Chicken in Prison:

Weakly Belief-Free Equilibria in Repeated Games

with Private Monitoring

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Abstract

The present paper introduces the notion of *weakly belief-free equilibria* for repeated games with imperfect private monitoring. This is a tractable class which subsumes, as a special case, the belief-free equilibria, which have played a major role in the existing literature (Ely and Valimaki (2002) and Ely, Horner, and Olszewski (2005)). An example is presented, where a simple weakly belief-free equilibrium outperforms the belief-free equilibria. The present paper also introduces the notion of *reduced games* of a repeated game and shows that weakly belief-free equilibria admit a recursive structure in the space of reduced games. The example embeds the chicken game, as the reduced games, in the repeated prisoners' dilemma with private monitoring.

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1 Introduction

The present paper shows a new way to construct an equilibrium in repeated games with imperfect private monitoring. I relax the notion of belief-free equilibria (Ely and Valimaki (2002) and Ely, Horner, and Olszewski (2005), EHO hereafter) and show that the resulting *weakly belief-free equilibria* continue to possess a nice recursive structure. I then apply this concept to a repeated prisoners' dilemma game with private monitoring and construct a simple equilibrium, which outperforms the belief-free equilibria in this game. The superior performance is due to the fact that the equilibrium in this example partially embodies the essential mechanism to achieve efficiency in imperfect public monitoring games (the asymmetric punishment scheme in Fudenberg, Levine, and Maskin (1994)). In addition, the equilibrium is robust in the sense that players are always taking a strict best reply actions.[1]

A repeated game is a dynamic game where the same set of agents play the same game (stage game) over an infinite time horizon. Economists and game theorists have successfully employed such a class of models to examine why and how self-interested agents manage to cooperate in a long-term relationship. A repeated game is said to have (imperfect) *private monitoring* if agents' actions are not directly observable and each agent receives imperfect private information (private signal) about the opponents' actions. This class of games has a number of important potential applications. A leading example is a price competition game where firms may offer secret price cut to their customers. In such a situation, each firm's sales level serves as the private signal, which imperfectly reveals the rivals' pricing behavior. Despite the wealth of potential applications, however, this class of games is not fully understood[2]. This is in sharp contrast to the case where players share the same information (repeated games with *perfect* or imperfect *public* monitoring), where the set of equilibria are fully characterized[3]. The main

[1] In contrast, it is essential that players are indifferent over a set of actions in the belief-free equilibria. Bhaskar (2000) argues that this is a problematic feature, because such an equilibrium may not be purified (in the sense of Harsanyi) by a plausible payoff purturbation. See Section 5 for more discussion.

[2] If communication is allowed, it is known that the folk theorm holds in private monitoring repeated games (Compte (1998) and Kandori and Matsushima (1998)). Recent literature, including the present article, mainly explorse the possibility of cooperation under no communication. This is important, because in a major applied area (collusion), communication is explicitly prohibited by the anti-trust laws.

[3] The *self-generation* condition of Abreu, Pearce and Stacchetti (1990), and the *folk theorem* of Fudenberg, Levine, and Maskin (1994) fully characterize the equilibria in an

difficulty in the private monitoring case comes from the fact that players have *diverse* information about each other's behavior.

In the perfect or public monitoring case, players always share a mutual understanding about what they are going to do in the future. In the private monitoring case, however, each player has to draw statistical inferences about the opponents' future action plans, because they depend on unobservable history of the opponents' private signals. The inferences quickly become complicated over time, even if players adopt relatively simple strategies. Hence, checking the equilibrium condition is in general a demanding task in repeated games with private monitoring (see Kandori (2002)).

To deal with this difficulty, the existing literature has adopted two alternative approaches. One is the *belief-based approach* (see, for example, Sekiguchi (1997), Bhaskar and van Damme (2002), Bhaskar and Obara (2002)) which looks at judiciously constructed model and equilibria, where the inference problem becomes tractable. The other approach, the *belief-free approach*, in contrast, by-passes this problem altogether by constructing equilibria where players do not have to draw the statistical inferences at all. Let us denote player i's action and private signal in period t by $a_i(t)$ and $\omega_i(t)$. Note that, in general, each player i's continuation strategy at time $t+1$ is determined by his *private history* $h_i^t = (a_i(1), \omega_i(1), ..., a_i(t), \omega_i(t))$. The belief-free approach constructs an equilibrium where player i's continuation strategy is a best reply to the opponents' continuation strategies *for any realization* of $h_{-i}^t = (a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t), \omega_{-i}(t))$, thereby making player i's belief over h_{-i}^t irrelevant. Such an equilibrium is called a *belief-free equilibrium* (EHO). The core of this approach was provided by the influential works by Piccione (2002), Obara (1999), and Ely and Valimaki (2002). This idea was later substantially generalized by Matsushima (2004), EHO (2005), Horner and Olszewski (2006), and Yamamoto (2006). EHO shows that the set of belief-free equilibria can be characterized by a simple recursive method, similar to Abreu, Pearce and Stacchetti (1990).

In the present paper, I propose a weakening of the belief-free conditions, leading to a set of equilibria which are still tractable and manage to sustain a larger payoff set. Note that the belief-free conditions imply that, at the beginning of period $t+1$, player i do not have to form beliefs over $h_{-i}^t = (a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t), \omega_{-i}(t))$. In contrast, I require that player i do not need to form beliefs over $(a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t))$, *omitting the last piece of information* $\omega_{-i}(t)$ *from the requirement.* This says that player i does not have to know the opponents histories *up to the previous actions.*

important class of strategies, known as public strategies.

However, player i need to understand correctly that, for each possible action profile $a(t)$, the private signals in the previous period are distributed according to $p(\omega(t)|a(t))$. I call equilibria with this property a *weakly belief-free equilibrium.*

To show that weakly belief-free equilibria have a recursive structure, I depart from the tradition of looking at the continuation *payoff sets*. In perfect or imperfect monitoring case (Abreu, Pearce, and Stacchetti (1990) as well as the belief-free approach in the private monitoring case (see EHO (2005)), it has been a common practice to keep track of the set of continuation payoffs to exploit the recursive structure. In contrast, I introduce the notion of *reduced games* and examine their recursive structure. A reduced game at time t is a game with the same set of players and actions as in the stage game. The reduced game payoff to player i under action profile a is defined to be i's continuation payoff when (i) players are adopting current continuation strategies and (ii) current action profile is a. When players use one-period memory strategies, current actions fully specifies the continuation strategies, so that the reduced game payoff to player i is represented as a simple function $u_i^t(a)$. In this case, the weakly belief-free equilibria can be characterized by the property that *players always play a correlated equilibrium of the reduced game* after any history. In general, players continuation strategies depends on the past history as well as current action. Let θ_i be a state variable to summarize player i's private history. In the general case, a reduced game payoff to player i is represented as $v_i^t(a|\theta_1, ..., \theta_N)$, and the weakly belief-free equilibria are characterized by the property that players always play a *Bayesian correlated equilibrium* of the reduced game.

The paper is organized as follows. Section 2 presents the basic model and defines weakly belief-free equilibria. Then, weakly belief-free equilibria are characterized by a recursive method, for one-period memory case (Section 3) and for the general case (Section 4). Section 5 presents an example of one-period memory weakly belief-free equilibrium, which outperforms the belief-free equilibria. This example 'embeds' the chicken game as the reduced games in the repeated prisoners' dilemma. Appendices A and B contain technical details of the example in Section 5.

2 The Model

Let us first define the stage game. Let A_i be the (finite) set of actions for player $i = 1, ..., N$ and define $A = A_1 \times \cdots \times A_N$. We mainly consider the case with imperfect private monitoring, where each player i observes her own action a_i and private signal $\omega_i \in \Omega_i$. (Our formulation, however, accommodates the imperfect public monitoring case: see footnote 4.) We denote $\omega = (\omega_1, ..., \omega_N) \in \Omega = \Omega_1 \times \cdots \times \Omega_N$ and let $p(\omega|a)$ be the probability of private signal profile ω given action profile a (we assume that Ω is a finite set). We also assume that each player cannot see which action is taken (or not taken) for sure; that is, we suppose, given any $a \in A$, each $\omega_i \in \Omega_i$ occurs with a positive probability.[4] We denote the marginal distribution of ω_i as $p_i(\omega_i|a)$. Player i's realized payoff is determined by her own action and signal, and denoted $\pi_i(a_i, \omega_i)$. Hence her *expected* payoff is given by

$$g_i(a) = \sum_{\omega \in \Omega} \pi_i(a_i, \omega_i) p(\omega|a).$$

Our formulation makes sure that the realized payoff π_i conveys no more information than a_i and ω_i do. The stage game is to be played repeatedly over infinite time horizon $t = 1, 2, ...$, and each player i's discounted payoff is $\sum_{t=1}^{\infty} g_i(a(t))\delta^{t-1}$, where $\delta \in (0, 1)$ is the discount factor and $a(t) \in A$ is the action profile at time t. A mixed action for player i is denoted by $\alpha_i \in \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions over A_i. With an abuse of notation, we denote the expected payoff and signal distribution under mixed action profile $\alpha = (\alpha_1, ..., \alpha_N)$ by $g_i(\alpha)$ and $p(\omega|\alpha)$ respectively.

A *private history* for player i up to time t is the record of player i's past actions and signals, $h_i^t = (a_i(1), \omega_i(1), ..., a_i(t), \omega_i(t)) \in H_i^t \equiv (A_i \times \Omega_i)^t$. To determine the initial action of each player, we introduce dummy initial history (or *null history*) h_i^0 and let H_i^0 be a singleton set $\{h_i^0\}$. A pure strategy s_i for player i is a function specifying an action after any history: formally, $s_i : H_i \to A_i$, where $H_i = \cup_{t \geq 0} H_i^t$. Similarly, a (behaviorally) mixed strategy for player i is denoted by $\sigma_i : H_i \to \Delta(A_i)$.

A *continuation strategy* for player i after private history h_i^t is denoted by $\sigma_i[h_i^t]$, defined as (i) $\sigma_i[h_i^t](h_i^0) = \sigma_i(h_i^t)$ and (ii) for any other history $h_i \neq h_i^0$, $\sigma_i[h_i^t](h_i) = \sigma_i(h_i^t h_i)$. For any given strategy profile $\sigma = (\sigma_1, ..., \sigma_N)$ and any private history profile $h^t = (h_1^t, ..., h_N^t)$, let $BR(\sigma_{-i}[h_{-i}^t])$ be the set of best reply strategies for player i against $\sigma_{-i}[h_{-i}^t]$. EHO defined a *belief-free* strategy profile as follows.

Definition 1 *A strategy profile σ is* belief-free *if for any h^t and i, $\sigma_i[h_i^t] \in BR(\sigma_{-i}[h_{-i}^t])$.*

[4] We do not require that the joint distribution of the private signals has full support. Our assumption accomodates the case of *imperfect publich monitoring*, where all players receive the same singnal with probability one (hence the event where players receive different signals has zero probability).

Note that the above requirement implies that the current continuation strategy for a player is a best reply, for *any realization of* private histories of other players. In this sense, in a belief-free equilibrium players never need to compute beliefs over opponents' private histories. EHO (2006) showed that belief-free equilibria are tractable in the sense that a recursive method similar to Abreu, Pearce and Stacchetti (1990) can be employed to obtain a complete characterization of belief-free equilibrium payoffs.

In the present paper, I propose a weakening of the belief-free conditions, leading to a set of equilibria which are still tractable and manage to sustain a larger payoff set. Note that the belief-free conditions imply that, at the beginning of period $t+1$, player i do not have to form beliefs over $h_{-i}^t = (a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t), \omega_{-i}(t))$. In contrast, I require that player i do not need to form beliefs over $(a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t))$, *omitting the last piece of information* $\omega_{-i}(t)$ *from the requirement.*

Now let us formalize the above idea. Fix any strategy profile σ and history profile $h^t = (a(1), \omega(1), ..., a(t), \omega(t))$. Consider a mixture of continuation strategy profiles of the opponents,

$$\sigma_{-i}[a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t), \omega'_{-i}(t)] \text{ for all } \omega'_{-i}(t) \in \Omega_{-i},$$

each of which is chosen with conditional probability $p_{-i}(\omega'_{-i}(t)|a(t), \omega_i(t))$. This would be player i's belief over the opponents' continuation strategies given his private history h_i^t (actually, the only relevant part in h_i^t is $(a_i(t), \omega_i(t))$), *if he knew* $(a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t))$. Let us denote the probability distribution thus defined over the opponents' continuation strategies by $\overline{\sigma}_{-i}[a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t)|h_i^t]$.[5]

Definition 2 *A strategy profile* σ *is* weakly belief-free *if for any* $h^t = (a(1), \omega(1), ..., a(t), \omega(t))$ *and* i*,* $\sigma_i[h_i^t] \in BR(\overline{\sigma}_{-i}[a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t)|h_i^t])$.

This definition says that, under a weakly belief-free strategy profile, player i in period $t+1$ does not have to know the opponents' histories *up to the previous actions* $(a_{-i}(1), \omega_{-i}(1), ..., a_{-i}(t))$ to calculate his optimal continuation strategy. He may, however, need to form some beliefs over the previous signals $\omega_{-i}(t)$. More precisely, he may need to understand correctly that, for each possible action profile $a(t)$, the private signals in the previous period are distributed according to $p(\omega(t)|a(t))$. In the subsequent sections, we characterize the set of weakly belief-free equilibria.

[5] Note that this is a *correlated strategy profile* of the opponents, when $N \geq 3$. Below we assume that the best reply correspondence BR_i is defined over the domain of correlated strategy profiles of the opponents.

3 One-Period Memory

In this section, we consider weak belief-free equilibria with one-period memory[6]. This is a particularly tractable class which subsumes a major part of the belief-free equilibrium identified by Ely and Valimaki (2002) and EHO (2005) as a special case. We say that player i's strategy has *one-period memory* if it specifies current (mixed) action $\alpha_i(t)$ depending only on $a_i(t-1)$ and $\omega_i(t-1)$.

Definition 3 *A* one-period memory strategy *for player i is defined by an initial (mixed) action $\alpha_i(1)$ and transition rules $m_i^t : A_i \times \Omega_i \to \Delta(A_i)$, $t = 1, 2, ...$ The probability of $a_i(t+1)$ given $a_i(t)$ and $\omega_i(t)$ under m_i^t is denoted by*

$$m_i^t(a_i(t+1)|a_i(t), \omega_i(t)).$$

The set of one-period memory transition rules for player i is denoted by M_i.

Under a one-period memory strategy profile, at each moment t, the current action profile $a(t)$ determines the continuation play (independent of previous history). (Continuation strategies after $t+1$ are determined, under a one-period memory strategy profile, by $a(t)$ and $\omega(t)$. As the latter is generated by $p(\omega(t)|a(t))$, $a(t)$ alone determines the contingent action plans of players after $t+1$.) Hence, we can define $u_i^t(a(t))$ as the (average) expected continuation payoff to player i. Function $u_i^t(a(t))$ can be regarded as a payoff in a game, which has the same action sets as the stage game. Let us call the game defined by $(u_i^t, A_i)_{i=1,...,N}$ a **reduced game**. This enables us to view a repeated game as *a sequence of reduced games*, and I will analyze its recursive structure. This is an important departure from the previous literature (Abreu, Pearce and Stacchetti (1990) and EHO (2005)) which views a repeated game as a sequence of continuation *payoff sets* and exploits its recursive structure.

Before stating our characterization, we need to define a couple of concepts. Consider a reduced game played with a (partial) correlation device $\omega \in \Omega$. We say that a probability distribution[7] q on $A \times \Omega$ is a *correlated equilibrium* of game $u : A \to \mathsf{R}^N$, when

$$\forall i \forall a_i \forall \omega_i \forall a_i' \sum_{a_{-i}, \omega_{-i}} u_i(a) q(a, \omega) \geq \sum_{a_{-i}, \omega_{-i}} u_i(a_i', a_{-i}) q(a, \omega). \quad (1)$$

[6]Deviations to general strategies (not neccessarily with one-period memory) are allowed, so that we are *not* weakening the usual equilibrium conditions.

[7]In the relevant case to our analysis, each player i first receives ω_i and then takes (possibly mixed) action, and this imposes certain restictions on the distriburtion of (a, ω). We chose not to exclude other distributions at this stage just to simplify the exposition.

In contrast to the standard definition of correlated equilibrium, which only consider a joint distribution of actions (interpreted as recommendations issued by the correlation device), this definition consider the situation where each player receives recommended action *and some additional information* ω_i. The condition (1) ensures that, in each of those situations, player i has an incentive to follow the recommended action a_i. The set of correlated equilibria of game u is denoted by $C(u)$:

$$C(u) \equiv \{q \in \Delta(A \times \Omega) | \text{ Condition (1) holds.}\}, \tag{2}$$

where $\Delta(A\times\Omega)$ is the set of probability distribution over $A\times\Omega$. A standard result for the set of correlated equilibria carries over to our formulation: From (1), we can see that $C(u)$ is convex. As it plays a vital role in what follows, we state it here.

Lemma 4 *For any $u : A \to \mathsf{R}^N$, $C(u)$ is convex.*

Now consider one-period memory transition rules $m = (m_1, ..., m_N) \in M = M_1 \times \cdots \times M_N$. Given such a profile and our monitoring structure $p(\omega|a)$, the probability of $(a(t+1), \omega(t))$ given $a(t)$ is determined as follows. This is going to play the role of correlation device for the reduced game for time t.

Definition 5 The action-signal distribution *given $a(t)$ under one-period memory strategy profile m is defined by*

$$q^m(a(t+1), \omega(t)|a(t)) \equiv \prod_{i=1}^{N} m_i(a_i(t+1)|a_i(t), \omega_i(t))p(\omega(t)|a(t)). \tag{3}$$

Its marginal distribution of $a(t+1)$ is the law of motion *under m and defined by*

$$p^m(a(t+1)|a(t)) \equiv \sum_{\omega(t)\in\Omega} q^m(a(t+1), \omega(t)|a(t)). \tag{4}$$

Now we are ready to introduce our equilibrium concept.

Definition 6 *A set of reduced games $U \subset \{u|u : A \to \mathsf{R}^N\}$ is* self-generating *if, for any $u \in U$, there exist $v \in U$ and a one-period memory transition rule profile $m \in M$ such that*

$$\forall a \;\; u(a) = (1-\delta)g(a) + \delta \sum_{a'\in A} v(a')p^m(a'|a) \tag{5}$$

and

$$\forall a \quad q^m(\cdot,\cdot|a) \in C(v), \tag{6}$$

where $C(u)$, q^m and p^m are defined by (2), (3), and (4).

Note that condition (6) is the key requirement that players are always playing a correlated equilibrium of the reduced game, *on and off the path of play.* We claim that the equilibrium payoffs associated with a self-generating set can be achieved in the repeated game equilibria. Let $N(u)$ be the Nash equilibrium payoff set associated with game u. Then one obtains the following complete characterization of one-period memory belief-free equilibria, which is similar to Abreu, Pearce and Stacchetti (1990) (note, however, that the present recursive characterization is given in terms of *reduced games*, in contrast to continuation *payoff sets* in APS).

Theorem 7 *Let $U \subset \{u | u : A \rightarrow \mathsf{R}^N\}$ be self-generating and bounded in the sense that there exists $K > 0$ such that $|u_i(a)| < K$ for all i, $u \in U$, and a. Then, any point in*

$$N(U) \equiv \bigcup_{u \in U} N(u)$$

can be achieved as the average payoff of a one-period memory weakly belief-free *sequential equilibrium. The set of all* one-period memory weakly belief-free *sequential equilibrium payoff profiles is given by $N(U^*)$, where U^* is the largest (in the sense of set inclusion) bounded self-generating set.*

Note that a one-period memory weakly belief-free sequential equilibrium has the following features; (i)for each player i, the distribution of $a_i(t)$ depends only on $\omega_i(t-1)$ and $a_i(t-1)$ in each stage $t > 1$ and (ii) players always play a correlated equilibrium of the repeated game after any history (on and off the path of play). Also note that, if (partial) correlation device is available at the beginning of the repeated game, the set of one-period memory belief-free sequential equilibrium is given by $C(U) \equiv \bigcup_{u \in U} C(u)$ (i.e., the correlated equilibria associated with reduced games $u \in U$).

Proof. For any $u \in U$, repeated applications of (5) induce sequence of reduced games $\{u^t\}$ and one-period memory strategies $\{m^t\}$ that satisfy

$$\forall a \quad u^t(a) = (1-\delta)g(a) + \delta \sum_{a' \in A} u^{t+1}(a') p^{m^t}(a'|a),$$

and

$$\forall a \quad q^{m^t}(\cdot,\cdot|a) \in C(u^{t+1}), \tag{7}$$

for $t = 1, 2, ...$ with $u^1 = u$. Hence, for any $T(> 2)$, we have

$$u(a) = (1-\delta)\left\{g(a) + E\left[\sum_{t=2}^{T-1} g(a(t))\delta^{t-1} + u^T(a(t+1))\delta^{T-1}\middle| a\right]\right\}.$$

The expectation $E[\cdot|a]$ presumes that the distribution of $a(t+1)$ given $a(t)$ is $p^{m^t}(a(t+1)|a(t))$ with $a(1) = a$. As u^T is bounded, we can take the limit $T \to \infty$ to get

$$u(a) = (1-\delta)\left\{g(a) + E\left[\sum_{t=2}^{\infty} g(a(t))\delta^{t-1}\middle| a\right]\right\}. \tag{8}$$

Hence $u(a)$ can be interpreted as the average payoff profile when the players choose a today and follow one-period memory strategy profile m^t, $t = 1, 2, ...$ Let α be a (possibly mixed) Nash equilibrium of game u, and let σ be the strategy where α is played in the first period and the players follow m^t, $t = 1, 2, ...$ By construction σ achieves average payoff $u(\alpha)$ (the expected payoff associated with α), and we show below that it is an sequential equilibrium because after any history no player can gain by one-shot unilateral deviation[8]. In the first period, no one can gain by one-shot unilateral deviation from α because it is a Nash equilibrium of game u. For stage $t > 0$, take any player i and any private history for her $(a_i^0(1),,,, a_i^0(t-1), \omega_i^0(1), ..., \omega_i^0(t-1))$. Let $\mu(a(t-1))$ be her belief about last period's action profile given her private history. Then, her belief about current signal distribution is

$$q(a(t), \omega(t)) = \sum_{a(t-1)\in A} q^{m^{t-1}}(a(t), \omega(t)|a(t-1))\mu(a(t-1)).$$

(Note that under σ other players' continuation strategies do not depend on their private histories *except* their current signals.) Let $v \in U$ be the continuation payoff in stage t (including stage t payoff). Then, condition (7) for self-generation, $q^m(\cdot, \cdot|a) \in C(v)$ for all a, and the convexity of the correlated equilibrium set $C(v)$ implies $q \in C(v)$. This means that player i cannot gain by one-shot unilateral deviation at this stage.

Conversely, given any one-period memory weakly belief-free sequential equilibrium, one can calculate a sequence of reduced games u^t, $t = 1, 2, ...$ It is straight forward to check $U' \equiv \{u^t | t = 1, 2, ...\}$ is a self-generating

[8] The standard dynamic programming result shows that this implies that no (possibly infinite) sequence of unilateral deviations is profitable.

set bounded by $K \equiv \max_{i.a} |g_i(a)|$. As a union of self-generating sets bounded by K is also self-generating and bounded by K, we conclude that the set of all one-period memory weakly belief-free sequential equilibrium payoff profiles is given by $N(U^K)$, where U^K is the largest (in the sense of set inclusion) self-generating set bounded by K. Now consider any self-generating set U which is bounded (not necessarily by K). The first part of this proof shows that U is actually bounded by K (as any $u \in U$ is an average payoff profile of the repeated game). This implies $U^K = U^*$, which completes the proof. ■

4 General Strategies

In this section, we consider weakly belief-free equilibria in fully general strategies. For the purpose of this section, it is convenient to represent a strategy in the following way. For each player i, we specify

- a set of states Θ_i
- an initial state $\theta_i(1) \in \Theta_i$
- (mixed) action choice for each state, $\rho_i : \Theta_i \to \Delta(A_i)$
- state transition $\tau_i : \Theta_i \times A_i \times \Omega_i \to \Delta(\Theta_i)$. This determines the probability distribution of the next state $\theta_i(t+1)$ based on the current state $\theta_i(t)$, current action $a_i(t)$, and current private signal $\omega_i(t)$.

We call $ms_i \equiv (\Theta_i, \theta_i(1), \rho_i, \tau_i)$ a *machine strategy*. All strategies can trivially be represented as a machine strategy, when we set Θ_i equal to the set of all histories for player i: $\Theta_i = H_i$.[9] The action choice and transition rule are assume to be time-independent, but this is without loss of generality. We can always include the current time in the state variable θ_i (as $\theta_i = (\widehat{\theta}_i, t)$).

Under a machine strategy profile $ms = (ms_i, ..., ms_N)$, if we fix a current state profile $\theta(t) \in \Theta \equiv \Theta_1 \times \cdots \times \Theta_N$, continuation strategies are fully specified. For each ms, we can compute the continuation payoff to player i, when (i) all players' continuation strategies are specified by ms given $\theta(t)$ and (ii) the current action profile is $a(t)$. Denote this by $v_i(a(t)|\theta(t))$. Note well that the function u_i is *defined over all* $a(t)$, some of which may be

[9] In this case, transition rule τ_i is deterministic: given $\theta_i(t) = h_i(t)$, $a_i(t)$, and $\omega_i(t)$, τ_i assigns probability one to $\theta_i(t+1) = (\theta_i(t), a_i(t), \omega_i(t))$. The initial state should be the null history $\theta_i(t) = h_i^0$.

outside of the support of the current mixed action specified by ms under $\theta(t)$. This makes v_i a useful tool to check the profitability of one-shot deviations from the given strategy profile ms. We call function $v : A \times \Theta \to \Re^N$ an **ex-post reduced game**.

If a machine strategy profile ms is a weakly belief-free equilibrium, in each period t, players are taking mutual best replies *for each* $(\theta(t-1), a(t-1))$.[10] Given $(\theta(t-1), a(t-1))$, the machine strategy profile under consideration provides some joint distribution of $\omega(t-1)$, $\theta(t)$, and $a(t)$, denoted by $r(\omega(t-1), \theta(t), a(t))$ Given a realization of $\omega_i(t-1)$, $\theta_i(t)$, and $a_i(t)$ (interpreted as a recommended action) of this distribution r, player i must be happy to choose $a_i(t)$. This can be regarded as a *correlated equilibrium of a Bayesian game*, where types θ, recommended actions a, and some additional information ω are generated by a *joint* distribution $r(\omega, \theta, a)$, and the ex-post payoff function is given by the ex-post reduced game $v_i(a|\theta)$.

Definition 8 *Probability distribution r over $\Omega \times \Theta \times A$ is a* Bayesian correlated equilibrium *of ex-post reduced game v when*

$$\forall i \forall a_i \forall \omega_i \forall \theta_i \forall a_i' \quad \sum_{a_{-i}, \omega_{-i}, \theta_{-i}} v_i(a|\theta) r(\omega, \theta, a) \geq \sum_{a_{-i}, \omega_{-i}, \theta_{-i}} v_i(a_i', a_{-i}|\theta) r(\omega, \theta, a). \tag{9}$$

The set of Bayesian correlated equilibria of ex-post reduced game v is denoted by

$$BC(v) = \{r \in \Delta(\Omega \times \Theta \times A) | \textit{Condition (9) holds.}\} \tag{10}$$

The defining condition (9) shows that BC is a convex set, which plays an important role in what follows. The following notation clarifies how $r(\omega(t-1), \theta(t), a(t))$ is determined in the repeated game.

Definition 9 *The* state-action-signal distribution *given $\theta(t-1)$, $a(t-1)$ under machine strategy profile ms is defined by*

$$q^{ms}(\omega(t-1), \theta(t), a(t) | \theta(t-1), a(t-1)) =$$

$$\sum_{\omega(t-1)} \prod_{i=1}^{N} \rho_i(a_i(t)|\theta_i(t)) \tau_i(\theta_i(t)|\theta_i(t-1), a_i(t-1), \omega_i(t-1)) p(\omega(t-1)|a(t-1)), \tag{11}$$

[10] This is because, under a machine strategy profile, previous history $(a(1), \omega(1), ..., a(t-2), \omega(t-2))$ affects the continuation strateies at time t only when it affects $\theta(t-1)$.

where ρ_i and τ_i are action choice and state transition rule of ms_i. Its marginal distribution of $(\theta(t), a(t))$ is the law of motion *under ms and defined by*

$$p^{ms}(\theta(t), a(t)|\theta(t-1), a(t-1)) \equiv \sum_{\omega(t-1)\in\Omega} q^{ms}(\omega(t-1), \theta(t), a(t)|\theta(t-1), a(t-1)). \tag{12}$$

Now we are ready to state our main characterization conditions.

Definition 10 *An ex-post reduced game $v_i(a|\theta)$, $i = 1, ..., N$ is* self-generating *if there exists a machine strategy profile ms (defined over states $\theta \in \Theta$) such that*

$$\forall i \forall a \forall \theta \quad v_i(a|\theta) = (1-\delta)g_i(a) + \delta \sum_{a' \in A} v_i(a'|\theta')p^{ms}(\theta', a'|\theta, a) \tag{13}$$

and

$$\forall a \forall \theta \quad q^{ms}(\cdot, \cdot, \cdot|\theta, a) \in BC(v), \tag{14}$$

where q^{ms} and p^{ms}, and $BC(v)$ are defined by (11), (12), and (10).

In contrast to the formulation in Section 3, where we considered set U of reduced games, here we consider a single function profile v. In Section 3, we needed to consider a set of reduced games to allow the possibility that the one-period memory transition rule is time-dependent (hence a set of reduced games $\{u^t | t = 1, 2, ...\}$ is associated with an equilibrium). Here, we can confine our attention to a single function profile v, because state θ can encode time (as $\theta = (\widehat{\theta}, t)$) and single function profile $v(\cdot|\theta)$ can represent potentially time-dependent ex-post reduced games.

Given an ex-post reduced game $v = v(a|\theta)$, let $N(v)$ be the set of Nash equilibrium payoff profiles of game $g(a) = v(a|\theta)$ for some θ. Suppose that v is self-generating and $w \in N(v)$ is obtained as a Nash equilibrium of game $g(a) = v(a|\theta)$. Then, w is obtained as a machine strategy equilibrium where the initial state is θ. Formally we obtain the following characterization result.

Theorem 11 *Let v be a self-generating ex-post reduced game, which is bounded in the sense that there exists $K > 0$ such that $|v_i(a|\theta)| < K$ for all i, a, and θ. Then any $w \in N(v)$ is a weakly belief-free equilibrium payoff profile. Conversely, any weakly belief-free equilibrium payoff profile is an element of $N(v)$, for some bounded self-generating ex-post reduced game v.*

The proof is basically the same as in Section 3. Condition (14) says that players are always playing a Bayesian correlated equilibrium of the ex-post reduced game *after any history (on and off the path of play)*, and it implies that one-shot deviations from the machine strategy profile do not pay. Hence, the standard dynamic programming argument shows that players are always choosing mutual best replies.

Remark 12 *Given a weakly belief-free equilibrium machine strategy profile, we can calculate the associated ex-post reduced game $v(a|\theta)$. The original (pure or mixed) equilibrium payoff is equal to a Nash equilibrium of $v(a|\theta(1))$, where $\theta(1)$ is the initial state profile of the given machine strategies. However, the weakly belief-free requirement implies that any Nash equilibrium payoff profile of $v(a|\theta)$ for* any *θ (not necessarily the initial one) is also an equilibrium payoff profile of the repeated game. This comes from the following fact. Consider the strategy profile defined as (i) the initial action profile is an equilibrium of game $g(a) = v(a|\theta)$ and (ii) the continuation play is given by the machine strategy profile. As the machine strategy profile is weakly belief-free, the strategy profile thus constructed satisfies the property that one-shot deviations are never profitable (hence it is an equilibrium).*

Remark 13 *We can extend Theorem 11 to the case where there is a correlation device at the beginning of the repeated game. It is also straightforward to incorporate public randomization device at each moment of time.*

Example: The belief-free (hence by definition weakly belief-free) equilibrium by Kandori and Obara (2006) (a private strategy equilibrium in a imperfect *public* monitoring game) is an example of Theorem 11. Unlike the equilibrium in Ely and Valimaki (2002), this equilibrium does not have the one-period memory property. It is a machine strategy profile where state space is a simple set $\Theta_i = \{P, R\}$. In this equilibrium, players are always playing an *ex-post equilibrium* (a special case of Bayesian correlated equilibrium) of $v(a|\theta)$. (More explanation to be added.)

5 An Example: Chicken in Prison

In this section, we present a simple example of one-period memory belief-free equilibrium, where set U in our characterization (Definition 6) is singleton. This example "embeds" the chicken game (as the reduced game) in a repeated prisoner's dilemma game. The stage game has the following prisoner's dilemma structure:

	C	D
C	$1, 1$	$-1/6, 3/2$
D	$3/2, -1/6$	$0, 0$

.

For computational purposes, I normalized the payoffs in such a way that the maximum and minimum payoffs are 1 and 0 respectively, but it may be easier to consider

	C	D
C	$6, 6$	$-1, 9$
D	$9, -1$	$0, 0$

,

which is proportional to the first payoff table. Each player's private signal has binary outcomes, $\omega_i = G, B, i = 1, 2$. The signal profile distribution depends on the current action profile and it is denoted by $p(\omega_1, \omega_2 | a_1, a_2)$. The relationship between current action and signal profiles is as follows:

$(C, C) \Longrightarrow$

ω_1Âω_2	G	B
G	$1/3$	$1/3$
B	$1/3$	0

$(D, C) \Longrightarrow$

ω_1Âω_2	G	B
G	$1/8$	$1/2$
B	$1/4$	$1/8$

$(C, D) \Longrightarrow$

ω_1Âω_2	G	B
G	$1/8$	$1/4$
B	$1/2$	$1/8$

$(D, D) \Longrightarrow$

ω_1Âω_2	G	B
G	0	$2/5$
B	$2/5$	$1/5$

Those set of distributions admits the following natural interpretation. When both players cooperate, they can avoid mutually bad outcome (B, B). If one player defects, with a high probability $(1/2)$, the defecting player enjoys a good outcome (G) while the other player receives a bad one (B). Finally, when both player defect, they cannot achieve mutually good outcome (G, G).

As I will argue below, I made some entries in the above tables equal to 0 (so that the example has "moving supports") to simplify the analysis, but this is inessential to the main results (i.e., similar results are obtained even though I make those entries non-zero, small numbers).

Let us consider the following simple (and intuitive) one-period memory transition rule:

$$a_i(t) = \begin{cases} C & \text{if } \omega_i(t-1) = G \\ D & \text{if } \omega_i(t-1) = B \end{cases}. \tag{15}$$

The *reduced game* payoff for profile a is defined to be the average payoff when a is played in the initial period and then players follow the above strategy. Let us denote the reduced game payoffs by

	C	D
C	x, x	α, β
D	β, α	y, y

.

Those payoffs are determined by the system of dynamic programming equations

$$\begin{cases} x = (1-\delta) + \delta\frac{1}{3}(x + \alpha + \beta) \\ y = \delta(\frac{1}{5}y + \frac{2}{5}(\alpha + \beta)) \\ \alpha = (1-\delta)(-\frac{1}{6}) + \delta(\frac{1}{8}x + \frac{1}{4}\alpha + \frac{1}{2}\beta + \frac{1}{8}y) \\ \beta = (1-\delta)\frac{3}{2} + \delta(\frac{1}{8}x + \frac{1}{2}\alpha + \frac{1}{4}\beta + \frac{1}{8}y) \end{cases}.$$

For example, $x = v_i(C, C)$ is associated with current payoff $1 = g_i(C, C)$, and given the current action profile (C, C) and the transition rule (15), the continuation payoff is $x = v_i(C, C)$, $\alpha = v_i(C, D)$, or $\beta = v_i(D, C)$ with probability $1/3$. Hence we have the first equality $x = (1-\delta) \times 1 + \delta\frac{1}{3}(x + \alpha + \beta)$. The rest admits a similar interpretation. When $\delta = 0.99$, we have the following solutions:

$$\begin{cases} x = 0.641\,26 \\ y = 0.627\,89 \\ \alpha = 0.629\,14 \\ \beta = 0.642\,5. \end{cases} \tag{16}$$

Note first that we have $\beta > x$ and $\alpha > y$, which means that the reduced game is a "Chicken Game". The high discount factor is responsible for the fact that those four payoffs are close to each other. This is because the transition rule (15) defines an irreducible and aperiodic Markov chain over the stage game action profiles (C, C), (C, D), (D, C), and (D, D), and the average payoff to player i given any initial action profile (i.e., x, y, α, or β) tends to, as $\delta \to 1$,

$$g^* = \sum_a g_i(a)\mu^*(a), \tag{17}$$

where μ^* is the unique (ergodic) stationary distribution of the Markov chain.

Hence, the reduced game coincides with the Prisoner's Dilemma game (i.e., the stage game) when $\delta = 0$ and all the four payoff profiles of the reduced game tend to the single point (g^*, g^*) given by (17), as $\delta \to 1$. Numerical computation shows that, when $\delta > 4/7$, the reduced game becomes a chicken game ($\alpha > y$, and $\beta > x$) (See Figure 1).

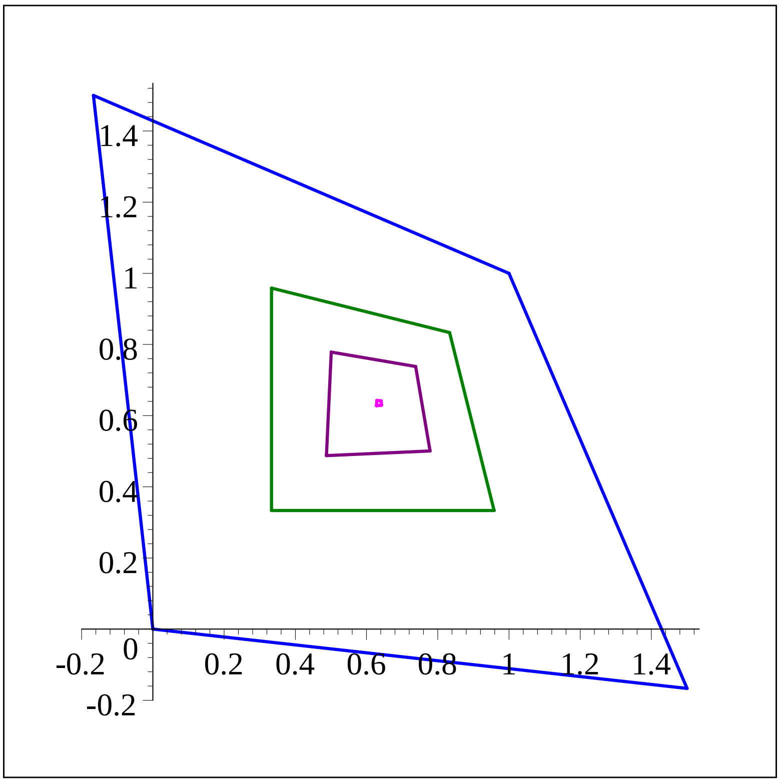


Figure 1 Reduced Games: From outer to inner, $\delta = 0$, $\delta = 4/7$, $\delta = 4/5$, $\delta = 0.99$.

I will show that, when $\delta \geq 0.989\,54$, the action profile distribution after any history becomes a correlated equilibrium of the reduced game (hence the one-period memory transition rule (15), coupled with a suitable initial action profile, is a weakly belief-free equilibrium).

Now let us check the incentive constraints. Note that the reduced game is strategically equivalent to

1\2	C	D
C	$0,0$	$\alpha-y,\beta-x$
D	$\beta-x,\alpha-y$	$0,0$

(18)

(In general, a game $g_i(a)$, $i=1,...,N$ is strategically equivalent to game $g_i(a)+K_i(a_{-i})$, which means that both games have the same best response correspondences and hence the same (Nash or correlated) equilibria.) For the reduced game to be a chicken game, we need to have

$$x<\beta \text{ and } y<\alpha. \tag{19}$$

The game (29) is in turn strategically equivalent to (just multiply the payoffs by $\frac{1}{\beta-x}$)

1\2	C	D
C	$0,0$	$z,1$
D	$1,z$	$0,0$

, (20)

where

$$z\equiv\frac{\alpha-y}{\beta-x}.$$

Hence the correlated equilibria are completely characterized by this single quantity z. Recall that, under transition rule (15), the strategy profile distribution is given by

1\2	C	D
C	$p(G,G\|a)$	$p(G,B\|a)$
D	$p(B,G\|a)$	$p(B,B\|a)$

,

where a is the action profile in the previous period. Let us now check when this is a correlated equilibrium of game (20) (hence, of the reduced game) *for all* a. Assuming that $p(G,G|a)+p(G,B|a)\neq 0$, when player 1 is called upon to play C, her posterior belief is that player 2 plays C with probability $p(G,G|a)/(p(G,G|a)+p(G,B|a))$ and D with probability $p(G,B|a)/(p(G,G|a)+p(G,B|a))$. Hence, player 1 would indeed like to play C in such a situation if

$$\frac{p(G,B|a)}{p(G,G|a)+p(G,B|a)}z\geq\frac{p(G,G|a)}{p(G,G|a)+p(G,B|a)}, \tag{21}$$

or simply[11]

$$p(G,B|a)z \geq p(G,G|a). \tag{22}$$

Similarly, we have the following incentive constraints for the proposed strategy profile distribution to be a correlated equilibrium.

$$\text{Condition for player 1 to choose } D\text{:} \quad p(B,G|a) \geq p(B,B|a)z \tag{23}$$

$$\text{Condition for player 2 to choose } C\text{:} \quad p(B,G|a)z \geq p(G,G|a) \tag{24}$$

$$\text{Condition for player 2 to choose } D\text{:} \quad p(G,B|a) \geq p(B,B|a)z \tag{25}$$

As we have $p(G,B|a) \neq 0$ and $p(B,G|a) \neq 0$, the correlated equilibrium conditions (22) - (25) reduce to

$$\min_a \min \left\{ \frac{p(G,B|a)}{p(B,B|a)}, \frac{p(B,G|a)}{p(B,B|a)} \right\} \geq z \geq \max_a \max \left\{ \frac{p(G,G|a)}{p(G,B|a)}, \frac{p(G,G|a)}{p(B,G|a)} \right\} \tag{26}$$

with the understanding that $\frac{p(G,B|a)}{p(B,B|a)}, \frac{p(B,G|a)}{p(B,B|a)} = \infty > z$ when $p(B,B|a) = 0$. Note that the minimum on the left hand side is attained by $a = (D,C)$, (C,D), and (D,D), and it is equal to $\frac{1/4}{1/8} = \frac{2/5}{1/5} = 2$. The maximum on the right hand side is attained by $a = (C,C)$, and it is equal to $\frac{1/3}{1/3} = 1$. Hence, action profile distributions are correlated equilibria (after all action profiles) of the reduced game iff

$$2 \geq z \geq 1, \tag{27}$$

and the incentive constraints (22) - (25) are satisfied with strict inequality when $2 > z > 1$. If $\delta = 0.99$, this is indeed satisfied, because we have $z = \frac{\alpha - y}{\beta - x} = 1.0027$. Numerical computation shows that the crucial equilibrium condition (27) is satisfied when $\delta \geq 0.98954$.

Note that the incentive constraint (26) is satisfied with strict inequalities when $\delta = 0.99$. This relationship is unchanged if I slightly modify the signal structure so that $p(\omega_1, \omega_2|a)$ is always strictly positive (because $z = \frac{\alpha - y}{\beta - x}$ is a continuous function of those parameters). In this sense the example here is robust: All major conclusions here holds if I make $p(\omega_1, \omega_2|a)$ strictly positive for all (ω_1, ω_2) and all a. More generally, all major results here

[11] An elementary pedagogical note: The example here satisfies $p(G,G|a) + p(G,B|a) \neq 0$ for all a, condisions (21) and (22) are equivalent. In general, (22) is more general because it covers the case with $p(G,G|a) + p(G,B|a) = 0$. Note that, when $p(G,G|a) + p(G,B|a) = 0$, we have $p(G,G|a) = p(G,B|a) = 0$, so that condition (22) is vacuously satisfied (both sides are equal to 0).

continue to hold even if we slightly perturb the stage game payoffs, the signal distributions, or the discount factor.

Since the incentive constraints (27) is satisfied with strict inequalities, each player is always taking a *strict* best reply action (given the future strategy profile). This is in sharp contrast to the equilibria obtained by Ely and Valimaki (2002) or EHO (2005), whose essential feature is that at least one player is indifferent between some actions. Bhaskar (2000) argues that such an equilibrium is somewhat unrealistic because it may not be justified by the Harsanyi-type purification argument (with independent perturbations to the stage payoffs). A follow-up paper by Bhaskar, Mailath and Morris (2006) partially confirms this conjecture. They consider one-period memory belief-free strategies *a la* Ely-Valimaki in *perfect monitoring* repeated *prisoners' dilemma* game (note that the Ely-Valimaki belief-free equilibrium applies to perfect as well as imperfect private monitoring). They show that those strategies cannot be purified by one-period memory strategies, but can be purified by infinite memory strategies. They conjecture that purification fails for any finite memory strategies (so that the purification is possible, but only with substantially more complex strategies). They also conjecture that similar results hold for the imperfect private monitoring case. The equilibrium here is free from the Bhaskar critique.

The best symmetric correlated equilibrium payoff associated with our reduced game is given by (32) in the Appendix A, and, when $\delta = 0.99$, it is equal to

$$\frac{z}{2+z}x + \frac{1}{2+z}\alpha + \frac{1}{2+z}\beta = 0.637\,64. \tag{28}$$

When we confine our attention to Nash equilibria (as opposed to correlated equilibria) of the repeated game, note that there are two asymmetric pure strategy Nash equilibria. Those corresponds to (D, C) and (C, D) in the reduced game, where players receive payoffs (α, β) or (β, α), where $\alpha = 0.629\,14$ or $\beta = 0.642\,5$. As is clear from (16) and the explanation thereafter, those payoffs are in any case close to the best correlated equilibrium payoff (28).

Appendix B shows that our equilibrium lies above the Pareto frontier of all belief-free equilibria in this game, identified by EHO. Let us summarize the results in this section as follows:

Proposition 14 *Let $v_i(a), i = 1, 2$ and $p(a(t+1)|a(t))$ be the reduced game payoffs and law of motion associated with the one-period memory transition rule (15). When $\delta \geq 0.989\,54$, $p(\cdot|a)$ is a correlated equilibrium of the reduced game v for each a, and hence any (Nash or correlated) equilibrium* *payoff profile of v is a weakly belief-free equilibrium payoff profile of the repeated game. Furthermore, when $\delta \geq 0.989\,54$,*

- *v is a chicken game, and*
- *any weakly belief-free equilibrium payoff profile with transition rule (15) lies above the Pareto frontier of the belief-free equilibrium payoff set.*

Why does the equilibrium here outperform the EHO equilibria? This is because the strategy here uses "transfer of continuation payoffs between players". In the EHO equilibria, a player's payoff is held constant when her opponent is being punished (note that the basic feature of the EHO equilibrium is that a player's payoff is solely determined by the opponent's strategy, and it is independent of whether the player is punishing the opponent or not). Intuitively, this amounts to throwing the total surplus away and results in the welfare losses that we have just calculated. On the other hand, if a player's payoff is increased when the opponent is punished, the loss of total surplus is mitigated (and if we do it right, the loss can completely vanish, as the Fudenberg-Levine-Maskin folk theorem (1994) shows). The equilibrium here embodies such transfers (although not so efficiently as the Fudenverg-Levine-Maskin equilibria do), and therefore it does better than the EHO equilibria. The asymmetric punishment mechanism is embodied in our example in the following way. If a player defects from (C, C), instead of going to a mutual punishment (D, D), our equilibrium transition rule (15) together with our information structure $p(\omega|a)$ imply that players alternate between (C, D) and (D, C) with a large probability.

6 Appendix A: The correlated equilibria in the chicken game

Note that the reduced game is strategically equivalent to

$1\backslash 2$	C	D
C	$0, 0$	$\alpha - y, \beta - x$
D	$\beta - x, \alpha - y$	$0, 0$

(29)

(In general, a game $g_i(a)$, $i = 1, ..., N$ is strategically equivalent to game $\gamma g_i(a) + K_i(a_{-i})$ $(\gamma > 0)$, which means that both games have the same best response correspondence and hence the same (Nash or correlated) equilibria.) For the reduced game to be a chicken game, we need $x < \beta$ and $y < \alpha$.

The game (29) is in turn strategically equivalent to (just multiply the payoffs by $\frac{1}{\beta-x}$)

$1\backslash 2$	C	D
C	$0,0$	$\frac{\alpha-y}{\beta-x},1$
D	$1,\frac{\alpha-y}{\beta-x}$	$0,0$

(30)

In general, when we have a chicken game

$1\backslash 2$	C	D
C	$0,0$	$z,1$
D	$1,z$	$0,0$

(31)

(where $z>0$), the extremal correlated equilibria are (more explanation to be added)

$1\backslash 2$	C	D
C	$\frac{z}{2+z}$	$\frac{1}{2+z}$
D	$\frac{1}{2+z}$	0

(32)

$1\backslash 2$	C	D
C	0	$\frac{z}{1+2z}$
D	$\frac{z}{1+2z}$	$\frac{1}{1+2z}$

(33)

$1\backslash 2$	C	D
C	0	1
D	0	0

(34)

$1\backslash 2$	C	D
C	0	0
D	1	0

(35)

$1\backslash 2$	C	D
C	$\frac{z^2}{(1+z)^2}$	$\frac{z}{(1+z)^2}$
D	$\frac{z}{(1+z)^2}$	$\frac{1}{(1+z)^2}$

(36)

Note that (34), (35), and (36) correspond to pure and mixed Nash equilibria of the reduced game. Hence the "best" correlated equilibrium (32) is

$1\backslash 2$	C	D
C	$\frac{\frac{\alpha-y}{\beta-x}}{2+\frac{\alpha-y}{\beta-x}}$	$\frac{1}{2+\frac{\alpha-y}{\beta-x}}$
D	$\frac{1}{2+\frac{\alpha-y}{\beta-x}}$	0

(37)

7 Appendix B: Comparison with the belief-free equilibria by Ely, Horner and Olszewski

In this section, we compare our "Chicken in Prison" example with the belief-free equilibrium payoffs identified by EHO (2005). To explain their characterization of the belief-free equilibrium payoffs, we first introduce the notion of *regime* $\mathcal{A}$ and an associated value $M_i^{\mathcal{A}}$. Using those concepts, we then find an upper bound for the belief-free equilibrium payoffs.

A regime $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is a product of non-empty subsets of the stage game action sets, $\mathcal{A}_i \subset A_i$, $\mathcal{A}_i \neq \emptyset$, $i = 1, 2$. In each period of a belief-free equilibrium, players typically have multiple best-reply actions and they are played with positive probabilities. A regime corresponds to the set of such actions. For each regime $\mathcal{A}$, define a number $M_i^{\mathcal{A}}$ as follows.

$$
\begin{aligned}
&M_i^{\mathcal{A}} = \sup v_i \\
&\text{such that for some mixed action } \alpha_{-i} \text{ whose support is } \mathcal{A}_{-i} \\
&\qquad \text{and } x_i : \mathcal{A}_{-i} \times \Omega_{-i} \to \Re_+ \\
&v_i \geq g(a_i, \alpha_{-i}) - \sum_{a_{-i}, \omega_{-i}} x_i(a_{-i}, \omega_{-i}) p_{-i}(\omega_{-i} | a_i, a_{-i}) \alpha_{-i}(a_{-i}) \\
&\text{for all } a_i \text{ with equality if } a_i \in \mathcal{A}_i,
\end{aligned}
$$

where $p_{-i}(\omega_{-i}|a_i, a_{-i})$ is the marginal distribution of ω_{-i} given action profile (a_i, a_{-i}). Intuitively, the positive number x_i represents the reduction in player i's future payoffs. Note that a belief-free equilibrium has the property that player i's payoff is solely determined by the opponent's strategy. This is why the reduction in i's future payoffs, x_i, depends the opponent's action and signal (a_{-i}, ω_{-i}). Note also that the opponent's action a_{-i} is restricted to the component $\mathcal{A}_{-i}$ of the current regime $\mathcal{A} = \mathcal{A}_i \times \mathcal{A}_{-i}$. The above set of inequalities ensures that player i's best reply actions in the current period correspond to set $\mathcal{A}_i$, a component of the regime $\mathcal{A} = \mathcal{A}_i \times \mathcal{A}_{-i}$. Hence, the value $M_i^{\mathcal{A}}$ is closely related to the best belief-free payoff when the current regime is $\mathcal{A}$ (more precise explanation will be given below).

Now let V^* be the limit set of belief-free equilibrium payoffs when $\delta \to 1$. EHO (2005) provides an explicit formula to compute V^*. For our purpose here, we only sketch the relevant part of their characterization to obtain a bound for V^*. In Section 4.1, EHO partitioned all games into three classes, *the positive*, *the negative*, and *the abnormal* cases (for our purpose here, we do not need to know their definitions). Their Proposition 6 shows that the abnormal case obtains *only if* one of the players has a dominant action in the stage game yielding the same payoff against all actions of the other player. Clearly, this is not the case in our example with the prisoner's dilemma stage game, so our example is in either the positive or negative case[12]. If it is in the negative case, EHO's Proposition 5 shows that the only belief-free equilibrium is the repetition of the stage game Nash equilibrium, yielding $(0, 0)$ in our example.

If our example is in the positive case, Proposition 5 in EHO implies that the limit set of belief-free equilibrium payoffs can be calculated as follows:

$$V^* = \bigcup_p \prod_{i=1,2} \left[\sum_{\mathcal{A}} p(\mathcal{A}) m_i^{\mathcal{A}},\ \sum_{\mathcal{A}} p(\mathcal{A}) M_i^{\mathcal{A}} \right], \tag{38}$$

where $m_i^{\mathcal{A}}$ is some number (for our purpose here, we do not need to know its definition) and p is a probability distribution over regimes $\mathcal{A}$. The union is taken with respect to all probability distributions p such that the intervals in the above formula (38) is well defined (i.e., $\sum_{\mathcal{A}} p(\mathcal{A}) m_i^{\mathcal{A}} \leq \sum_{\mathcal{A}} p(\mathcal{A}) M_i^{\mathcal{A}}$, $i = 1, 2$). The point to note is that V^* is a union of product sets (rectangles), and the efficient point (upper-right corner) of each rectangle is a convex combination of $(M_1^{\mathcal{A}}, M_2^{\mathcal{A}})$.

The above characterization (38) of V^* implies, in the positive case, the belief-free equilibrium payoffs satisfy the following bound

$$(v_1, v_2) \in V^* \Longrightarrow v_1 + v_2 \leq \max_{\mathcal{A}} M_1^{\mathcal{A}} + M_2^{\mathcal{A}}, \tag{39}$$

where maximum is taken over all possible regimes (i.e., for all $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ such that $\mathcal{A}_i \subset A_i$, $\mathcal{A}_i \neq \emptyset$, $i = 1, 2$).

In what follows, we estimate $M_1^{\mathcal{A}} + M_2^{\mathcal{A}}$ for each regime $\mathcal{A}$. In our example, $A_i = \{C, D\}$, so that $\mathcal{A}_i = \{C\}, \{D\}$, or $\{C, D\}$. Before examining each regime, we first derive some general results. Consider a regime $\mathcal{A}$ where $C \in \mathcal{A}_i$. In this case, the incentive constraint in the definition of $M_i^{\mathcal{A}}$ reduces to

$$\begin{aligned} v_i &= g(C, \alpha_{-i}) - \sum_{a_{-i}, \omega_{-i}} x_i(a_{-i}, \omega_{-i}) p_{-i}(\omega_{-i} | C, a_{-i}) \alpha_{-i}(a_{-i}) & (40) \\ &\geq g(D, \alpha_{-i}) - \sum_{a_{-i}, \omega_{-i}} x_i(a_{-i}, \omega_{-i}) p_{-i}(\omega_{-i} | D, a_{-i}) \alpha_{-i}(a_{-i}). & (41) \end{aligned}$$

[12] With some calculation, we can determine which case applies to our example, but this is not necessary to derive our upper bound payoff.

This inequality (41) can be rearranged as

$$\sum_{a_{-i},\omega_{-i}} x_i(a_{-i},\omega_{-i})p_{-i}(\omega_{-i}|C,a_{-i})\left(\frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}-1\right)\alpha_{-i}(a_{-i})$$
$$\geq \; g(D,\alpha_{-i})-g(C,\alpha_{-i}). \tag{42}$$

Now let

$$L^* = \max_{\omega_{-i},a_{-i}} \frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}$$

be the maximum likelihood ratio to detect player i's deviation from C to D. The preceding inequality (42) and $L^*-1>0$ imply[13]

$$\sum_{a_{-i},\omega_{-i}} x_i(a_{-i},\omega_{-i})p_{-i}(\omega_{-i}|C,a_{-i})\alpha_{-i}(a_{-i}) \geq \frac{g(D,\alpha_{-i})-g(C,\alpha_{-i})}{L^*-1}.$$

Plugging this into the definition (40) of v_i, we obtain

$$v_i \leq g(C,\alpha_{-i}) - \frac{g(D,\alpha_{-i})-g(C,\alpha_{-i})}{L^*-1}.$$

This is essentially the formula identified by Abreu, Milgrom and Pearce (1991). The reason for welfare loss (the second term on the right hand side), is that players are sometimes punished simultaneously in belief-free equilibria. The welfare loss associated with simultaneous banishment was originally pointed out by Radner, Myerson, and Maskin (1986). Recall that $M_i^{\mathcal{A}}$ is obtained as the supremum of v_i with respect to x_i and α_{-i} whose support is $\mathcal{A}_{-i}$. (Note that the right hand side of the above inequality, in contrast, does not depend on x_i.) Hence, we have

$$M_i^{\mathcal{A}} \leq \sup g(C,\alpha_{-i}) - \frac{g(D,\alpha_{-i})-g(C,\alpha_{-i})}{L^*-1}, \tag{43}$$

where the supremum is taken *over all* α_{-i} *whose support is* $\mathcal{A}_{-i}$.

Now we calculate the maximum likelihood ratio L^* and determine the right hand side of above inequality (43). In our example, when $a_{-i} = C$, $\max_{\omega_{-i}} \frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}$ is equal to (as our example is symmetric, consider $-i=2$ without loss of generality)

$$\frac{p_2(\omega_2=B|D,C)}{p_2(\omega_2=B|C,C)} = \frac{\frac{1}{2}+\frac{1}{8}}{1/3} = \frac{15}{8}.$$

[13] Note that, as long as player i's action affects the distribution of the opponent's signal (which is certainly the case in our example), there must be some ω_{-i} which becomes more likely when player i deviates from C to D. Hence, we have $L^*>1$.

When $a_{-i} = D$, $\max_{\omega_{-i}} \frac{p_{-i}(\omega_{-i}|D,a_{-i})}{p_{-i}(\omega_{-i}|C,a_{-i})}$ is equal to

$$\frac{p_2(\omega_2 = B|D,D)}{p_2(\omega_2 = B|C,D)} = \frac{2/5 + 1/5}{1/4 + 1/8} = \frac{8}{5}.$$

As the former is larger, we conclude $L^* = \frac{15}{8}$. Plugging this into (43), we obtain the following upper bounds of $M_i^{\mathcal{A}}$.

1. When $C \in \mathcal{A}_i$ and $\mathcal{A}_{-i} = \{C\}$,

$$\begin{aligned} M_i^{\mathcal{A}} &\leq g(C,C) - \frac{g(D,C) - g(C,C)}{\frac{15}{8} - 1} \\ &= 1 - \frac{1/2}{\frac{15}{8} - 1} = \frac{3}{7}. \end{aligned}$$

2. When $C \in \mathcal{A}_i$ and $\mathcal{A}_{-i} = \{D\}$,

$$\begin{aligned} M_i^{\mathcal{A}} &\leq g(C,D) - \frac{g(D,D) - g(C,D)}{\frac{15}{8} - 1} \\ &= -\frac{1}{6} - \frac{1/6}{\frac{15}{8} - 1} = -\frac{5}{14}. \end{aligned}$$

3. When $C \in \mathcal{A}_i$ and $\mathcal{A}_{-i} = \{C, D\}$, the larger upper bound in the above two cases applies, so that we have

$$M_i^{\mathcal{A}} \leq \frac{3}{7}.$$

Given those bounds, we are ready to estimate $M_1^{\mathcal{A}} + M_2^{\mathcal{A}}$ for each regime $\mathcal{A}$.

Case (i), where $C \in \mathcal{A}_i$ for $i = 1, 2$: The above analysis (Cases 1 and 3) shows

$$M_1^{\mathcal{A}} + M_2^{\mathcal{A}} \leq \frac{6}{7}.$$

Case (ii), where $C \in \mathcal{A}_i$ and $\mathcal{A}_{-i} = \{D\}$: Our Case 2 shows $M_i^{\mathcal{A}} \leq -\frac{5}{14}$. In contrast, $M_{-i}^{\mathcal{A}}$ is simply achieved by $x_{-i} \equiv 0$ (as D is the dominant strategy in the stage game) so that

$$M_{-i}^{\mathcal{A}} = \sup_{\alpha_i} g(D, \alpha_i) = g(D, C) = \frac{3}{2}.$$

Hence, we have

$$M_1^{\mathcal{A}} + M_2^{\mathcal{A}} \leq \frac{3}{2} - \frac{5}{14} = \frac{8}{7}.$$

Case (iii), $\mathcal{A} = \{D\} \times \{D\}$: Since D is the dominant action in the stage game, $M_i^{\mathcal{A}}$ is achieved by $x_i \equiv 0$. Moreover, the opponent's action is restricted to $\mathcal{A}_{-i} = \{D\}$, so that we have $M_i^{\mathcal{A}} = g(D, D) = 0$. Hence,

$$M_1^{\mathcal{A}} + M_2^{\mathcal{A}} = 0.$$

Let me summarize our discussion above. If our example is in the negative case as defined by EHO, the only belief-free equilibrium payoff is $(0, 0)$. Otherwise, our example is in the positive case, where the sum of belief-free equilibrium payoffs $v_1 + v_2$ (in the limit as $\delta \to 1$) is bounded above by the maximum of the upper bounds found in Cases (i)-(iii), which is equal to $\frac{8}{7}$. Altogether, those results show that any limit belief-free equilibrium payoff profile (as $\delta \to 1$) $(v_1, v_2) \in V^*$ satisfies $v_1 + v_2 \leq \frac{8}{7}$.

To complete our argument, we now examine the belief free equilibrium payoffs *for a fixed discount factor* $\delta < 1$. Let $V(\delta)$ be the set of belief-free equilibrium payoff profiles for discount factor $\delta < 1$. The standard argument[14] shows that this is monotone increasing in δ ($V(\delta) \subset V(\delta')$ if $\delta < \delta'$). Hence, we have $V(\delta) \subset V^*$, so that for any discount factor δ, all belief-free equilibrium payoffs $(v_1, v_2) \in V(\delta)$ satisfy $v_1 + v_2 \leq \frac{8}{7}$. Now recall that in our example, our one-period memory transition rule (15) is an equilibrium if $\delta \geq 0.989\,54$, with reduced game given by

	C	D
C	x, x	α, β
D	β, α	y, y

. (44)

The numerical analysis in Subsection 7.2 shows $x, y, \alpha, \beta > 0.6$ for $\delta \geq 0.989\,54$. Hence, the total payoff *in any entry* in our reduced game payoff table (44) exceeds 1.2, which is larger than the upper bound for the total

[14] The proof is as follows. Suppose we terminate the repeated game under $\delta' > \delta$ randomly in each period with probability $1 - \frac{\delta}{\delta'}$ and start a new game (and repeat this procedure). In this way, we can decompose the repeated game under δ' into a series of randomly terminated repeated games, each of which has effective discount factor equal to $\delta' \times \frac{\delta}{\delta'} = \delta$. Hence, any equilibrium (average) payoff under δ can also be achieved under $\delta' > \delta$. This argument presupposes that public randomization is available (to terminate the game). Even without public radomization, however, our conclusion $V(\delta) \subset V^*$ also holds, because (i) the set of belief-free payoff set $V(\delta)$ is smaller without public randomization and (ii) the *same* limit payoff set V^* obtains with or without public randomization (see the online appendix to EHO (2004)).

payoffs associated with the belief-free equilibria, $\frac{8}{7}$ + 1.14. This implies that *all of our equilibria lie above the Pareto frontier of the belief-free equilibrium payoff set.*

The reason is as follows. Our analysis shows that, by choosing an equilibrium of the reduced game (44) in the first period and then following our one-period memory transition rule (15), (i) we obtain a (strongly recursive) equilibrium of the repeated game and (ii) the average repeated game payoffs are equal to the chosen equilibrium payoff of the reduced game. When $\delta \geq 0.989\,54$, the reduced game (44) is a chicken game. Hence, if we assume that there is no correlation device, (C, D) and (D, C) (the Nash equilibrium of the chicken game (44)) correspond to our equilibria. When public randomization or partial correlation device is available at the beginning of the game, we can also achieve additional outcomes, i.e., the correlated equilibria of the reduced game (44). In any case, all those equilibria lie above the Pareto frontier of the belief-free equilibrium payoff set.

7.1 Impossibility of the review strategy equilibria in our example

Matsushima (2004) shows a larger payoff set can be sustained by extending the idea of the belief-free equilibrium by means of *review strategies*. A review strategy equilibrium treats T consecutive stage games as if they were a single stage game, or a *block stage game*, and apply the belief-free equilibrium technique to the sequence of such block stage games. Matsushima showed that, under certain conditions, the folk theorem can be obtained by the review strategies, even if the observability is quite limited. In particular, Matsushima showed that this is true in repeated prisoner's dilemma game, provided that the private signals are independently distributed conditional on the action profile and (unobservable) common shock. This requirement is expressed as

$$p(\omega_1, \omega_2|a) = \sum_{\theta \in \Theta} q_1(\omega_1|a_1, a_2, \theta) q_2(\omega_2|a_1, a_2, \theta) f(\theta|a_1, a_2), \qquad (45)$$

where $\theta \in \Theta$ is the hidden common shock, *and* for all i and a_i,

$$q_i(\cdot|a_i, a_{-i}, \theta), \text{ for } a_{-i} \in A_{-i} \text{ and } \theta \in \Theta, \text{ are linearly independent.}$$

The latter requirement is satisfied only if $|\Omega_i| \geq |A_{-i}| \times |\Theta|$. Since we have $|\Omega_i| = |A_{-i}| = 2$ in our example, the requirement is satisfied only if $|\Theta| = 1$. In such a case, the first requirement (45) implies that the private signals are conditionally independent $p(\omega_1, \omega_2|a) = q_1(\omega_1|a_1, a_2)q_2(\omega_2|a_1, a_2)$, and this is clearly not the case in our example. Hence, Matsushima's review strategy results do not apply to our example.

7.2 The reduced game payoffs for $\delta \geq 0.98954$

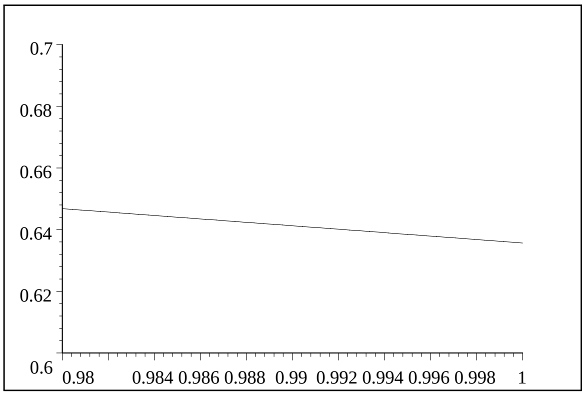
In our example, (15) can be a weakly belief-free equilibrium transition rule when $\delta \geq 0.989\,54$. Here we numerically show that for this range all equilibrium payoffs dominate the belief-free equilibria. The reduced game payoffs in table (44) is obtained as a solution to the system of dynamic programming equations

$$\begin{cases} x = (1-\delta) + \delta\frac{1}{3}(x+\alpha+\beta) \\ y = \delta(\frac{1}{5}y + \frac{2}{5}(\alpha+\beta)) \\ \alpha = (1-\delta)(-\frac{1}{6}) + \delta(\frac{1}{8}x + \frac{1}{4}\alpha + \frac{1}{2}\beta + \frac{1}{8}y) \\ \beta = (1-\delta)\frac{3}{2} + \delta(\frac{1}{8}x + \frac{1}{2}\alpha + \frac{1}{4}\beta + \frac{1}{8}y) \end{cases}.$$

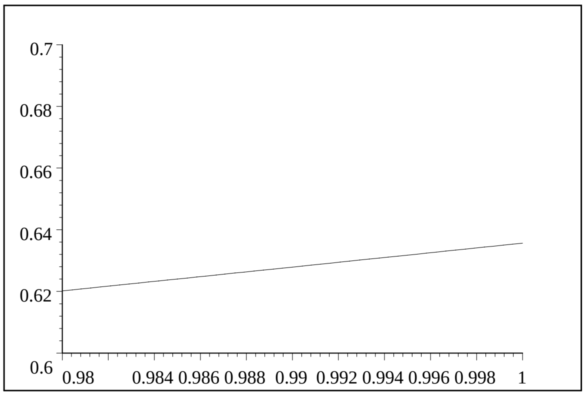
Using Maple, we obtain

$x = \frac{1}{3}\frac{7\delta^2+91\delta-180}{17\delta-60}$, $\beta = -\frac{1}{6}\frac{7\delta^3+285\delta^2-1632\delta+2160}{(17\delta-60)(4+\delta)}$, $y = \frac{2}{3}(7\delta-48)\frac{\delta}{17\delta-60}$, and $\alpha = -\frac{1}{6}\frac{1448\delta-395\delta^2-240+7\delta^3}{(17\delta-60)(4+\delta)}$.

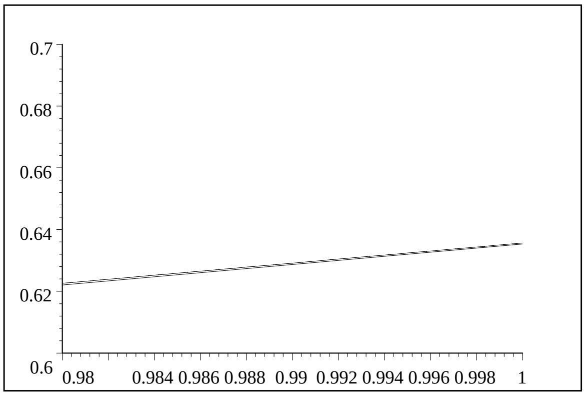
Plotting $x(\delta) = \frac{1}{3}\frac{7\delta^2+91\delta-180}{17\delta-60}$



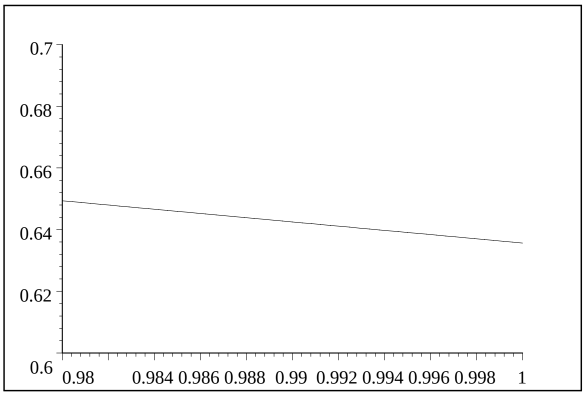
Plotting $y(\delta) = \frac{2}{3}(7\delta - 48)\frac{\delta}{17\delta - 60}$



Plotting $\alpha(\delta) = -\frac{1}{6}\frac{1448\delta - 395\delta^2 - 240 + 7\delta^3}{(17\delta - 60)(4 + \delta)}$



Plotting $\beta(\delta) = \beta = -\frac{1}{6}\frac{7\delta^3 + 285\delta^2 - 1632\delta + 2160}{(17\delta - 60)(4 + \delta)}$



Hence, we have numerically confirmed that $x, y, \alpha, \beta > 0.6$ holds for $\delta \geq 0.989\,54$.

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