orchestrate the whole plan of the deviation by assigning a new CPNE to each subgame so that the target allocation (by the deviation) would be attained as PCPNE of the reduced game $\Gamma(T, \sigma_{-T})$.

In general, it is hard to see the properties of PCPNE of lobby formation game with common agency including its existence of equilibrium. However, in public good provision problem, we can assure existence of equilibrium and provide a characterization of PCPNE in its equilibrium outcome set. Consider a PCPNE (σ^*, a^*). An **outcome allocation** for (σ^*, a^*) is a list $(S, a^*, u) \in 2^N \times A \times \mathbb{R}^N \times \mathbb{R}$, where $S = \{i \in N : \sigma_i^{1*} = 1\}$ and (u, u_G) is the resulting utility allocation for players and the agent such that for all $i \notin S$, $u_i = v_i(a^*)$. From the facts obtained in common agency game, (S, a^*, u) satisfies $a^* = a^*(S)$, and for any $T \subseteq S$, $\sum_{i \in T} u_i \leq W_\Gamma(S) - W_\Gamma(S \backslash T)$. The agant's payoff u_G is implicitly determined by $\sum_{i \in S} u_i + u_G = W_\Gamma(S)$.

3 A Public Good Provision Problem

In this section, we consider a case in which all players' interests are in the same direction, while the intensity of their interests can be heterogeneous. A stylized public good model can be viewed as a special class of the above game. Agenda is a public good provision level, and is one-dimensional: $A = \mathbb{R}_+$, and the provision cost of public good is described by a C^2 cost function $C : A \to \mathbb{R}_+$ with $C(0) = 0$, $C'(a) > 0$ and $C''(a) > 0$ (for uniqueness: for simplicity). Player i's utility function is quasi linear in private good net consumption x and is written as $v_i(a) - x$, where $v_i : A \to \mathbb{R}_+$ is $v_i(0) = 0$, $v_i'(a) > 0$ and $v_i''(a) \leq 0$. In order to guarantee the existence of solution, we assume the Inada condition on the cost function: $\lim_{a \to 0} C'(a) = 0$ and $\lim_{a \to \infty} C'(a) = \infty$.

We will analyze PCPNE of our two stage game in this economy. First, we will define an intuitive but not well-grounded hybrid solution concept, *free-riding-proof core (FRP-core)*, which is the set of Foley-core allocations[21] that are immune to free-riding incentives and is Pareto-optimal in a constrained sense. The FRP-Core is always nonempty in the public good provision problem. Second, by an example, we investigate how the FRP-Core looks like. Finally, we prove that the set of outcomes of PCPNE is equivalent to the FRP-Core.

[21] The Foley core of our public good economy is the standard core concept assuming that deviating coalitions have to provide public good by themselves. That is, it assumes that there is no spillover of public good across the groups.

3.1 Free-Riding-Proof Core: A Hybrid Solution Concept

A public good provision problem determines two things: (i) which group provides public good and how much, and (ii) how to allocate the benefits from providing public good among the members of the group (or how to share the cost). Let $S \subseteq N$ with $S \neq \emptyset$. For $T \subseteq S$, let

$$V(S) \equiv \max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right],$$

and

$$a^*(S) \equiv \arg\max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right].$$

An **allocation for** S is $(S, a^*(S), u)$ such that $u \in \mathbb{R}^N_+$, $\sum_{i \in S} u_i \leq V(S)$,[22] and $u_j = v_j(a)$ for all $j \notin S$ (utility allocation). That is, $N \backslash S$ are passive free-riders, and they do not contribute at all. Given that S is the lobby group, a natural way to allocate utility among the members is to use the core (Foley, 1970). A **core allocation for** S, $(S, a^*(S), u)$, is an allocation for S such that $\sum_{i \in T} u_i \geq V(T)$ holds for all $T \subseteq S$.

However, a core allocation for S may not be immune to free-riding incentives by its members of S. So, we will define a hybrid solution concept of cooperative and noncooperative games. A **free-riding-proof core allocation for** S is a core allocation $(S, a^*(S), u)$ for S such that

$$u_i \geq v_i(a^*(S \backslash \{i\})) \text{ for all } i \in S.$$

A free-riding-proof core allocation is immune to unilateral deviations of the members of S. Note that, given the nature of public good provision problem, we can allow a coalitional deviation from S at no cost (since one person deviation is the most profitable). Let $Core^{FRP}(S)$ be the set of all free-riding-proof core allocations for S. Note that $Core^{FRP}(S)$ may be empty for large group S, while for small groups it is nonempty (especially, for singleton groups it is always nonempty). We collect free-riding-proof core allocations for all S, and take their Pareto frontiers: the set of **free-riding-proof core**

[22] Note that we have $V(S) = W_\Gamma(S) - W_\Gamma(\emptyset)$ in our public good provision problem.

is defined as

$$\begin{aligned} Core^{FRP} &= \big\{(S, a^*(S), u) \in \cup_{S' \in 2^N} Core^{FRP}(S') : \\ &\quad \forall T \in 2^N, \forall u' \in Core^{FRP}(T), \exists i \in N \text{ with } u_i > u'_i \big\}. \end{aligned}$$

That is, an element of $Core^{FRP}$ is a free-riding-proof core allocation for some S that is not weakly dominated by any other free-riding-proof core allocation for any T. *Note that $Core^{FRP}$ is **not** a subsolution of $Core(N)$: it only achieves constrained efficiency due to free-riding incentives, since we often have $Core^{FRP}(N) = \emptyset$.* Note that there always exists a free-riding-proof core allocation. since for all singleton set $S = \{i\}$, $Core^{FRP}(S)$ is nonempty.

Proposition 1. $Core^{FRP} \neq \emptyset$.

In the next section, a simple example illustrate the properties of free-riding-proof core allocations.

3.2 Linear-Utility and Quadratic-Cost Case

Let $v_i(a) = \theta_i a$ for all $i \in N$ and $C(a) = \frac{1}{2}a^2$, where $\theta_i > 0$ is a parameter.[23] With this setup, for group S, the optimal public good provision is determined by the first order condition, $\sum_{i \in S} \theta_i - a = 0$: i.e.,

$$a^*(S) = \sum_{i \in S} \theta_i.$$

Thus, the value of S is written as

$$\begin{aligned} V(S) &= \sum_{i \in S} \theta_i \left(\sum_{i \in S} \theta_i\right) - \frac{1}{2}\left(\sum_{i \in S} \theta_i\right)^2 \\ &= \frac{\left(\sum_{i \in S} \theta_i\right)^2}{2}. \end{aligned}$$

For an outsider $j \in N \backslash S$, the payoff is

$$v_j(a^*(S)) = \theta_j \left(\sum_{i \in S} \theta_i\right).$$

[23] Coefficient 1/2 of $C(a)$ function is just matter of normalization. For any $k > 0$ with $C(a) = ka^2$, we get isomorphic results.

Consider the following example.

Example 1. Let $N = \{1, 3, 5, 11\}$ with $\theta_i = i$ for each $i \in N$.

Suppose first that a simultaneous voluntary contribution of public goods are done, instead of our two stage public good provision process. Then, only $i = 11$ contributes, and all others free-ride. The public good provision level is $a = 11$.

Now, let us move to our problem. First suppose that $S = N$. Then, we have $a^*(N) = \sum_{i \in N} i = 20$, and $V(N) = \frac{20^2}{2} = 200$. However, in order to have free-riding-proofness, we need to give each player the following payoff at the very least:

$$\begin{aligned} v_{11}(a^*(N \backslash \{11\})) &= (20 - 11) \times 11 = 99, \\ v_5(a^*(N \backslash \{5\})) &= (20 - 5) \times 5 = 75, \\ v_3(a^*(N \backslash \{3\})) &= (20 - 3) \times 3 = 51, \\ v_1(a^*(N \backslash \{1\})) &= (20 - 1) \times 1 = 19. \end{aligned}$$

The sum of all the above values exceeds the value of the grand coalition $V(N)$. As a result, we can conclude $Core^{FRP}(N) = \emptyset$. Next, consider $S = \{11, 5\}$. Then, $a^*(S) = 16$, and $V(S) = 128$. In order to check if the free-riding-proof core for S is nonempty, first again check the free-riding-incentives.

$$\begin{aligned} v(a^*(S \backslash \{11\})) &= (16 - 11) \times 11 = 55, \\ v(a^*(S \backslash \{5\})) &= (16 - 5) \times 5 = 55. \end{aligned}$$

Thus, if there is a free-riding-proof core allocation $u = (u_{11}, u_5)$ for S, u must satisfy

$$\begin{aligned} u_{11} + u_5 &= 128, \\ u_{11} &\geq 55, \\ u_5 &\geq 55, \\ u_{11} &\geq \frac{11 \times 11}{2} = 60.5, \\ u_5 &\geq \frac{5 \times 5}{2} = 12.5. \end{aligned}$$

The last two conditions are obtained by the core requirement. Thus, we have[24]

$$Core(\{11,5\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 12.5, \\ \tilde{u}_3 = 48,\ \tilde{u}_2 = 32,\ \tilde{u}_1 = 16 \end{array} \right\},$$

and

$$Core^{FRP}(\{11,5\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 55, \\ \tilde{u}_3 = 48,\ \tilde{u}_2 = 32,\ \tilde{u}_1 = 16 \end{array} \right\}.$$

As is easily seen, $Core^{FRP}(\{11,5\}) \neq \emptyset$, but it is a smaller set than $Core(\{11,5\})$. That is, the first observation is obvious:

- "*Free-riding-proof constraints may narrow the set of attainable core allocations.*"

Now, let us consider a simultaneous move voluntary public good provision game by Bergstrom, Blume and Varian (1986). Each player i chooses her monetary contribution $m_i \geq 0$ to provide public good. The public good provision level is determined by $a(m) = \sqrt{2\sum_{i \in N} m_i}$ reflecting the cost function of public good production. Consider player i. Given that others are contributing M_{-i} together, player i maximizes $\theta_i\sqrt{2(m_i + M_{-i})} - m_i$. Thus, the best response for player i is $m_i^* = \max\left\{\frac{i^2}{2} - M_{-i}, 0\right\}$. This implies that only player 11 contributes, and the public good provision level is 11. Thus, by forming a contribution group in the first stage, it is possible to increase the public good provision level in equilibrium.[25]

Now, let us characterize the free-riding-proof core, the FRP-core. Since the FRP-core requires Pareto-efficiency on the union of free-riding-proof cores

[24] For notational simplicity, without confusion, we abuse notations by dropping irrelevant arguments of allocations. Thus, in this subsection, allocations are utility allocations.

[25] In relation to this, the readers may wonder about the Lindahl equilibrium allocation for $S = \{11,5\}$. Unfortunately, this example is not very useful since utility function is quasi-linear. The result would totally dependent on how the profits are distributed as is seen below. The Lindahl prices are $p_{11} = 11$ and $p_5 = 5$ given $\theta_{11} = 11$ and $\theta_5 = 5$, since $a^*(\{11,5\}) = 16$ means marginal cost is $16(= 11 + 5)$. Since there are pure profits in producing public goods (cost function is strictly convex), we need to specify the way to allocate the profits 128. If they are distributed equally, then both get 64 each as profit share, and this is the only source of their utilities. If they are distributed according to players' willingnesses-to-pay, then players get 88 and 40. In the former case, the free-riding-proof conditions are satisfied, but in the latter case, they are not satisfied.

for all subsets of the players, we need to find free-riding-proof core for each S, in order to find the set of free-riding-proof core. The following lemma helps us to do the task.

Lemma 1. In the linear-utility-quadratic-cost public good problem, the free-riding-proof core for S is nonempty if and only if S satisfies $\Phi(S) \equiv \sum_{i \in S} \theta_i a^*(S) - \frac{1}{2}(a^*(S))^2 - \sum_{i \in S} \theta_i a^*(S \backslash \{i\}) \geq 0$ (aggregated "no free riding condition").

Even in this simple setup, we can make a few interesting observations.

Example 1. (continued) The free-riding-proof core allocations are attained by groups $\{11, 5, 1\}$, $\{11, 3, 1\}$, $\{11, 5\}$, $\{11, 3\}$, and $\{5, 3\}$.

First by Lemma1, we can easily check for which S, $Core^{FRP}(S) \neq \emptyset$ holds. There are 12 such contribution groups: $\{11, 5, 1\}$, $\{11, 3, 1\}$, $\{11, 5\}$, $\{11, 3\}$, $\{11, 1\}$, $\{5, 3\}$, $\{5, 1\}$, $\{3, 1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$.

Note that $S = \{11, 5, 3\}$ does not have nonempty free-riding-proof core for S. Let $S = \{11, 5, 3\}$. Then, $a^*(S) = 19$ and $W(S) = 180.5$. Now, $11v(a^*(S \backslash \{11\})) = 88$, $5v(a^*(S \backslash \{5\}) = 70$ and $3v(a^*(S \backslash \{3\})) = 48$. Since $88 + 70 + 48 > 180.5$, there is no free-riding-proof core allocation for $S = \{11, 5, 3\}$. Thus, $\{11, 5, 1\}$ is the group that achieves the highest level of public good provision, and has nonempty free-riding-proof core.[26] This analysis gives an interesting observation:[27]

- *(Even the largest) group that achieves a free-riding-proof core allocation may not be consecutive.*

The intuition of this result is simple. Suppose $\Phi(S)$ is positive (say, $S = \{11, 5\}$). Then by Lemma 1, there is an internally stable allocation for S. Now, we may try to find $S' \supset S$ that still keeps $\Phi(S') \geq 0$. If the value of $\Phi(S)$ is positive yet the value is not so large, then adding high θ player (say, player 3) may make $\Phi(S') < 0$, since adding such a player may increase $a^*(S')$ a lot, making free-riding problem severer. However, if low θ player

[26] As is seen below, group $\{11, 5, 1\}$ supports some allocations in $Core^{FRP}$.

[27] Although the context and approach are very different, in political science and sociology, formation of such non-consecutive coalitions is of a tremendous interest. For a game theoretical treatment of this line of literature (known and "Gamson's law"), see Le Breton et al. (2007).

(say, player 1) is added, the free-rider problem does not become too severe, and $\Phi(S') \geq 0$ may be satisfied relatively easily.

Among the above 12 groups, it is easy to see that groups $\{5,1\}$, $\{3,1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$ do not survive the test of Pareto-domination by free-riding-proof core allocations for other groups. For example, consider $S = \{11,5\}$ and $u' = (73, 55, 48, 32, 16) \in Core^{FRP}(\{11,5\})$.[28] Since the payoff of 11 by free-riding is $v_{11}(a) = 11a$, every allocation for the above groups are dominated by the above u'. On the other hand, $\{5,3\}$ is not dominated, since player 11 gets 88 by free-riding, respectively. Thus, player 11 would not join a deviation (11 can obtain maximum 73 in a free-riding-proof core allocation for $S \ni 11$). Without player 11's cooperation, there is no free-riding core allocation that dominates those of $\{5,3\}$.

By the same reasons, free-riding-proof core allocations for $S = \{11,1\}$ are dominated by the one for $S' = \{11,5\}$. Under $S = \{11,1\}$, player 5 gets 60, but S' can attain $u' = (63, 65, 48, 32, 16)$.[29] However, free-riding-proof core allocations for $S = \{11,3,1\}$ and $\{11,3\}$ cannot be beaten by the ones for $S' = \{11,5\}$, since player 5 gets 70 even under $\{11,3\}$.[30]

Finally, $S = \{11,5\}$, $\{11,3\}$. The free-riding-proof core allocations for $S = \{11,5\}$ is characterized by $u_{11} + u_5 = 128$, $u_{11} \geq 60.5$ and $u_5 \geq 55$, with $u_3 = 48$, $u_2 = 32$ and $u_1 = 16$. Now, consider $S' = \{11,5,1\}$. The free-riding-proof core allocations for S' is characterized by $u'_{11} + u'_5 + u'_1 = 144.5$, $u'_1 \geq 66$, $u'_5 \geq 60$ and $u'_1 \geq 16$, with $u'_3 \geq 51$ and $u'_2 \geq 34$. Thus, S' can attain $u'_{11} + u'_5 = 144.5 - 16 = 128.5$ as long as $u'_{11} \geq 66$ and $u'_5 \geq 60$. Thus, if $u \in Core^{FRP}(\{11,5\})$ satisfies $u_{11} + u_5 = 128$, $60.5 \leq u_{11} \leq 68.5$, and $55 \leq u_5 \leq 62.5$, then u is improved upon by an allocation in $Core^{FRP}(\{11,5,1\})$. However, if $u \in Core^{FRP}(\{11,5\})$ satisfies $u_{11} + u_5 = 128$, $u_{11} > 68.5$, or $u_5 > 62.5$, then u cannot be improved upon by forming group $\{11,5,1\}$. Free-riding-proof core allocations for $S = \{11,3\}$ has a similar property with possible deviations by group $S' = \{11,3,1\}$. This phenomenon illustrates another interesting observation:

- *An expansion of group definitely increases the total value of the group,*

[28]The best allocation for player 11 in $Core^{FRP}(\{11,5\})$. See the characterization of $Core^{FRP}(\{11,5\})$ in Example 1. Other players are free-riders, and their payoffs are directly generated from $a^*(\{11,5\}) = 16$.

[29]Under $S = \{11,2\}$, player 11 can get at most 62.5 in order to satisfy the free-riding-proofness for player 2 ($v_2(\{11\}) = 22$).

[30]Since $V(\{11,5\}) = 128$, and player 5 demands at least 70, player 11 can get at most 58. However, $V(\{11\}) = 60.5$. Thus, involving player 5 is not feasible.

while it gives less flexibility in allocating it since free-riding incentives are strengthened by having more public good. As a result, some unequal free-riding-proof core allocations for the original group may not be improved upon by expanding the group.

In summary, the free-riding-proof core is *union* of the following sets of allocations attained by five different groups.

1. $S = \{11, 5, 1\}$, then $a^*(S) = 17$ and all free-riding-proof core allocations for S are attained:

$$Core^{FRP}(\{11,5,1\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_5 + \tilde{u}_1 = 144.5,\ \tilde{u}_3 = 51,\ \tilde{u}_2 = 34, \\ 66 \leq \tilde{u}_{11},\ 60 \leq \tilde{u}_5,\ 16 \leq \tilde{u}_1 \end{array} \right\}$$

2. $S = \{11, 3, 1\}$, then $a^*(S) = 15$ and all free-riding-proof core allocations for S are attained:

$$Core^{FRP}(\{11,3,1\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_3 + \tilde{u}_1 = 112.5,\ \tilde{u}_5 = 75,\ \tilde{u}_2 = 30, \\ 60.5 \leq \tilde{u}_{11},\ 36 \leq \tilde{u}_3,\ 14 \leq \tilde{u}_1 \end{array} \right\}$$

3. $S = \{11, 5\}$, then $a^*(S) = 16$ and only subset of free-riding-proof core allocations for S can be attained:

$$\begin{array}{cl} & \left\{ \tilde{u} \in Core^{FRP}(\{11,5\}) : \tilde{u}_{11} > 68.5,\ \text{or}\ \tilde{u}_5 > 62.5 \right\} \\ = & \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_5 = 128,\ \tilde{u}_3 = 48,\ \tilde{u}_2 = 32,\ \tilde{u}_1 = 16, \\ [68.5 < \tilde{u}_{11} \leq 73 \text{ and } 55 \leq \tilde{u}_5 < 59.5] \\ \text{or}\ [62.5 < \tilde{u}_5 \leq 67.5 \text{ and } 60.5 \leq \tilde{u}_{11} < 65.5] \end{array} \right\} \end{array}$$

4. $S = \{11, 3\}$, then $a^*(S) = 14$ and only subset of free-riding-proof core allocations for S can be attained:

$$\begin{array}{cl} & \left\{ \tilde{u} \in Core^{FRP}(\{11,3\}) : \tilde{u}_{11} > 62.5 \right\} \\ = & \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_{11} + \tilde{u}_3 = 98,\ \tilde{u}_5 = 70,\ \tilde{u}_2 = 28,\ \tilde{u}_1 = 14, \\ [62.5 < \tilde{u}_{11} \leq 65 \text{ and } 33 \leq \tilde{u}_3 < 35.5] \end{array} \right\} \end{array}$$

5. $S = \{5, 3\}$, then $a^*(S) = 8$ and all free-riding-proof core allocations for S are attained:

$$Core^{FRP}(\{5,3\}) = \left\{ \begin{array}{c} \tilde{u} \in \mathbb{R}^5_+ : \tilde{u}_5 + \tilde{u}_3 = 32,\ \tilde{u}_{11} = 88,\ \tilde{u}_2 = 16,\ \tilde{u}_1 = 8, \\ 15 \leq \tilde{u}_5,\ 15 \leq \tilde{u}_3 \end{array} \right\}$$

We can observe that in the last two groups, the levels of public good provision are less than the Nash equilibrium provision level of the standard voluntary contribution game (recall that $a = 11$ by player 11's contribution only is the unique Nash equilibrium.):

- *There may be free-riding-proof core allocations that achieve less public good provision than Nash equilibrium one of a simple voluntary contribution game by Bergstrom, Blume and Varian (1986).*

This occurs since in our setup, player 11 can commit to being an outsider in the first stage. In a simultaneous move voluntary contribution game, this cannot happen. Finally, needless to say, we have:

- *The free-riding-proof core may be a highly nonconvex set.*

3.3 Result for Public Good Provision Problem

Now, we can state our main result of this paper. In this public good provision problem, Fact 3 shown in Laussel and Le Breton (2001) will be useful. Note that the core of $(W_\Gamma(T))_{T\subseteq S}$ is equivalent to $Core(S)$ in our game. It can be seen as follows. Since in a public good provision problem preferences are **comonotonic**, i.e., $v_i(a) \geq v_i(a')$ if and only if $v_j(a) \geq v_j(a')$ for all $i, j \in S$ and all $a, a' \in A$, $(W_\Gamma(T))_{T\subseteq S}$ is a convex game (Laussel and Le Breton, 2001). Thus, no rent property $u_G = W_\Gamma(\emptyset)$ holds, and $W_\Gamma(\emptyset) = 0$ in public good game. This implies

$$\sum_{i\in S} u_i = W_\Gamma(S) - W_\Gamma(\emptyset) = W_\Gamma(S).$$

Therefore, the second stage CPNE outcomes coincide $Core(S)$.[31] This gives us some insight in our two-stage noncooperative game. Given the setup of our lobby formation game in the first stage, if a CPNE outcome u in a subgame S can realize as the equilibrium outcome (on-equilibrium path), it is *necessary* to have $u \in Core^{FRP}(S)$, since otherwise, some member of S would deviate in the first stage and obtain a secured free-riding payoff. This observation is useful in our analysis in the equivalence theorem.

[31] Actually, with no rent property, CPNE and strong Nash equilibrium (Aumann, 1959, but with weakly improving deviations) are equivalent in common agency game. See Konishi, Le Breton and Weber (1999).

Theorem 1. In public good provision problem, an allocation $(S, a^*(S), u)$ is in the FRP-core, if and only if there is a PCPNE (σ^*, a^*) of which outcome is $(S, a^*(S), u)$.

Proof. First, we show that a free-riding-proof core allocation u^* is supportable by a PCPNE. Suppose that $u^* \in Core^{FRP}(S^*)$. The first step is to assign a CPNE utility profile to each subgame S' (although this does not happen in the equilibrium, it matters when deviations are considered).

1. For any S' with $S' \cap S^* = \emptyset$, we assign a CPNE that achieves an extreme point of the core for S'. For an arbitrarily selected order ω over S', we assign payoff vector $u_{\omega(1)} = W(\{\omega(1)\}) - W(\emptyset)$, $u_{\omega(2)} = W(\{\omega(1), \omega(2)\}) - W(\{\omega(1)\})$,... etc. following Shapley (1971).[32]

2. For any S' with $S' \cap S^* \neq \emptyset$, we assign the following CPNE: Let $\omega : |S' \backslash S^*| \to S' \backslash S^*$ be an arbitrary bijection, and let $u_{\omega(1)} = W(\{\omega(1)\})$, $u_{\omega(2)} = W(\{\omega(1), \omega(2)\}) - W(\{\omega(1)\})$,..., $u_{\omega(|S' \backslash S^*|)} = W(S' \backslash S^*) - W(S' \backslash S^* \backslash \{\omega(|S' \backslash S^*|)\})$. The rest $W(S') - W(S' \backslash S^*)$ goes to $S' \cap S^*$ in the following manner. For each $i \in S' \cap S^*$, let $\bar{u}_i = \max\{u_i^*, v_i(S' \backslash \{i\})\}$, and player i gets[33]
$$u_i = \bar{u}_i + \frac{1}{|S' \cap S^*|} \left(W(S') - W(S' \backslash S^*) - \sum_{j \in S' \cap S^*} \bar{u}_j \right).$$

3. The above payoff vectors in 1 and 2 are achieved by truthful strategies played by S'.

Now, suppose to the contrary that coalition T profitably and credibly deviates from the equilibrium. Note that in the reduced game by T, it is a PCPNE deviation. In equilibrium, S^* is the lobby, any player in $N \backslash T$ would not change their strategies. This implies that $(N \backslash S^*) \backslash T$ play 0 in the first stage and they free-ride, while $S^* \backslash T$ play 1 in the first stage and they play the same strategies (menus contingent to formed lobbies) in the second stage.

[32] Let the ordering be a bijection $\omega : \{1, 2, ..., |S'|\} \to S'$. Let $u_{\omega(i)} = W(\{\omega(1), ..., \omega(i)) - W(\{1, ..., \omega(i-1)\})$. The allocation $(u_{\omega(i)})_{i=1}^{|S'|}$ is in the core for S', since the game W is convex.

[33] The contents of the parenthesis can be positive or negative. If it were positive, all players in $S' \backslash S^*$ are better off by joining S', and there is no free-riding incentive for them.

Note that $T\backslash S^*$ play 1 in the first period (by definition), while $T \cap S^*$ may or may not play 1. Some may choose to free-ride by switching to 0, while others stay in the lobby with adjustment of their strategies in the second stage.

Let S' be the lobby formed by T's deviation (see Figure 1). Then, there are five groups of players:

(i) the members of $S^*\backslash S' \subset T$ free-ride after the deviation,

(ii) the members of $S'\backslash S^* \subset T$ join the lobby,

(iii) the members of $(S^* \cap S')\backslash T \subset S'$ do not change their strategies in any stage,

(iv) the members of $(S^* \cap S') \cap T \subset S'$ change strategies in the second stage,

(v) the members of $N\backslash(S' \cup S^*)$ are outsiders before or after the deviation.

Let the resulting allocation be $(S', a^*(S'), u')$. Since T is a profitable deviation, the members in (i), (ii) and (iv) are better-off after T deviates. That is,

$$\begin{aligned} v_i(a^*(S')) &\geq u_i^* \text{ for all } i \in S^*\backslash S', \\ u_i' &\geq u_i^* \text{ for all } i \in S'\backslash S^*, \\ u_i' &\geq u_i^* \text{ for all } i \in (S^* \cap S') \cap T, \end{aligned}$$

must hold. Note that, since u_i' is achieved in a CPNE in the second stage game, we have

$$\sum_{i \in S'\backslash S^*} u_i' \geq W(S'\backslash S^*).$$

By the construction of CPNE in subgame, we also know:

$$u_i' = \bar{u}_i + \frac{1}{|S'\backslash S^*|}\left(W(S') - W(S'\backslash S^*) - \sum_{j \in S'\backslash S^*} \bar{u}_j\right) \text{ for all } i \in (S^* \cap S')\backslash T.$$

It is because players in (iii) use truthful strategies. Thus, we have

$$\sum_{i \in (S^* \cap S')\backslash T} u_i' = \sum_{i \in (S^* \cap S')\backslash T} \bar{u}_i + \frac{|(S' \cap S^*)\backslash T|}{|S' \cap S^*|}\left(W(S') - W(S'\backslash S^*) - \sum_{j \in S' \cap S^*} \bar{u}_j\right).$$

This together with the feasibility of new lobby, $\sum_{i\in S'} u'_i = W(S')$, we have

$$\begin{aligned}\sum_{i\in(S^*\cap S')\cap T} u'_i &\leq W(S') - W(S'\backslash S^*) \\ &\quad - \sum_{i\in(S^*\cap S')\backslash T} \bar{u}_i - \frac{|(S'\cap S^*)\backslash T|}{|S'\cap S^*|}\left(W(S') - W(S'\backslash S^*) - \sum_{j\in S'\cap S^*} \bar{u}_j\right) \\ &= \sum_{i\in(S^*\cap S')\cap T} \bar{u}_i + \frac{|(S'\cap S^*)\cap T|}{|S'\cap S^*|}\left(W(S') - W(S'\backslash S^*) - \sum_{j\in S'\cap S^*} \bar{u}_j\right).\end{aligned}$$

This is the sum of payoffs of players in (ii) and (iv). Since the deviation is profitable, and is immune to further (unilateral) deviation, $u'_i \geq \bar{u}_i$ for all $i \in (S^*\cap S')\cap T$. Hence, we conclude

$$W(S') - W(S'\backslash S^*) - \sum_{j\in S'\cap S^*} \bar{u}_j \geq 0.$$

This guarantees that players in (iii) are also better off.

Moreover, after the deviation, players in (iii) have no incentive to free-ride by the construction of $\bar{u}_i$. And since members of (ii) are better off, we have $a^*(S') > a^*(S^*)$.[34] This implies that the members of (v) are better off, too. This proves that all players in (i), (ii), (iii), (iv) and (v) are better off, and have no incentive for the members of S' to free-ride ((ii) by CPNE). However, this is a contradiction to the presumption $u^* \in Core^{FRP}(S^*)$. Hence, $(S^*, a^*(S^*), u^*)$ is supportable with a PCPNE.

Second, we will show that a PCPNE achieves a free-riding-proof core allocation. It is easy to see that the outcome $(S, a^*(S), u)$ of a PCPNE is a free-riding-proof core allocation for S, since otherwise the resulting allocation will not be a subgame perfect Nash equilibrium (a player who has free-riding incentive certainly deviates in the first stage). Suppose that $u \notin Core^{FRP}$. Then, there is an free-riding-proof core allocation $u' \in Core^{FRP}$ with $u' > u$. Consider a coalitional deviation N by preparing a PCPNE that achieves u' by using the first part of the proof of the theorem. This implies that there is a credible coalitional deviation from u. This is a contradiction. Thus, every PCPNE achieves a free-riding-proof core allocation.□

[34]Note that (ii) is nonempty, since otherwise, $S' \subset S^*$ holds, and coalitional deviation cannot be profitable.

4 Pure Conflict of Interests Problem: A Binary Agenda Case

In the previous section, we considered the case in which consumers have comonotonic preferences: if costs are forgotten, everybody is happier by having more public good. Here, we consider the other extreme: there are two groups of consumers whose preferences are completely opposite. There are only two agendas $A = \{0, 1\}$, and there are two groups of players $N = N^+ \cup N^-$: each player $i \in N^+$ has utility function $v_i(a) = \theta_i a$, and each player $j \in N^-$ has utility function $v_j(a) = -\theta_j a$, where $\theta_i, \theta_j > 0$. That is, players in group N^+ prefer $a = 1$ to $a = 0$, while ones in N^- prefer $a = 0$ to $a = 1$. An example that may fall in this category is the situation in which the government is discussing if it should introduce a free trade agreement with another country. Import competing industries would oppose the policy, while competitive exporting industries would favor it. The government's utility function is $u_G(a) = 0$ for all $a \in \{0, 1\}$, that is, the government is impartial.[35] For $S \subset N$, we have

$$a^*(S) = \begin{cases} 0 & \text{if } \sum_{i \in N^+ \cap S} \theta_i < \sum_{j \in N^- \cap S} \theta_j \\ \{0,1\} & \text{if } \sum_{i \in N^+ \cap S} \theta_i = \sum_{j \in N^- \cap S} \theta_j \\ 1 & \text{if } \sum_{i \in N^+ \cap S} \theta_i > \sum_{j \in N^- \cap S} \theta_j \end{cases}$$

The government gets contributions of the losing groups' utility sum in each case.

Consider a lobby group $S \subseteq N$. In a Nash equilibrium of a common agency game, (τ_S^*, a^*), we have the following.

Proposition 2. In the pure conflict of interests problem, in the second stage common agency game, in every Nash equilibrium for lobby group S, (τ_S^*, a^*), achieves an efficient allocation $a^* = a^*(S)$.[36] It is a CPNE if and only if $a^* = a^*(S)$ and

1. when $\sum_{i \in N^+ \cap S} \theta_i < \sum_{j \in N^- \cap S} \theta_j$,

[35] It is very easy to introduce an partial government.

[36] The set of Nash equilibria is strictly larger than the set of CPNEs. For example, consider the case of $\sum_{j \in N^- \cap S} \theta_j > \sum_{i \in N^+ \cap S} \theta_i$, a strategy profile such that (i) $\tau_i^*(0) = \tau_i^*(1) = 0$ for all $i \in N^+ \cap S$, (ii) $\tau_j^*(1) = 0$ for all $j \in N^- \cap S$ and $\sum_{j \in N^- \cap S} \tau_j^*(0) = \max_{i \in N^+ \cap S} \theta_i$ with $\tau_j^*(0) \in [0, \theta_j]$ for all $j \in N^- \cap S$. Obviously, the minority player cannot reverse the result unilaterally.

(a) $\sum_{j\in N^-\cap S}\tau_j^*(0)=\sum_{i\in N^+\cap S}\theta_i$ and $\tau_i^*(0)=0$ for all $i\in N^+\cap S$,

(b) $\tau_j^*(1)=0$ for all $j\in N^-\cap S$ and $\tau_i^*(1)=\theta_i$ for all $i\in N^+\cap S$,

2. when $\sum_{i\in N^+\cap S}\theta_i=\sum_{j\in N^-\cap S}\theta_j$,

   (a) $\tau_j^*(0)=\theta_j$ and $\tau_j^*(1)=0$ for all $j\in N^-\cap S$,

   (b) $\tau_i^*(0)=0$ and $\tau_i^*(1)=\theta_i$ for all $i\in N^+\cap S$,

3. when $\sum_{i\in N^+\cap S}\theta_i<\sum_{j\in N^-\cap S}\theta_j$,

   (a) $\sum_{i\in N^+\cap S}\tau_i^*(0)=\sum_{j\in N^-\cap S}\theta_j$ and $\tau_j^*(1)=0$ for all $j\in N^-\cap S$,

   (b) $\tau_i^*(0)=0$ for all $i\in N^+\cap S$ and $\tau_j^*(0)=\theta_j$ for all $j\in N^-\cap S$.

Proof. We start with the efficiency of Nash equilibrium. First, if $\sum_{i\in N^+\cap S}\theta_i=\sum_{j\in N^-\cap S}\theta_j$, then either outcome is efficient. Thus, we assume that the equality does not hold. We consider, without loss of generality, that $\sum_{i\in N^+\cap S}\theta_i<\sum_{j\in N^-\cap S}\theta_j$, and we will show that efficiency of Nash equilibrium is efficient ($a^*=0$). Suppose to the contrary that $a^*=1$ holds. Then, we have

$$\sum_{j\in N^-\cap S}\tau_j^*(0)+\sum_{i\in N^+\cap S}\tau_i^*(0)\le\sum_{j\in N^-\cap S}\tau_j^*(1)+\sum_{i\in N^+\cap S}\tau_i^*(1).$$

Moreover, in Nash equilibrium, an equality must hold, since any player who is contributing for $a=1$ has an incentive to reduce $\tau_i^*(1)$ (without changing the outcome $a^*=1$) if the above inequality is strict. Since $\sum_{j\in N^-\cap S}\tau_j^*(1)=0$ (otherwise, since $a^*=1$ and Nash), we have

$$\sum_{j\in N^-\cap S}\tau_j^*(0)+\sum_{i\in N^+\cap S}\tau_i^*(0)=\sum_{i\in N^+\cap S}\tau_i^*(1).$$

Since in Nash equilibrium, all players get nonnegative payoffs, $\sum_{i\in N^+\cap S}\tau_i^*(1)\le\sum_{i\in N^+\cap S}\theta_i<\sum_{j\in N^-\cap S}\theta_j$ holds, thus we have

$$\sum_{j\in N^-\cap S}\tau_j^*(0)<\sum_{j\in N^-\cap S}\theta_j.$$

This implies that there is at least one $j\in N^-\cap S$ such that $\tau_j^*(0)<\theta_j$. Recall j gets $0-\tau_j^*(1)=0$ in equilibrium. However, if j increases $\tau_j(0)$ from

$\tau_j^*(0)$ by a small $\epsilon > 0$, then she can obtain $\theta_j - \tau_j^*(0) - \epsilon > 0$. This is a contradiction to our assumption $a^* = 1$. Thus, $a^* = 0$ must hold, and Nash equilibrium is always efficient.

Second, we strengthen the equilibrium concept to CPNE. First consider the case of $\sum_{i \in N^+ \cap S} \theta_i < \sum_{j \in N^- \cap S} \theta_j$ (case 1). It is obvious that there is a CPNE with the equilibrium described in the statement of proposition (Facts 1 and 2: a truthful strategy profile). It is easy to see $\sum_{j \in N^- \cap S} \tau_j^*(0) \leq \sum_{i \in N^+ \cap S} \theta_i$. Otherwise, the group $N^- \cap S$ can reduce the payments for $a = 0$ jointly without changing the outcome. We will show that the equality actually holds. Suppose to the contrary that $\sum_{j \in N^- \cap S} \tau_j^*(0) < \sum_{i \in N^+ \cap S} \theta_i$. However, then, the other side group $N^+ \cap S$ can change the result (from $a^* = 0$ to $a^* = 1$) by choosing $\sum_{i \in N^+ \cap S} \tau_i^*(1) \in \left(\sum_{j \in N^- \cap S} \tau_j^*(0), \sum_{i \in N^+ \cap S} \theta_i \right)$ jointly. Their payoffs are improved by this deviation. This is a contradiction. Thus, we have $\sum_{j \in N^- \cap S} \tau_j^*(0) = \sum_{i \in N^+ \cap S} \theta_i$. Thus, there is no other CPNE that is described in the statement of the proposition. The same argument applies to other cases (cases 2 and 3).□

Now, we go back to the first stage. In order to characterize the PCPNE of our two stage game. An **outcome allocation for** S is $(S, a^*(S), u)$ satisfies $u \in \mathbb{R}_+^N$ with $\sum_{i \in S} u_i = W_\Gamma(S) - W_\Gamma(\emptyset) = W_\Gamma(S)$, and $u_j = v_j(a)$ for all $j \notin S$. A **free-riding-proof allocation for** S is an allocation $(S, a^*(S), u)$ for S such that

$$u_i \geq v_i(a^*(S \backslash \{i\})) \text{ for all } i \in S.$$

A lobby group S is **free-riding-proof** if there is a free-riding-proof allocation for S. It is easy to characterize free-riding-proof lobby group S. A subset S is a **maximal free-riding-proof lobby group** if there is no proper superset of S that is free-riding-proof. We have the following characterization of maximal free-riding-proof lobby groups (the proof is obvious, so omitted). An outcome allocation $(S, a^*(S), u)$ is **maximal free-riding-proof outcome allocation** if it is an outcome allocation, and if S is a maximal free-riding-proof lobby group. It is easy to see from Fact 2 that a maximal free-riding-proof outcome allocation is characterized as follows.

Proposition 3. In the pure conflict of interests problem, the maximal free-riding-proof outcome allocation $(S, a^*(S), u)$ is characterized by

1. when $\sum_{i \in N^+} \theta_i < \sum_{j \in N^-} \theta_j$,

(a) $N^+ \cap S = N^+$, $\sum_{i\in N^+} \theta_i \leq \sum_{j\in N^-\cap S} \theta_j$, and for any $j' \in N^- \cap S$, $\sum_{j\in N^-\cap(S\setminus\{j'\})} \theta_j < \sum_{i\in N^+} \theta_i$.

(b) $a^*(S) = 0$, $u_i = 0$ for all $i \in N^+$, $\sum_{j\in N^-\cap S} u_j = \sum_{j\in N^-\cap S} \theta_j - \sum_{i\in N^+} \theta_i$.

2. when $\sum_{i\in N^+} \theta_i = \sum_{j\in N^-} \theta_j$,

(a) $S = N$.

(b) $a^*(S) \in \{0, 1\}$, $u_i = 0$ for all $i \in N^+$ and $u_j = 0$ for all $j \in N^-$.

3. when $\sum_{j\in N^-} \theta_j < \sum_{i\in N^+} \theta_i$,

(a) $N^- \cap S = N^-$, $\sum_{j\in N^-} \theta_j \leq \sum_{i\in N^+\cap S} \theta_i$, and for any $i' \in N^+ \cap S$, $\sum_{i\in N^+\cap(S\setminus\{i'\})} \theta_i < \sum_{j\in N^-} \theta_j$.

(b) $a^*(S) = 1$, $u_j = 0$ for all $j \in N^-$ and $\sum_{i\in N^+\cap S} u_i = \sum_{i\in N^+\cap S} \theta_i - \sum_{j\in N^-} \theta_j$.

It is clear from the above characterization, any maximal free-riding-proof lobby group contains a pivotal winning group and the entire losing group. Now, we can characterize PCPNE in the pure conflict of interests problem.

Theorem 2. In the pure conflict of interests problem, an allocation $(S, a^*(S), u)$ satisfies the conditions described in Proposition 3, if and only if there is a PCPNE (σ^*, a^*) of which outcome is $(S, a^*(S), u)$.

Proof. Pick an allocation that satisfies the conditions in Proposition 3. If there were a deviation, it must happen in the first stage since in the second stage the allocation is supported by a CPNE. It is easy to see that in a subgame, there is no coalition that move out of lobby group S. Suppose that there is a coalitional deviation $T \subset S$. If it were an improving deviation, the members move out of S without changing the outcome (a weakly improving deviation). In order to do this, $T \cap N^- \neq \emptyset$ and $T \cap N^+ \neq \emptyset$, and one of these two parties can save their contributions. Without loss of generality, $a^*(S) = 0$, thus $\sum_{j\in N^-\cap S} \theta_j \geq \sum_{i\in N^+\cap S} \theta_i$ and $\sum_{j\in N^-\cap(S\setminus T)} \theta_j \geq \sum_{i\in N^+\cap(S\setminus T)} \theta_i$. Then, there is $T' \subset T \cap N^+$ such that $\sum_{j\in N^-\cap(S\setminus T)} \theta_j < \sum_{i\in N^+\cap(S\setminus T)\cup T'} \theta_i$. Thus, by this further deviation T', they can change the outcome to $a^* = 1$. Thus, the original deviation T is not credible. Thus, the allocation that satisfies the conditions in Proposition 3 can be supported by a PCPNE. By repeating the same argument for T', it is easy to see that if S is not maximal, there is a subset of $N \backslash S$ from the opposition party that can change the equilibrium outcome. Thus, maximality of S is necessary, too. This completes the proof.□

This result has an interesting implication. Unlike in the case of conflicts of interests, the size of equilibrium lobby group is large, and the government collects a big rent. Thus, weak free-riding incentives are achieved through the government's power to raise contributions due to the conflict of interests.

5 Concluding Remarks

This paper added players' participation decisions to common agency games. The solution concept we used is a natural extension of coalition-proof Nash equilibrium to a dynamic game, perfectly coalition-proof Nash equilibrium (PCPNE). We considered two special classes of common agency games: environments without conflict of interest and with pure conflicts of interests. In the former case, we show that PCPNE is equivalent to an intuitive hybrid solution, free-riding-proof core Theorem 1). In a simple public good example, we found that the equilibrium lobby group may not be consecutive (with respect to willingness-to-pay), and public good can be underprovided. In contrast, in the latter case, we show that PCPNE is equivalent to the set of maximal free-riding-proof allocations, in which lobby group contains all the losing party and pivotal winning parties (Theorem 2). The size of lobby is large, and the government collects a lot of rents.

Theorems 1 and 2 highlight the importance of the presence of conflict of interests. If there is no conflict of interests (public good provision problem), there is no rent for the government, while free-riding incentive is very strong and the equilibrium lobby group is small. In contrast, if there is pure conflict of interests in a binary case, the rent for the government is very large, while free-riding incentive is weak and equilibrium lobby group is large. Although Theorem 2 brings a sharp result, it is valid only for two agenda case. It would be interesting to extend the binary choice case to a Hotelling-like one dimensional agenda choice case. This is a future projects.

Appendix

Lemma 1. In the linear utility- quadratic cost public good problem, the free-riding-proof core for S is nonempty if and only if S satisfies $\sum_{i\in S}\theta_i a^*(S) - \frac{1}{2}(a^*(S))^2 \geq \sum_{i\in S}\theta_i a^*(S\backslash\{i\})$ (aggregated "no free riding conditions").

Proof. If the above condition is violated, there is no allocation that satisfies no free riding for S. Thus, we only need to show that if the above condition is satisfied then we can find a core allocation that satisfies $\sum_{i\in T} u_i \geq V(T) = \sum_{i\in T}\theta_i a^*(T) - \frac{1}{2}(a^*(T))^2$. To be instructive, we will not explicitly solve $a^*(T)$ for a while. The strategy we take is to construct an allocation, and verify that it is in the core. Let $u_S \in \mathbb{R}^S_+$ be such that for all $i \in S$

$$u_i = \theta_i a^*(S\backslash\{i\}) + \frac{\theta_i}{\sum_{j\in S}\theta_j}\left(\sum_{i\in S}\theta_i a^*(S) - k(a^*(S))^2 - \sum_{j\in S}\theta_j a^*(S\backslash\{j\})\right).$$

Notice that the contents of the parenthesis is the aggregated "no free riding" surplus: given the no free riding conditions, the most surplus the lobby group S can distribute for their members. The above formula distribute this surplus proportionally according to members' willingnesses-to-pay θs. Obviously, we have $\sum_{i\in S} u_i = V(S) = \sum_{i\in S}\theta_i a^*(S) - k(a^*(S))^2$, and $u_i \geq \theta_i a^*(S\backslash\{i\})$. Thus, we only need to check condition 2. For a coalition $T \subsetneq S$, we have

$$\begin{aligned}
&\sum_{i\in T} u_i - V(T) \\
= &\sum_{i\in T}\theta_i a^*(S\backslash\{i\}) + \frac{\sum_{i\in T}\theta_i}{\sum_{j\in S}\theta_j}\left(\sum_{j\in S}\theta_j a^*(S) - \frac{1}{2}(a^*(S))^2 - \sum_{j\in S}\theta_j a^*(S\backslash\{j\})\right) \\
&-\left(\sum_{i\in T}\theta_i a^*(T) - \frac{1}{2}(a^*(T))^2\right) \\
= &\frac{\sum_{i\in T}\theta_i}{\sum_{j\in S}\theta_j}\left(\sum_{j\in S}\theta_j a^*(S) - \frac{1}{2}(a^*(S))^2\right) - \left(\sum_{i\in T}\theta_i a^*(T) - \frac{1}{2}(a^*(T))^2\right) \\
&+\sum_{i\in T}\theta_i a^*(S\backslash\{i\}) - \frac{\sum_{i\in T}\theta_i}{\sum_{j\in S}\theta_j}\sum_{j\in S}\theta_j a^*(S\backslash\{j\}).
\end{aligned}$$

We want this to be nonnegative for all $T \subset S$. Now, we use quadratic cost and linear utility. The first order condition for optimal public good provision is

$$a^*(S) = \sum_{i\in S} \theta_i.$$

Thus, we have

$$\sum_{i\in S} \theta_i a^*(S) - k(a^*(S))^2 = \frac{\left(\sum_{i\in S}\theta_i\right)^2}{2},$$

and

$$\theta_i a^*(S\backslash\{i\}) = \theta_i \left(\sum_{j\in S}\theta_j - \theta_i\right).$$

Thus, we have

$$\begin{aligned}
&\sum_{i\in T} u_i - V(T)\\
=\ & \frac{\sum_{i\in T}\theta_i}{2\sum_{j\in S}\theta_j}\left(\sum_{j\in S}\theta_j\right)^2 - \frac{1}{2}\left(\sum_{i\in T}\theta_i\right)^2 + \sum_{i\in T}\theta_i \sum_{j\neq i, j\in S}\theta_j - \frac{\sum_{i\in T}\theta_i}{\sum_{i\in S}\theta_i}\sum_{i\in S}\theta_i\sum_{j\neq i,j\in S}\theta_j\\
=\ & \frac{1}{2}\left(\sum_{i\in T}\theta_i\right)\left(\sum_{j\in S}\theta_j\right) + \sum_{i\in T}\theta_i\left(\sum_{j\in S}\theta_j - \theta_i\right) - \frac{\sum_{i\in T}\theta_i}{\sum_{i\in S}\theta_i}\sum_{i\in S}\theta_i\left(\sum_{j\in S}\theta_j - \theta_i\right)\\
=\ & \frac{1}{2}\left(\sum_{i\in T}\theta_i\right)\left(\sum_{j\in S}\theta_j\right) + \sum_{i\in T}\theta_i\left(\sum_{j\in S}\theta_j\right) - \sum_{i\in T}\theta_i^2 - \sum_{i\in T}\theta_i\left(\sum_{j\in S}\theta_j\right) + \frac{\sum_{i\in T}\theta_i}{\sum_{i\in S}\theta_i}\sum_{i\in S}\theta_i^2\\
=\ & \frac{1}{2}\left(\sum_{i\in T}\theta_i\right)\left(\sum_{j\in S}\theta_j\right) - \sum_{i\in T}\theta_i^2 + \frac{\sum_{i\in T}\theta_i}{\sum_{i\in S}\theta_i}\sum_{i\in S}\theta_i^2\\
=\ & \left(\sum_{i\in T}\theta_i\right)\left[\frac{\sum_{j\in S}\theta_j}{2} - \frac{\sum_{i\in T}\theta_i^2}{\sum_{i\in T}\theta_i} + \frac{\sum_{i\in S}\theta_i^2}{\sum_{i\in S}\theta_i}\right]\\
=\ & \left(\sum_{i\in T}\theta_i\right)\left[\frac{\sum_{j\in S}\theta_j}{2} - \sum_{j\in T}\frac{\theta_j}{\sum_{i\in T}\theta_i}\times\theta_j + \sum_{j\in S}\frac{\theta_j}{\sum_{i\in S}\theta_i}\times\theta_j\right].
\end{aligned}$$

The second term is the only negative term, and it takes maximum absolute value when T is composed by the players with the highest values of θ_j. Let us call such value $\theta_{\max}$. Suppose that $\sum_{i\in S} u_i - V(T) < 0$. Then, by focusing the first two terms, we know $\theta_{\max} > \frac{1}{2}\sum_{i\in S}\theta_i$. However, if it is the case, we have

$$
\begin{aligned}
& \frac{\sum_{j\in S}\theta_j}{2} - \sum_{j\in T}\frac{\theta_j}{\sum_{i\in T}\theta_i}\times\theta_j + \sum_{j\in S}\frac{\theta_j}{\sum_{i\in S}\theta_i}\times\theta_j \\
\geq\ & \frac{\sum_{j\in S}\theta_j}{2} - \theta_{\max} + \sum_{j\in S}\frac{\theta_j}{\sum_{i\in S}\theta_i}\times\theta_j \\
\geq\ & \frac{\theta_{\max}}{2} - \theta_{\max} + \frac{\theta_{\max}}{\sum_{i\in S}\theta_i}\times\theta_{\max} \\
>\ & \frac{\theta_{\max}}{2} - \theta_{\max} + \frac{1}{2}\times\theta_{\max} = 0.
\end{aligned}
$$

This is a contradiction. Therefore, u is in $Core^{FRP}(S)$.■

References

[1] Aumann, R., Acceptable Points in General Cooperative N-Person Games, in "Contributions to the Theory of Games IV," H.W. Kuhn and R.D. Luce eds. (1959), Princeton University Press (Princeton), 287-324.

[2] Bergstrom, T., L. Blume, and H. Varian, 1986, On the Private Provision of Public Goods, Journal of Public Economics 29, 25-49.

[3] Bernheim, B.D., B. Peleg, and M.D. Whinston, 1987, Coalition-Proof Nash Equilibria I: Concepts, Journal of Economic Theory 42, 1-12.

[4] Bernheim, B.D., and M.D. Whinston, 1986, Menu Auctions, Resource Allocation, and Economic Influence, Quarterly Journal of Economics 101, 1-31.

[5] Bloch, F., Non-Cooperative Models of Coalition Formation in Games with Spillovers, in "New Directions in the Economic Theory of the Environment," C. Carraro and D. Siniscalco eds. (1997), Cambridge University Press (Cambridge, U.K.), 311-352.

[6] Bombardini, M., Firm Heterogeneity and Lobby Participation, UBC Working Paper, University of British Columbia

[7] Conley, J.P., and H. Konishi, 2002, Migration-Proof Tiebout Equilibrium: Existence and Asymptotic Efficiency, Journal of Public Economics 86, 241-260.

[8] d'Aspremont, C., A. Jacquemin, J.J. Gabszevicz, and J.A. Weymark, 1983, On the Stability of Collusive Price Leadership, Canadian Journal of Economics 16, 17-25.

[9] Foley, D.K., 1970, Lindahl's Solution and the Core of an Economy with Public Goods, Econometrica 38, 66-72.

[10] Gomes, A., and P. Jehiel, 2005, Dynamic Processes of Social and Economic Interactions: On the Persistence of Inefficiencies, Journal of Political Economy 113, 626-667.

[11] Grossman, G., and E. Helpman, 1994, Protection for Sale, American Economic Review 84, 833-850.

[12] Groves T., and J. Ledyard, 1977, Optimal Allocation of Public Goods: a Solution to the "Free Rider" Problem, Econometrica 45, 783-811.

[13] Hart, S., and M. Kurz, 1983, Endogeneous Formation of Coalitions, Econometrica 51, 1047-1064.

[14] Hurwicz, L., 1979, Outcome Functions Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points, Review of Economic Studies 46, 217-225.

[15] Konishi, H., M. Le Breton and S. Weber, 1999, On Coalition-Proof Nash Equilibria in Common Agency Games, Journal of Economic Theory 85, 122-139.

[16] Konishi, H., and D. Ray, 2003, Coalition Formation as a Dynamic Process, Journal of Economic Theory 110, 1-41.

[17] Konishi, H., and M.U. Ünver, 2006, Credible Group-Stability in Many-to-Many Matching Problems, Journal of Economic Theory 129, 57-80 (2006).

[18] Laussel, D., and M. Le Breton, 1998, Efficient Private Provision of Public Goods under Common Agency, Games and Economic Behavior 25, 194-218.

[19] Laussel, D., and M. Le Breton, 2001, Conflict and Cooperation: the Structure of Equilibrium Payoffs in Common Agency, Journal of Economic Theory 100, 93-128.

[20] Le Breton, M., and F. Salaniè, 2003, Lobbying under Political Uncertainty, Journal of Public Economics 87, 2589-2610.

[21] Mitra, D., 1999, Endogenous Lobby Formation and Endogenous Protection: A Long-Run Model of Trade Policy Determination, American Economic Review 89, 1116-1134.

[22] Maruta, T., and A. Okada, 2005, Group Formation and Heterogeneity in Collective Action Games, Hitotsubashi University Discussion Paper #2005-7.

[23] Nishimura, Y., and R. Shinohara, 2006, A Voluntary Participation Game through a Unit-by-Unit Cost Share Mechanism of a Non-Excludable Public Good, Working Paper, Shinshu University.

[24] Palfrey, R.T., and H. Rosenthal, 1984, Participation and the Provision of Discrete Public Goods: a Strategic Analysis, Journal of Public Economics 24, 171-193.

[25] Paltseva, E., 2006, Protection for Sale to Oligopolists, IIES Working Paper, Stockholm University.

[26] Ray, D., 1989, Credible Coalitions and the Core, International Journal of Game Theory 18, 185-187.

[27] Ray, D., and R. Vohra, 2001, Coalitional Power and Public Goods, Journal of Political Economy 109, 1355–1384.

[28] Saijo, T., and T. Yamato, 1999, A Voluntary Participation Game with a Non-Excludable Public Good, Journal of Economic Theory 84, 227-242.

[29] Samuelson, P.A., 1954, The Pure Theory of Public Expenditure, Review of Economics and Statistics 36, 387-389.

[30] Shapley, L., 1971, Core of Convex Games, International Journal of Game Theory 1, 11-26.

[31] Shinohara, R., 2003, Coalition-proof Equilibria in a Voluntary Participation Game, Working Paper, Hitotsubashi University.

[32] Thoron, S., 1998, A Coalition-Proof Stable Cartel, Canadian Journal of Economics 31, 63-76.

[33] Walker, M., 1981, A Simple Incentive Compatible Scheme for Attaining Lindahl Allocations, Econometrica 49, 65-71.

[34] Yi., S.-S., 1996, Endogenous Formation of Customs Unions under Imperfect Competition: Open Regionalism Is Good, Journal of International Economics 41, 153-177.