Inductive Game Theory: A Basic Scenario*†

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Abstract

The aim of this paper is to present the new theory called "inductive game theory". A paper, published by one of the present authors with A. Matsui, discussed some part of inductive game theory in a specific game. Here, we will give a more developed discourse of the theory. The paper is written to show one entire picture of the theory: From individual raw experiences, short-term memories to long-term memories, inductive derivation of individual views, classification of such views, decision making or modification of behavior based on a view, and repercussion from the modified play in the objective game. We focus on some clear-cut cases, forgetting a lot of possible variants, but will still give a lot of results. In order to show one possible discourse as a whole, we will ask the question of how Nash equilibrium is emerging from the viewpoint of inductive game theory, and will give one answer.

1. Introduction

1.1. General Motivations

In game theory and economics it is customary to assume, often implicitly and sometimes explicitly, that each player has well formed beliefs/knowledge of the game he plays. Various frameworks have been prepared for explicit analyses of this subject. However, the more basic question of where a personal understanding of the game comes from is left unexplored. In some situations, such as parlour games, it might not be important

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to ask the source of a player's understanding. The rules of parlour games are often described clearly in a rule book. However, in social and economic situations, which are main target areas for game theory, the rules of the game are not clearly specified anywhere. In those cases, players need some other sources for their beliefs/knowledge. They may have learned the structure of the game from other people, perhaps, through education, but then we meet the same question of how those educators obtained their understanding of the game. One ultimate source for a player's understanding is his individual experiences of playing the game. The purpose of this paper is to develop and to present a theory about the origin and emergence of individual beliefs/knowledge from the individual experiences of players with bounded cognitive abilities.

People often behave naturally and effectively without much conscious effort to understand the world in which they live. For example, we work, socialize, exercise, eat, sleep, without thinking about the structure of our social situation. The accumulation of experiences gained by these activities will, nevertheless, influence our understanding and thoughts about society. On some level of abstraction, the social situations we interact in can be viewed as games that are repeated over time. By behaving and experiencing in these games, individuals accumulate information, which will in turn influence their beliefs about the games they play.

Treating particular experiences as the ultimate source of general beliefs/knowledge is an inductive process. Induction is differentiated from deduction in the way that induction is a process of deriving a general statement from a finite number of observations, while deduction is a process of deriving conclusions with the same or less logical content with well-formed inference rules from given premises. Formation of beliefs/knowledge about social games from individual experiences is typically an inductive process. Thus, we will call our theory *inductive game theory*, as was done in Kaneko-Matsui [16]. In fact, economic theory has had a long tradition of using arguments about learning by experiences to explain how players come to know the structure of their economy. Even in introductory microeconomics textbooks, the scientific method of analysis is discussed: collecting data, formulating hypotheses, predicting, behaving, checking, and updating. Strictly speaking, these steps are applied to economics as a science, but also sometimes, less scientifically, to ordinary peoples' economic activities.

Our theory formalizes some part of an inductive process of an individual decision maker. In particular, we describe how a player might use his experiences to form a hypothesis about the rules and structure of the game. In the starting point of our theory, a player has no *a priori* beliefs/knowledge about the structure of the particular game. All beliefs/knowledge about the structure of the particular game are derived from his experiences and memories.

In our theory, a player tends to follow some regular behavior, but he occasionally experiments by taking some trials in order to learn about the game he plays. Game theorists and scientists might wonder how a player can act regularly or conduct ex-

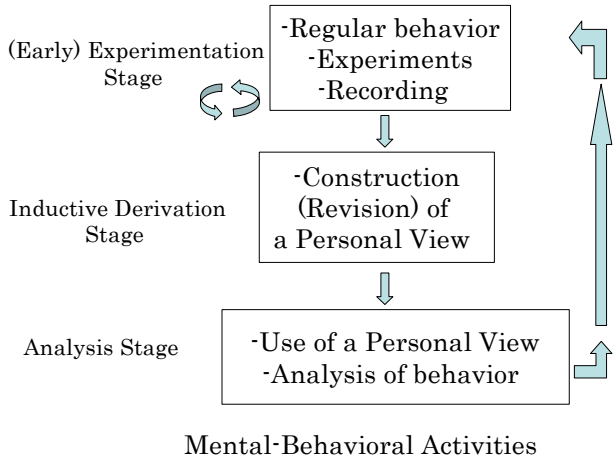

Figure 1.1:

periments initially without any beliefs or knowledge. Recall, as mentioned above, that many of our activities do not involve high brow analytical thoughts, but we simply act. Accordingly, we presume in our analysis that some well defined *default action* is known to a player, and whenever he faces a situation he has not thought about, he chooses this action. Initially, the default action describes his regular behavior. We may interpret this default action as a norm in society. The early experimental trials are not well developed experiments, but rather trials taken to see what happens. By taking these trials and observing the outcomes, a player will start to learn more about the other possibilities and the game overall.

The theory we propose has three main stages illustrated in Figure 1.1: the (early) *experimentation stage*; the *inductive derivation stage*; and the *analysis stage*. In the (early) experimentation stage, the player accumulates experiences by choosing his default action or previously prescribed actions and maybe occasionally by trying some alternatives, which may already require conscious effort. This stage may take quite some time and involve many repetitions before a player moves on to the inductive stage. In the inductive derivation stage he constructs a view of the game based on the accumulated experiences. In the analysis stage, he uses his derived view to analyze and optimize his behavior. If a player successfully passes through these three stages, then he brings back his optimizing behavior to the objective situation in the form of a strategy and behaves accordingly. In this paper, we will stop at various points to discuss what might happen when a player meets some difficulty at some stage. However, one inten-

tion is to give a entire scenario, so we will move on to each stage. After passing through all three stages, the player may start to experiment again with other behaviors and the experimentation stage starts again. Experimentation is no longer early since the player now has a belief about the game being played. Armed with his belief, a player may now potentially learn more from his experiments.

While we will take one player through all the stages in our theory, we emphasize that other players will experiment and move through the stages also at different times or even at the same time. The precise timing of this movement is not given rigorously. In Section 7.2 we give an example of how this process of moving through these stages might occur. We emphasize that experiments are still infrequent occurrences, and the regular behavior is crucial for a player to gain some information from his experiments. Indeed, if all players experiment too frequently, little would be learned.

We should distinguish our theory from some approaches in extant game theory. First, we take up the type-space approach of Harsanyi [9], which has been further developed by Mertens-Zamir [22] and Brandenburger-Dekel [3]. In this approach, a player starts with a set of parameter values describing the possible games a player might be playing, and considers a description of each player's beliefs about those parameters. In our theory, we ask the origin and emergence of the beliefs of players with bounded cognitive abilities. Here, a belief is not about parameter values but about the structure of the game he is playing. We look for the sources for the origin and emergence in the individual experiences. In this sense, our approach is quite different from the type-space approach. One benefit of our approach is to able to consider how a player may formulate his basic beliefs about the game structure from his experiences. On the other hand, when player's have more information about the set of possible parameters, a type-space approach may be appropriate.

Next, we distinguish our theory from the behavioral game theories that fall under the terms of evolution/learning/experiment (cf., Weibull [28], Fudenberg-Levine [6], Kalai-Lehrer [11], and more generally, Camerer [4]) and the cased based decision theory of Gilboa-Schmeidler [7]. The behavioral game theories are typically interested in adjustment/convergence of behavior to some equilibrium. They do not address questions on how a player learns the rules/structure of the game. By rules/structure of the game, we do not mean the "rules of behavior" used by behavioral game theorists to describe what are more commonly called strategies in game theory. Instead, by those terms, we mean the order of moves, players, information structure, payoffs, etc. In contrast to behavioral game theory, a main target of our theory is the emergence of a player's beliefs about the rules/structure. Case based decision theory looks more similar to ours. This theory focuses on how a player uses his past experiences to predict the consequences of an action in similar games. Unlike our theory, it does not discuss the emergence of beliefs/knowledge on social structures.

We now discuss the inductive derivation stage in more detail. A player may, from

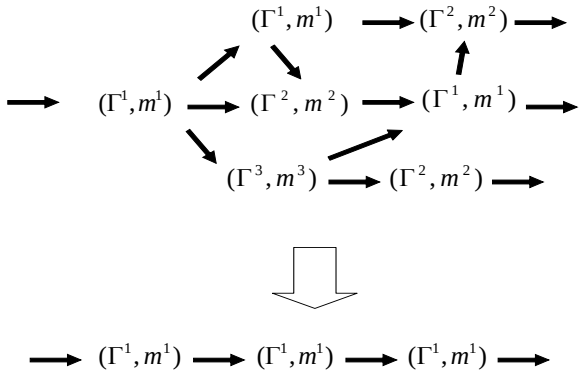

Figure 1.2:

time to time, construct a personal view to better understand the structure of some objective game he has been repeatedly playing. His view depends on his past interactions. The entire dynamics of a player's interactions is conceptually illustrated in the upper figure of Figure 1.2. Here, each particular game is assumed to be described by a pair $(\Gamma, \mathfrak{m})$ of an n-person extensive game Γ and *memory functions* $\mathfrak{m} = (\mathfrak{m}_1, ..., \mathfrak{m}_n)$, which will be explained in detail in Sections 1.2 and 2.2. Different superscripts denote different games a player might face and the arrows represent the passing of time. This diagram expresses the fact that a player will interact in different games with different players, and sometimes repeat the same games.

We will describe local memories in an extensive game Γ with the concept of a memory function $\mathfrak{m}_i$. In the standard theory of extensive games, the memory ability of a player, such as the perfect recall condition, is expressed in terms of information sets (cf. Kuhn [19]), but this does not explicitly describe the memory ability of a player. Since memory itself plays a central role in our inductive game theory, we need an explicit formulation of local memories, which is given by a memory function.

We assume that a player focuses on a particular game situation such as $(\Gamma^1, \mathfrak{m}^1)$, but he does not try to understand the entire dynamics of repeated plays of all games depicted in the upper diagram of Figure 1.2. The situation $(\Gamma^1, \mathfrak{m}^1)$ occurs occasionally, and we assume that when it occurs, the player notices it and his behavior depends only upon its occurrence. By these assumptions, the entire game is effectively reduced into the lower diagram of Figure 1.2. His target is the particular situation $(\Gamma^1, \mathfrak{m}^1)$. In the remainder of the paper, we denote a particular situation $(\Gamma^1, \mathfrak{m}^1)$ under our scrutiny by $(\Gamma^o, \mathfrak{m}^o)$, where the superscript "o" means "*objective*".

The induction in our theory is made from the various accumulated experiences of a player i. From his accumulated experiences, player i inductively derives his general belief $(\Gamma^i, \mathfrak{m}^i)$ about the structure of $(\Gamma^o, \mathfrak{m}^o)$. His general belief $(\Gamma^i, \mathfrak{m}^i)$ is called a *personal view* of player i. The first component in his personal view is the game Γ^i which he believes to be playing. The second component is the memory function $\mathfrak{m}^i$ he believes he has in the game.

The *objective* memory function $\mathfrak{m}_i^o$ of player i describes how the raw experiences of playing the game are perceived in his mind. We refer to these memories as *short-term* ones. Each short-term memory may be used to make an on the spot decision. For a more global picture of the game structure, a player needs to accumulate the memories of multiple different experiences.

A personal view $(\Gamma^i, \mathfrak{m}^i)$ is constructed based only on the part of the game player i has experienced, and only on the corresponding memories he has accumulated as *long-term memories*. What/how short-term memories are accumulated over repeated plays will be discussed in Section 1.2.

1.2. Treatments of Memories and Inductive Processes

The memories of a player are based on what he observes in each play of the game. These memories are occurring in his mind. We start with the short-term memories which occur as the game is played. We presume that the player might observe *information pieces* and *actions* in each play of the game. The information pieces correspond to what in game theory are typically called "information sets", and they convey information to the player about the set of available actions at the current move and perhaps some other details about the current environment.

A short-term memory $\mathfrak{m}_i(x)$ for a player at his node (move) x consists of sequences of pairs of information pieces and actions as depicted in Figure 1.3. In the figure it consists of three sequences and describes what, a player thinks, might have happened prior to the node x in the current play of the game. In his mind, any of these sequences might have happened; the multiplicity may be due to forgetfulness. We will use the terms *memory thread* for such a sequence, and *memory yarn* for the value ("set of memory threads") of the memory function at a point of time.

Now we consider the long-term memories. Recall that the players are in the repeated situation described as the lower diagram of Figure 1.2. Objectively speaking, the entire history of short-term memories for player i is a long sequence. We assume, rather than keeping this entire sequence of short-term memories, that only one incidence of each short-term memory, i.e., a set of memory threads and a set of memory yarns, may possibly remain in the mind of player i as a long-term memory. As we are interested in players with limited cognitive abilities, we presume that such players retain only some incidences of short-term memories in their long-term memories.

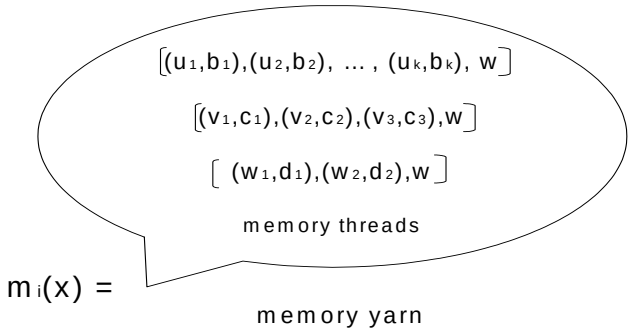

Figure 1.3:

We use the concept of *habituation* to motivate the types of short-term memories that are likely to remain as long-term ones. The process of habituation is the process of memorizing something by repeating it frequently enough. While a player may not retain the exact frequency of having had a short-term memory, the more frequently he has it, the more likely he is to retain it. Things occurring on some regular basis will become long-term memories. In addition, a player may use some *conscious effort* to keep some memories as long-term ones. In particular, when he makes experiments, he might try to recall the results. Conscious effort might still require some repetition, but not as much as without it. The transition from short-term to long-term memories will be discussed in Section 3, first in a heuristic manner before our mathematization of an inductive derivation. We refer to the long-term memories as *accumulated memories.*

We will focus on two sets of accumulated memories: the set of *accumulated memory threads* and the set of *accumulated memory yarns.* The first set is used by a player i to inductively derive the subjective game Γ^i. The second set is used by player i to construct his subjective memory function $\mathfrak{m}^i$. We refer to the pair of a set of accumulated threads and a set of accumulated yarns as a *memory kit.* This is loosely described in Diagram 1.4.

The memory kit brings us to the process of induction, by which a player transforms his accumulated memories into a personal view. In this paper we consider one specific procedure for the induction process, which we call the *initial segment procedure.* In this process, the player focuses on the maximal threads in his memory as the source for his view. This initial segment procedure will be discussed formally in Section 4. Since the aim of this paper is to present a basic scenario, we give this clear-cut case. In separate

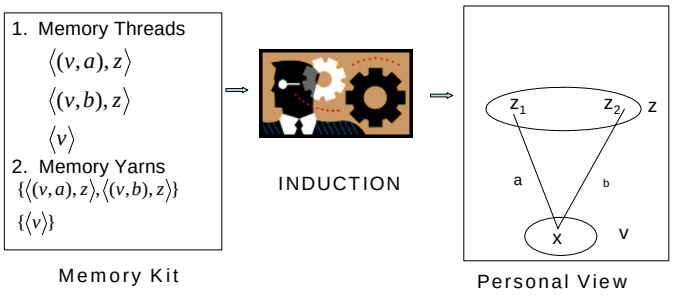

Figure 1.4:

papers, we will give more general procedures.

Our work may be compared to some of the philosophy literature on induction. Francis Bacon [1] discussed a process of gradual induction as a means of learning about the laws of nature. Hume's [10] work is also related to what we are doing since it also emphasizes that individual experience is the ultimate source of our understanding nature. Both Hume and Bacon focus on learning about cause and effect from past experiences. Our theory includes learning about cause and effect by experiences, but it is used specifically to learn about the rules of a particular game, rather than the laws of nature. It is relevant to emphasize that in our theory, a player often constructs a false view, may find some falsities, and it would be difficult for him to remove such falsities entirely. Thus, our discourse does not give a simple progressive view for induction. It is more close to Thomas Kuhn's [20] discourse of scientific revolution (cf. also Harper-Schulte [8] for a concise survey of related works).

1.3. The Structure of the Present Paper

This paper is divided into three parts:

Part I: **Background, and Basic Concepts of Inductive Game Theory** - -Section 1 to Section 3. Section 1 is now describing the motivation, background, and a rough sketch of our new theory. We will attempt, in this paper, to give a basic scenario of our entire theory. The mathematical structure of our theory is based on extensive games.

Section 2 gives the definition of an extensive game in two senses: strong and weak. This distinction will be used to distinguish the objective description of a game from a player's inductively derived description. The derivation of a view is based on a player's experiences and memories. Section 3 gives an informal theory of accumulating long-term memories, and a formal description of the long-term memories as a memory kit.

Part II: Inductive Derivation of a Personal View - - Section 4 to Section 6. In Section 4, we define an inductively derived personal view. We do not describe the induction process entirely. Rather, we give conditions that determine whether on not a personal view might be inductively derived from a memory kit. Because we have so many potential views, we define a direct view in Section 5, which turns out to be a representative of all the views a player might inductively derive (Section 6).

Part III: Decision Making using an Inductively Derived View - - Section 7 to Section 9. In this part, we consider each player's use of his derived view for his decision making. We consider a specific memory kit which allows each player to formulate his decision problem as a 1-person game. Nevertheless, this situation serves as an experiential foundation of Nash equilibrium. This Nash equilibrium result, and more general issues of decision making, are discussed in Sections 7 and 8.

2. Extensive Games, Memory, Views, and Behavior

To describe a basic situation like $(\Gamma^1, \mathfrak{m}^1)$ in Figure 1.2, we will use an n-person extensive game Γ^1 and memory functions $\mathfrak{m}^1 = (\mathfrak{m}_1^1, ..., \mathfrak{m}_n^1)$. We follow essentially Kuhn's [19] formulation of an extensive game to represent Γ^1.[1] The memory functions $\mathfrak{m}_1^1, ..., \mathfrak{m}_n^1$ are new and will be described in Section 2.2. Because we use the same tools to model the inductively derived view of a player, we will give an extensive game in the strong and weak senses in Section 2.1. One in the weak sense is needed since a player may only experience some part of the game being played. We do adopt an extensive game in the strong sense to describe the objective situation. After describing extensive games and memory functions, we formally define an objective description $(\Gamma^1, \mathfrak{m}^1)$ and a personal view $(\Gamma^i, \mathfrak{m}^i)$ of player i in Section 2.2. In Section 2.3 we give a formal definition of a behavior pattern (strategy configuration) for the players.

2.1. Extensive Games

First, we define an extensive game Γ in the strong sense, and then we define an extensive game in the weak sense. Our definition of an extensive game in the strong sense differs

[1] There are various formulations of extensive games such as in Selten [27], Dubey-Kaneko [5], Osborne-Rubinstein [25] and Ritzberger [26]. Those are essentially the same formulations, while Dubey-Kaneko [5] give a simultaneous move form.

from that of Kuhn [19] mainly in that the information sets of Kuhn are replaced by information pieces. Although this difference is inessential from the objective point of view, it is essential, indeed, from the subjective point of view, which will be explained later. An extensive game in the weak sense differs more substantially from an extensive game of Kuhn.

For notational simplicity, we sometimes make use of a function with the empty domain, which we call an *empty function*. When the empty domain and some (possibly nonempty) region are given, the empty function is uniquely determined. The reason for a slightly twisted definition of K33 will be found in K33$_0$.

Definition 2.1 (Extensive Games). An *extensive game in the strong sense* $\Gamma = ((X, <), (\lambda, W), \{(\varphi_x, A_x)\}_{x \in X}, (\pi, N), h)$ is defined as follows:

K1(*Game Tree*): $(X, <)$ is a finite forest (in fact, a tree by K14);

K11: X is a finite non-empty set of nodes, and $<$ is a partial ordering over X;

K12: $\{x \in X : x < y\}$ is totally ordered with $<$ for any $y \in X$;[2]

K13: X is partitioned into the set X^D of *decision nodes* and the set X^E of *endnodes* so that every node in X^D has at least one successor, and every node in X^E has no successors;[3]

K14: X has the smallest element x_0, called the *root*.[4]

K2(*Information Function*): W is a finite set of information pieces and $\lambda : X \to W$ is a surjection with $\lambda(x) \neq \lambda(z)$ for any $x \in X^D$ and $z \in X^E$;

K3(*Available Action Sets*): A_x is a finite set of *available actions* for each $x \in X$;

K31: $A_x = \varnothing$ for all $x \in X^E$;

K32: for all $x, y \in X^D$, $\lambda(x) = \lambda(y)$ implies $A_x = A_y$;

K33: for any $x \in X$, φ_x is a bijection from the set of immediate successors of x to A_x;

K4(*Player Assignment*): N is a finite set of players and $\pi : W \to 2^N$ is a player assignment with two conditions;

K41: $|\pi(w)| = 1$ if $w \in \{\lambda(x) : x \in X^D\}$ and $\pi(w) = N$ if $w \in \{\lambda(x) : x \in X^E\}$;

K42: for all $j \in N$, $j \in \pi(w)$ for some $w \in \{\lambda(x) : x \in X^D\}$;

[2]The binary relation $<$ is called a *partial ordering* on X iff it satisfies (i)(irreflexivity): $x \not< x$; and (ii)(transitivity): $x < y$ and $y < z$ imply $x < z$. It is a *total* ordering iff it is a partial ordering and satisfies (iii)(totality): $x < y$, $x = y$ or $y < x$ for all $x, y \in X$.

[3]We say that y is a *successor* of x iff $x < y$, and that y is an *immediate* successor of x, denoted by $x <^I y$, iff $x < y$ and there is no $z \in X$ such that $x < z$ and $z < y$.

[4]A node x is called the *smallest* element in X iff $x < y$ or $x = y$ for all $y \in X$.

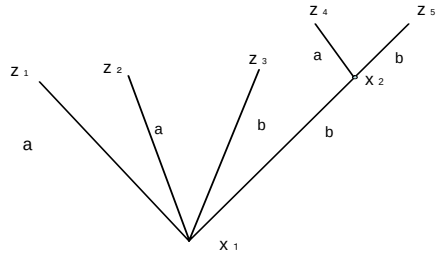

Figure 2.1:

K5(*Payoff functions*): $h = \{h_i\}_{i \in N}$, where $h_i : \{\lambda(x) : x \in X^E\} \to R$ is a payoff function for player $i \in N$.

Bijection φ_x associates an action with an immediate successor of x. Game theoretically, it names each branch at each node in the tree. When x is an endnode, φ_x is the empty function. Since A_x is empty, too, by K31, φ_x is a bijection.

When K14 (root) is dropped, and K33 (bijection) and K5 (payoffs) are replaced by the following weaker requirements, we say that Γ is an *extensive game in the weak sense*:

K33$_0$: for any $x \in X$, φ_x is a function from the set of immediate successors of x to A_x.

K5$_0$: $h : \{\lambda(x) : x \in X^E\} \to R$ is a payoff function for player i.

Since X may not have the smallest element, $(X, <)$ is not necessarily a tree. However, $(X, <)$ is divided into several connected parts. We can prove that a maximal connected subset of $(X, <)$ is a tree. Thus, $(X, <)$ is a class of trees, i.e., a forest. For any $x \in X$, there is a unique *path* to x, i.e., each maximal set $\{x_1, ..., x_{m+1}\}$ with $x_t < x_{t+1}$ for $t = 1, ..., m$ and $x_{m+1} = x$. When x is an endnode, we will call the path to x a *play*.

In an extensive game in the weak sense, an action a at a node x may not uniquely determine an immediate successor. See Figure 2.1, which will be discussed as a derived view in Section 4.1. The converse, however, that an immediate successor determines a unique action, does hold by K33$_0$. Thus, we can define: $x <^I_a y$ iff $x <^I y$ and $\varphi_x(y) = a$, which means that y is an immediate successor of x via action a. Then, we define $x <_a y$ iff there is some y' such that $x <^I_a y'$ and ($y' = y$ or $y' < y$).

We will use an extensive game in the strong sense as an objective description of a social situation we target, e.g., $\Gamma^o = \Gamma^1$ in Figure 1.2. An extensive game in the weak sense will be used for a personal view inductively derived from experiences. The latter differs from the former in several respects, besides the one mentioned above. First, we take the payoffs as personal and assume that a player's personal view does not include the payoffs of other players. Hence, condition K5 is weakened to K5$_0$ for a personal

view. Dropping the root assumption and weakening K33 are more substantial changes. We will see why such changes are needed when we derive a personal view in Section 4. These are alternative assumptions and we will discuss other alternatives in separate papers.

For an extensive game in the weak or strong sense, condition K32 implies that the set of available actions at a node x is determined by the information piece $w = \lambda(x)$. Thus, we may write A_w or $A_{\lambda(x)}$ rather than A_x.

An extensive game in the strong sense is the same as that given in Kuhn [19], except that we use information pieces W, rather than information sets. When the structure of Γ is known, information sets are defined by information pieces, i.e., $\{x : \lambda(x) = w\}$ for $w \in W$. In this sense, our definition of an extensive game is essentially the same as Kuhn's formulation from the objective point of view. However, the replacement of information sets by information pieces is substantive from the subjective point of view for our inductive game theory.

For the purpose of comparisons, we first mention the standard interpretation of the theory of extensive games due to Kuhn [19] (also, cf., Luce-Raiffa [21], Section 3.6). The interpretation is summarized as follows:

(i)(Cognizance): each player is fully cognizant of the game structure;

(ii)(*Ex Ante* Decision): each player makes a strategy choice before the actual play of the game.

Under (i), when a player receives an information piece w, he can infer the information set $\{x : \lambda(x) = w\}$ corresponding to piece w.

In the inductive context described in Section 1, the assumption (i) is dropped. Instead, players learn some part of the game structure by playing the game. Early on, a player may not infer at all the set of possible nodes having information piece w. To explain such differences, we use one small example of an extensive game, which we will repeatedly use to illustrate new concepts.

Example 2.1. Consider the extensive game depicted in Figure 2.2. It is an example of a 2-person extensive game. Player 1 moves at the root x_0, and then at the node x_3 if it is reached. Player 2 moves at x_1 or x_2 depending on the choice of player 1 at x_0. At endnode z_4 the payoffs to players 1 and 2 are $(h_1(z_4), h_2(z_4)) = (0, 1)$. The information function λ assigns $\lambda(x_0) = w$, $\lambda(x_1) = \lambda(x_2) = v$, $\lambda(x_3) = u$. At the endnodes, z_1, z_2, z_3, z_4, z_5, the information function is the identity function, i.e., $\lambda(z_t) = z_t$ for $t = 1, ..., 5$. In Kuhn's interpretation, each player has the knowledge of the game tree. In Figure 2.2, for example, when player 2 receives information piece v, he can infer that either x_1 or x_2 is possible, which means that he knows the information set $\{x_1, x_2\}$. He can also infer from his structural knowledge that player 1 chose either a or b at x_0.

Under the inductive interpretation, when player 2 receives information piece v, he may not come to either of the conclusions mentioned in the previous paragraph. He

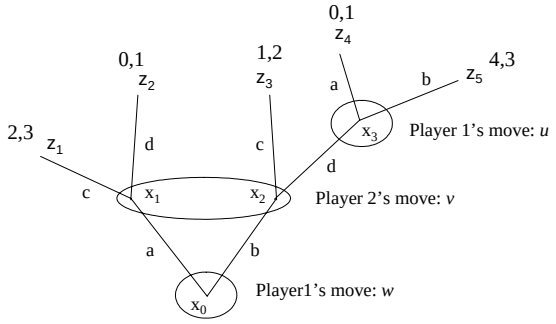

Figure 2.2:

might not even be aware of the existence of player 1. In such a case, piece v does not imply the information set $\{x_1, x_2\}$ and the choices by player 1 either. Thus, in the inductive situation, receiving information piece v may be totally different from knowing the corresponding information set.

The above consideration suggests that there are multiple interpretations of the knowledge a player gets from an information piece. Here, we specify the minimal content a player gets from each information piece w in Γ:

M1: the set A_w of available actions;

M2: the value $\pi(w)$ of the player assignment π if w is a decision piece;

M3: his own payoff $h_i(w)$ (as a numerical value) if w is an endpiece.

These are interpreted as being written on each piece w. These conditions will be discussed further when we consider some specific memory functions in Section 2.2 and the inductive derivation of a view in Section 4.

Let us return to (i) and (ii) of the standard interpretation of an extensive game given by Kuhn [19]. In our inductive game theory, since we drop the cognizance assumption (i), the *ex ante* decision making of (ii) does not make sense before an individual constructs a view of the game. We presume that until he constructs a view, he follows some regular behavior and makes occasional trials in an effort to learn the game he is playing. At some point of time, he will try to construct a view based on his accumulated memories of his experiences. Once a view is constructed, it may then be used by the player to construct an optimal strategy for future plays.

2.2. Memory Functions and Views

It is standard in the literature of extensive games to describe the memory ability of a player in terms of information sets (cf. Kuhn [19]). This does not separate the role of an information piece (set) as information transmission from the role of individual memory capabilities. In our inductive game theory, the treatment of memories plays a central role, and thus, we need an explicit formulation of individual memories in addition to an extensive game. For this reason, we will introduce the concept of a memory function, which describes short-term memories of a player within an extensive game.

A memory function expresses a player's short-term memory about the history of the current play of a game. Let $\Gamma = ((X, <), (\lambda, W), \{(\varphi_x, A_x)\}_{x \in X}, (\pi, N), h)$ be an extensive game in the weak or strong sense. Recall that for each node $x \in X$, there is a unique path to x which is denoted by $\langle x_1, ..., x_{m+1} \rangle$ with $x_{m+1} = x$. Also, the actions taken at $x_1, ..., x_m$ on the path to x are uniquely determined, i.e., for each $t = 1, ..., m$, there is a unique $a_t \in A_{x_t}$ satisfying $\varphi_{x_t}(x_{t+1}) = a_t$. We define the *complete history of information pieces and actions up to* x by

$$\theta(x) = \langle (\lambda(x_1), a_1), ..., (\lambda(x_m), a_m), \lambda(x_{m+1}) \rangle. \tag{2.1}$$

The history $\theta(x)$ consists of observable elements for players, while the path $\langle x_1, ..., x_{m+1} \rangle$ to x consists of unobservables for players. A memories will be defined in terms of these observable elements.

A short-term memory consists of memory threads that look somewhat like the historical sequence $\theta(x)$. However, we allow a player to be forgetful, which is expressed by incomplete threads or multiple threads. Formally, a *memory thread* is a finite sequence

$$\mu = \langle (v_1, a_1), ..., (v_m, a_m), v_{m+1} \rangle, \tag{2.2}$$

where

$$v_t \in W, \ a_t \in A_{v_t} \text{ for all } t = 1, ..., m \text{ and } v_{m+1} \in W. \tag{2.3}$$

Each component (v_t, a_t) $(t = 1, ..., m)$ or v_{m+1} in μ is called a *memory knot*. A finite nonempty set of memory threads is called a *memory yarn*. See Figure 1.3 for an illustration of these concepts. Now, we have the definition of a memory function.

Definition 2.2 (Memory Functions). We say that a function $\mathfrak{m}_i$ is a *memory function* of player i iff for each node $x \in X_i := \{x \in X : i \in \pi \cdot \lambda(x)\}$, $\mathfrak{m}_i(x)$ is a memory yarn satisfying:

$$w = \lambda(x) \text{ for all } \langle \zeta, w \rangle \in \mathfrak{m}_i(x). \tag{2.4}$$

The memory function $\mathfrak{m}_i$ gives a memory yarn consisting of a finite number of memory threads at each node for player i. The multiplicity of threads in a yarn describes uncertainty at a point in time about the past.

In Figure 1.3, the memory yarn $\mathfrak{m}_i(x)$ consists of three memory threads. The first one is a long one, the second and third are memory threads of short lengths. Condition (2.4) states that the tails of any memory threads at a node x are identical to the correct piece $\lambda(x)$. This is interpreted as meaning that the player correctly perceives the current information piece.

Here, we mention five specific examples of memory functions. In the first memory function, which is the exact perfect recall memory function, player i recalls what information he received during the current game and what actions he took, but nothing about the other players. For this example, we define player i's own history: For a node $x \in X_i$, let $\theta(x) = \langle(\lambda(x_1), a_1), ..., (\lambda(x_m), a_m),\ \lambda(x_{m+1})\rangle$, and let $\langle x_{k_1}, ..., x_{k_l}, x_{k_{l+1}}\rangle$ be the subsequence of nodes in the path $\langle x_1, ..., x_m, x_{m+1}\rangle$ to x belonging to player i, i.e, $i \in \pi \cdot \lambda(x_{k_t})$ for $t = 1, ..., l+1$. Then we define *player i*'s (objective) *history of information pieces and actions up to x* by:

$$\theta(x)_i = \langle(\lambda(x_{k_1}), a_{k_1}), ..., (\lambda(x_{k_l}), a_{k_l}), \lambda(x_{k_{l+1}})\rangle. \tag{2.5}$$

(1) Exact Perfect Recall Memory Function: It is formulated as follows:

$$\mathfrak{m}_i^{EPR}(x) = \{\theta(x)_i\} \text{ for each } x \in X_i. \tag{2.6}$$

With the memory function $\mathfrak{m}_i^{EPR}$, player i recalls his own information pieces and actions taken in the current play of the game. This memory function will have a special status in the discourse of this paper.

In Figure 2.2, the memory function $\mathfrak{m}_1^{EPR}$ for player 1 is given as:

$$\begin{aligned} \mathfrak{m}_1^{EPR}(x_0) &= \{\langle w\rangle\}, \text{ and } \mathfrak{m}_1^{EPR}(x_3) = \{\langle(w,b),u\rangle\}; \\ \mathfrak{m}_1^{EPR}(z_t) &= \{\langle(w,a),z_t\rangle\} \text{ for } t = 1,2 \text{ and } \mathfrak{m}_1^{EPR}(z_3) = \{\langle(w,b),z_3\rangle\}; \\ \mathfrak{m}_1^{EPR}(z_4) &= \{\langle(w,b),(u,a),z_4\rangle\} \text{ and } \mathfrak{m}_1^{EPR}(z_5) = \{\langle(w,b),(u,b),z_5\rangle\}. \end{aligned} \tag{2.7}$$

At node x_3, player 1 receives piece u and recalls his choice b at w. By the minimal requirement M1, he knows the available actions $A_w = \{a, b\}$ and $A_u = \{a, b\}$. Without adding any other source than $\mathfrak{m}_1^{EPR}$, player 2 does not appear in the scope of player 1. It will be discussed that Figure 2.1 is an inductively derived view in this example.

The next example is the Markov memory function. As its name suggests, a player forgets the entire history after he makes a decision. This memory function is defined as follows:

(2) Markov Memory Function: It is formulated as

$$\mathfrak{m}_i^{M}(x) = \{\lambda(x)\} \text{ for each } x \in X_i. \tag{2.8}$$

It gives only the present information piece. Nonetheless, by the minimal requirement M1, the player can extract his available action set $A_{\lambda(x)}$ whenever he receives an information piece $\lambda(x)$.

For both $\mathfrak{m}_i^{EPR}$ and $\mathfrak{m}_i^M$, we would have no difficulty in presuming that each player only receives his own information pieces and gets the minimal information described by M1, M2 and M3. As we will see now, some other memory functions provide a player with information about some other players' information pieces and actions. The first such memory function is the perfect information memory function.

(3) Perfect Information Memory Function: This is formulated as

$$\mathfrak{m}_i^{PI}(x) = \{\theta(x)\} \text{ for each } x \in X_i. \tag{2.9}$$

Recall that $\theta(x)$ is given by (2.1). Thus, if player i has this memory function, he recalls the perfect history even including the other players' pieces. By M1 and M2, he also knows the available actions and the player who moves at each decision piece. There are at least two possible interpretations of how he comes to know the perfect history.

One interpretation is that player i observes other players' moves as the game is played. Another interpretation is that player i's information pieces contain the complete history, i.e., $\theta(x)$ is written on piece $\lambda(x)$. Under either interpretation, a player gets more than the minimal amount of information described in M1-M3.

The next memory function typically gives a player less information than the perfect information memory function.

(4) Classical Memory Function: This memory function is formulated as:

$$\mathfrak{m}_i^{C}(x) = \{\theta(y) : y \in X_i \text{ and } \lambda(y) = \lambda(x)\} \text{ for each } x \in X_i. \tag{2.10}$$

Observe that this function gives player i the set of complete histories up to nodes with his current information piece. The multiplicity of memory threads can be interpreted as some ambiguity about the past. This memory function can also be interpreted in the ways suggested for $\mathfrak{m}_i^{PI}$. We should mention yet another interpretation which is the motivation for the name "classical" memory function. In this interpretation, player i knows the structure of the extensive game. Consequently, he can infer the set of possible complete histories compatible with the present information piece. The classical memory function together with this interpretation is less compatible with our inductive game theory than the other memory functions. Since it is still mathematically allowed and is closer to the classical game theory, we consider it.

The general definition of a memory function allows it to even involve false components. We give one example of false memories using the following simple extensive

game. Consider the 1-person extensive game $(\Gamma, \mathfrak{m}_1)$ depicted as Figure 2.3 with the identity information function.

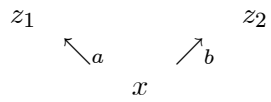

Figure 2.3

A false memory function $\mathfrak{m}_1$ is given as:

$$\mathfrak{m}_1(x) = \{\langle x \rangle\}, \mathfrak{m}_1(z_1) = \{\langle (x, a), z_1 \rangle\} \text{ and } \mathfrak{m}_1(z_2) = \{\langle (x, a), z_2 \rangle\}. \tag{2.11}$$

This $\mathfrak{m}_1$ takes a false value at z_2, at which player 1 incorrectly recalls having chosen a at x though he actually chose b at x.

Having described an extensive game and memory functions, we now have the basic ingredients for objective descriptions and (subjective) personal views. Both are described by extensive games and memory functions.

(Objective Description): A pair $(\Gamma^o, \mathfrak{m}^o)$ is called an *objective description* iff Γ^o is an extensive game in the strong sense and $\mathfrak{m}^o = (\mathfrak{m}_1^o, ..., \mathfrak{m}_n^o)$ is an n-tuple of memory functions in Γ^o.

The objective description is not available to a player. We use the superscript o to denote the objective description. We will put a superscript i to denote a personal view of player i.

(**Personal View**): A pair $(\Gamma^i, \mathfrak{m}^i)$ is a *personal view* for player i iff Γ^i is an extensive game in the weak sense specifying only the payoff function of player i, and $\mathfrak{m}^i$ is a memory function for player i in Γ^i.

A personal view $(\Gamma^i, \mathfrak{m}^i)$ of player i describes the game player i believes he is playing. Since his belief is based on his experiences, we do not include the memory functions or payoffs of other players. We regard payoff values and memory values as personal.[5]

2.3. Behavior Patterns

Let $\Gamma = ((X, <), (\lambda, W), \{(\varphi_x, A_x)\}_{x \in X}, (\pi, N), h)$ be an an extensive game in the weak or strong sense and let $\mathfrak{m}_i$ be a memory function for player $i \in N$. The extensive game and memory function may be either the objective description or a personal view. We give a definition of a behavior pattern that works in both cases.

[5] As stated several times, we regard this as an alternative assumption adopted in the present discourse. This can be extended to include other players.

We say that a function σ_i on the set of nodes $X_i^D := \{x \in X^D : i \in \pi \cdot \lambda(x)\}$ is a *behavior pattern* (*strategy*) of player i iff it satisfies conditions (2.12) and (2.13):

$$\text{for all } x \in X_i^D,\ \sigma_i(x) \in A_x; \tag{2.12}$$

$$\text{for all } x, y \in X_i^D,\ \mathfrak{m}_i(x) = \mathfrak{m}_i(y) \text{ implies } \sigma_i(x) = \sigma_i(y). \tag{2.13}$$

Condition (2.12) means that a behavior pattern σ_i prescribes an action to each decision node. Condition (2.13) means that a strategy depends upon local memories. These are standard conditions for the definition of a strategy. We denote, by Σ_i, the *set of all behavior patterns* for player i in Γ. We say that an n-tuple of strategies $\sigma = (\sigma_1, ..., \sigma_n)$ is a *profile of behavior patterns*.

We use the term behavior pattern (strategy) to acknowledge that the behavior of a player may initially represent some default behavior with no strategic considerations. Once, a player has gathered enough information about the game, his behavior may become strategic. This will be discussed in a remark in the end of Section 3.2.

In order to evaluate a behavior pattern, we introduce the concepts of compatible endnodes and compatible endpieces. All evaluations of strategies in this paper will be done in terms of compatible endpieces. Each behavior profile $\sigma = (\sigma_1, ..., \sigma_n)$ determines the set of *compatible endnodes*:

$$\begin{aligned} z(\sigma) \quad = \quad & \{z \in X^E : \theta(z) = \langle(\lambda(x_1), \sigma(x_1)), ..., (\lambda(x_k), \sigma(x_k)), \lambda(x_{k+1})\rangle \\ & \text{for the path } \langle x_1, ..., x_k, x_{k+1}\rangle \text{ to } z\}. \end{aligned} \tag{2.14}$$

Thus, the actions in the history $\theta(z)$ were prescribed by the behavior profile $\sigma = (\sigma_1, ..., \sigma_n)$. Each behavior profile σ also determines the set of *compatible endpieces*:

$$\lambda(\sigma) = \{w : \lambda(x) = w \text{ for some } x \in z(\sigma)\}. \tag{2.15}$$

When Γ is an extensive game in the strong sense, $z(\sigma)$ and $\lambda(\sigma)$ are singleton sets. However, for extensive games in the weak sense, these sets may have multiple elements. (They may be empty since all actions may not be used in the extensive game in the weak sense, but this difficulty will not be discussed in the discourse of this paper.)

3. Bounded Memory Abilities and Accumulation of Short-term Memories

In this section, we first define a domain of accumulation of short-term memories. This definition is based on the presumption that a player has a quite restricted ability of memory. Theoretically, however, there are still many other possibilities. In Section 3.2, we will give one informal theory about the accumulation of short-term memories

as long-term ones. This informal theory suggests a particular domain which we call the *unilateral trial (UT) domain*, which turns out to be linked to Nash equilibrium behavior, as will be shown in Section 7.2. Informal discussions of this type are intended to provoke further discussions and debates over the appropriate domain(s) for consideration. As well described by von Neumann-Morgenstern [29], p.7,

> ... This preliminary stage is necessarily *heuristic*, i.e., the phase of transition from unmathematical plausibility considerations to the formal procedure of mathematics. ...

we need a careful premathematical consideration for our theory.

3.1. The Objective Recurrent Situation and Domains of Accumulation of Local Memories

Let an extensive game $\Gamma^o = ((X^o, <^o), (\lambda^o, W^o), \{(\varphi^o_x, A^o_x)\}_{x\in X}, (\pi^o, N^o), h^o)$ in the strong sense and a profile $\mathfrak{m}^o = (\mathfrak{m}^o_1, ..., \mathfrak{m}^o_n)$ of memory functions be the description of the objective situation. The present purpose is to consider the accumulation of memories from playing in $(\Gamma^o, \mathfrak{m}^o)$ repeatedly.

>From the objective point of view, an individual player, i, has been experiencing short-term memories:

$$\begin{array}{ccc} \cdots \quad | \quad \mathfrak{m}_i(x^t_1), ..., \mathfrak{m}_i(x^t_{\ell_t}) \mid \mathfrak{m}_i(x^{t+1}_1), ..., \mathfrak{m}_i(x^{t+1}_{\ell_{t+1}}) \mid \cdots \\ (\Gamma^o, \mathfrak{m}^o) \text{ at } t \qquad\qquad (\Gamma^o, \mathfrak{m}^o) \text{ at } t+1 \end{array} \tag{3.1}$$

where $\langle x^t_1, ..., x^t_{\ell_t}\rangle$ is the realized sequence of player i's nodes in the occurrence of $(\Gamma^o, \mathfrak{m}^o)$ at time t. Due to bounded memory, player i will only accumulate some part of these as long-term memories.

In the extensive game $(\Gamma^o, \mathfrak{m}^o)$, the *domain of accumulation* for player i is a non-empty subset D_i of the set $X^o_i = \{x \in X^o : \pi^o(x) \ni i\}$ of nodes for player i. Player i is *relevant in his own domain* D_i iff it contains at least one decision node for player i. This definition will be important later in this paper.

We say that a pair $(T_{D_i}, \mathcal{Y}_{D_i})$ of a set of memory threads and a set of memory yarns is a *memory kit* for domain D_i iff

$$T_{D_i} = \bigcup_{x\in D_i} \mathfrak{m}^o_i(x); \tag{3.2}$$

$$\mathcal{Y}_{D_i} = \{\mathfrak{m}^o_i(x) : x \in D_i\}. \tag{3.3}$$

A memory kit is determined by both the domain of accumulation D_i and the objective memory function $\mathfrak{m}^o_i$ of player i. It will be the source for an inductive construction of a personal view. The set T_{D_i} of memory threads is used to construct a skeleton of the

tree for a personal view. The set $\mathcal{Y}_{D_i}$ of yarns is used to construct a perceived memory function. Mathematically speaking, the latter set gives the former, but we keep those two sets to emphasize that they have different usages.

It may be appropriate to compare our intended use of a memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ with a jigsaw puzzle. Treating the memory threads as pieces in a jigsaw puzzle, a player will first try to reconstruct the extensive game he is playing. If he succeeds at this task, he may next treat the memory yarns as the pieces in his jigsaw puzzle, and try to put memory yarns to appropriate nodes. It is important to keep in mind that unlike a standard jigsaw puzzle, the pieces in his memory kit may typically be incomplete and some are missing. In putting the pieces together, he acts as an artist, adding and ignoring somewhat as necessary. Consequently, the picture he constructs from his memory kit may not resemble closely the original even though it could be more understandable and, perhaps, more understandable.

For a memory kit, we assume that player i has accumulated some incidences of short-term memories as both threads and yarns. A kit does not include a sequence of short-term memories and/or frequencies. In Section 3.2, we will discuss one rationale for this treatment.

Here, we give three domains of accumulation. The first two are trivial ones, and the third example is the one we are going to explore in this paper.

(1): **Full Domain**: This is simply given as the entire set $D_i^F = X_i^o$ of player i's nodes. When the game is small, is repeated often enough, and when the accumulation ability of player i is strong enough, this domain may be appropriate.

(2): **Cane Domain**: A cane domain is a complete set of nodes for player i on one play. Formally, let $\langle x_0, ..., x_m \rangle$ be the path to an endnode x_m. Then the *cane domain of player i to x_m* is given as $\{x_0, ..., x_m\} \cap X_i^o$. A cane domain may arise if every player behaves always following some regular behavior pattern with no deviations.

Now, let $\sigma^o = (\sigma_1^o, ..., \sigma_n^o)$ be a profile of behavior patterns in the extensive game $(\Gamma^o, \mathfrak{m}^o)$. Then, this σ^o determines a unique path to an endnode. Hence, the cane domain for player i is uniquely determined, which is denoted by $D_i^c(\sigma^o)$. Using this concept, we can define the unilateral trial domain relative to a profile of behavior patterns.

(3): **Unilateral Trial (UT) Domain**: The UT-domain relative to a profile $\sigma^o = (\sigma_1^o, ..., \sigma_n^o)$ of behavior patterns for player i is given as

$$D_i^U(\sigma^o) = \bigcup_{\sigma_i \in \Sigma_i^o} D_i^c(\sigma_i, \sigma_{-i}^o). \tag{3.4}$$

Here, Σ_i^o is the *set of all behavior patterns* for player i in $(\Gamma^o, \mathfrak{m}^o)$ and $(\sigma_i, \sigma_{-i}^o)$ is the profile obtained from σ^o by substituting σ_i for σ_i^o in σ^o. That is, the UT-domain $D_i^U(\sigma^o)$ is the set of nodes for player i that are reached by unilateral deviations of player i.

For a unified treatment of the above domains, we introduce one condition. We say that a domain D_i for player i is *closed* iff D_i is expressed as some union of cane domains of player i. The above three examples of domains are closed. A domain which is not closed is the set X^{oE} of endnodes.

Example 3.1. Let us continue with the example of Figure 2.2. Let the regular behavior be given by $\sigma_1^o(x_0) = \sigma_1^o(x_3) = a$ and $\sigma_2^o(x_1) = \sigma_2^o(x_2) = c$. The cane domain and UT-domain of player 1 determined by σ^o are given as

$$D_1^c(\sigma^o) = \{x_0, z_1\} \text{ and } D_1^U(\sigma^o) = \{x_0, z_1, z_3\}.$$

The full domain is simply given as $D_1^F = X_1^o = \{x_0, x_3, z_1, z_2, z_3, z_4, z_5\}$.

The memory kit of player 1 depends also on his objective memory function $\mathfrak{m}_1^o$. For the three domains mentioned above and the Markov, EPR memory functions, we have a total of six memory kits. We mention two and leave the reader to consider the other four.

For the EPR memory function $\mathfrak{m}_1^o = \mathfrak{m}_1^{EPR}$ and the cane domain, we have $T_{D_1^c(\sigma^o)} = \{\langle w\rangle, \langle (w,a), z_1\rangle\}$, and $\mathcal{Y}_{D_1^c(\sigma^o)} = \{\{\langle w\rangle\}, \{\langle (w,a), z_1\rangle\}\}$.

For the Markov memory function $\mathfrak{m}_1^o = \mathfrak{m}_1^M$ and the UT-domain, we have $T_{D_1^U(\sigma^o)} = \{\langle w\rangle, \langle z_1\rangle, \langle z_3\rangle\}$ and $\mathcal{Y}_{D_1^U(\sigma^o)} = \{\{\langle w\rangle\}, \{\langle z_1\rangle\}, \{\langle z_3\rangle\}\}$.

3.2. An Informal Theory of Behavior and Accumulation of Memories

Our mathematical theory starts with a memory kit. Behind a memory kit, there is some underlying process of behavior and accumulation of short-term memories that generates this kit. We now describe one such underlying process informally which justifies the UT-domain of accumulation. This description is given in terms of some informal postulates.

(1): Postulates for Behavior and Trials: The first postulate is the rule-governed behavior of each player in the recurrent situation $..., (\Gamma^o, \mathfrak{m}^o), ..., (\Gamma^o, \mathfrak{m}^o), ...$

Postulate BH1 (Regular Behavior): Each player typically behaves regularly following his behavior pattern σ_i^o.

Player i may have adopted his regular behavior for some time without thinking, perhaps since he found worked well in the past or he was taught to follow it. Without assuming regular behavior and/or patterns, a player may not be able to extract any causal pattern from his experiences. In essence, learning requires some regularity.

To learn some other part than that regularity experienced in the extensive game, the players need to make some trial deviations. We postulate that such deviations take place in the following manner.

Postulate BH2 (Occasional Deviations): Once in a while (infrequently), each

player unilaterally and independently makes a trial deviation $\sigma_i \in \Sigma_i^o$ from his regular behavior σ_i^o and then returns to his regular behavior.

Early on, such deviations may be unconscious or not well thought out. Nevertheless, a player might find that a deviation leads to a better outcome, and he may start making deviations consciously in the future. Once he has become conscious of his behavior-deviation, he might make more and/or different trials.

The set of trial deviations for a player is not yet well specified. In the remainder of this paper, we explore one extreme case where he tries every possible behavior. The following postulate is made for simplicity in our discourse and since it connects our theory to standard game theory.

Postulate BH3 (All Possible Trials): Each player experiments over all his possible behaviors.

Postulate BH3 is an extreme case that each player tries all his alternative behaviors. We do not take this as basic. The choice of a smaller set of trial deviations is very relevant, since a player might not have prior knowledge of his available behaviors. This aspect will be explored in a separate paper.

(2): Epistemic Postulate: Each player may learn something through his regular behavior and deviations. What he learns in an instant is described by his short-term memory. For the transition from short-term memories to long-term memories, there are various possibilities. Here we list some postulates based on bounded memory abilities that suggest only the UT-domain of accumulation.

The first postulate states that if a short-term memory does not occur frequently enough, it will disappear from the mind of a player. We give this as a postulate for a cognitive bound on a player.

Postulate EP1 (Forgetfulness): If experiences are not frequent enough, then they would disappear from a player's mind.

In the face of this cognitive bound, only some memories become lasting. The first type of memories that become lasting are the regular ones since they occur quite frequently. The process of making a memory last by repetition is known as habituation.

Postulate EP2 (Habituation): A short-term (local) memory becomes lasting as a long-term memory in the mind of a player by habituation, i.e., if he experiences something frequently enough, it remains in his memory as a long-term memory even without conscious effort.

By EP2, when all players follow their regular behavior patterns, the short-term memories given by them will become long-term memories by habituation.

The remaining possibilities for long-term memories are the memories of trials made by some players. We postulate that a player may consciously spend some effort to

memorize the outcomes of his own trials.

Postulate EP3 (Conscious Memorization Effort): A player makes a conscious effort to memorize the result of his own trials. These efforts are successful if they occur frequently enough relative to his trials.

Postulate EP3 means that when a player makes a trial deviation, he also makes a conscious effort to record his experience in his long-term memory. These memories are more likely to be successful if they are repeated frequently enough relative to his trials. Since the players are presumed to behave independently, the trial deviations involving multiple players will occur infrequently, even relative to one player's trials. Thus, the memories associated with multiple players' trials do not remain as long-term memories. This has the implication that our experiential foundation is typically incompatible with the subgame perfect concept of Selten [27], which will be discussed again in Section 9.

In sum, postulates EP1 to EP3 and BH1 to BH3 suggest that we can concentrate on the UT-domain of a player.

Some other domains such as a cane domain and the full domain might emerge as candidates in slightly different situations. For example, if no trials are made, then EP2(Habituation) gives the cane domain corresponding to σ^o. Alternatively, if the game is small enough and if it is repeated enough, then each player has experienced every outcome. And if he has an ability to recall all the incidences, then we would get the full domain. The additional assumption of full recall seems plausible for small games.

Remark (Default Decision and all the Possible Behaviors): One may criticize our treatments in that:

(1) σ_i^o has the total domain X_i^o and

(2) σ_i varies over the entire Σ_i^o of (3.4),

since these might conflict with the assumption of no *a priori* knowledge of the structure of the game for player i.

We can answer (1) by interpreting one action at every decision node as a default action. When a player receives an unknown information piece, he just takes the default action. This assumption avoids a player's need to plan for his behavior over the entire domain.

We take (2) as a legitimate criticism, particularly, when the game is large. We have chosen (3.4) as a working assumption in this paper. We will develop a theory corresponding to this criticism in a future paper. Nevertheless, the case treated in the present paper helps to connect our theory to the standard game theory.

4. Inductively Derived Views

In this section, we will give a definition of an inductively derived (personal) view, which we abbreviate as an i.d.view. Here, player i uses only his memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ as a summary of his experiences to construct an i.d.view. Before the definition, we will talk about our basic principles to be adopted in this paper. After the definition, we will consider various examples to see the details of the definition.

4.1. Observables, Observed, and Additional Components

The central notion in the present paper is the process of inductive inferences. An inductive inference is distinguished from a deductive inference in that the former allows some generalization of observations by adding some hypothetical components, while the latter changes expressions following the well-formed inference rules and keeps the same or less contents. A player, i, having a memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ may add some hypothetical components to the kit in his inductive process to develop a personal view.

The need for this addition of hypothetical components may be found in the assumption that a player can only observe some elements of the objective extensive game Γ^o. As remarked in Section 2.2, only information pieces and actions are observable for each player, while nodes are hypothetical and unobservables. In addition, many or some pieces and actions that are observable will not end up in the memory kit. Pieces and actions only along some of the paths in a game tree will likely be observed for players. Moreover, the bounds on their memory capabilities will allow them to accumulate memories of only some of what they have observed. The memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ is the collection of observed parts which are effectively remaining in the mind of player i.

Since player i describes his view $(\Gamma^i, \mathfrak{m}^i)$ as an extensive game in the weak sense with a memory function, he will need to invent a tree structure by adding hypothetical nodes. In this sense he already goes beyond deductive inferences. To construct a coherent view, a player may add other components, e.g., more information pieces, actions, and possible histories to his memories. In this paper, however, we adhere to the basic principle that only elements in the memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ can be used as the observables in $(\Gamma^i, \mathfrak{m}^i)$. In Section 4.2, we will adopt a specific inductive process called the *initial-segment procedure* and use this procedure to define an i.d.view. With this procedure, a player forms the underlying skeletal structure of his view by adding hypothetical nodes. In a separate paper, we consider a more loose restriction on inductive inferences.

4.2. Definition and Examples

Now, consider the recurrent situation of $(\Gamma^o, \mathfrak{m}^o)$ illustrated in Figure 1.2. Here, $\Gamma^o = ((X^o, <^o),\ (\lambda^o, W^o), \{(\varphi^o_x, A^o_x)\}_{x \in X^o}, (\pi^o, N^o), \{h^o_i\}_{i \in N^o})$ is an extensive game in the strong sense and $\mathfrak{m}^o = (\mathfrak{m}^o_1, ..., \mathfrak{m}^o_n)$ is an n-tuple of memory functions. Recall that a

personal view is given as a pair $(\Gamma^i, \mathfrak{m}^i)$, where $\Gamma^i = ((X^i, <^i), (\lambda^i, W^i), \{(\varphi^i_x, A^i_x)\}_{x \in X^i}, (\pi^i, N^i), h^i)$ is an extensive game in the weak sense specifying only the payoff function h^i of player i and $\mathfrak{m}^i$ is a memory function for player i in that game. We assume that player i uses his memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ in the sense of (3.2) and (3.3) to construct his personal view $(\Gamma^i, \mathfrak{m}^i)$.

Strictly speaking, we will not consider the precise process of inductive derivation of a view $(\Gamma^i, \mathfrak{m}^i)$. Instead, we consider possible candidates of $(\Gamma^i, \mathfrak{m}^i)$ for the result of inductive derivation. For the definition of such a candidate, we need a bridge between $(T_{D_i}, \mathcal{Y}_{D_i})$ and $(\Gamma^i, \mathfrak{m}^i)$. We can think of various procedures to have such bridges, but we will use one procedure, called the *initial-segment procedure,* as stated in Section 4.1. It will become clear shortly why we have chosen this name.

First, for a given candidate $(\Gamma^i, \mathfrak{m}^i)$, we can define the set $\Theta(\Gamma^i)$ of possible histories in Γ^i:

$$\Theta(\Gamma^i) = \{\theta^i(y) : y \in X^i\}, \tag{4.1}$$

where $\theta^i(y) = \langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ is the complete history up to y in Γ^i. We would like to have a procedure to yield this set from his experiences captured in T_{D_i}. Each complete history $\theta^i(y) \in \Theta(\Gamma^i)$ is an initial segment of a maximal history in Γ^i, that is, $\Theta(\Gamma^i)$ is described as the set of initial segments of maximal histories. To connect $\Theta(\Gamma^i)$ to T_{D_i}, we use the set of initial segments of threads in T_{D_i}.

For the sake of rigor, we make the following definitions. First, a subsequence of $[(w_1, a_1), ..., (w_m, a_m)]$ is simply defined in the standard manner by regarding each (w_t, a_t) as a component of the sequence. Second, $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ is said to be a *subsequence* of $\langle (v_1, b_1), ..., (v_k, b_k), v_{k+1} \rangle$ iff $[(w_1, a_1), ..., (w_m, a_m), (w_{m+1}, a)]$ is a subsequence of $[(v_1, b_1), ..., (v_k, b_k), (v_{k+1}, a)]$ for some a. A supersequence is defined in the dual manner. We say that $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ is a *maximal sequence* in a given set of sequences iff there is no proper supersequence in that set. An *initial segment* of $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ is a subsequence of the form $\langle (w_1, a_1), ..., (w_k, a_k), w_{k+1} \rangle$ and $k \leq m$.

Now, we can define the set of initial segments of memory threads in T_{D_i} as:

$$T^*_{D_i} := \{\langle \xi, v \rangle : \langle \xi, v \rangle \text{ is an initial segment of some maximal sequence in } T_{D_i}\}. \tag{4.2}$$

We require $\Theta(\Gamma^i)$ to be the same as $T^*_{D_i}$ for Γ^i to be inductively derived from T_{D_i}. It should now be clear why this is called the *initial-segment procedure.* A player uses all his initial segments in T_{D_i} and nothing more to construct the histories in Γ^i.

We now give the full set of requirements for an inductively derived personal view based on this procedure. As mentioned above, we will give a more general definition of an i.d.view in another paper, which will allow for other inductive procedures. In the following definition, we assume that player i is relevant in his own domain D_i, i.e., D_i contains at least one decision node of player i.

Definition 4.1 (Inductively Derived View). A personal view $(\Gamma^i, \mathfrak{m}^i)$ for player i is *inductively derived* from the memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ iff

P1(Construction of an Extensive Game): Γ^i is an extensive game in the weak sense satisfying:

(a)(Preservation of the Informational Structure): $\Theta(\Gamma^i) = T^*_{D_i}$;

(b)(Action Sets): $A^i_x = A^o_{\lambda^i(x)}$ for each $x \in X^i$;

(c)(Player Assignment at Decision Nodes): $\pi^i \cdot \lambda^i(x) = \pi^o \cdot \lambda^i(x)$ for all $x \in X^{iD}$;

(d)(Own Payoffs): $h^i \cdot \lambda^i(x) = h^o_i \cdot \lambda^i(x)$ for each $x \in X^{iE}$;

P2(Construction of a Memory Function): $\mathfrak{m}^i$ is a memory function on $X^i_i = \{x \in X^i : i \in \pi^i \cdot \lambda^i(x)\}$ satisfying:

(a)(Preservation of Memory Yarns): $\{\mathfrak{m}^i(x) : x \in X^i_i\} \subseteq \mathcal{Y}_{D_i}$;

(b)(Internal Consistency): $\theta^i(x) \in \mathfrak{m}^i(x)$ for any $x \in X^i_i$;

(c)(History Determined Memories): if $\theta^i(x) = \theta^i(y)$, then $\mathfrak{m}^i(x) = \mathfrak{m}^i(y)$.

We abbreviate an inductively derived view as an *i.d.view*.

Since the memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ is completely determined by the domain of accumulation D_i and the objective memory function $\mathfrak{m}^o_i$ of player i, we may sometimes speak of an i.d.view for D_i or just an i.d.view when there appears to be no source for confusion.

For an i.d.view, the extensive game Γ^i is constructed based on the set $T^*_{D_i}$ of initial segments of maximal memory threads in T_{D_i}. P1a states that the game tree is based on $T^*_{D_i}$. Conditions P1b, P1c, P1d are the minimum requirements M1, M2, M3 stated in Section 2.1. By P1c and K42, the player set for Γ^i is determined as

$$N^i = \{j \in N^o : j \in \pi^i \cdot \lambda^i(x) \text{ for some } x \in X^{iD}\}.$$

Since λ^i is a surjection from X^i to W^i by K2, and since $\Theta(\Gamma^i) = T^*_{D_i}$ by P1a, we have $W^i \subseteq W^o$. Hence, P1b and P1c are well-defined. For the well-definedness of P1d, it should hold that for any $x \in X^{iE}$, the associated piece $\lambda^i(x)$ is an endpiece in the objective game Γ^o. Thus, if $\lambda^i(x)$ is not an endpiece in the objective game Γ^o for some $x \in X^{iE}$, then $(\Gamma^i, \mathfrak{m}^i)$ is not an i.d.view.

The personal memory function $\mathfrak{m}^i$ is constructed based on the set $\mathcal{Y}_{D_i}$ of memory yarns. Condition P2a states that the player uses only his accumulated memory yarns in his construction, though he might not use all of them. Condition P2b is a consistency requirement that each yarn $\mathfrak{m}^i(x)$ should contain the complete history $\theta^i(x)$. This is made since $(\Gamma^i, \mathfrak{m}^i)$ is now in the mind of player i and can be seen by player i as the objective observer. Still, P2b is just one alternative among several internal consistency

requirements. Condition P2c is a more basic principle about the view a player inductively constructs. It states that a player should not have different memories, unless two nodes have a different history in his view Γ^i.

To see how an i.d.view is obtained, we look at several examples.

Example 4.1 (Exact Perfect Memory Function $\mathfrak{m}_1^{EPR}$). Observe that for this memory function, any i.d.view will be a one player game played by player 1. Consider this memory function on the cane domain described in Example 3.1. Recall that the memory kit is given as

$$T_{D_1^c(\sigma^o)} = \{\langle w\rangle, \langle (w,a), z_1\rangle\}, \text{ and } \mathcal{Y}_{D_1^c(\sigma^o)} = \{\{\langle w\rangle\}, \{\langle (w,a), z_1\rangle\}\}.$$

Then $T_{D_1^c(\sigma^o)} = T^*_{D_1^c(\sigma^o)}$, and an i.d.view is given as Figure 4.1. It consists of the set of nodes $X^1 = \{y_0, y_1\}$, $\lambda^1(y_0) = w, \lambda^1(y_1) = z_1, \pi^1(w_0) = \pi^1(z_1) = \{1\}, h^1(z_1) = 2$ and his memory function satisfies $\mathfrak{m}^1(y_1) = \{\langle (w,a), z_1\rangle\}$ and $\mathfrak{m}^1(y_0) = \{\langle w\rangle\}$. Since $A^1_{y_0} = A^o_w = \{a, b\}$ by P1b, condition K33 (bijection requirement) is violated, but K33_0 is satisfied.

Figure 4.1.

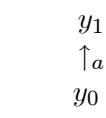

Figure 4.2.

Now, we consider some multiplicity of i.d.views. Observe that in Definition 4.1, the connection between $(T_{D_i}, \mathcal{Y}_{D_i})$ and $(\Gamma^i, \mathfrak{m}^i)$ are made based on observable information pieces and actions. The nodes are unobservable in the original game $(\Gamma^o, \mathfrak{m}^o)$ and the nodes are auxiliary in the derived game $(\Gamma^1, \mathfrak{m}^1)$. We can use different symbols for y_0 and y_1 without changing the informational structure of the game. In addition, the game having the duplication of $(\Gamma^1, \mathfrak{m}^1)$ described in Figure 4.2 satisfies all the requirements of Definition 4.1. Thus, we have multiple personal views. We will introduce the concept of a game theoretic p-morphism in Section 6 as a means for dealing with some types of multiplicity.

Example 4.2 (Markov Memory Function $\mathfrak{m}_1^M$: Failure). Let player i have the Markov memory function $\mathfrak{m}_i^M$. The domain D_i contains a decision node y of player i in Γ^o, as noted before Definition 4.1. Let $\lambda^o(y) = w$. Since $\mathfrak{m}_i^M$ is the Markov memory function, $\langle w\rangle$ is a maximal sequence in $T^*_{D_i}$. Let $(\Gamma^i, \mathfrak{m}^i)$ be a personal view. Then, by P1a, $\langle w\rangle$ is a maximal sequence in $\Theta(\Gamma^i)$. This means there must be an endnode $y' \in X^{iE}$ such that $\lambda^i(y') = w$. But since $h_i^o(w)$ is not defined, $(\Gamma^i, \mathfrak{m}^i)$ cannot be an i.d.view.

Example 4.3 (Perfect Information Memory Function $\mathfrak{m}_1^{PI}$: Full Recoverability). Let player 1 have the perfect information memory function $\mathfrak{m}_1^{PI}$ and let the domain

be the full domain $D_1^F = X_1^o$ in the game of Figure 2.2. In this case, player 1 can reconstruct the objective game Γ^o from his memory kit, except for player 2's payoffs and memory function. When player 1 has the classical memory function $\mathfrak{m}_1^C$, we also have the full recoverability result when we use the full domain D_1^F.

We give one last example which shows some difficulty that may be encountered with the classical memory function on some closed domains.

Example 4.4 (Classical Memory $\mathfrak{m}_1^C$ with the Cane Domain: Failure). Let player 1 have the classical memory function $\mathfrak{m}_1^C$ on the cane domain D_1^c described in Example 3.1. Then one candidate is described as Figure 4.3, which violates condition K31.

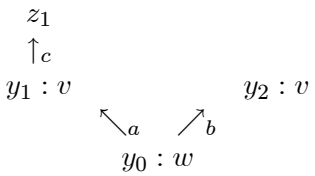

Figure 4.3.

In fact, there is no i.d.view. To see this, let $(\Gamma^1, \mathfrak{m}^1)$ be a candidate for a personal view of player 1. With the classical memory function on the cane domain, player 1 will have the memory thread $\langle (w, b), v \rangle$ as a maximal one in $T_{D_1}^*$. This means that Γ^1 must have an endnode x, and $\lambda^1(x) = v$. But $h_1^o(v)$ is not defined since v is not an endpiece in Γ^o.

5. Direct Views

In Section 4, we gave the definition of an inductively derived view for a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ and found that there may be many i.d.views for each $(T_{D_i}, \mathcal{Y}_{D_i})$. In this section, we single out one of those views which we call the direct view. We will argue that it has a special status among i.d.views or simply among views. Here, we give some results for a direct view to be an i.d.view. In Section 6, we will show that our analysis of direct views is sufficient to describe the game theoretic contents of any i.d.view.

A direct view for a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ is constructed by treating each thread in $T_{D_i}^*$ as a node in the derived game. As in Section 4, we assume that player i is relevant in his own domain D_i.

Definition 5.1 (Direct View). A *direct view* $(\Gamma^d, \mathfrak{m}^d) = ((X^d, <^d), (\lambda^d, W^d), \{(\varphi_x^d,\ A_x^d)\}_{x \in X^d}, (\pi^d, N^d), h^d), \mathfrak{m}^d)$ from a memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ is defined in the following manner:

d1: $X^d = T_{D_i}^*$;

d2: $\langle \xi, v\rangle <^d \langle \eta, w\rangle$ iff $\langle \xi, v\rangle$ is a proper initial segment of $\langle \eta, w\rangle$;

d3 (Information Function): $\lambda^d\langle \xi, v\rangle = v$ for all $\langle \xi, v\rangle \in X^d$; and $W^d = \{v : \langle \xi, v\rangle \in X^d \text{ for some } \xi\}$;

d4 (Action sets): $A^d_{\langle \xi, v\rangle} = A^o_v$ for all $\langle \xi, v\rangle \in X^d$; and if $\langle \xi, v\rangle \in X^{dD}$, then $\varphi^d_{\langle \xi, v\rangle}\langle \xi, (v, a), u\rangle = a$ for each immediate successor $\langle \xi, (v, a), u\rangle$ of $\langle \xi, v\rangle$;

d5 (Player Assignment): $\pi^d(v) = \pi^o(v)$ for all $\langle \xi, v\rangle \in X^{dD}$; and $\pi^d(v) = N^d$ for all $\langle \xi, v\rangle \in X^{dE}$, where $N^d = \{j : j \in \pi^o(v) \text{ for some } \langle \xi, v\rangle \in X^{dD}\}$;

d6 (Payoff Function): for any $\langle \xi, v\rangle \in X^{dE}$, if $\lambda^o(x) = v$ for some $x \in X^{oE}$, then $h^d(v) = h^o_i(v)$; and otherwise, $h^d(v)$ is arbitrary;

d7 (Memory Function): for any node $\langle \xi, v\rangle$ in X^d_i, if some $\mathfrak{y} \in \mathcal{Y}_{D_i}$ contains $\langle \xi, v\rangle$, then $\mathfrak{m}^d\langle \xi, v\rangle$ is such a $\mathfrak{y} \in \mathcal{Y}_{D_i}$; and otherwise, $\mathfrak{m}^d\langle \xi, v\rangle = \{\langle \xi, v\rangle\}$.

In the following, $\Gamma^d = ((X^d, <^d), (\lambda^d, W^d), \{(\varphi^d_x, A^d_x)\}_{x\in X^d}, (\pi^d, N^d), h^d)$ defined by d1 to d6 is called a *direct structure*, and $\mathfrak{m}^d$ defined by d7 is a *direct memory function*.

Conditions d6 and d7 have some arbitrariness and allow multiple direct views for a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$. However, when a direct structure is an extensive game in the weak sense, it is uniquely determined, though there may remain multiple memory functions.

A direct view $(\Gamma^d, \mathfrak{m}^d)$ for $(T_{D_i}, \mathcal{Y}_{D_i})$ may not be a personal view; specifically, conditions K21 and K31 may be violated. Example 4.4 violates K21 and K31, and also, when the objective memory function is the Markov, a direct view typically violates K31. In Theorem 5.2, we will give a condition for a direct view to be a personal view as well as an i.d.view.

Another important comment is about the constructive nature of a direct view. It is directly constructed using the components in the memory kit, focusing the initial segments of memory threads in T_{D_i}. In Section 6, it will be further shown that it is the smallest among i.d.views for a given $(T_{D_i}, \mathcal{Y}_{D_i})$.

By condition d1, the nodes in X^d are actually threads in $T^*_{D_i}$. Consequently, the complete history up to each node $x \in X^d$ is the same as x itself. This fact is stated as a lemma.

Lemma 5.1. For any direct structure Γ^d, $\theta^d(x) = x$ for all $x \in X^d$.

Proof. Let $x \in X^d$. Then $x = \langle \xi, v\rangle = \langle (w_1, a_1), ..., (w_k, a_k), v\rangle$ is an initial segment of a maximal thread in T_{D_i}. The path to $\langle \xi, v\rangle$ is $\langle w_1\rangle, \langle (w_1, a_1), w_2\rangle, ..., \langle (w_1, a_1), ..., (w_{k-1}, a_{k-1}), w_k\rangle, \langle \xi, v\rangle$. The complete history up to $\langle \xi, v\rangle$ is the sequence $\langle (w_1, a_1), ..., (w_{k-1}, a_{k-1}), (w_k, a_k), v\rangle$, which is x itself. ■

Let us now look at an example of a direct view.

Example 4.1 (continued): In Figures 4.1 and 4.2 we gave two examples of i.d.views

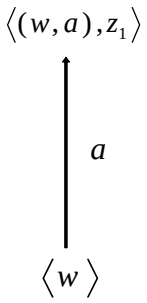

Figure 5.1:

for player 1. This example has a unique direct view, which is given in Figure 5.1 and is an i.d.view with the memory function $\mathfrak{m}^i(x) = \{x\}$ for all $x \in X^d$.

A direct view for a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$, however, may not be an i.d.view. The memory kits in Examples 4.2 and 4.4 lead to this type of problem. In either of these examples, the direct structure is not an extensive game in the weak sense.

Now, we give conditions for a direct view to be an i.d.view. Recall the assumption that player i is relevant for his own domain D_i.

Theorem 5.2 (Conditions for a Direct View to be I.D.): Let $(T_{D_i}, \mathcal{Y}_{D_i})$ be a memory kit.

(i): The direct structure Γ^d for $(T_{D_i}, \mathcal{Y}_{D_i})$ is uniquely determined and is an extensive game in the weak sense satisfying P1a-P1d if and only if for any maximal $\langle \xi, v \rangle$ in $T^*_{D_i}$, $v = \lambda^o(x)$ for some $x \in X^{oE}$.

(ii): Let Γ^d be a direct structure for T_{D_i}. There there is a direct memory function $\mathfrak{m}^d$ for Γ^d satisfying P2a-P2c if and only if for any $\langle \xi, w \rangle \in T^*_{D_i}$ with $i \in \pi^o(w)$, there is an $x \in D_i$ such that $\langle \xi, w \rangle \in \mathfrak{m}^o_i(x)$.

This theorem will be proved at the end of this section. The part (i) states that a condition for the unique determination of a direct structure is that every maximal thread in $T^*_{D_i}$ occurs at an endnode in the objective game. The part (ii) gives a necessary and sufficient condition for a direct memory function prescribed by d7 to satisfy P2a-P2c. When both of these conditions are satisfied, there is a direct view that is i.d., but there is still, however, some arbitrariness in the memory function, which allows for multiple direct views, which is shown by Example 5.1.

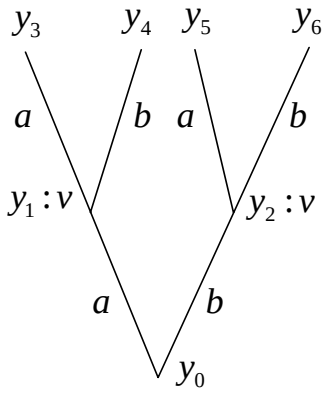

Figure 5.2:

Example 5.1. Consider the objective 1-person sequential move game of Figure 5.2. Here, the information function is given by $\lambda^o(y_t) = v$ for $t = 1, 2$, and it is the identity function everywhere else. Suppose that the domain of accumulation is the full domain $D_1^F = X_1^o = X^o$. Let the objective memory function $\hat{\mathfrak{m}}_1^o$ be defined by:

$$\hat{\mathfrak{m}}_1^o(y_t) = \begin{cases} \{\theta^o(y_t)\} & \text{if } t \neq 1 \text{ and } t \neq 2; \\ \{\langle (y_0, a), v\rangle, \langle (y_0, b), v\rangle, \langle v\rangle\} & \text{if } t = 1; \\ \{\langle (y_0, a), v\rangle, \langle (y_0, b), v\rangle\} & \text{if } t = 2. \end{cases} \tag{5.1}$$

In this example, the direct structure Γ^d is uniquely determined, which has the same structure as Figure 5.2 consisting of nodes $\theta^o(y_1), ..., \theta^o(y_6)$. However, a memory function has some arbitrariness at the nodes $\theta^o(y_1)$ and $\theta^o(y_2)$. For example, assigning the memory $\mathfrak{m}^d(\theta^o(y_1)) = \mathfrak{m}_1^o(y_2)$ and $\mathfrak{m}^d(\theta^o(y_2)) = \mathfrak{m}_1^o(y_1)$, together with $\mathfrak{m}^d(\theta^o(y_t)) = \mathfrak{m}_i^o(y_t)$ for $t \neq 1$ and $t \neq 2$, gives one i.d.direct view. In this view, the player mixes up his memories at y_1 and y_2. In Section 8.2, we will see how this mixing up may create some difficulties. Another view is where he assigns his memory yarns correctly. Still two other views are obtained if he assigns one memory yarn to each of those nodes.

We now introduce two conditions on a memory function, that we will use in combination with Theorem 5.2 to provide a sufficient condition for the uniqueness of a direct view.

(Recall of Past Memories - RPM): for all $x, y \in X_i^o$, if $\langle \xi, w\rangle \in \mathfrak{m}_i^o(x)$ and $x <^o y$, then $\langle \xi, w\rangle$ is a proper initial segment of some $\langle \eta, v\rangle \in \mathfrak{m}_i^o(y)$.

(Single Thread Yarns - STY): $|\mathfrak{m}_i^o(x)| = 1$ for all $x \in X_i^o$.

The first condition states that every memory thread occurring at a node x of player i, will occur as a subsequence of a thread at any later node y of player i. This is interpreted as meaning that player i recalls what past memories he had in the current play of the game. The second condition is simply that each yarn consists of a single thread.

The following corollary gives a sufficient condition for the unique determination of a direct view, which guarantees that it is an i.d.view.

Corollary 5.3. Let $(T_{D_i}, \mathcal{Y}_{D_i})$ be a memory kit having a closed domain D_i and derived from $\mathfrak{m}_i^o$ satisfying conditions RPM and STY. Furthermore, suppose the latter part of Theorem 5.2.(ii). Then, the direct view $(\Gamma^d, \mathfrak{m}^d)$ is uniquely determined by d1-d7, and $\mathfrak{m}^d(x) = \{x\}$ for all $x \in X_i^d$. Moreover, $(\Gamma^d, \mathfrak{m}^d)$ is an i.d.view.

It is straightforward to check that the exact perfect recall memory function $\mathfrak{m}_i^{EPR}$ and the perfect information memory function $\mathfrak{m}_i^{PI}$ on a closed domain satisfy the conditions of Corollary 5.3. Thus, in those cases, we can speak of a unique direct view. We prove this corollary after proving Theorem 5.2.

Proof of Theorem 5.2.(i)(If): Suppose that for any maximal $\langle \xi, v \rangle$ in $T_{D_i}^*$, $v = \lambda^o(x)$ for some $x \in X^{oE}$. Under this supposition, we first show that the direct structure is a uniquely determined extensive game in the weak sense.

Let Γ^d be a direct structure satisfying d1 to d7. First, observe that the verification of each of K11 to K13 is straightforward by d1, d2, the non-emptiness of D_i and the finite number of threads for each yarn of the memory function $\mathfrak{m}_i^o$. Condition K2 follows from K2 for Γ^o, d1, d2, d3, condition (2.3) for $\mathfrak{m}_i^o$, and the supposition of the *if* part. Condition K31 also follows from the supposition of the *if* part together with K31 on Γ^o and d4. Conditions K32 and K33$_0$ follow from d1, d2, d3, and d4. K4 uses d5 and d6. Finally, condition K5$_0$ follows from d6. The supposition of the *if* part implies the payoff function h_i^d is uniquely determined by d6. Thus, we have shown that the direct structure Γ^d is determined uniquely as an extensive game in the weak sense.

Next we show that P1a holds. By Lemma 5.1, $\Theta(\Gamma^d) = X^d$, and by d1, $X^d = T_{D_i}^*$. Hence, $\Theta(\Gamma^d) = T_{D_i}^*$. The other parts of P1 follow immediately from the definition of a direct structure.

(Only-if): Suppose that there is a maximal $\langle \xi, v \rangle$ in $T_{D_i}^*$ and $v = \lambda^o(x)$ for some $x \in X^{oD}$. By K33 for Γ^o, $A_x^o \neq \varnothing$. By d4, we have $A^d_{\langle \xi, v \rangle} = A_x^o \neq \varnothing$. However, $\langle \xi, v \rangle \in X^{dE}$ since $\langle \xi, v \rangle$ is maximal in $T_{D_i}^*$. Hence, K31 is violated for Γ^d, and thus Γ^d is not an extensive game in the weak sense.

(ii)(If): Suppose that for any $\langle \xi, w \rangle \in T_{D_i}^*$ with $i \in \pi^o(w)$, there is an $x \in D_i$ such that $\langle \xi, w \rangle \in \mathfrak{m}_i^o(x)$. Then we can define $\mathfrak{m}^d \langle \xi, w \rangle = \mathfrak{m}_i^o(x)$. This is a direct memory function of player i for the direct structure Γ^d, since it associates a memory yarn from $\mathcal{Y}_{D_i}$ to each $\langle \xi, w \rangle \in T_{D_i}^* = X_i^d$. Then, P2a and P2b are satisfied since by Lemma 5.1,

$\theta^d\langle\xi, w\rangle = \langle\xi, w\rangle$. Finally, $\mathfrak{m}^d$ satisfies P2c, since by Lemma 5.1, $\theta^d\langle\xi, w\rangle = \theta^d\langle\eta, v\rangle$ implies $\langle\xi, w\rangle = \langle\eta, v\rangle$.

(Only-if): If $\mathfrak{m}^d$ is a direct memory function for Γ^d, then the result follows by P2a and P2b for $\mathfrak{m}^d$. ■

Proof of Corollary 5.3. The right-hand side of Theorem 5.2.(i) is equivalent to that if $\langle\xi, w\rangle \in T^*_{D_i}$ and $\lambda^o(x) = w$ for some decision node $x \in D_i$, then $\langle\xi, w\rangle$ is not maximal in $T^*_{D_i}$. Let $\langle\xi, w\rangle \in T^*_{D_i}$ and suppose that $\lambda^o(x) = w$ for some decision node $x \in D_i$. Then either $\langle\xi, w\rangle$ is a proper initial segment of some $\langle\eta, v\rangle \in T^*_{D_i}$, or $\langle\xi, w\rangle \in T_{D_i}$. In the first case, $\langle\xi, w\rangle$ cannot be maximal in $T^*_{D_i}$. Suppose that $\langle\xi, w\rangle \in T_{D_i}$. Then, $\langle\xi, w\rangle \in \mathfrak{m}^o_i(x')$ for some $x' \in D_i$. By K2, (2.4), and the supposition that $\lambda^o(x) = w$ for some decision node $x \in D_i$, it follows that x' must also be a decision node in D_i. Then, by closedness we have a $z \in D_i$ with $x' <^o z$. By RPM, there is a $\langle\eta, v\rangle \in \mathfrak{m}^o_i(z)$ such that $\langle\xi, w\rangle$ is a proper subsequence of $\langle\eta, v\rangle$. Thus, $\langle\xi, w\rangle$ is not maximal in $T^*_{D_i}$.

By Theorem 5.2.(i), the direct structure Γ^d is uniquely determined and is an extensive game in the weak sense satisfying P1a-P1d. It remains to be shown that the memory function $\mathfrak{m}^d(x) = \{x\}$ is the only memory function for Γ^d that satisfies P2. By the supposition in the corollary that for any $\langle\xi, w\rangle \in T^*_{D_i}$ with $i \in \pi^o(w)$, there is an $x \in D_i$, it follows by Theorem 5.2.(ii) that there is a direct memory function for Γ^d that satisfies P2. By STY, $\mathfrak{m}^d(x) = \{x\}$ is the only possible memory function for Γ^d. ■

6. Game Theoretical p-Morphisms: Comparisons of Views

In this section, we will show that for any i.d.view $(\Gamma^i, \mathfrak{m}^i)$ for a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$, there is a direct and inductively derived view $(\Gamma^d, \mathfrak{m}^d)$ having the same game theoretical structure. This result reduces the multiplicity of i.d.views, and allows us to concentrate on the direct views for our analysis of i.d.views. For example, the existence of an i.d.view is equivalent to the existence of a direct i.d.view. This consideration will be possible by introducing the concept of a game theoretical p-morphism, which is a modification of a p-morphism in the modal logic literature (cf. Ono [24] and Blackburn-de Rijke-Venema [2]). We call it simply a g-morphism.

6.1. Definition and Results

In the following definition, we abbreviate the superscript i for each component of $(\Gamma^i, \mathfrak{m}^i)$ and $(\hat{\Gamma}^i, \hat{\mathfrak{m}}^i)$ to avoid unnecessary complications.

Definition 6.1 (Game Theoretical p-Morphism): Let $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be personal views of player i. A function ψ from X to $\hat{X}$ is called a *g-morphism* (game theoretical p-morphism) iff

g0: ψ is a surjection from X to $\hat{X}$;

g1: for all $x, y \in X$ and $a \in A_x$, $x <_a y$ implies $\psi(x) \hat{<}_a \psi(y)$;

g2: for all $\hat{x}, \hat{y} \in \hat{X}, y \in X$ and $a \in \hat{A}_{\hat{x}}$,
$\hat{x} \hat{<}_a \hat{y}$ and $\hat{y} = \psi(y)$ imply $x <_a y$ and $\hat{x} = \psi(x)$ for some $x \in X$;

g3 (Information Pieces): $\hat{\lambda} \cdot \psi(x) = \lambda(x)$ for all $x \in X$;

g4 (Action Sets): $\hat{A}_{\psi(x)} = A_x$ for all $x \in X$;

g5 (Player Assignment): $\hat{\pi} \cdot \hat{\lambda} \cdot \psi(x) = \pi \cdot \lambda(x)$ for all $x \in X$;

g6 (Payoff Function): $\hat{h} \cdot \hat{\lambda} \cdot \psi(x) = h \cdot \lambda(x)$ for all $x \in X^E$;

g7 (Memory Function): $\hat{\mathfrak{m}} \cdot \psi(x) = \mathfrak{m}(x)$ for all $x \in X_i$.

We say that $(\Gamma, \mathfrak{m})$ is *g-morphic* to $(\hat{\Gamma}, \hat{\mathfrak{m}})$, denoted by $(\Gamma, \mathfrak{m}) \rightarrowtail (\hat{\Gamma}, \hat{\mathfrak{m}})$, iff there is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

A g-morphism ψ compares one personal view to another one. When a g-morphism exists from $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$, the set of nodes in Γ is mapped onto the set of nodes in $\hat{\Gamma}$, while the game theoretic components of $(\Gamma, \mathfrak{m})$ are preserved. Since ψ is a surjection from X to $\hat{X}$, we cannot take the direct converse of g1, but we take a weak form, g2, which requires that the image $(\hat{\Gamma}, \hat{\mathfrak{m}})$ should not have any additional structure. In sum, the mapping ψ embeds $(\Gamma, \mathfrak{m})$ into $(\hat{\Gamma}, \hat{\mathfrak{m}})$ without losing the game structure. Nevertheless, a g-morphism allows a comparison of quite different games.

In the modal logic literature, the concept of a p-morphism is used to compare two Kripke models and their validities. As mathematical objects, Kripke models and extensive games have some similarity in that their basic structures are expressed as some graphs (or trees) (cf., Ono [24] and Blackburn at el [2]). In our case, the other game theoretical components including a memory function are attached over the extensive game. Therefore, we require our g-morphism to preserve those components, i.e., g3-g7. It will be seen that this concept is very useful for comparisons of i.d.views for a given memory kit.

Let us consider a few examples to understand g-morphisms.

Example 6.1 (Infinite Number of P.V.'s g-morphic to a given P.V.). Given personal view $(\Gamma, \mathfrak{m})$, we can construct a larger personal view by simply replicating $(\Gamma, \mathfrak{m})$. The replicated game with twice as many nodes is g-morphic to $(\Gamma, \mathfrak{m})$. By this method, we can construct personal views of any size that are g-morphic to $(\Gamma, \mathfrak{m})$. Thus, there are an infinitely many personal views g-morphic to $(\Gamma, \mathfrak{m})$.

We will see from the following example that the set of personal views g-morphic to a given personal view is not only many but also is more complicated than what Example 6.1 describes.

Example 6.2. Figure 6.1 gives a g-morphism between two 1-person games, where the memory function for each personal view is assumed to be the perfect information

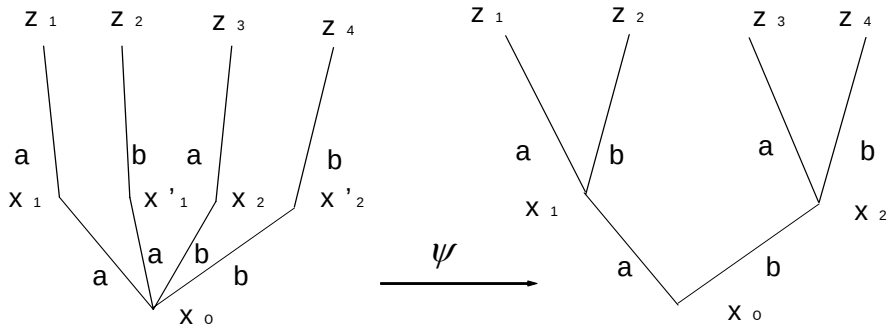

Figure 6.1:

memory function $\mathfrak{m}^{PI}$. Define ψ as the identity mapping everywhere except that $\psi(x'_1) = x_1$ and $\psi(x'_2) = x_2$. Then this ψ is a g-morphism from the left game to the right game.

It is easy to find two personal views between which there are no g-morphisms. Here, we give such an example, in which no g-morphism is caused by memory functions.

Example 6.3 (Negative Example). Consider the objective description of Example 5.1. In this case, the player has four distinct direct views, each of which is an i.d.view. The direct structure is uniquely determined, but there are four possible direct memory functions. No g-morphisms are admitted between each pair of direct views.

Now, we show that a g-morphism fully preserves the i.d.property. The theorems presented in this subsection will be proved in Section 6.2.

Theorem 6.1 (Preservation of the I.D. Property). Suppose that $(\Gamma, \mathfrak{m})$ is g-morphic to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then, $(\Gamma, \mathfrak{m})$ is an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$ if and only if $(\hat{\Gamma}, \hat{\mathfrak{m}})$ is an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$.

It follows from this theorem and Example 6.1 that if a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ admits at least one i.d.view, then there are, in fact, an infinite number of i.d.views for $(T_{D_i}, \mathcal{Y}_{D_i})$. Thus, we should consider which i.d.views are more appropriate than others. We will see that the direct views have a special status among the i.d.views. Before that, we give the following simple but basic observations, which can be proved just by looking at the definitions carefully.

Lemma 6.2.(1): The g-morphic relation $\rightarrowtail$ satisfies reflexivity and transitivity.
(2): Suppose that $(\Gamma, \mathfrak{m}) \leftarrowtail\!\!\rightarrowtail (\hat{\Gamma}, \hat{\mathfrak{m}})$. Then the g-morphism ψ from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$ satisfies

g0*: ψ is a bijection from X to $\hat{X}$;

g1*: for all $x, y \in X$ and $a \in A_x$, $x <_a y$ if and only if $\psi(x) \hat{<}_a \psi(y)$.

Here we write $(\Gamma, \mathfrak{m}) \longleftrightarrow (\hat{\Gamma}, \hat{\mathfrak{m}})$ iff $\Gamma, \mathfrak{m}) \rightarrowtail (\hat{\Gamma}, \hat{\mathfrak{m}})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}}) \rightarrowtail (\Gamma, \mathfrak{m})$. By (1), the relation $\longleftrightarrow$ is an equivalence relation over personal views. We can use this relation to consider the equivalence classes of personal views. Any two views in one equivalence class are isomorphic in the sense of g0*,g1* and g3-g7, where g2 is included in g1*. These two views are identical in our game theoretical sense except for the names of nodes.

In the next theorem we show that every i.d.view is g-morphic to a direct view.

Theorem 6.3 (g-Morphism to a Direct Personal View). Let $(T_{D_i}, \mathcal{Y}_{D_i})$ be a memory kit. For each i.d.view $(\Gamma, \mathfrak{m})$, there is a direct view $(\Gamma^d, \mathfrak{m}^d)$ such that $(\Gamma^d, \mathfrak{m}^d)$ is a personal view and $(\Gamma, \mathfrak{m})$ is g-morphic to $(\Gamma^d, \mathfrak{m}^d)$.

The direct view $(\Gamma^d, \mathfrak{m}^d)$ given in Theorem 6.3 is also an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$ by Theorem 6.1. This has the implication that we can focus our attention on direct views without loss of generality. The following corollary states that the existence of an i.d.view is characterized by the existence of a direct i.d.view which in turn is characterized by Theorem 5.2.

Corollary 6.4. **(Existence of an i.d.view).** Let $(T_{D_i}, \mathcal{Y}_{D_i})$ be a memory kit. There is an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$ if and only if there is a direct view that is an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$.

In particular, this result suggests that when a direct view for a memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ is unique and is i.d., the direct view $(\Gamma^d, \mathfrak{m}^d)$ is really the smallest, except for the isomorphic ones such as mentioned in Lemma 6.2. In general, however, there may still be several distinct direct i.d.views for a given memory kit, as was pointed out in Example 5.2. Even then, the class of all i.d.views can be partitioned into corresponding subclasses so that every view in each subclass is g-morphic to the unique direct one in that subclass. Also, the direct views have a constructive nature by definition. Thus, a direct view passes two criteria:

(i) it should be small;

(ii) it should be constructive.

When there are multiple direct i.d.views, however, a player still may face some difficulties, which will be discussed in Section 8.

6.2. Proofs of the Results

First, we start with giving a simple observation.

Lemma 6.5. Let ψ be a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then $x \in X^D$ if and only if $\psi(x) \in \hat{X}^D$.

Proof. Let $x \in X^D$. Then x has an immediate successor. Thus, $A_x \neq \varnothing$ by K33$_0$, which implies $\hat{A}_{\psi(x)} \neq \varnothing$ by g4. By K31, $\psi(x) \in \hat{X}^D$. The converse follows by tracing back this argument starting with $\psi(x) \in \hat{X}^D$. ■

The next lemma translates g1 and g2 into the corresponding $\bar{\text{g}}$1 and $\bar{\text{g}}$2 in terms of the immediate successor relation $<_a^I$.

Lemma 6.6. Suppose that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then:

$\bar{\text{g}}$1: for all $x, y \in X$ and $a \in A_x$, $x <_a^I y$ implies $\psi(x) \hat{<}_a^I \psi(y)$;

$\bar{\text{g}}$2: for all $\hat{x}, \hat{y} \in \hat{X}, y \in X$ and $a \in \hat{A}_{\hat{x}}$,
$\hat{x} \hat{<}_a^I \hat{y}$ and $\hat{y} = \psi(y)$ imply $x <_a^I y$ and $\hat{x} = \psi(x)$ for some $x \in X$.

Proof. $\bar{\text{g}}$1: Let $x <_a^I y$ for some $x, y \in X$. Now, on the contrary, suppose that $\psi(x) \hat{<}_a \hat{z} \hat{<}_b \psi(y)$ for some $\hat{z}$ and b. Then, by g2, there is some $z \in X$ such that $\psi(z) = \hat{z}$ and $z <_b y$. By K12 for Γ, we have $x <_a z <_b y$ or $z <_b x <_a y$. The first case, $x <_a z <_b y$, is impossible since it contradicts $x <_a^I y$. In the second case, we have $\hat{z} \hat{<}_b \psi(x)$ by g1, and then, by $\psi(x) \hat{<}_a \hat{z}$, we have $\hat{z} \hat{<} \hat{z}$ by transitivity (K11) for $\hat{\Gamma}$, which contradicts irreflexivity (K11) for $\hat{\Gamma}$. Thus, we must have $\psi(x) \hat{<}_a^I \psi(y)$.

$\bar{\text{g}}$2: Let $\hat{x} \hat{<}_a^I \hat{y}$ and $\hat{y} = \psi(y)$ for some $\hat{x}, \hat{y} \in \hat{X}$, $y \in X$ and $a \in \hat{A}_{\hat{x}}$. By g2, there is some $x \in X$ such that $x <_a y$ and $\hat{x} = \psi(x)$. Now, on the contrary, suppose that $x <_a z <_b y$ for some z and b. Then, by g1, we have $\psi(x) \hat{<}_a \psi(z) \hat{<}_b \psi(y)$, which is a contradiction to $\hat{x} \hat{<}_a^I \hat{y}$. Thus, we must have $x <_a^I y$. ■

The next lemma makes use of the previous one.

Lemma 6.7. Suppose that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then:

(1): If $\langle x_1, ..., x_m \rangle$ is a path in $(\Gamma, \mathfrak{m})$, then $\langle \psi(x_1), ..., \psi(x_m) \rangle$ is a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$ and $\theta(x_t) = \hat{\theta} \cdot \psi(x_t)$ for $t = 1, ..., m$.

(2): If $\langle \hat{x}_1, ..., \hat{x}_m \rangle$ is a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$, then there is a path $\langle x_1, ..., x_m \rangle$ in $(\Gamma, \mathfrak{m})$ such that $\psi(x_t) = \hat{x}_t$ and $\theta(x_t) = \hat{\theta}(\hat{x}_t)$ for $t = 1, ..m$.

Proof.(1): Let $\langle x_1, ..., x_m \rangle$ is a path in $(\Gamma, \mathfrak{m})$. Then there are $a_1, ..., a_{m-1}$ such that $x_t <_{a_t}^I x_{t+1}$ for $t = 1, ..., m-1$. Thus, $\psi(x_t) \hat{<}_{a_t}^I \psi(x_{t+1})$ for $t = 1, ..., m-1$ by $\bar{\text{g}}$1 of Lemma 6.6. This means that $\langle \psi(x_1), ..., \psi(x_m) \rangle$ is a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$ and $\theta(x_t) = \hat{\theta} \cdot \psi(x_t)$ for $t = 1, ..., m$.

(2) Let $\langle \hat{x}_1, ..., \hat{x}_m \rangle$ be a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then there are $a_1, ..., a_{m-1}$ such that $\hat{x}_t \hat{<}_{a_t}^I \hat{x}_{t+1}$ for $t = 1, ..., m-1$. Then, we can choose an $x_m \in X$ with $\psi(x_m) = \hat{x}_m$. Then, applying $\bar{\text{g}}$2 of Lemma 6.6 to the last pair $(\hat{x}_{m-1}, \hat{x}_m)$ and $\psi(x_m) = \hat{x}_m$, there is an $x_{m-1} \in X$ such that $\psi(x_{m-1}) = \hat{x}_{m-1}$ and $x_{m-1} <_{a_{m-1}}^I x_m$. Repeating this argument (exactly speaking, by mathematical induction), we construct $\langle x_1, ..., x_m \rangle$ with $x_t <_{a_t}^I$

x_{t+1} for $t = 1, ..., m-1$ and $\psi(x_t) = \hat{x}_t$ for $t = 1, ..., t$. This is a path in $(\Gamma, \mathfrak{m})$ having the required properties. ■

We have the immediate result from Lemma 6.7 that the mapping ψ preserves the complete histories of information pieces and actions, and the values of the memory yarns.

Lemma 6.8. Suppose that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then:

(a): $\Theta(\Gamma) = \Theta(\hat{\Gamma})$;

(b): $\{\mathfrak{m}^i(x) : x \in X_i^i\} = \{\hat{\mathfrak{m}}^i(x) : x \in \hat{X}_i^i\}$.

Proof.(a): Lemma 6.7.(1) states that $\theta(x) = \hat{\theta} \cdot \psi(x)$ for all $x \in X$. Thus, $\Theta(\Gamma) \subseteq \Theta(\hat{\Gamma})$. Conversely, take any $\hat{x} \in \hat{X}$. Lemma 6.7.(2) states that there is an x such that $\theta(x) = \hat{\theta}(\hat{x})$. Thus, $\Theta(\hat{\Gamma}) \subseteq \Theta(\Gamma)$.

(b): By g7, we have $\{\mathfrak{m}^i(x) : x \in X_i^i\} \subseteq \{\hat{\mathfrak{m}}^i(x) : x \in \hat{X}_i^i\}$. The converse inclusion follows from the surjectivity of ψ by g0. ■

Now, we prove Theorem 6.1. Actually, we prove a more precise claim than the theorem: when there is a g-morphism ψ from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$, each of P1a-P1d and P2a-P2c for $(\Gamma, \mathfrak{m})$ is equivalent to the corresponding one for $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

Proof of Theorem 6.1. Suppose that there is a g-morphism ψ from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. As stated above, we prove that each requirement of P1a-P1d and P2a-P2c for $(\Gamma, \mathfrak{m})$ is equivalent to the corresponding one for $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

P1a: By Lemma 6.8.(a), we have $\Theta(\Gamma) = \Theta(\hat{\Gamma})$. P1a holds for Γ, i.e., $T_{D_i}^* = \Theta(\Gamma)$, if and only if $T_{D_i}^* = \Theta(\hat{\Gamma})$, i.e., P1a for $\hat{\Gamma}$.

P1b: Let P1b hold for Γ, i.e., $A_x = A^o_{\lambda(x)}$. Consider any $\hat{x} \in \hat{X}$. Then we have some $x \in X$ with $\psi(x) = \hat{x}$. By g4, $\hat{A}_{\hat{x}} = A_x$. Thus, $\hat{A}_{\hat{x}} = A^o_{\lambda(x)}$. Since $\lambda(x) = \hat{\lambda}(\hat{x})$ by g3, we have $\hat{A}_{\hat{x}} = A^o_{\hat{\lambda}(\hat{x})}$. The converse can be proved similarly.

P1c: Suppose that P1c holds for $\hat{\Gamma}$, i.e., $\hat{\pi} \cdot \hat{\lambda}(\hat{x}) = \pi^o \cdot \hat{\lambda}(\hat{x})$ for any $\hat{x} \in \hat{X}$. Let $x \in X$. By g3,g5 and P1c for $\hat{\Gamma}$, we have $\pi \cdot \lambda(x) = \hat{\pi} \cdot \hat{\lambda} \cdot \psi(x) = \pi^o \cdot \hat{\lambda} \cdot \psi(x) = \pi^o \cdot \lambda(x)$. Thus, we have P1c for Γ. The converse is similar.

P1d: Suppose that P1d for Γ. Consider any $\hat{x} \in \hat{X}$. We should show $\hat{h} \cdot \hat{\lambda}(\hat{x}) = h_i^o \cdot \hat{\lambda}(\hat{x})$. By g3,g6 and P1d for Γ, we have $\hat{h} \cdot \hat{\lambda}(\hat{x}) = \hat{h} \cdot \hat{\lambda} \cdot \psi(x) = h \cdot \lambda(x) = h_i^o \cdot \lambda(x) = h_i^o \cdot \hat{\lambda} \cdot \psi(x) = h_i^o \cdot \hat{\lambda}(\hat{x})$. Thus, P1d for $\hat{\Gamma}$. The converse is similar.

P2a: By Lemma 6.8.(b), $\{\hat{\mathfrak{m}}(\hat{x}) : \hat{x} \in \hat{X}_i\} = \{\mathfrak{m}(x) : x \in X_i\}$. Hence, $\mathfrak{m}$ satisfies P2a if and only if $\hat{\mathfrak{m}}$ does.

P2b: By g7 and Lemma 6.7, $\mathfrak{m}$ satisfies P2b if and only if $\hat{\mathfrak{m}}$ does.

P2c: Suppose P2c for $\hat{\mathfrak{m}}$. Let $\hat{\theta}(\hat{x}) = \hat{\theta}(\hat{y})$. Since ψ is a surjection, we have some $x, y \in X$ such that $\psi(x) = \hat{x}$ and $\psi(y) = \hat{y}$. By Lemma 6.7, $\theta(x) = \hat{\theta}(\hat{x})$ and $\theta(y) = \hat{\theta}(\hat{y})$. Hence

$\mathfrak{m}(x) = \mathfrak{m}(y)$ by P2c for $\mathfrak{m}$. Then, by g7, $\hat{\mathfrak{m}}(\hat{x}) = \mathfrak{m}(x)$ and $\hat{\mathfrak{m}}(\hat{y}) = \mathfrak{m}(y)$. Thus, P2c holds for $\hat{\mathfrak{m}}$. The converse is similar. ■

The next target is to prove Theorem 6.3. We take two steps to have the assertion of the theorem: under the supposition that $(\Gamma, \mathfrak{m})$ is an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$, (1) we can find a direct view so that it is a personal view; and (2) it is g-morphic to $(\Gamma, \mathfrak{m})$. The first part is given as a lemma, and the second is given as the proof of the theorem.

Lemma 6.9. Suppose that $(\Gamma, \mathfrak{m})$ is an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$. Then $(\Gamma^d, \mathfrak{m}^d)$ is a personal view where Γ^d is the unique direct structure for $(T_{D_i}, \mathcal{Y}_{D_i})$ and $\mathfrak{m}^d$ is defined by:

$$\text{for all } x \in X_i^d,\ \mathfrak{m}^d(x) = \mathfrak{m}(y_x) \text{ for some } y_x \in X_i \text{ satisfying } \theta(y_x) = x. \tag{6.1}$$

Proof. Let $(\Gamma, \mathfrak{m})$ be an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$. We first show the right hand side of Theorem 5.2.(i). This implies that Γ^d is the unique direct structure for $(T_{D_i}, \mathcal{Y}_{D_i})$ and Γ^d is an extensive game in the weak sense. We next show that (6.1) defines a memory function for Γ^d, from which it follows that $(\Gamma^d, \mathfrak{m}^d)$ is personal view.

Suppose, on the contrary, that there is some maximal thread $\langle \xi, v \rangle \in T^*_{D_i}$ such that $v = \lambda^o(x)$ for some $x \in X^{oD}$. Then, $A^o_v \neq \varnothing$ by K33 for Γ^o. Since $(\Gamma, \mathfrak{m})$ is an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$, we have $\Theta(\Gamma) = T^*_{D_i}$ by P1a. Also, since $\langle \xi, v \rangle$ is maximal in $T^*_{D_i}$, there exists $y \in X^E$ such that $\theta(y) = \langle \xi, v \rangle$. Then, by P1b, $A_y = A^o_v \neq \varnothing$. This contradicts that y is an endnode in Γ. Hence, the right hand side of Theorem 5.2.(i) holds.

Now let us see that $\mathfrak{m}^d$ is defined by (6.1) is a memory function for Γ^d. By P1a, $W = W^d$. Then by c4 and P1b, $\mathfrak{m}^d$ is a memory function for Γ^d since $\mathfrak{m}$ is a memory function for Γ. ■

Proof of Theorem 6.3. Let $(\Gamma, \mathfrak{m})$ be an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$. By Lemma 6.9, $(\Gamma^d, \mathfrak{m}^d)$ is a personal view, where Γ^d is the unique direct structure for $(T_{D_i}, \mathcal{Y}_{D_i})$ and $\mathfrak{m}^d$ is defined by (6.1). First we show that $(\Gamma^d, \mathfrak{m}^d)$ is a direct view. Since Γ^d is the unique direct structure, we need only to show that $\mathfrak{m}^d$ satisfies d7. Let $x \in X^d_i$. By (6.1) and P2b for $\mathfrak{m}$, $x = \theta(y_x) \in \mathfrak{m}(y_x) = \mathfrak{m}^d(x)$ for some $y_x \in X_i$.

We define the function ψ from $(\Gamma, \mathfrak{m})$ to $(\Gamma^d, \mathfrak{m}^d)$ by:

$$\psi(x) = \theta(x) \text{ for all } x \in X. \tag{6.2}$$

The proof will be completed if we show that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\Gamma^d, \mathfrak{m}^d)$.
g0: We have $X^d = T^*_{D_i}$ by d1, and also $\Theta(\Gamma) = T^*_{D_i}$ by P1a for $(\Gamma, \mathfrak{m})$. Thus, $X^d = \Theta(\Gamma)$ and so ψ is a surjection from X to X^d.
g1: Let $x < y$. Then, $\theta(x)$ is an initial segment of $\theta(y)$, i.e., $\psi(x) = \theta(x) <^d \theta(y) = \psi(y)$

by d2.

g2: Let $\hat{x} <^d \hat{y}$ and $\hat{y} = \psi(y)$. Then, $\hat{x}$ and $\hat{y}$ can be written as $\langle \xi, v \rangle$ and $\langle \eta, w \rangle$ respectively, and by d2, $\langle \xi, v \rangle$ is an initial segment of $\langle \eta, w \rangle$. Since $\hat{y} = \psi(y) = \theta(y) = \langle \eta, w \rangle$, and $\langle \xi, v \rangle$ is an initial segment of $\langle \eta, w \rangle$, we can find a unique x on the path to y with $\theta(x) = \langle \xi, v \rangle$. Thus, $x < y$ and $\psi(x) = \theta(x) = \hat{x}$.

For g3-g7 we will use the generic history $\theta(x) = \langle \xi, v \rangle$ for the node x in question.

g3: Let $x \in X$. Then $\psi(x) = \theta(x) = \langle \xi, v \rangle$. Hence, $\lambda^d \cdot \psi(x) = \lambda^d \langle \xi, v \rangle = v$ where the last equality follows from d3. Hence, we have shown that $\lambda^d \cdot \psi(x) = \lambda(x)$.

g4: Let $x \in X$. Then, by d4, $A^c_{\langle \xi, v \rangle} = A^o_v$. By P1b, we have $A_x = A^o_{\lambda(x)} = A^o_v$. Hence, $A^c_{\psi(x)} = A_x$.

g5: Let $x \in X$. By d3, $\pi^d \cdot \lambda^d \cdot \psi(x) = \pi^c(v)$. If $x \in X^D$, then by P1c, $\pi \cdot \lambda(x) = \pi^o \cdot \lambda(x) = \pi^o(v)$. Also, since $x \in X^D$, it follows by Lemma 6.2 that $\langle \xi, v \rangle \in X^{dD}$. Hence, by d5, $\pi^d(v) = \pi^o(v)$. Thus, for $x \in X^D$ we have the desired result that $\pi^d \cdot \lambda^d \cdot \psi(x) = \pi \cdot \lambda(x)$. Next consider $x \in X^E$. Then by K42, $\pi \cdot \lambda(x) = \{j : j \in \pi \cdot \lambda(y) \text{ for some } y \in X^D\}$. By Lemma 6.5, g0 and d5, it follows that this set is equivalent to $\pi^d(v)$.

g6: $\hat{h} \cdot \hat{\lambda} \cdot \psi(x) = h \cdot \lambda(x)$ for all $x \in X^E$: Let $x \in X^E$. By P1a and P1d, $v = \lambda(x) = \lambda^o(y)$ for some $y \in X^{oE}$, and $h(v) = h^o_i(v)$. By Lemma 6.2 and g3, $\psi(x) \in X^{dE}$ and $\lambda \cdot \psi(x) = v = \lambda^o(y)$ for some $y \in X^{oE}$. So, by d6, $h^d(v) = h^o_i(v)$. Hence, we have shown that $h^d \cdot \lambda^d \cdot \psi(x) = h \cdot \lambda(x)$.

g7: Let $x \in X_i$. Then by the definition of ψ, (6.1) and P2c for $\mathfrak{m}$, it follows that $\mathfrak{m}^d \cdot \psi(x) = \mathfrak{m}^d \cdot \theta(x) = \mathfrak{m}(y) = \mathfrak{m}(x)$. ■

7. Decision Making and Prescribed Behavior in Inductive Game Theory

The inductive derivation of an individual view from past experiences is not the end of the entire scenario of our theory. The next step is to use an i.d.view for decision making and to bring the prescribed (or modified) behavior back to the objective situation. This is the third stage of Figure 1.1. Because this paper aims to present a basic and entire scenario of our theory, we will here concentrate on a clear-cut case. Specifically, we assume in this section that the objective memory function $\mathfrak{m}^o_i$ for each player i is given as the exact perfect memory function $\mathfrak{m}^{EPR}_i$, and that player i has the unilateral trial domain $D^U_i(\sigma^o)$. Then, we will discuss how he can use the inductively derived view for his decision making as well as how the prescribed behavior helps his objective behavior. This gives an experiential foundation for Nash equilibrium.

7.1. Decision Making using a Personal View

Our analysis of decision making using a personal view is restricted to players who are relevant in their own domains of accumulation. Players who are not relevant in their

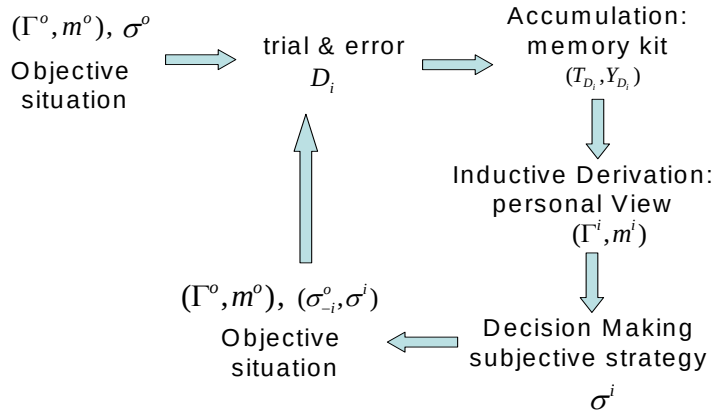

Figure 7.1:

own domains will have no decision making to consider. We presume in this subsection that each player i is relevant in his own domain and has:

(7a): the EPR memory function $\mathfrak{m}_i^o = \mathfrak{m}_i^{EPR}$;

(7b): behavior pattern σ_i^o;

(7c): the UT domain $D_i^U(\sigma^o)$;

(7d): the direct view $(\Gamma^d, \mathfrak{m}^d)$.

In this case, it is already proved in Corollary 5.3 that there is a unique direct i.d.view for each player i. We assume that player i adopts the unique direct i.d.view $(\Gamma^d, \mathfrak{m}^d)$. In Section 8, we will argue how the consideration here is extended to other cases, i.e., where he adopts another i.d.view.

Figure 7.1 describes the steps from experimentation (trial and error) to decision making using the i.d.view. One basic question is whether the i.d.view helps the player for his decision making, as well as whether the decision can be used in the objective situation when he brings it back there. The following lemma states that when 7a-7d hold, the direct i.d.view $(\Gamma^d, \mathfrak{m}^d)$ consists of an extensive game in the strong sense and a memory function that uses all the yarns in $\mathcal{Y}_{D_i}$. The proofs of the results will be given in the end of this subsection.

Lemma 7.1. The direct view $(\Gamma^d, \mathfrak{m}^d)$ for $(T_{D_i}, \mathcal{Y}_{D_i}) = (T_{D_i^U(\sigma^o)}, \mathcal{Y}_{D_i^U(\sigma^o)})$ is uniquely determined and is an i.d.view satisfying:

(a): Γ^d is a 1-person extensive game in the strong sense with $N^d = \{i\}$;

(b): $\mathfrak{m}^d$ satisfies P2a with equality, i.e., $\{\mathfrak{m}^d(x) : x \in X^d\} = \mathcal{Y}_{D_i^U(\sigma^o)}$.

For the consideration of utility maximization of a behavior pattern σ_i, player i needs to consider the sets of compatible endnodes for various behavior patterns. Recall from (2.15) that $\lambda(\sigma)$ denotes the set of compatible endpieces for a profile of behavior patterns $\sigma = (\sigma_1, ..., \sigma_n)$. Since Γ^d is a 1-person extensive game in the strong sense, the set of compatible endpieces will be a singleton set for each behavior pattern σ_i of player i. Consequently, we will use $\lambda^d(\sigma_i)$ here to denote the compatible endpiece in Γ^d for σ_i.

Then, player i has a subjective strategy σ_i^d in Γ^d to maximize h^d in the following sense:

$$h^d \cdot \lambda^d(\sigma_i^d) \geq h^d \cdot \lambda^d(\sigma_i) \text{ for all } \sigma_i \in \Sigma_i^d. \tag{7.1}$$

Once again, we emphasize that this decision is made in the personal view of player i, i.e., in the mind of player i. This is conceptually not the same as the payoff maximization in the objective situation, which is now the subject to be considered.

After the choice of the subjective strategy in (7.1), player i brings back σ_i^d to the objective situation $(\Gamma^o, \mathfrak{m}^o)$, adjusting his behavior pattern σ_i^o with σ_i^d. The adjustment from his objective behavior σ_i^o into σ_i^1 is as follows: for all $x \in X_i^o$,

$$\sigma_i^1(x) = \begin{cases} \sigma_i^d\langle \xi, v\rangle & \text{if } \mathfrak{m}_i^o(x) = \{\langle \xi, v\rangle\} \text{ for some } \langle \xi, v\rangle \in X^d; \\ \\ \sigma_i^o(x) & \text{if } \mathfrak{m}_i^o(x) \notin \mathcal{Y}_{D_i^U(\sigma^o)}. \end{cases} \tag{7.2}$$

That is, player i follows σ_i^d whenever a memory yarn in $\mathcal{Y}_{D_i^U(\sigma^o)}$ occurs; and otherwise, he keeps the old behavior pattern. Mathematically, $\sigma_i^1 \in \Sigma_i^o$, i.e., this adjustment produces a behavior pattern for player i in Γ^o. The next theorem states that the modified strategy σ_i^1 of player i defined by (7.2) is actually objectively utility maximizing for player i in Γ^o when the other players follow their regular behavior σ_{-i}^o in Γ^o.

Before the next theorem, we give a small remark. Since the objective game Γ^o is also an extensive game in the strong sense, the set of compatible endpieces $\lambda^o(\sigma_i, \sigma_{-i}^o)$ will also be a singleton for player i's behavior pattern σ_i and the other players' behavior patterns σ_{-i}^o. We follow the convention of using $\lambda^o(\sigma_i, \sigma_{-i}^o)$ to denote the compatible endpiece, not the set of compatible endpieces.

Theorem 7.2 (One-Person Utility Maximization in the n-Person Game): The strategy σ_i^1 defined by (7.2) satisfies the objective payoff maximization for player i, i.e.,

$$h_i^o \cdot \lambda^o(\sigma_i^1, \sigma_{-i}^o) \geq h_i^o \cdot \lambda^o(\sigma_i, \sigma_{-i}^o) \text{ for all } \sigma_i \in \Sigma_i^o. \tag{7.3}$$

This theorem has various points to be emphasized: First, this is not the utility maximization obtained directly in the objective situation. Instead, the utility maximization is made in his i.d.view $(\Gamma^d, \mathfrak{m}^d)$, and then the modified strategy σ_i^1 is brought to the objective situation $(\Gamma^o, \mathfrak{m}^o)$. It happens that it actually maximizes his objective utility

function. We emphasize that this process of obtaining the objective utility maximization is indirect, and occurs only after many repetitions of collecting data to construct his view.

A second point is that when the modified strategy σ_i^1 is brought to the objective situation, it gives a sufficient prescription to player i to behave in his expected world. One reason is that $\mathfrak{m}^d$ satisfies condition P2a with equality. Otherwise, the subjective memory function may not prescribe actions for all possible memory yarns.

Thus, we have succeeded in having individual utility maximization in the well-defined form in both subjective and objective senses. Nevertheless, once we leave the case of 7a-7d, player i would have many difficulties at various levels in Figure 7.1. These problems will be discussed in Section 8.2 and in separate papers.

Proof of Lemma 7.1.(a): The condition $N^d = \{i\}$ follows immediately since $\mathfrak{m}_i^o = \mathfrak{m}_i^{EPR}$. By Corollary 5.3, it suffices to show that Γ^d satisfies K14 and K33.

K14: Since Γ^o is an extensive game in the strong sense, each strategy combination determines a unique play. Let $\langle x_1, ..., x_m \rangle$ be the unique play determined by σ^o, and let x_t be the first node of player i in this play, i.e., $i \in \pi^o \cdot \lambda^o(x_t)$ and $i \notin \pi^o \cdot \lambda^o(x_s)$ for all $s < t$. Then $\theta^o(x_t) = \langle (\lambda^o(x_1), \sigma^o_{j_1}(x_1)), ..., (\lambda^o(x_{t-1}), \sigma^o_{j_{t-1}}(x_{t-1})), \lambda^o(x_t) \rangle$ where $j_1, ..., j_{t-1}$ denote the players moving at $x_1, ..., x_{t-1}$ respectively. Let $(\sigma_i, \sigma^o_{-i})$ be any other strategy combination where all the players other than player i choose according to σ^o. Then, the first t nodes in the play determined by this strategy combination must also be $x_1, ..., x_t$. Hence, for any play determined on the unilateral trial domain, x_t is the first node of player i. Thus, x_t determines the smallest node $\theta^o(x_t)$ in X^d.

K33: We show that for each $\langle \xi, v \rangle \in X^{dD}$, the function $\varphi^d_{\langle \xi, v \rangle}$ defined in d4 is a bijection. Let $\langle \xi, v \rangle \in X^{dD}$ and let a be an arbitrary action in $A_{\langle \xi, v \rangle}$. Since $\langle \xi, v \rangle \in X^{dD}$ and the memory function is EPR, $\langle \xi, v \rangle = \theta^o(x)_i$ for some $x \in X_i^o$, and x is on the path determined by some $(\sigma_i, \sigma^o_{-i})$. Consider the strategy σ_i' defined by:

$$\sigma_i'(y) = \begin{cases} \sigma_i(y) \text{ if } \mathfrak{m}_i^o(y) \neq \mathfrak{m}_i^o(x); \\ \\ a \qquad \text{if } \mathfrak{m}_i^o(y) = \mathfrak{m}_i^o(x). \end{cases}$$

Since $\mathfrak{m}_i^o = \mathfrak{m}_i^{EPR}$, $\mathfrak{m}_i^o(y) \neq \mathfrak{m}_i^o(x)$ for any $y \in X_i^{oD}$ on the path to x. Hence x is on the play determined by $(\sigma_i', \sigma^o_{-i})$. Since the other players follow their strategies in σ^o, the action a determines a unique immediate successor $\langle \xi, (v, a), u \rangle \in X^d$. A different action $a' \in A_{\langle \xi, v \rangle}$ determines a different immediate successor $\langle \xi, (v, a'), u' \rangle \in X^d$. Thus, the mapping $\varphi^d_{\langle \xi, v \rangle}$ from $\langle \xi, (v, a), u \rangle$ to a is a bijection.

(b): Let $x \in D_i$. We show that $\mathfrak{m}_i^o(x) \in \{\mathfrak{m}^d(y) : y \in X^d\}$. Since $\mathfrak{m}_i^o = \mathfrak{m}_i^{EPR}$, we have $T_{D_i} = T^*_{D_i}$. Since $\mathfrak{m}_i^o(x) = \{\theta^o(x)_i\}$, it follows that $\theta^o(x)_i \in T^*_{D_i} = X^d$. Corollary 5.3 states that the direct view $(\Gamma^d, \mathfrak{m}^d)$ exists uniquely and $\mathfrak{m}^d(y) = \{y\}$ for all $y \in X_i^d$.

Hence, $\mathfrak{m}^d(\theta^o(x)_i) = \{\theta^o(x)_i\} = \mathfrak{m}^o_i(x)$. ■

Proof of Theorem 7.2. Consider any $\sigma_i \in \Sigma^o_i$. Recall that the endnode determined by $(\sigma_i, \sigma^o_{-i})$ in Γ^o is denoted by $z(\sigma_i, \sigma^o_{-i})$. Let $x = z(\sigma_i, \sigma^o_{-i})$. Consider the history of player i at x, i.e., $\theta^o(x)_i = \langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ with $w_{m+1} = \lambda^o(x)$, and also, let the corresponding history of nodes be given as $\langle x_1, ..., x_m, x_{m+1} \rangle$ with $x_{m+1} = x$. Then, $\lambda^o(x_t) = w_t$ and $\sigma_i(x_t) = a_t$ for all $t = 1, ..., m$. Hence, we choose a strategy τ^d_i having the property that $\tau^d_i \langle (w_1, a_1), ..., (w_{t-1}, a_{t-1}), w_t \rangle = \sigma_i(x_t)$ for $t = 1, ..., m$. Then, the compatible endpiece $\lambda^o(\sigma_i, \sigma^o_{-i}) = \{v\}$ is the same as $\lambda^d(\tau^d_i)$. Hence, $\lambda^o(\sigma^1_i, \sigma^o_{-i}) = \lambda^d(\tau^d_i)$. If we apply this procedure to σ^1_i, then we have σ^d_i satisfying (7.1). Hence, we have $\lambda^o(\sigma^1_i, \sigma^o_{-i}) = \lambda^d(\sigma^1_i)$.

By d7 and using the above result, we have $h^o_i \cdot \lambda^o(\sigma^1_i, \sigma^o_{-i}) = h^d \cdot \lambda^d(\sigma^d_i) \geq h^d \cdot \lambda^d(\tau^d_i) = h^o_i \cdot \lambda^o(\sigma_i, \sigma^o_{-i})$. ■

7.2. An Experiential Foundation for Nash Equilibrium

Mathematically speaking, it is easy to extend the result of Theorem 7.2 to every player in the player set of players in N^o who are relevant in their own domains, and then we obtain a Nash equilibrium. Here, we will still state this theorem, and then consider its implications. For it, however, we need some more notation and one more definition.

First, since our discussion will involve more than one i.d.view, we put subscript "i" to the direct i.d.view of player i, i.e., $(\Gamma^d_i, \mathfrak{m}^d_i)$. Second, for each player i who is relevant in his own domain, we define the *induced strategy* σ^d_i of σ^o to the direct i.d.view $(\Gamma^d_i, \mathfrak{m}^d_i)$ for $(T_{D^U_i(\sigma^o)}, \mathcal{Y}_{D^U_i(\sigma^o)})$ by: for all $\langle \xi, w \rangle \in X^d_i$,

$$\sigma^d_i \langle \xi, w \rangle = \sigma^o_i(x) \text{ for any } x \in X^o_i \text{ with } \theta^o(x) = \langle \xi, w \rangle. \tag{7.4}$$

The well-definedness of (7.4) is verified as follows. First, by the properties of the EPR memory function, for each $\langle \xi, w \rangle \in X^d_i$, there is an $x \in X^o_i$ such that $\theta^o(x) = \langle \xi, w \rangle$. Secondly, since $\theta^o(x) = \theta^o(y)$ implies $\mathfrak{m}^{EPR}_i(x) = \mathfrak{m}^{EPR}_i(y)$, the strategy defined by (7.4) does not depend upon he choice of x. Finally, we verify (2.12) and (2.13) for σ^d_i. The condition (2.12) follows from d4. Condition (2.13) is also satisfied since by Corollary 5.3, the direct memory function of player i is uniquely determined as $\mathfrak{m}^d_i \langle \xi, w \rangle = \{\langle \xi, w \rangle\}$.

Then we have the following theorem, which is a straightforward implication of Theorem 7.2

Theorem 7.3 (Experimental Foundation for Nash Equilibrium): A profile σ^o of behavior patterns is a Nash equilibrium in $(\Gamma^o, \mathfrak{m}^o)$ if and only if for each player $i \in N^o$ who is relevant in his own UT-domain $D^U_i(\sigma^o)$, the induced strategy σ^d_i of σ^o to the direct view $(\Gamma^d_i, \mathfrak{m}^d_i)$ for the memory kit $(T_{D^U_i(\sigma^o)}, \mathcal{Y}_{D^U_i(\sigma^o)})$ satisfies condition (7.1).

First, let us recall that we have adopted the assumptions 7a-7d in this section. Under these assumptions, each player makes his decision in his 1-person derived view. The

theorem states that the behavior pattern σ^o is a Nash equilibrium in the the objective game $(\Gamma^o, \mathfrak{m}^o)$ if and only the induced strategy for each player i maximizes his utility in the direct view $(\Gamma_i^d, \mathfrak{m}_i^d)$. Thus, this theorem decomposes the Nash equilibrium in $(\Gamma^o, \mathfrak{m}^o)$ into utility maximizations in n one-person games.

As discussed in Section 3, the accumulation of $(T_{D_i^U(\sigma^o)}, \mathcal{Y}_{D_i^U(\sigma^o)})$ and the inductive derivation of $(\Gamma_i^d, \mathfrak{m}_i^d)$ need many repetitions of the game $(\Gamma^o, \mathfrak{m}^o)$. Also, in the present scenario, each player revises his behavior over $D_i^U(\sigma^o)$, and other players may be influenced by his revision, and may change their personal views. This process of changing may continue. The above theorem describes a stationary state in this revision process.

The revision process may take a long time to reach a Nash equilibrium or even may not reach a Nash equilibrium. Furthermore, we did not explicitly consider the case where the players' trials and errors are restricted. If we take these limitations over experimentations, the above "Nash equilibrium" is understood as a Nash equilibrium relative to the restricted domains of actions.

In the above senses, Theorem 7.3 is one characterization of Nash equilibrium from the experiential viewpoint. In separate papers, we will discuss other characterizations of Nash equilibrium and/or difficulties arising for them.

Finally, we give one example to suggest the nonconvergence of the process of revising behavior via constructed personal views. If the objective game $(\Gamma^o, \mathfrak{m}^o)$ has no Nash equilibria, then the above process does not converge. The following example has a Nash equilibrium.

Example 7.1. (Nonconvergence): Consider the 2-person simultaneous game which is described as Figure 7.2 and its payoffs are given in Figure 7.3. The bold arrow is the regular path $(\mathbf{s}_{12}, \mathbf{s}_{22})$ and each player is presumed to have the EPR memory function.

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	$(3,3)^{NE}$	$(2,2)$	$(2,2)$
$\mathbf{s}_{12}$	$(2,2)$	$(4,2)$	$(2,4)$
$\mathbf{s}_{13}$	$(2,2)$	$(2,4)$	$(4,2)$

Figure 7.3

Player 1's direct i.d.view is the 1-person game summarized by the matrix form of Figure 7.4, and player 2's i.d.view is the 1-person game summarized in Figure 7.5.

$\mathbf{s}_{11}$	2
$\mathbf{s}_{12}$	4
$\mathbf{s}_{13}$	2

Figure 7.4

$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
2	2	4

Figure 7.5

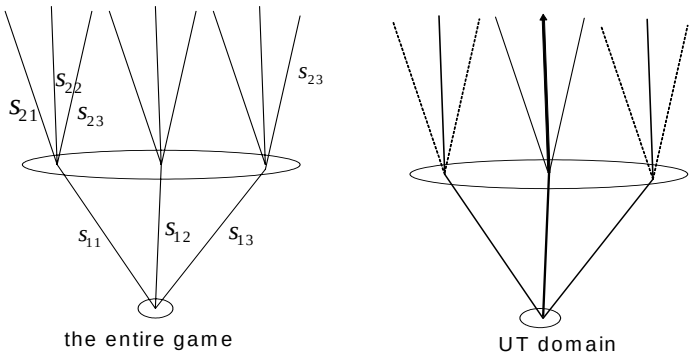

Figure 7.2:

In this case, player 1 maximizes his utility in his i.d.view by choosing $\mathbf{s}_{12}$. Thus, he has no incentive to change his objective behavior from the regular pattern. However, player 2 maximizes his utility in his i.d.view by changing from $\mathbf{s}_{22}$ to $\mathbf{s}_{23}$.

By this revision, the regular behavior becomes $(\mathbf{s}_{12}, \mathbf{s}_{23})$. After experiencing this pair as well as some trials, the personal views of the player's will be revised to the 1-person games summarized by the matrices of Figures 7.6 and 7.7.

$\mathbf{s}_{11}$	2
$\mathbf{s}_{12}$	2
$\mathbf{s}_{13}$	4

Figure 7.6

$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
2	2	4

Figure 7.7

With this new view, player 1 now finds that he should change his behavior, while player 2 does not. The revised behavior becomes $(\mathbf{s}_{13}, \mathbf{s}_{23})$. In this manner, the players move cyclically through the four regular behaviors depicted in the bottom right corner of Figure 7.8, and never converge to the Nash equilibrium $(\mathbf{s}_{11}, \mathbf{s}_{21})$.

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	$(3,3)$	$(2,2)$	$(2,2)$
$\mathbf{s}_{12}$	$(2,2)$	$(4,2)$	$\rightarrow (2,4) \downarrow$
$\mathbf{s}_{13}$	$(2,2)$	$\uparrow (2,4)$	$\leftarrow (4,2)$

Figure 7.8

8. *g*-Morphism Analysis of Decision Making

In Section 7, we assumed that each player makes a decision using the direct view $(\Gamma^d, \mathfrak{m}^d)$. In Section 6, we showed, using the concept of a g-morphism, that the direct view can be regarded as a representative one. In this section, we apply the g-morphism analysis to the decision making of a player. The concept of a g-morphism helps us analyze decision making within all i.d.views, not just direct ones. Here we do not restrict ourselves to the memory kits based on the EPR memory function $\mathfrak{m}_i^{EPR}$ and on the unilateral trial domain $D_i^U(\sigma^*)$. Although the g-morphism analysis works well, we will find certain difficulties in decision making with personal views and in transitions from subjective optimality to objective behavior.

8.1. Subjective Optimality and *g*-Morphism Analysis

Let $(\Gamma, \mathfrak{m})$ be a personal view of player i. We assume that Γ satisfies $N = \{i\}$, i.e., it is a 1-person game. We call such a view a *purely personal view*.

We compare subjective optimality across g-morphic views of player i. For this purpose, let $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be two purely personal views of player i, and let $\sigma_i \in \Sigma_i$ and $\hat{\sigma}_i \in \hat{\Sigma}_i$. Here, we follow the convention that each notion in $(\hat{\Gamma}, \hat{\mathfrak{m}})$ is distinguished from the corresponding one in $(\Gamma, \mathfrak{m})$ by the "cap", e.g., Σ_i and $\hat{\Sigma}_i$ are the sets of strategies of $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$, respectively. We say that σ_i and $\hat{\sigma}_i$ are *endpiece-equivalent* iff

$$\lambda(\sigma_i) = \hat{\lambda}(\hat{\sigma}_i). \tag{8.1}$$

Recall that $\lambda(\sigma_i)$ is the set of compatible endpieces for σ_i, which was defined in (2.15). Endpiece-equivalent strategies σ_i and $\hat{\sigma}_i$ lead to the same endpieces in $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Using this notion, we show that it suffices to consider the direct view in order to study subjective optimality in a personal view.

The first theorem guarantees that the strategic consideration in either view of $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ can be carried over to the other as far as the compatible endpieces are concerned. The proofs will be given in the end of this subsection.

Theorem 8.1 (*g*-Morphism and Behavior). Let $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be two purely personal views of player i, and let ψ be a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

(a): For each $\hat{\sigma}_i \in \hat{\Sigma}_i$, the function σ_i defined by (8.2) is a strategy in Σ_i and is endpiece-equivalent to $\hat{\sigma}_i$: for all $x \in X_i^D$,

$$\sigma_i(x) = \hat{\sigma}_i \cdot \psi(x). \tag{8.2}$$

(b): For each $\sigma_i \in \Sigma_i$, the function $\hat{\sigma}_i$ defined by (8.3) is a strategy in $\hat{\Sigma}_i$ and is endpiece-equivalent to σ_i: for each $\hat{x} \in \hat{X}_i^D$,

$$\hat{\sigma}_i(\hat{x}) = \sigma_i(x) \text{ for some } x \in X_i^D \text{ with } \psi(x) = \hat{x}. \tag{8.3}$$

Although Theorem 8.1 itself (as well as the following corollaries) does not refer to i.d.views, the theorem is more relevant in the context of i.d.views. Recalling the results of Section 6, Theorem 8.1 guarantees that our strategic analysis on a direct view is carried over to any other i.d.view that is g-morphic to that direct view.

We should consider the implications of Theorem 8.1 in two respects. One is in terms of subjective optimality, and the other is about when player i brings back his modified behavior in the objective situation. From the viewpoint of g-morphisms, everything works well even in these respects. However, there are some difficulties intrinsic to these two respects, which are not captured by g-morphisms. These will be discussed in Section 8.2.

(1): g-Morphism and Subjective Optimality: Here since we do not assume that $\mathfrak{m}_i^o$ is the EPR memory function and the domain is the unilateral-trial one $D_i^U(\sigma^*)$, some i.d.views may be extensive games in the properly weak sense. In such cases, the utility maximization (7.3) in Section 7 needs some modification. Here, we give one possible modification.

Let $(\Gamma, \mathfrak{m})$ be a purely personal view of player i. A strategy σ_i is *subjectively optimal* in $(\Gamma, \mathfrak{m})$ iff

$$\min_{w \in \lambda(\sigma_i)} h(w) \geq \min_{w' \in \lambda(\sigma_i')} h(w') \text{ for all } \sigma_i' \in \Sigma_i. \tag{8.4}$$

Optimality of σ_i means that the worst outcome compatible with this strategy is better than or equal to the worst outcome of any other strategy. That is, we assume the maximin criterion for his decision making.

Corollary 8.2 (g-Morphism and Subjective Optimality). Let $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be two purely personal views of player i, and let ψ be a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

(a): If σ_i satisfies (8.4) in $(\Gamma, \mathfrak{m})$, then the endpiece-equivalent strategy $\hat{\sigma}_i$ defined by (8.3) satisfies (8.4) in $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

(b): If $\hat{\sigma}_i$ satisfies (8.4) in $(\hat{\Gamma}, \hat{\mathfrak{m}})$, then the endpiece-equivalent strategy σ_i defined by (8.2) satisfies (8.4) in $(\Gamma, \mathfrak{m})$.

Again, we talk about the corollary in the context of i.d.views. By the results of Section 6, we can regard $(\hat{\Gamma}, \hat{\mathfrak{m}})$ as a direct one. By this result, we lose nothing in terms of subjective optimality by focusing on a direct view.

(2): g-Morphism and Objective Behavior: After his decision making in an i.d.view, a player modifies his behavior pattern by his subjective strategy, and brings it back to the objective situation. This modification, in theory, might depend upon the particular i.d.view of the player. In fact, we will show that the prescriptions for objective strategies are not different across g-morphic i.d.views. This implies that we can focus on the direct view even in the step of taking the prescription back to the objective world.

For the above consideration, we first modify (7.2) in the following way. Let $(\Gamma, \mathfrak{m})$ be a purely personal view of player i and let σ_i satisfy (8.4). We define the *prescribed* behavior of player i in $(\Gamma^o, \mathfrak{m}^o)$ by: for all $x \in X_i^o$,

$$\sigma_i^1(x) = \begin{cases} \sigma_i(x') & \text{if } \mathfrak{m}_i^o(x) = \mathfrak{m}(x') \text{ for some } x' \in X; \\ \\ \sigma_i^o(x) & \text{if } \mathfrak{m}_i^o(x) \neq \mathfrak{m}(x') \text{ for any } x' \in X. \end{cases} \tag{8.5}$$

This strategy prescribes the same behavior as (7.2) in the more restrictive case considered in Section 7. The next corollary states that g-morphic views give the same prescriptions for behavior in the objective situation.

Corollary 8.3 (g-Morphism and Modified Behavior). Let $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be two purely personal views of player i, and let ψ be a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

(a): Let σ_i be a strategy in $(\Gamma, \mathfrak{m})$, and let $\hat{\sigma}_i$ be the endpiece-equivalent strategy defined by (8.3). Then σ_i and $\hat{\sigma}_i$ prescribe the same behavior to player i in $(\Gamma^o, \mathfrak{m}^o)$, that is, the modified behaviors defined by (8.5) with σ_i and $\hat{\sigma}_i$ are the same.

(b): Let $\hat{\sigma}_i$ be a strategy in $(\hat{\Gamma}, \hat{\mathfrak{m}})$, and let σ_i be the endpiece-equivalent strategy defined by (8.2). Then σ_i and $\hat{\sigma}_i$ prescribe the same behavior to player i in $(\Gamma^o, \mathfrak{m}^o)$.

In this corollary, we did not refer to the optimization condition (8.4). Of course, we can assume that σ_i in (a) or $\hat{\sigma}_i$ in (b) satisfies (8.4). Although Corollary 8.2 states that subjective optimality is invariant with personal views, subjective optimality may not guarantee, in general, the objective optimality of the prescribed behavior in contrast to Theorem 7.2. This as well as some other difficulties arising in subjective thinking will be discussed in Section 8.2.

Now we prove Theorem 8.1 and the corollaries.

Proof of Theorem 8.1.(a): Let $\hat{\sigma}_i \in \hat{\Sigma}_i$. Consider the function σ_i defined by (8.2). We show that σ_i satisfies (2.12) and (2.13) for $(\Gamma, \mathfrak{m})$, and that it is endpiece-equivalent to $\hat{\sigma}_i$.

Since $\hat{\sigma}_i \cdot \psi(x) \in \hat{A}_{\psi(x)}$ by (2.12) for $\hat{\sigma}_i$ and g4 requires $\hat{A}_{\psi(x)} = A_x$, we have $\sigma_i(x) \in A_x$, i.e., σ_i satisfies (2.12). To prove (2.13) for σ_i, consider $x, y \in X_i^D$ with $\mathfrak{m}(x) = \mathfrak{m}(y)$. Then, by g7, $\hat{\mathfrak{m}} \cdot \psi(x) = \hat{\mathfrak{m}} \cdot \psi(y)$. Since $\hat{\sigma}_i$ satisfies (2.13), we have $\hat{\sigma}_i \cdot \psi(x) = \hat{\sigma}_i \cdot \psi(y)$, which means $\sigma_i(x) = \sigma_i(y)$.

Next we show that the two strategies are endpiece-equivalent. This has two parts, $\lambda(\sigma_i) \subseteq \hat{\lambda}(\hat{\sigma}_i)$ and $\hat{\lambda}(\hat{\sigma}_i) \subseteq \lambda(\sigma_i)$.

First, let $w \in \lambda(\sigma_i)$. Then, there is a play $\langle x_1, ..., x_k, x_{k+1} \rangle$ in Γ with $\lambda(x_{t+1}) = w$ and $\theta(x_{k+1}) = \langle (\lambda(x_1), \sigma_i(x_1)), ..., (\lambda(x_k), \sigma_i(x_k)), \lambda(x_{k+1}) \rangle$. We denote $\psi(x_t)$ by $\hat{x}_t$ for $t = 1, ..., k+1$. By Lemma 6.7, $\langle \hat{x}_1, ..., \hat{x}_{k+1} \rangle$ is a play in $\hat{\Gamma}$ and $\hat{\theta}(\hat{x}_{t+1}) = \theta(x_{k+1})$. By g3, $\hat{\lambda}(\hat{x}_t) = \lambda(x_t)$ for $t = 1, ..., k+1$, and by (8.2), $\hat{\sigma}_i(\hat{x}_t) = \sigma_i(x_t)$ for $t = 1, ..., k$. Hence, $\hat{\theta}(\hat{x}_{t+1}) = \langle (\hat{\lambda}(\hat{x}_1), \hat{\sigma}_i(\hat{x}_1)), ..., (\hat{\lambda}(\hat{x}_1), \hat{\sigma}_i(\hat{x}_k)), \hat{\lambda}(\hat{x}_{k+1}) \rangle$, which means $w \in \hat{\lambda}(\hat{\sigma}_i)$.

Conversely, let $w \in \hat{\lambda}(\hat{\sigma}_i)$. Then, there is a play $\langle \hat{x}_1, ..., \hat{x}_k, \hat{x}_{k+1} \rangle$ in $\hat{\Gamma}$ with $\hat{\lambda}(\hat{x}_{t+1}) = w$ and $\hat{\theta}(\hat{x}_{t+1}) = \langle (\hat{\lambda}(\hat{x}_1), \hat{\sigma}_i(\hat{x}_1)), ..., (\hat{\lambda}(\hat{x}_1), \hat{\sigma}_i(\hat{x}_k)), \hat{\lambda}(\hat{x}_{t+1}) \rangle$. By Lemma 6.7, there is a play $\langle x_1, ..., x_k, x_{k+1} \rangle$ in $(\Gamma, \mathfrak{m})$ such that $\psi(x_t) = \hat{x}_t$ and $\hat{\theta}(\hat{x}_{t+1}) = \theta(x_{k+1})$ for $t = 1, ..., k+1$. By g3, $\hat{\lambda}(\hat{x}_t) = \lambda(x_t)$ for $t = 1, ..., k+1$, and by (8.2), $\hat{\sigma}_i(\hat{x}_t) = \sigma_i(x_t)$ for $t = 1, ..., k$. Hence, $\theta(x_{k+1}) = \langle (\lambda(x_1), \sigma_i(x_1)), ..., (\lambda(x_k), \sigma_i(x_k)), \lambda(x_{k+1}) \rangle$, which means $w \in \lambda(\sigma_i)$.

(b): Let $\sigma_i \in \Sigma_i$. Then, it holds that for all $x, y \in X^D$,

$$\text{if } \psi(x) = \psi(y), \text{ then } \sigma_i(x) = \sigma_i(y). \tag{8.6}$$

Suppose $\psi(x) = \psi(y)$. Then by g7, $\mathfrak{m}(x) = \mathfrak{m}(y)$. This and (2.13) imply $\sigma_i(x) = \sigma_i(y)$.

Next, we show that $\hat{\sigma}_i$ given in (8.3) is well-defined as a function over $\hat{X}_i^D$. Take an arbitrary $\hat{x} \in \hat{X}_i^D$. Since ψ is a surjection, there is some $x \in X$ satisfying $\psi(x) = \hat{x}$. By Lemma 6.5, $x \in X_i^D$. By (8.6), the particular x chosen to satisfy $\psi(x) = \hat{x}$ does not affect the strategy $\hat{\sigma}_i(\hat{x})$. Hence, (8.3) defines a unique function $\hat{\sigma}_i$ for player i.

Once we show that $\hat{\sigma}_i$ satisfies (2.12) and (2.13), we will conclude that $\hat{\sigma}_i$ is a strategy in $(\hat{\Gamma}, \hat{\mathfrak{m}})$. By (2.12) for σ_i and g4, we have that $\hat{\sigma}_i(\hat{x}) \in \hat{A}_{\hat{x}} = A_x$, i.e., $\hat{\sigma}_i$ satisfies (2.12). Next, let $\hat{x}, \hat{y} \in \hat{X}_i^D$ and $\hat{\mathfrak{m}}(\hat{x}) = \hat{\mathfrak{m}}(\hat{y})$. Then, by g7, for any $x, y \in X_i^D$, if $\psi(x) = \hat{x}$ and $\psi(y) = \hat{y}$, then $\mathfrak{m}(x) = \mathfrak{m}(y)$. Since σ_i satisfies (2.13), we have $\sigma_i(x) = \sigma_i(y)$, and $\hat{\sigma}_i(\hat{x}) = \hat{\sigma}_i(\hat{y})$.

We can prove that $\hat{\sigma}_i$ and σ_i are endpiece-equivalent in almost the same way as in the proof of (a) using (8.3) in place of (8.2). ■

Proof of Corollary 8.2.(a): Let σ_i satisfy (8.4) in $(\Gamma, \mathfrak{m})$, and let $\hat{\sigma}_i$ be the endpiece-equivalent strategy defined by (8.3). By g3, g6, and endpiece-equivalence of σ_i and $\hat{\sigma}_i$, we have $\min_{\hat{w} \in \hat{\lambda}(\hat{\sigma}_i)} \hat{h}(\hat{w}) = \min_{w \in \lambda(\sigma_i)} h(w)$. For each $\hat{\sigma}_i' \in \hat{\Sigma}_i$, Theorem 8.1 guarantees that there is an endpiece-equivalent strategy $\sigma_i' \in \Sigma_i$ defined by (8.2) and $\min_{\hat{w}' \in \hat{\lambda}(\hat{\sigma}_i')} \hat{h}(\hat{w}') = \min_{w' \in \lambda(\sigma_i')} h(w')$. Hence, since σ_i satisfies (8.4) in $(\Gamma, \mathfrak{m})$, we have, $\min_{\hat{w} \in \hat{\lambda}(\hat{\sigma}_i)} \hat{h}(\hat{w}) \geq \min_{\hat{w}' \in \lambda(\hat{\sigma}_i')} h(\hat{w}')$ for all $\hat{\sigma}_i' \in \hat{\Sigma}_i$.

Part (b) is proved in almost the same way. ■

Proof of Corollary 8.3.(a): Let σ_i satisfy (8.4) in $(\Gamma, \mathfrak{m})$, and let $\hat{\sigma}_i$ be the strategy defined by (8.3). By Corollary 8.3, $\hat{\sigma}_i$ satisfies (8.4) in $(\hat{\Gamma}, \hat{\mathfrak{m}})$. We let $\sigma_i^1(x)$ and $\hat{\sigma}_i^1(x)$ denote the behavior prescribed by (8.5) in $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ respectively. Let $x \in X_i^o$. If $\mathfrak{m}_i^o(x) = \mathfrak{m}(x')$ for some $x' \in X$, then by g0 there is an $\hat{x}' \in \hat{X}$ where $\hat{x}' = \psi(x')$. By (8.3), $\hat{\sigma}_i(\hat{x}') = \sigma_i(x')$, so $\sigma_i^1(x) = \hat{\sigma}_i^1(x)$. If, alternatively, $\mathfrak{m}_i^o(x) \neq \mathfrak{m}(x')$ for any $x' \in X$, then $\sigma_i^1(x) = \sigma_i^o(x) = \hat{\sigma}_i^1(x)$. Part (b) is proved in almost the same way as (a). ■

8.2. Difficulties involved in Subjective Thinking and in Playing in the Objective Situation

In Section 7, we assumed that player i has the memory function $\mathfrak{m}_i^o = \mathfrak{m}_i^{EPR}$ and the unilateral trial domain $D_i^U(\sigma^o)$. Then, he succeeds in having the unique direct view, in finding an optimal strategy in $(\Gamma^d, \mathfrak{m}^d)$ as well as in bringing it back to the objective situation. However, if we drop these assumptions, then a subjectively optimal strategy may not help him behave properly in the objective situation. We can find many difficulties in decision making here, but we restrict ourselves to only some of them.

(1): Difficulty in Subjective Thinking: We start with a difficulty involved in subjective thinking. In Corollary 5.3, we gave a necessary and sufficient condition for a direct view to be unique and inductively derived. When the direct view is uniquely determined, the results of Section 6 state that it is essentially the smallest i.d.view. Also, the results of Section 8.1 imply that decision making is invariant for the choice of a personal view.

Problems may arise, not because of multiplicity of i.d.views, but rather because possible multiplicity of direct views for a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$. In this case, player i faces a difficulty first in choosing an i.d.view.

In Example 5.1 there are four direct i.d.views, which all differ in terms of the memory function. Figure 8.1 gives two of those direct i.d.views with only the relevant memory yarns listed, and the payoffs are now attached. In the left view, the memory yarns are mixed up at the nodes $\langle (y_0, a), v \rangle$ and $\langle (y_0, b), v \rangle$ as $\mathfrak{m}_1^o(y_2)$ and $\mathfrak{m}_1^o(y_1)$, while in the right view, he expects the same memory yarn $\mathfrak{m}_1^o(y_1)$ at each of his second decision nodes. In the right view, he does not use the memory yarn $\mathfrak{m}_1^o(y_2)$ in $\mathcal{Y}_{D_i}$. This multiplicity of views causes some difficulty for the player in deciding which view to use for his decision making. His choice of a view may influence his decision making since, e.g., in the left view he can make different choices at $\langle (y_0, a), v \rangle$ and $\langle (y_0, b), v \rangle$, while in the right view, he is required to make the same choice.

(2): Difficulty in Objective Optimality: Suppose that player i has chosen an i.d.direct view and a behavior pattern for it that is subjectively optimal in the sense of (8.4). Consider the left direct view of Figure 8.1. One subjectively optimal strategy is defined by σ_1 which chooses action a at the root node and if his memory node where his memory is $\mathfrak{m}_1^o(y_2)$, while choosing b if his memory yarn is $\mathfrak{m}_1^o(y_1)$. When he modifies his regular behavior in the objective game by this strategy σ_1 and brings it back to the objective situation, he receives the payoff 0. Thus he fails to behave optimally in the objective situation. In this case, he may want to revise his behavior or his view.

Next, consider the right direct view. In this view, he has a subjectively optimal strategy prescribing the choice of b at the root node. If he takes this strategy to the objective world, he will receive the memory yarn $\mathfrak{m}_1^o(y_2)$, which he does not expect and, indeed, is not contained in his constructed personal view. Thus, the player finds

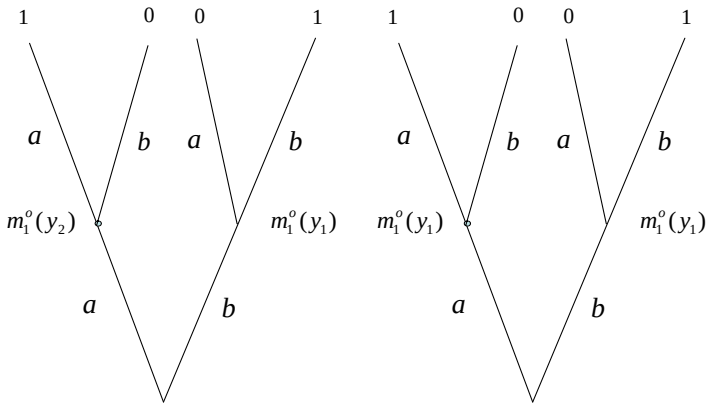

Figure 8.1:

a further difficulty with his view and a reason to revise his behavior or his view. This difficulty is caused by the weak inclusion condition of P2a, allowing the possibility of $\{\mathfrak{m}^i(x) : x \in X_i^i\} \subsetneq \mathcal{Y}_{D_i}$. By strengthening P2a to equality, this difficulty could be avoided as in the right view of Figure 8.1. Nevertheless, the multiplicity of views remains, and so does the difficulty that a subjectively optimal strategy may not be objectively optimal.

Thus, when there are multiple i.d.direct views, player i may meet some difficulties both subjectively and objectively. Either of these difficulties gives a player a reason to revise his behavior or his view. In this paper, however, we do not consider those revisions. In a separate paper (Kaneko-Kline [15]), we will discuss these types of difficulties.

9. Concluding Comments

In this paper, we have given a discourse of inductive game theory by confining ourselves to clear-cut cases. It would be, perhaps, appropriate to start this section with comments on the discourse of this paper. Then we will discuss some implications for extant game theory of the considerations given in our discourse.

9.1. Comments on the Discourse of this Paper

We have made particular choices of assumptions and definitions for our discourse. One important methodological choice is to adopt extensive games in the strong and weak

senses for objective and subjective descriptions. First, we will give some comments on this choice, and then, we will discuss the definition of an inductively derived view given in Section 4 based on the initial segment procedure.

As pointed out in Section 4, an extensive game contains observable and unobservable elements. The nodes with the successor relation are unobservable for the players and even for the outside observer, in which sense those are highly hypothetical. The components in a memory kit are all observables and actually observed. Thus, our definition of the inductive derivation of a personal view from a memory kit extends the observed observables by adding hypothetical elements. This may be interpreted as an "inductive" process of adding unobservable elements to observed data. However, this freedom of adding hypothetical elements leads us a proliferation of possible views. To prevent this proliferation, we need some criterion to choose a view from many possible ones. In this paper, we have used the concept of a g-morphism (game theoretical p-morphism) to choose a smallest one.

Conceptually speaking, the choice of a personal view is done by a player, rather than us. While the definition of an inductive derivation allows many views, a player cannot construct a large one because of his bounded cognitive ability. Thus, the criteria of smallness and constructiveness are important from this point of view. The direct view defined in Section 5 has a constructive nature as well as being a smallest one for a given memory kit. In this sense, the direct view has a special status among those possible views.

Nevertheless, Definition 4.1 may admit no inductively derived view for a given memory kit, as characterized by Theorem 5.2. In fact, the initial segment procedure adopted in Definition 4.1 still gives a strong restriction on the addition of hypothetical elements. If we allow more freedom in using hypothetical elements in an inductive derivation, we could avoid the nonexistence result. For example, if we allow a player to add "nature nodes" to his personal view, we could even avoid the use of an extensive game in the weak sense. On the other hand, this creates vast arbitrariness in inductive derivations; and we expect serious difficulties in finding natural criteria to narrow down the use of "nature nodes". Until we find natural criteria, we should refrain from the cheap use of "nature nodes".

The above conclusion may sound negative to any extension of our definition of an inductive derivation, but we have different opinions. We could actually have a more general procedure to construct a personal view than the initial segment procedure. Since this paper is intended to provide an entire scenario, we have chosen the initial segment procedure as a clear-cut case. In separate papers, we will discuss less restrictive definitions.

Another comment should be given on the choice of extensive games. In fact, we can avoid the adoption of extensive games; instead, the present authors ([14]) have developed a theory of *information protocols*, which avoids the use of nodes and describes game

situations directly in terms of information pieces and actions together with a history-event relation. If we adopt this theory, then we could avoid a proliferation of personal views generated by the use of hypothetical nodes. In the theory of information protocols it may be easier to discuss extensions of inductive derivations. One reason for our adoption of extensive games is their familiarity within our profession. Nevertheless, the choice of extensive games makes the distinction between observables and unobservables explicit, which is another reason for our choice.

We expect gradual developments of inductive game theory to come about by deeper analysis and alternative approaches to the various stages mentioned in the diagram of Figure 1.1. By such gradual developments, we may find natural criteria for steps such as the use of nature nodes, and some experimental tests of inductive game theory.

9.2. Some Implications to Extant Game Theory

It is a main implication of the discourse of inductive game theory that a good individual view on society is difficult to construct from the experiential point of view. There are many places for a player to get stuck in his inductive process and analysis process. Nevertheless, we gave a characterization theorem of Nash equilibrium in Section 7. We will discuss some implications of our discourse on inductive game theory to extant game theory and economics chiefly with respect to Nash equilibrium.

There are various interpretations of Nash equilibrium (cf. Kaneko [13], Act 4). Nash [23] himself described his concept from purely *ex ante* decision making, but in economic applications, it is typically more natural to interpret Nash equilibrium as a strategically stable stationary state in a recurrent situation. The characterization given in Section 7 is along this line of interpretations.

Our characterization of Nash equilibrium is a full description of the stationary state interpretation including *ex ante* decision making in a player's constructed personal view. To reach Nash equilibrium, which may not be the case, it takes a long time. Also, the process of trial and error may not allow all possible available actions. The Nash equilibrium reached should be regarded as a Nash equilibrium in the game with respect to the actually experienced domains. Thus, the characterization of Nash equilibrium in Section 7 should not merely be interpreted as a positive result. It means that the characterization would be obtained if all those processes go through well and if reservations about restrictions on trials are taken into account.

>From the same point of view, the subgame perfect equilibrium of Selten [27] involves even deeper difficulties from our experiential point of view, which was already pointed out in Kaneko-Matsui [16]. The reason is that subgame perfection requires higher order experimentations. When one player deviates from his regular behavior, other players in turn need, again, to make experimentations from regular behavior. This second or higher order experimentation is already problematic and violates some

principles discussed in the informal theory in Section 3.2. In fact, a similar criticism is applied to Nash equilibrium, as already stated. Nash equilibrium itself is regarded as one limit notion, and subgame perfection is a higher limit concept.

Taking the above criticism seriously, one important problem would arises. The complexities, in a certain sense, of an inductively derived view as well as of experimentations are measured and restricted. In the epistemic logic context, Kaneko-Suzuki [18] introduced the concept contentwise complexity, which measures "contentwise complexity" of a single instance of a game. This notion can be converted to our inductive game theory. Then, we will be able to give restrictions on individual views as well as experiments. In this manner, our inductive game theory will be developed in the direction of "bounded rationalities".

We have restricted our attention to the purely experiential sources. In our society, usually, we have different sources of beliefs/knowledge such as from other people or through education. These suggest that a player may get more beliefs/knowledge on the social structure, but do not suggest that he can guess other people's thinking, which has usually been assumed in the standard game theory (cf., Harsanyi [9] for incomplete information game and Kaneko [12] for the epistemic logic approach). At least, the assumption of common knowledge is far beyond experiences. If we restrict interpersonal thinking to very shallow levels, deductive game theory may have some connections to inductive game theory (cf. Kaneko-Suzuki [17] for such a direction of deductive game theory).

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