Necessary and Sufficient Conditions for Efficient Risk-Sharing Rules

Chiaki Hara*

Institute of Economic Research, Kyoto University

July 6, 2006

Abstract

We show that for every collection of strictly increasing risk-sharing rules and every strictly increasing and strictly concave expected utility function, there exists a collection of strictly increasing and strictly concave expected utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with the corresponding representative consumer's utility function. This result shows that the efficiency property imposes no restriction on the risk-sharing rules beyond the comonotonicity, or on the state-pricing rule beyond the positivity and antimonotonicity. We also obtain contrasting results when the individual consumers are assumed to exhibit hyperbolic absolute risk aversion.

JEL Classification Codes: D51, D61, D81, G12, G13.

Keywords: Risk-sharing rule, representative consumer, expected utility, risk aversion, Inada condition, complete markets.

1 Introduction

We consider an exchange economy under uncertainty with a single good and a single consumption period, consisting of consumers who all have expected utility functions with respect to a homogeneous probabilistic belief but their expected utility functions may exhibit heterogenous risk attitudes. As usual, we assume that all individual consumers prefer more to less and are averse to risk, which means that their utility functions are strictly increasing and strictly

*Email: hara@kier.kyoto-u.ac.jp

concave. Then, a Pareto-efficient allocation can be described by a collection of efficient *risk-sharing rules*, one for each consumer, and its supporting (or, decentralizing) price system can be represented by the expected utility function for the *representative consumer*. Specifically, an individual consumer's risk-sharing rule is a deterministic function that maps realized aggregate consumption levels to realized individual consumption levels; and the representative consumer's marginal utility at a given aggregate consumption level is the price for unit consumption in the event that the realized aggregate consumption equals the given level, divided by the probability of that event.

A couple of properties are well known for risk-sharing rules and the representative consumer's utility function. First, every individual consumer's efficient risk-sharing rule is a strictly increasing function. This property is called the *comonotonicity* because it is equivalent to the property that the consumers' consumption levels are monotonically increasing functions of one another. Also, the representative consumer's utility function is strictly increasing and strictly concave. Since his marginal utility function gives a state-price deflator, these properties are equivalent to the property that the price for unit consumption in the event that the realized aggregate consumption is at any given level is positive and, once divided by the probability of the event, a decreasing function of aggregate consumption levels. We shall therefore call them the *positivity* and *antimonotonicity* of the *state-pricing rule*.

In some special cases, more can be said of risk-sharing and state-pricing rules. For example, if all individual consumers have a common constant relative risk aversion, then their efficient risk-sharing rules are all linear and the representative consumer also has the same constant relative risk aversion. Hara, Huang, and Kuzmics (2005) closely investigated the case in which the individual consumers exhibit hyperbolic absolute risk aversion but the cautiousness (which is the derivative of the reciprocal of absolute risk aversion and is therefore constant) differ across them. They found, in this case, that no individual consumer's efficient risk sharing rule is linear and the representative consumer's cautiousness is strictly increasing with aggregate consumption levels. Other than these special cases, little is known on the general properties of efficient risk-sharing rules and the representative consumer's risk attitudes.

In this paper, we show (Theorem 2 and Corollary 5) that the efficiency implies no property other than that the comonotonicity of the risk-sharing rules the positivity and antimonotonicity of the state-pricing rule. More specifically, for every collection of strictly increasing risk-sharing rules, one for each consumer, there exists a collection of strictly increasing and strictly concave utility functions, one for each consumer, for which the given risk-sharing rules are efficient. We moreover show that these efficient risk-sharing rules can be coupled with an arbitrary positive and antimonotone state-pricing rule. That is, for every collection of strictly increasing risk-sharing rules and every strictly increasing and strictly concave utility function, there exists a collection of strictly increasing and strictly concave utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with the corresponding representative consumer's utility function. We also show that once a collection of risk-sharing rules and a positive and antimonotone state-pricing rule is given, each individual consumer's utility function are essentially uniquely determined.

What can we learn from this result? First, it shows the comonotonicity of risk-sharing rules and the positivity and antimonotonicity of the state-pricing rule exhaust all the implications of efficiency, when the consumers have expected utility functions with respect to a homogeneous probabilistic belief but no other condition is imposed on their utility functions. Our result also clarifies the degree of freedom in the choice of individual utility functions generating the given risk-sharing rules and the state-pricing rule.

Second, it shows that the joint assumption of complete asset markets and expected utility functions with a homogeneous probabilistic belief imposes no restriction on equilibrium asset prices and transaction volumes. To see this point, recall first that the first and second welfare theorems hold for complete asset markets, that any arbitrage-free asset prices correspond to a unique state-pricing rule, and that any asset allocation gives rise to a risk-sharing rule, one for each consumer. Hence, the arbitrariness of the state-pricing rule implies the arbitrariness of the equilibrium asset prices, and the arbitrariness of the risk-sharing rules implies the arbitrariness of the equilibrium transaction volumes. Regarding viable restrictions on equilibrium state prices, Kreps (1981) established (his Theorem 1) necessary and sufficient conditions for consistency with a preference-maximizing consumer in a much more general setting than ours, but our result, in spirit, refines his Theorem 1 in two respects. One is that we ask, and solves, the question of consistency in a more demanding setting of expected utility functions exhibiting risk aversion. The other is that by allowing for multiple consumers, we can show that any choice of arbitrary-free asset prices is viable at equilibrium with any patterns of transaction volumes.

Third, our result shows that if there is any value added to the use of non-expected utility functions and incomplete asset markets (including the presence of transaction costs) to improve the prediction of equilibrium asset prices, it must be because there is some a priori class of expected utility functions that are regarded as being reasonable, for whatever reason. The best example for the type of quest for a better prediction of equilibrium asset prices based on "reasonable" utility functions is the equity premium puzzle, first proposed by Mehra and Prescott (1985). As has been well documented in Kocherlakota (1996), the subsequent literature has indeed been well aware that the issue here is whether it is possible to reproduce empirically observed equity premium by utility functions exhibiting reasonable risk aversion, not just by any expected utility functions. Our result justifies this approach of pursuing reasonable risk aversion, by showing that if no such restriction is imposed, any arbitrage-free asset prices may emerge. But it also suggests a weakness of this approach: since the criterion for reasonable utility function is often ad hoc and ambiguous, and since any arbitrage-free asset prices are in principle viable at equilibrium, the virtue of any model incorporating non-expected utility functions or incomplete markets hinges critically on whether the criterion employed is appropriate.

Let us now turn to the related literature. The problem of characterizing efficient risk-sharing rules the representative consumer's risk attitudes is classical. A pioneering work is Wilson (1968), who related the individual consumers' risk attitudes to the representative consumer's counterpart and their efficient risk-sharing rules. His analysis started with a given set of utility functions for the individual consumers and did not explicitly exhaust all the possible implications of efficiency on risk-sharing rules. Hara, Huang, and Kuzmics (2005) obtained some interesting properties of efficient risk-sharing rules allowing for significant heterogeneity in the individual consumers' risk attitudes. However, their analysis also started with a given set of risk-sharing rules and did not try to construct utility functions for a given set of risk-sharing rules.

The most closely related paper is Dana and Meilijson (2003). They also constructed a collection of increasing and concave utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with the corresponding representative consumer's utility function. There are two additional properties for the utility functions that they did not guarantee, which nonetheless we do by employing a rather different proof method and imposing extra conditions. The first one is the Inada condition, which says that the marginal utility spans from zero to infinity. The second condition is the differentiability of arbitrarily many times. The Inada condition is obtained by assuming that the risk-sharing rule is an onto function, covering the entire domain of the utility function for the consumer. The differentiability is obtained by assuming that the risk-sharing rule and the utility function given for the representative consumer are appropriately many times differentiable. Extra conditions as they are, it is worth imposing them to guarantee the two properties. Although the existence of an interior optimal consumption is crucial for many applications, it would not be guaranteed without the Inada condition. Many interesting properties of the curvature of risk-sharing rules, such as those in Hara, Huang, and Kuzmics (2005), cannot be established without four times differentiability of utility functions.

A related strand of literature is on the empirical testing of the full insurance hypothesis. As our result indicates, if no restriction is imposed on the utility functions other than the strict increasingness and strict concavity, then for any consumption allocation generated by a collection strictly increasing risk-sharing rules, we do not reject the hypothesis that the allocation is efficient. This appears to be too lenient a test for efficiency. The existing literature therefore imposed additional conditions on utility function to derive more restrictions on risk-sharing rules. Townsend (1994), Mace (1991), and Kohara, Ohtake, and Saito (2002) conducted tests for efficiency for the cases where all consumers have the same utility function that exhibits either constant absolute risk aversion or constant relative risk aversion. In this case, all the efficient risk-sharing rules are linear, and hence the hypothesis of an efficient allocation is rejected whenever the observed data set is inconsistent with linearity. This apparently provides a rather stringent test for efficiency; and the hypothesis that the observed consumption allocation is efficient is often rejected in the literature. The tests of Ogaki and Zhang (2001) relaxed the assumption of the common constant relative risk aversion. In this case, the hypothesis of an efficient allocation is often not rejected. In the face of this type of evidence, the literature tends to conclude that consumers are likely to have non-constant (decreasing, specifically), rather than constant, relative risk aversion.

The result of this paper shows that the linearity versus non-linearity of the risk-sharing rules does not really matter for the nature of utility functions. Recall that our result is that for every collection of strictly increasing risk-sharing rules and every strictly increasing and strictly concave utility function, there exists a collection of strictly increasing and strictly concave utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with the corresponding representative consumer's utility function. Just with a small trick, we can also show (Corollary 3) that for every collection of strictly increasing risk-sharing rules and every strictly increasing and strictly concave utility function, there exists a collection of strictly increasing and strictly concave utility functions for which the given risk-sharing rules are efficient and the given utility function coincides with one, say the first, individual consumer's (rather than the representative consumer's) utility function. In short, any observed consumption allocation in no way restricts one individual consumer's utility function.

As can be seen from reviewing the literature on the empirical testing of the full insurance hypothesis, in most applications all individual consumers are assumed to exhibit hyperbolic absolute risk aversion. We identify (Theorems 4 and 5) the nature of the risk-sharing rules and the state-pricing rule in such an environment by characterizing a class of utility functions that is closed under aggregation. What should be contrasted with the result (Theorem 2) for the general case is Proposition 4, which claims that the risk-sharing rule is essentially uniquely determined by the state-pricing rule, or, equivalently, the utility function for the representative consumer. The risk-sharing rules, one for each consumer, can be completely recovered from the state-pricing rule in such an environment.

This paper is organized as following. Section 2 lays out the setting of this paper and formulates our problem. Section 3 states and proves the main result (Theorem 2) of this paper, and its corollaries are presented in Section 4. Section 5 deals with the special but commonly used case in which individual consumers exhibit hyperbolic absolute risk aversion. Section 6 gives analytical examples of implications of the main result. Section 7 gives concluding remarks and suggestion directions of future research.

2 Setting

There are I consumers, $i \in \{1, \dots, I\}$. Consumer i has a von-Neumann Morgenstern (also known as Bernoulli) utility function $u_i : D_i \to \boldsymbol{R}$, where D_i is a nonempty open interval of $\boldsymbol{R}$, which may or may not be bounded from below or above. We assume that u_i is of class C^r with $r \geq 2$ and satisfies $u_i'(x_i) > 0$ and $u_i''(x_i) < 0$ for every $x_i \in D_i$. We also assume that u_i satisfies the Inada condition, that is, $u_i'(x_i) \to \infty$ as $x_i \to \inf D_i$ and $u_i'(x_i) \to 0$ as $x_i \to \sup D_i$.

For each $\lambda = (\lambda_1, \ldots, \lambda_I) \in \boldsymbol{R}^I_{++}$ and each $x \in \sum_i D_i$, consider the following maximization problem:

$$\begin{aligned} \max_{(x_1,\ldots,x_I)\in D_1\times\cdots\times D_I} \quad & \sum \lambda_i u_i(x_i), \\ \text{subject to} \quad & \sum x_i = x. \end{aligned} \tag{1}$$

By the strict concavity and the Inada condition, for each x, there exists exactly one solution to this problem, which we denote by $f_\lambda(x) = (f_{\lambda 1}(x), \ldots, f_{\lambda I}(x))$. Then, for every λ, the mapping $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is well defined. It follows from the first-order condition and the implicit function theorem that f_λ is of class C^{r-1}, and that $f_{\lambda i}'(x) > 0$ for every $x \in \sum_j D_j$, $f_{\lambda i}(x) \to \inf D_i$ as $x \to \inf \sum_j D_j$, and $f_{\lambda i}(x) \to \sup D_i$ as $x \to \sup \sum_j D_j$, for every i. Define $u_\lambda : \sum_i D_i \to \boldsymbol{R}$ as the value function of this problem that is, $u_\lambda(x) = \sum_i \lambda_i u_i(f_{\lambda i}(x))$. By the envelope theorem, $u_\lambda'(x) = \lambda_i u_i'(f_{\lambda i}(x))$ for every i and every $x \in \sum_j D_j$. This shows that $u_\lambda'(x) > 0$, u_λ satisfies the Inada condition, and u_λ' is of class C^{r-1}. Hence u_λ is of class C^r and $u_\lambda''(x) = \lambda_i u_i''(f_{\lambda i}(x)) f_{\lambda i}'(x) < 0$.

The importance of f_λ and u_λ lies in the following fact, subject to appropriate integrability conditions:[1] In an economy under uncertainty described by the probability measure space Ω with the aggregate endowment e, which is a random variable defined on Ω, an allocation $c = (c_1, \ldots, c_I)$ of random variables is a Pareto-efficient allocation of e if and only if $c = f_\lambda(e)$ for some $\lambda \in \boldsymbol{R}^I_{++}$. Moreover, then, the linear functional on $\mathscr{L}^2(\Omega)$ whose Riesz representation is $u_\lambda'(e)$ is the unique price system, modulo strictly positive scalar multiplications, that decentralizes c. In other words, f_λ tells us how the aggregate consumption level is distributed across consumers and u_λ tells us how to price assets whose payoffs are determined by the aggregate consumption levels.

The argument so far can be more formally summarized as follows.

Denote by $\mathscr{D}$ the set of all nonempty open intervals of $\boldsymbol{R}$. For each $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$ and each $D_i \in \mathscr{D}$, let $\mathscr{U}^r_{D_i}$ be the set of all functions $u_i : D_i \to \boldsymbol{R}$ that satisfy the following four conditions:

[1] It can be traced back to Borch (1962, p. 428) and Wilson (1968), and is nicely explained in Kreps (1990, Section 5.4).

C^r u_i is of class C^r

INC $u_i'(x_i) > 0$ for every $x_i \in D_i$

CONC $u_i''(x_i) < 0$ for every $x_i \in D_i$

INADA $u_i'(x_i) \to \infty$ as $x_i \to \inf D_i$ and $u_i'(x_i) \to 0$ as $x_i \to \sup D_i$

Write $D = (D_1, \ldots, D_I) \in \mathscr{D}^I$ and define $\mathscr{U}_D^r = \mathscr{U}_{D_1}^r \times \cdots \times \mathscr{U}_{D_I}^r$. Then, specifying a list of consumers is equivalent to specifying an $I \in \boldsymbol{Z}_{++}$, a $D \in \mathscr{D}^I$, and a $(u_1, \ldots, u_I) \in \mathscr{U}_D^2$.

For each $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, each integer $I \in \boldsymbol{Z}_{++}$, and each $D = (D_1, \ldots, D_I) \in \mathscr{D}^I$, let $\mathscr{F}_D^{r-1}$ as the set of all functions $f = (f_1, \ldots, f_I) : \sum_j D_j \to D_1 \times \cdots \times D_I$, with $f_i : \sum_j D_j \to D_i$ for every i, that satisfy the following four conditions:

C^{r-1} f_i is of class C^{r-1} for every i

COMONO $f_i'(x) > 0$ for every i and every $x \in \sum_i D_i$

ONTO $f_i(x) \to \inf D_i$ as $x \to \inf \sum_j D_j$ and $f_i(x) \to \sup D_i$ as $x \to \sup \sum_j D_j$ for every i.

SUM $\sum_i f_i(x) = x$ for every $x \in \sum_i D_i$

What we have stated above can then be restated as follows:

Theorem 1 *For every $I \in \boldsymbol{Z}_{++}$, every $D = (D_1, \ldots, D_I) \in \mathscr{D}^I$, every $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, every $(u_1, \ldots, u_I) \in \mathscr{U}_D^r$, and every $\lambda \in \boldsymbol{R}_{++}^I$, if $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the solution and $u_\lambda : \sum_i D_i \to \boldsymbol{R}$ is the value function of (1), then $f_\lambda \in \mathscr{F}_D^{r-1}$ and $u_\lambda \in \mathscr{U}_{\sum_i D_i}^r$.*

3 Main Result

The main result of this paper is the converse of Theorem 1:

Theorem 2 *For every $I \in \boldsymbol{Z}_{++}$, every $D = (D_1, \ldots, D_I) \in \mathscr{D}^I$, every $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, every $\lambda \in \boldsymbol{R}_{++}^I$, every $f \in \mathscr{F}_D^{r-1}$, and every $u \in \mathscr{U}_{\sum_i D_i}^r$, there exists a $(u_1, \ldots, u_I) \in \mathscr{U}_D^r$ such that if $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the solution and $u_\lambda : \sum_i D_i \to \boldsymbol{R}$ is the value function of (1), then $f = f_\lambda$ and $u = u_\lambda$. Moreover, for every $(v_1, \ldots, v_I) \in \mathscr{U}_D^r$, $(v_1, \ldots, v_I)$ has the same property as this $(u_1, \ldots, u_I)$ if and only if $u_i - v_i$ is constant for every i and $\sum_i \lambda_i (u_i - v_i) = 0$.*

The first part of this theorem is the converse of Theorem 1, which establishes the existence of the profile $(u_1, \ldots, u_I) \in \mathscr{U}_D^r$ of utility functions satisfying $f_\lambda = f$ and $u_\lambda = u$. The second part is the uniqueness of the profile up to constants. We do not have the degree of freedom with respect to the scalar multiplication, because we require the identity $u = u_\lambda$, which is cardinal in nature. Since the constants are relevant for the risk attitudes, this theorem implies that f and u uniquely determines each individual consumer's risk attitudes.

Proof of Theorem 2 As in the statement of this theorem, let $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, $I \in \boldsymbol{Z}_{++}$, $D \in \mathscr{D}^I$, $\lambda \in \boldsymbol{R}^I_{++}$, $f \in \mathscr{F}_D^{r-1}$, and $u \in \mathscr{U}^r_{\sum_i D_i}$. Let $d \in \sum_i D_i$. For each i, define $u_i : D_i \to \boldsymbol{R}$ by

$$u_i(x_i) = \frac{1}{\lambda_i}\left(\int_{f_i(d)}^{x_i} u'\left(f_i^{-1}(z_i)\right)\, dz_i + \frac{u(d)}{I}\right) \tag{2}$$

for every $x_i \in D_i$. This is indeed well defined by Conditions COMONO and ONTO. We shall prove that $(u_1, \ldots, u_I)$ is what we want.

Since $u' \circ f_i^{-1}$ is of class C^{r-1}, u_i is of class C^r. Since

$$u_i'(x_i) = \frac{1}{\lambda_i} u'\left(f_i^{-1}(x_i)\right) \tag{3}$$

for every $x_i \in D_i$, u_i satisfies Conditions INC and INADA. Hence

$$u_i''(x_i) = -\frac{1}{\lambda_i} u''\left(f_i^{-1}(x_i)\right) \frac{1}{f_i'\left(f_i^{-1}(x_i)\right)} < 0 \tag{4}$$

for every $x_i \in D_i$. Thus u_i satisfies Condition CONC, and $(u_1, \ldots, u_I) \in \mathscr{U}_D^r$.

By (3), $\lambda u_1'(f_1(x)) = \cdots = \lambda u_I'(f_I(x))$. Thus, the first-order condition for the solution of (1) is satisfied by $f(x)$ for every $x \in \sum_i D_i$. By Condition SUM, $f = f_\lambda$.

This implies that $u_\lambda(x) = \sum_i \lambda_i u_i\left(f_{\lambda i}(x)\right)$ for every $x \in \sum_i D_i$. By (3),

$$u_\lambda'(x) = \sum_i \lambda_i u_i'\left(f_{\lambda i}(x)\right) f_i'(x) = \sum_i \lambda_i \frac{1}{\lambda_i} u'\left(f_i^{-1}\left(f_{\lambda i}(x)\right)\right) f_i'(x) = u'(x).$$

Thus $u_\lambda(x) - u(x)$ does not depend on x. Moreover,

$$u_\lambda(d) = \sum_i \lambda_i \frac{1}{\lambda_i}\left(0 + \frac{u(d)}{I}\right) = u(d).$$

Hence $u = u_\lambda$. This completes the existence part of this theorem.

To prove the second part, let $(v_1, \ldots, v_I) \in \mathscr{U}_D^r$. Suppose first that for every i, there is a constant κ_i such that $u_i(x_i) - v_i(x_i) = \kappa_i$ for every $x_i \in D_i$ and $\sum_i \lambda_i \kappa_i = 0$. Then

$$\lambda_i u_i' \left(f_i(x)\right) = \lambda_i v_i' \left(f_i(x)\right)$$

for every $x \in \sum_j D_j$ and hence f is the solution to (1) when the u_i are replaced by the v_i. Denote the corresponding value function by v_λ, then

$$\begin{aligned} v_\lambda(x) = \sum_i \lambda_i v_i \left(f_i(x)\right) &= \sum_i \lambda_i u_i \left(f_i(x)\right) + \sum_i \lambda_i \left(v_i \left(f_i(x)\right) - u_i \left(f_i(x)\right)\right) \\ &= \sum_i \lambda_i u_i \left(f_i(x)\right) - \sum_i \lambda_i \kappa_i = u_\lambda(x) \end{aligned}$$

for every $x \in \sum_i D_i$. Thus $(v_1, \ldots, v_I) \in \mathscr{U}_D^r$ has the same property as $(u_1, \ldots, u_I)$.

Suppose conversely that $(v_1, \ldots, v_I) \in \mathscr{U}_D^r$ has the same property as $(u_1, \ldots, u_I)$. Then, by the envelope theorem,

$$\lambda_i u_i' \left(f_i(x)\right) = u'(x) = \lambda_i v_i' \left(f_i(x)\right)$$

for every i and $x \in D_i$. By $\lambda_i > 0$, $u_i' \left(f_{\lambda i}(x)\right) = v_i' \left(f_{\lambda i}(x)\right)$ for every i and $x \in D$. By Condition ONTO, this implies that $u_i' \left(x_i\right) = v_i' \left(x_i\right)$ for every i and $x_i \in D_i$. Thus $u_i - v_i$ is constant. Moreover

$$\begin{aligned} \sum_i \lambda_i u_i \left(f_i(x)\right) = u_\lambda(x) &= \sum_i \lambda_i v_i \left(f_i(x)\right) \\ &= \sum_i \lambda_i u_i \left(f_i(x)\right) + \sum_i \lambda_i \left(v_i \left(f_i(x)\right) - u_i \left(f_i(x)\right)\right). \end{aligned}$$

Thus $\sum_i \lambda_i \left(v_i - u_i\right) = 0$. ///

4 Corollaries of Theorem 2

In this section, we present corollaries of Theorem 2 in various directions.

4.1 Infinitely Many Times Differentiability

Define $\mathscr{U}^{\infty}_{D_i} = \bigcap_{r=2}^{\infty} \mathscr{U}^{r}_{D_i}$ and $\mathscr{F}^{\infty}_{D} = \bigcap_{r=2}^{\infty} \mathscr{F}^{r-1}_{D}$. Then these sets consist of infinitely many times differentiable utility functions and risk-sharing rules. By applying Theorems 1 and 2 for all $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, we obtain the following corollaries:

Corollary 1 *For every $I \in \boldsymbol{Z}_{++}$, every $D = (D_1, \dots, D_I) \in \mathscr{D}^I$, every $(u_1, \dots, u_I) \in \mathscr{U}^{\infty}_{D}$, and every $\lambda \in \boldsymbol{R}^I_{++}$, if $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the solution and $u_\lambda : \sum_i D_i \to \boldsymbol{R}$ is the value function of (1), then $f_\lambda \in \mathscr{F}^{\infty}_{D}$ and $u_\lambda \in \mathscr{U}^{\infty}_{\sum_i D_i}$.*

Corollary 2 *For every $I \in \boldsymbol{Z}_{++}$, every $D = (D_1, \dots, D_I) \in \mathscr{D}^I$, every $\lambda \in \boldsymbol{R}^I_{++}$, every $f \in \mathscr{F}^{\infty}_{D}$, and every $u \in \mathscr{U}^{\infty}_{\sum_i D_i}$, there exists a $(u_1, \dots, u_I) \in \mathscr{U}^{\infty}_{D}$ such that if $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the solution and $u_\lambda : \sum_i D_i \to \boldsymbol{R}$ is the value function of (1), then $f = f_\lambda$ and $u = u_\lambda$. Moreover, for every $(v_1, \dots, v_I) \in \mathscr{U}^{\infty}_{D}$, $(v_1, \dots, v_I)$ has the same property as this $(u_1, \dots, u_I)$ if and only if $u_i - v_i$ is constant for every i and $\sum_i \lambda_i (u_i - v_i) = 0$.*

4.2 Utility Function for an Individual Consumer

The following corollary shows that the utility function of one individual consumer, rather than the representative consumer, can be chosen arbitrarily.

Corollary 3 *For every $I \in \boldsymbol{Z}_{++}$ with $I \geq 2$, every $D = (D_1, \dots, D_I) \in \mathscr{D}^I$, every $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, every $\lambda \in \boldsymbol{R}^I_{++}$, every $f \in \mathscr{F}^{r-1}_{D}$, and every $u_1 \in \mathscr{U}^{r}_{D_1}$, there exists a $(u_2, \dots, u_I) \in \mathscr{U}^{r}_{(D_2,\dots,D_I)}$ such that if $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the solution of (1), then $f = f_\lambda$. Moreover, for every $(v_2, \dots, v_I) \in \mathscr{U}^{r}_{(D_2,\dots,D_I)}$, $(v_2, \dots, v_I)$ has the same property as this $(u_2, \dots, u_I)$ if and only if $u_i - v_i$ is constant for every $i \geq 2$.*

The second part of this corollary establishes the essential uniqueness of the other consumers' utility functions. The corollary, as a whole, says that although one consumer's utility function cannot at all be specified by the efficient risk-sharing rules, once it is specified, the other consumers' utility functions can be identified up to a scalar addition, which is completely irrelevant for the consumer choice.

Proof of Corollary 3 As in the statement of this corollary, let $I \in \boldsymbol{Z}_{++}$ with $I \geq 2$, $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, $D \in \mathscr{D}^I$, $\lambda \in \boldsymbol{R}^I_{++}$, $f \in \mathscr{F}^{r-1}_{D}$, and $u_1 \in \mathscr{U}^{r}_{D_1}$. Let $d \in \sum_i D_i$. Define $u : \sum_i D_i \to \boldsymbol{R}$

by

$$u(x) = \lambda_1 \int_d^x u_1' (f_1(z))\, dz \tag{5}$$

for every $x \in \sum_i D_i$. Since $u_1' \circ f_1$ is of class C^{r-1}, u is of class C^r. Since

$$u'(x) = \lambda_1 u_1' (f_1(x)) \tag{6}$$

for every $x \in \sum_i D_i$, u_1 satisfies Conditions INC and INADA, and f_1 satisfies Conditions COMONO and ONTO, u satisfies Conditions INC and INADA. Moreover,

$$u''(x) = \lambda_1 u_1'' (f_1(x))\, f' (x) < 0 \tag{7}$$

for every $x \in \sum_i D_i$. Thus u satisfies Condition CONC, and $u \in \mathscr{U}^r_{\sum_i D_i}$.

By Theorem 2, therefore, there exists a $(\hat{u}_1, \dots, \hat{u}_I) \in \mathscr{U}^r_D$ such that if $\hat{f}_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the solution and $\hat{u}_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the value function of (1) with the u_i replaced by the $\hat{u}_i$, then $f = \hat{f}_\lambda$ and $u = \hat{u}_\lambda$.

By the envelope theorem and (5),

$$\lambda_1 \hat{u}_1' (f_1(x)) = u'(x) = \lambda_1 u_1' (f_1(x))$$

for every $x \in \sum_i D_i$. Since $\lambda_1 > 0$ and f_1 satisfies ONTO, this implies that $u_1 - \hat{u}_1$ is constant. Denote this value by α. For each $i \geq 2$, define $u_i : D_i \to \boldsymbol{R}$ by

$$u_i(x_i) = \hat{u}_i(x_i) - \frac{\lambda_1}{\lambda_i} \frac{\alpha}{I - 1}$$

for every $x_i \in D_i$. Then $u_i - \hat{u}_i$ is constant for every i and $\sum_{i \geq 1} \lambda_i (u_i - \hat{u}_i) = 0$. By Theorem 2, therefore, if $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is the solution and $u_\lambda : \sum_i D_i \to \boldsymbol{R}$ is the value function of (1) with this $(u_1, u_2, \dots, u_I)$, then $f_\lambda = \hat{f}_\lambda = f$ and $u_\lambda = \hat{u}_\lambda = u$.

The If-Part of the second part of this corollary follows from the fact that the solution to (1) is unaffected by any scalar addition to u_i. As for its Only-If part, assume that $(v_2, \dots, v_I)$ has the same property as $(u_2, \dots, u_I)$. Denote by $v_\lambda : \sum_i D_i \to \boldsymbol{R}$ the value function of (1) with

$(u_2, \ldots, u_I)$ replaced by $(v_2, \ldots, v_I)$. Then, by the envelope theorem and (5),

$$v_\lambda(x) = \lambda_1 u_1' \left(f_1(x)\right) = u_\lambda'(x)$$

for every $x \in \sum_i D_i$. Thus $u_\lambda - v_\lambda$ is constant. Again by the envelope theorem,

$$v_i' \left(f_i(x)\right) - u_i' \left(f_i(x)\right) = \frac{1}{\lambda_i} \left(v_\lambda'(x) - u_\lambda'(x)\right) = 0$$

for every $i \geq 2$ and $x \in \sum_i D_i$. Since f_i satisfies Condition ONTO, this implies that $u_i - v_i$ is constant. ///

Corollary 3 can of course be modified for the case of $f \in \mathscr{F}_D^\infty$ and $u_1 \in \mathscr{U}_{D_1}^\infty$, but we shall not give its formal statement here to save space.

4.3 Marginal Utility Functions

In some cases, for example when we deal with utility functions of hyperbolic absolute risk version, it is more convenient to deal with marginal utility functions, rather than the utility functions themselves, as they possess a more tractable aggregation property and the marginal utility function of the representative consumer is nothing but a state-price deflator. We shall now provide an alternative formulation of the risk-sharing rules and the representative consumer's utility function in terms of marginal utility functions.

For each $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$ and each $D_i \in \mathscr{D}$, let $\mathscr{M}_{D_i}^{r-1}$ be the set of all functions $\pi_i : D_i \to \boldsymbol{R}_{++}$ that satisfy the following three conditions:

$\boldsymbol{C^{r-1}}$ π_i is of class C^{r-1}

DEC $\pi_i' \left(x_i\right) < 0$ for every $x_i \in D_i$

INADA $\pi_i(x_i) \to \infty$ as $x_i \to \inf D_i$ and $\pi_i(x_i) \to 0$ as $x_i \to \sup D_i$

Then, for every C^2 function $u_i : D_i \to \boldsymbol{R}$, $u_i \in \mathscr{U}_{D_i}^r$ if and only if $u_i' \in \mathscr{M}_{D_i}^{r-1}$. For $D = (D_1, \ldots, D_I) \in \mathscr{D}^I$, define $\mathscr{M}_D^{r-1} = \mathscr{M}_{D_1}^{r-1} \times \cdots \times \mathscr{M}_{D_I}^{r-1}$. Then, specifying an element of $\mathscr{M}_D^{r-1}$ is equivalent to an element of $\mathscr{U}_D^r$ up to scalar addition to each component function. Define $\mathscr{M}_{D_i}^\infty = \bigcap_{r=2}^\infty \mathscr{M}_{D_i}^{r-1}$ and $\mathscr{M}_D^\infty = \bigcap_{r=2}^\infty \mathscr{M}_D^{r-1}$.

Theorem 1 shows that if $(u_1', \dots, u_I') \in \mathscr{M}_D^{r-1}$, then $u_\lambda' \in \mathscr{M}_{\sum_i D_i}^{r-1}$. Note here that by the envelope theorem, $u_\lambda'(x) = \lambda_i u_i'(f_{\lambda i}(x))$ for every i and $x \in \sum_i D_i$, where $f_\lambda \in \mathscr{F}_D^{r-1}$ is the risk-sharing rule giving the solution to (1). Let $\pi = u_i'$ and $\pi_\lambda = u_\lambda'$, we can rewrite this as

$$\pi_\lambda(x) = \lambda_i \pi_i (f_{\lambda i}(x)) \tag{8}$$

for every $x \in \sum_i D_i$.

Now let $x \in \sum_i D_i$ and $z \in \boldsymbol{R}_{++}$ satisfy $z = u_\lambda'(x)$. Then $x = (u_\lambda')^{-1}(z)$ and

$$f_{\lambda i}(x) = (u_i')^{-1}\left(\frac{z}{\lambda_i}\right).$$

By Condition SUM,

$$(u')^{-1}(z) = \sum_i (u_i')^{-1}\left(\frac{z}{\lambda_i}\right),$$

or, equivalently,

$$\pi_\lambda^{-1}(z) = \sum_i \pi_i^{-1}\left(\frac{z}{\lambda_i}\right) \tag{9}$$

for every $z > 0$. This is the relationship that directly relate the individual consumers' marginal utility functions to the representative consumer's counterpart.

Theorems 1 and 2 can then be restated as follows.

Corollary 4 *For every $I \in \boldsymbol{Z}_{++}$, every $D = (D_1, \dots, D_I) \in \mathscr{D}^I$, every $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, every $(\pi_1, \dots, \pi_I) \in \mathscr{M}_D^{r-1}$, and every $\lambda \in \boldsymbol{R}_{++}^I$, if $\pi_\lambda : \sum_i D_i \to \boldsymbol{R}$ is defined by (9) and $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is then defined by (8), then $f_\lambda \in \mathscr{F}_D^{r-1}$ and $\pi_\lambda \in \mathscr{M}_{\sum_i D_i}^{r-1}$.*

Corollary 5 *For every $I \in \boldsymbol{Z}_{++}$, every $D = (D_1, \dots, D_I) \in \mathscr{D}^I$, every $r \in \boldsymbol{Z}_{++}$ with $r \geq 2$, every $\lambda \in \boldsymbol{R}_{++}^I$, every $f \in \mathscr{F}_D^{r-1}$, and every $\pi \in \mathscr{M}_{\sum_i D_i}^{r-1}$, there exists a unique $(\pi_1, \dots, \pi_I) \in \mathscr{M}_D^{r-1}$ such that if $\pi_\lambda : \sum_i D_i \to \boldsymbol{R}$ is defined by (9) and $f_\lambda : \sum_i D_i \to D_1 \times \cdots \times D_I$ is then defined by (8), then $f = f_\lambda$ and $u = u_\lambda$.*

5 The Case of Hyperbolic Absolute Risk Aversion

In this section we consider the special case in which the consumers exhibit *hyperbolic absolute risk aversion*. Formally, for a utility function $u_i \in \mathscr{U}_{D_i}^2$ on $D_i \in \mathscr{D}$, the *absolute risk aversion*

$a_i : D_i \to \boldsymbol{R}_{++}$ is defined by $a_i(x_i) = -u_i''(x_i)/u_i'(x_i)$ for every $x_i \in D_i$, and the absolute risk aversion a_i is *hyperbolic* if there exist a $\gamma_i \in \boldsymbol{R}$ and a $\tau_i \in \boldsymbol{R}$ such that $a_i(x_i) = (\gamma_i x_i + \tau_i)^{-1}$ for every $x_i \in D_i$. The constant γ_i is referred to as the *cautiousness* of u_i. When the domain $D_i = \left(\underline{d}_i, \overline{d}_i\right)$ is determined so that $\gamma_i x_i + \tau_i > 0$ if and only if $x_i \in D_i$, u_i satisfies Condition INADA. In fact, this is the only choice of D_i for which the utility function u_i exhibiting hyperbolic absolute risk aversion satisfies Condition INADA. If D_i is chosen in this way and if the cautiousness γ_i is strictly positive, then $\underline{d}_i = -\tau_i/\gamma_i > -\infty$ and $\overline{d}_i = \infty$. Conversely, if $\underline{d}_i > -\infty$, $D_i = (\underline{d}_i, \infty)$, and $u_i \in \mathscr{U}^2_{D_i}$ exhibits hyperbolic absolute risk aversion $a_i(x_i) = (\gamma_i x_i + \tau_i)^{-1}$, then $\gamma_i > 0$ and $\tau_i = -\gamma_i \underline{d}_i$. In short, each utility function u_i satisfying Condition INADA and exhibiting hyperbolic absolute risk aversion with a strictly positive cautiousness can be completely identified with the value of cautiousness, γ_i, and the minimum subsistence level, $\underline{d}_i$. In this case, then,

$$a_i\left(x_i\right) = \frac{1}{\gamma_i\left(x_i - \underline{d}_i\right)} \tag{10}$$

for every $x_i > \underline{d}_i$. The *relative risk aversion* $b_i(x_i)$ is defined by $b_i(x_i) = a_i(x_i)x_i$ for $x_i > 0$. Then

$$b_i\left(x_i\right) = \frac{1}{\gamma_i}\frac{x_i}{x_i - \underline{d}_i}.$$

Hence u_i exhibits constant relative risk aversion γ_i^{-1} if $\underline{d}_i = 0$, strictly decreasing relative risk aversion if $\underline{d}_i > 0$, and strictly increasing relative risk aversion if $\underline{d}_i < 0$.

We saw earlier that if $I \in \boldsymbol{Z}_{++}$, $r \in \boldsymbol{Z}_{++}$, $D = (D_1, \ldots, D_I) \in \mathscr{D}^I$, $(u_1, \ldots, u_I) \in \mathscr{U}^r_D$, and $\lambda = (\lambda_1, \ldots, \lambda_I) \in \boldsymbol{R}^I_{++}$, then the value function of (1), u_λ, belongs to $\mathscr{U}^r_{\sum_i D_i}$. For each $r \geq 2$, write $\mathscr{U}^r = \bigcup_{D \in \mathscr{D}} \mathscr{U}^r_D$. We can then define a mapping

$$R : \bigcup_{I \in \boldsymbol{Z}_{++}} \big(\underbrace{\mathscr{U}^2 \times \cdots \times \mathscr{U}^2}_{I\text{-times}} \times \boldsymbol{R}^I_{++}\big) \to \mathscr{U}^2$$

by letting $R\left(u_1, \ldots, u_I; \lambda\right) = u_\lambda$. Then, if $u_i \in \mathscr{U}^r_{D_i}$ for every i, then $R\left(u_1, \ldots, u_I; \lambda\right) \in \mathscr{U}^r_{\sum_i D_i}$.

We also introduce a similar operation on $\mathscr{M}^{r-1}_D$. Specifically, for each $r \geq 2$, write $\mathscr{M}^{r-1} = \bigcup_{D \in \mathscr{D}} \mathscr{M}^{r-1}_D$. Then define

$$Q : \bigcup_{I \in \boldsymbol{Z}_{++}} \big(\underbrace{\mathscr{M}^1 \times \cdots \times \mathscr{M}^1}_{I\text{-times}} \times \boldsymbol{R}^I_{++}\big) \to \mathscr{M}^1$$

by letting $\pi = Q(\pi_1, \ldots, \pi_I; \lambda)$ satisfy

$$\pi^{-1}(z) = \sum_i \pi_i^{-1}\left(\frac{z}{\lambda_i}\right) \tag{11}$$

for every $z \in \boldsymbol{R}_{++}$. Then, if $\pi_i \in \mathscr{M}^{r-1}_{D_i}$ for every i, then $R(\pi_1, \ldots, \pi_I; \lambda) \in \mathscr{M}^{r-1}_{\sum_i D_i}$. The relationship between R and Q is given by the following proposition.

Proposition 1 *Let $(u_1, \ldots, u_I) \in \mathscr{U}^r_D$, $u \in \mathscr{U}^r_{\sum_i D_i}$, and $\lambda \in \boldsymbol{R}^I_{++}$. If $u = R(u_1, \ldots, u_I; \lambda)$, then $u' = Q(u'_1, \ldots, u'_I; \lambda)$. Conversely, if $u' = Q(u'_1, \ldots, u'_I; \lambda)$, then $u - R(u_1, \ldots, u_I; \lambda)$ is constant.*

Proof of Proposition 1 The first part can be shown as in Section 4.3. Suppose that $u' = Q(u'_1, \ldots, u'_I; \lambda)$. Write $u_\lambda = R(u_1, \ldots, u_I; \lambda)$, then we can show, as before, that

$$\left(u'_\lambda\right)^{-1}(z) = \sum_i \left(u'_i\right)^{-1}\left(\frac{z}{\lambda_i}\right).$$

By assumption,

$$\sum_i \left(u'_i\right)^{-1}\left(\frac{z}{\lambda_i}\right) = \left(u'\right)^{-1}(z).$$

Thus $(u')^{-1} = (u'_\lambda)^{-1}$, and hence $u' = u'_\lambda$. Thus $u - u_\lambda$ is constant. ///

We now introduce the concept of closedness under aggregation.

Definition 1 A subset $\mathscr{W}$ of $\mathscr{U}^2$ is *closed under aggregation* if $I \in \boldsymbol{Z}_{++}$, $u_i \in \mathscr{W}$ for every $i = 1, \ldots, I$, and $\lambda \in \boldsymbol{R}^I_{++}$, then $R(u_1, \ldots, u_I; \lambda) \in \mathscr{W}$.

We can give an analogous definition for marginal utility functions.

Definition 2 A subset $\mathscr{L}$ of $\mathscr{M}^1$ is *closed under aggregation* if $I \in \boldsymbol{Z}_{++}$, $\pi_i \in \mathscr{L}$ for every $i = 1, \ldots, I$, and $\lambda \in \boldsymbol{R}^I_{++}$, then $Q(\pi_1, \ldots, \pi_I; \lambda) \in \mathscr{L}$.

Elements of $\mathscr{M}^1$ of particular interest are the sums of power functions with negative exponents. Formally, for each $d \in \boldsymbol{R}$ and each $N \in \boldsymbol{Z}_{++}$, let $\mathscr{K}^N_d$ be the set of all $\pi \in \mathscr{M}^\infty_{(d,\infty)}$ for which there exist a $c \in \boldsymbol{R}^N_{++}$ and a $p \in \boldsymbol{R}^N_{++}$ such that

$$\pi^{-1}(z) = \sum_{n=1}^N c_n z^{-p_n} + d \tag{12}$$

for every $z > 0$. Indeed, since the right hand side is an infinitely many times differentiable function of $z \in \boldsymbol{R}_{++}$ that has strictly negative first derivatives and ranges from ∞ to d, the inverse function theorem guarantees that $\pi \in \mathscr{M}^{\infty}_{(d,\infty)}$. Note that if π belongs to $\mathscr{K}^N_d$, then any positive multiplication of π also belongs to $\mathscr{K}^N_d$, and also that $\pi \in \mathscr{K}^N_0$ if and only if there exist a $c \in \boldsymbol{R}^N_{++}$ and a $p \in \boldsymbol{R}^N_{++}$ such that

$$\pi^{-1}(z) = \sum_{n=1}^{N} c_n z^{-p_n} \tag{13}$$

for every $z > 0$. Thus, for every $\pi \in \mathscr{M}^{\infty}_{(d,\infty)}$, $\pi \in \mathscr{K}^N_d$ if and only if $\hat{\pi} \in \mathscr{K}^N_0$, where $\hat{\pi}$ is defined by $\hat{\pi}(y) = \pi(y + d)$ for every $y > 0$.

Since the p_n are allowed to take equal values, $\mathscr{K}^N_d \subset \mathscr{K}^{N+1}_d$ for every $d \in \boldsymbol{R}_{++}$ and every $N \in \boldsymbol{Z}_{++}$. For each $d \in \boldsymbol{R}$, define $\mathscr{K}_d = \bigcup_{N=1}^{\infty} \mathscr{K}^N_d$. For each $N \in \boldsymbol{Z}_{++}$, define $\mathscr{K}^N = \bigcup_{d\in\boldsymbol{R}} \mathscr{K}^N_d$. Define $\mathscr{K} = \bigcup_{N=1}^{\infty} \bigcup_{d\in\boldsymbol{R}} \mathscr{K}^N_d$.

We also give analogous definitions for $\mathscr{U}^2$. For each $d \in \boldsymbol{R}$ and each $N \in \boldsymbol{Z}_{++}$, define $\mathscr{V}^N_d$ the set of all $u \in \mathscr{U}^{\infty}_{(d,\infty)}$ such that $u' \in \mathscr{K}^N_d$. Then $\mathscr{V}^N_d \subset \mathscr{V}^{N+1}_d$ because $\mathscr{K}^N_d \subset \mathscr{K}^{N+1}_d$, for every N. For each $d \in \boldsymbol{R}$, define $\mathscr{V}_d = \bigcup_{N=1}^{\infty} \mathscr{V}^N_d$. For each $N \in \boldsymbol{Z}_{++}$, define $\mathscr{V}^N = \bigcup_{d\in\boldsymbol{R}} \mathscr{V}^N_d$. Define $\mathscr{V} = \bigcup_{N=1}^{\infty} \bigcup_{d\in\boldsymbol{R}} \mathscr{V}^N_d$. If u belongs to $\mathscr{V}^N_d$, then any positive multiplication and any scalar addition of u also belongs to $\mathscr{V}^N_d$. Note that for every $u \in \mathscr{U}^2$, u exhibits hyperbolic absolute risk aversion with a positive cautiousness if and only if $u \in \mathscr{V}^1$; and u exhibits constant relative risk aversion if and only if $u \in \mathscr{V}^1_0$.

To familiarize ourselves with the notion of closedness under aggregation, we can paraphrase the mutual fund theorem as follows:

Theorem 3 (Mutual Fund Theorem)

1. *For every $p \in \boldsymbol{R}_{++}$, the set of all $u \in \mathscr{U}^2$ for which there exist a $d \in \boldsymbol{R}$ and a $c \in \boldsymbol{R}_{++}$ such that $u \in \mathscr{U}^2_{(d,\infty)}$ and $(u')^{-1}(z) = cz^{-p} + d$ for every $z \in \boldsymbol{R}_{++}$ is closed under aggregation.*

2. *For every $p \in \boldsymbol{R}_{++}$, the set of all $u \in \mathscr{U}^2_{\boldsymbol{R}_{++}}$ for which there exists a $c \in \boldsymbol{R}_{++}$ such that $(u')^{-1}(z) = cz^{-p}$ for every $z \in \boldsymbol{R}_{++}$ is closed under aggregation.*

The first part of the above theorem is the mutual fund theorem for the consumers exhibiting hyperbolic absolute risk aversion with the common cautiousness p. The second part is the mutual fund theorem for the special case where the consumers has the same constant relative risk aversion p^{-1}. Our own first results are the following.

Proposition 2 *1. For every $u \in \mathscr{V}$, there exist an $I \in \boldsymbol{Z}_{++}$, a $u_i \in \mathscr{V}^1$ for each $= 1, \ldots, I$, and a $\lambda \in \boldsymbol{R}_{++}$ such that $u = R(u_1, \ldots, u_I; \lambda)$.*

2. For every $u \in \mathscr{V}_0$, there exist an $I \in \boldsymbol{Z}_{++}$, a $u_i \in \mathscr{V}_0^1$ for each $= 1, \ldots, I$, and a $\lambda \in \boldsymbol{R}_{++}$ such that $u = R(u_1, \ldots, u_I; \lambda)$.

Proposition 3 *1. For every $\pi \in \mathscr{K}$, there exist an $I \in \boldsymbol{Z}_{++}$, a $\pi_i \in \mathscr{K}^1$ for each $= 1, \ldots, I$, and a $\lambda \in \boldsymbol{R}_{++}$ such that $\pi = Q(\pi_1, \ldots, \pi_I; \lambda)$.*

2. For every $\pi \in \mathscr{K}_0$, there exist an $I \in \boldsymbol{Z}_{++}$, a $\pi_i \in \mathscr{K}_0^1$ for each $= 1, \ldots, I$, and a $\lambda \in \boldsymbol{R}_{++}$ such that $\pi = Q(\pi_1, \ldots, \pi_I; \lambda)$.

The first part of Proposition 2 states that every utility function, of which the inverse function of marginal utilities is a sum of power functions possibly with a constant term, can be the utility function for the representative consumer of the economy consisting of individual consumers exhibiting hyperbolic absolute risk aversion. The second part deals with the special case of constant relative risk aversion: If no constant term is added to the sum of power functions, then we can find individual consumers exhibiting constant relative risk aversion. Proposition 3 gives the same results in term of marginal utilities and a state-pricing rule.

Proof of Proposition 3 1. let $I \in \boldsymbol{Z}_{++}$ and $d \in \boldsymbol{R}$ be such that $\pi \in \mathscr{V}_d^I$. Then let $(c_1, \ldots, c_I) \in \boldsymbol{R}_{++}^I$ and $(p_1, \ldots, p_I) \in \boldsymbol{R}_{++}^I$ be such that

$$\pi^{-1}(z) = \sum_{i=1}^{I} c_i z^{-p_i} + d$$

for every $z \in \boldsymbol{R}_{++}$. For each i, define $\pi_i \in \mathscr{V}_{d/I}^1$ by letting

$$\pi_i^{-1}(z) = \left(c_i \lambda_i^{-p_i}\right) z^{-p_i} + \frac{d}{I}$$

for every $z \in \boldsymbol{R}_{++}$. Then

$$\sum_i \pi_i^{-1}\left(\frac{z}{\lambda_i}\right) = \sum_i \left(c_i \lambda_i^{-p_i}\right)\left(\frac{z}{\lambda_i}\right)^{-p_i} + \sum_i \frac{d}{I} = \pi^{-1}(z).$$

2. This part can be established by replacing the d_i and d by 0 in the proof of part 1. ///

Proof of Proposition 2 This theorem follows from Theorem 5 and Proposition 3. ///

It is convenient to state Proposition 2 by explicitly referring to the representative consumer's utility function.

Corollary 6 *1. For every $u \in \mathscr{V}$, there exist an $I \in \boldsymbol{Z}_{++}$ and a $(u_1 \dots, u_I) \in \mathscr{U}^2 \times \dots \times \mathscr{U}^2$ such that u_i exhibits hyperbolic absolute risk aversion for every i, and $u_\lambda = u$, where u_λ is the value function of (1).*

2. For every $u \in \mathscr{V}_0$, there exist an $I \in \boldsymbol{Z}_{++}$ and a $(u_1 \dots, u_I) \in \mathscr{U}_0^2 \times \dots \times \mathscr{U}_0^2$ such that u_i exhibits constant relative risk aversion for every i, and $u_\lambda = u$, where u_λ is the value function of (1).

The main theorem of this section is the following.

Theorem 4 *1. The set $\mathscr{V}$ is the smallest subset of $\mathscr{U}^2$ that is closed under aggregation and includes $\mathscr{V}^1$.*

2. The set $\mathscr{V}_0$ is the smallest subset of $\mathscr{U}^2$ that is closed under aggregation and includes $\mathscr{V}_0^1$.

The above result will be derived from the corresponding result for the set $\mathscr{K}^1$.

Theorem 5 *1. The set $\mathscr{K}$ is the smallest subset of $\mathscr{M}^1$ that is closed under aggregation and includes $\mathscr{K}^1$.*

2. The set $\mathscr{K}_0$ is the smallest subset of $\mathscr{M}^1$ that is closed under aggregation and includes $\mathscr{K}_0^1$.

Proof of Theorem 5 1. To show that $\mathscr{V}$ is closed under aggregation, let $I \in \boldsymbol{Z}_{++}$, $\pi_i \in \mathscr{V}$ for every $i = 1, \dots, I$, and $\lambda \in \boldsymbol{R}^I_{++}$. For each i, let $N_i \in \boldsymbol{Z}_{++}$ and $d_i \in \boldsymbol{R}$ be such that $\pi \in \mathscr{V}^{N_i}_{d_i}$. Then let $(c_{i1}, \dots, c_{iN_i}) \in \boldsymbol{R}^{N_i}_{++}$ and $(p_{i1}, \dots, p_{iN_i}) \in \boldsymbol{R}^{N_i}_{++}$ be such that

$$\pi_i^{-1}(z) = \sum_{n=1}^{N_i} c_{in} z^{-p_{in}} + d_i$$

for every $z \in \boldsymbol{R}_{++}$. Then

$$Q\left(\pi_1, \ldots, \pi_I; \lambda\right) = \sum_{i=1}^{I} \left(\sum_{n=1}^{N_i} c_{in} \left(\frac{z}{\lambda_i} \right)^{-p_{in}} + d_i \right) = \sum_{i=1}^{I} \sum_{n=1}^{N_i} \left(c_{in} \lambda_i^{p_{in}} \right) z^{-p_{in}} + \sum_{i=1}^{I} d_i.$$

Thus $Q\left(\pi_1, \ldots, \pi_I; \lambda\right) \in \mathscr{K}_{\sum_i d_i}^{\sum_i N_i} \subset \mathscr{K}$.

It follows from Proposition 3 that every subset of $\mathscr{U}^2$ that is closed under aggregation and includes $\mathscr{V}^1$ also includes $\mathscr{V}$. Since $\mathscr{V}$ is itself closed under aggregation, it is the smallest subset of $\mathscr{U}^2$ that is closed under aggregation and includes $\mathscr{V}^1$

2. This part can be established by replacing the d_i and d by 0 in the proof of part 1. ///

Proof of Theorem 4 This theorem follows from Theorem 5 and Proposition 1. ///

The following proposition establishes the essential uniqueness of individual consumers' utility functions and risk-sharing rules for a given utility function for the representative consumer, when the individual consumers exhibit hyperbolic absolute risk aversion.

Proposition 4 *1. Let $N \in \boldsymbol{Z}_{++}$, $d \in \boldsymbol{R}$, and $(p, c) \in \boldsymbol{R}_{++}^N \times \boldsymbol{R}_{++}^N$. Assume that $p_1, \ldots, p_N$ are all distinct and define $\pi \in \mathscr{K}_d^N$ by (12). Let $I \in \boldsymbol{Z}_{++}$, $(\gamma, \underline{d}) \in \boldsymbol{R}_{++}^I \times \boldsymbol{R}^I$, and, for each i, $u_i \in \mathscr{U}_{(\underline{d}_i, \infty)}^{\infty}$ satisfy (10). Let $\lambda \in \boldsymbol{R}_{++}^I$ and $f_\lambda \in \mathscr{F}_{((\underline{d}_1, \infty), \ldots, (\underline{d}_I, \infty))}^{\infty}$ be the efficient risk-sharing rule giving the solution to (1) and $u_\lambda \in \mathscr{U}_{(\sum_i \underline{d}_i, \infty)}^{\infty}$ be its value function, that is, $u_\lambda = R\left(u_1, \ldots, u_I; \lambda\right)$. If $d = \sum_i \underline{d}_i$ and $u_\lambda'(x)/\pi(x)$ does not depend on $x > d$, then $\{\gamma_1, \ldots, \gamma_I\} = \{p_1, \ldots, p_N\}$. Moreover, for each i, there exists a $\theta_i > 0$ such that $\sum_{\{i | \gamma_i = p_n\}} \theta_i = 1$ for every $n = 1, \ldots, N$ and*

$$f_{\lambda i}(x) = \theta_i c_n \left(\pi\left(x\right)\right)^{-p_n} + \underline{d}_i \tag{14}$$

for every i and n with $\gamma_i = p_n$ and every $x > d$.

2. Let $N \in \boldsymbol{Z}_{++}$ and $(p, c) \in \boldsymbol{R}_{++}^N \times \boldsymbol{R}_{++}^N$. Assume that $p_1, \ldots, p_N$ are all distinct and define $\pi \in \mathscr{K}_0^N$ by (13). Let $I \in \boldsymbol{Z}_{++}$, $\gamma \in \boldsymbol{R}_{++}^I$, and, for each i, $u_i \in \mathscr{U}_{\boldsymbol{R}_{++}}^{\infty}$ satisfy (10) with $\underline{d}_i = 0$. Let $\lambda \in \boldsymbol{R}_{++}^I$ and $f_\lambda \in \mathscr{F}_{(\boldsymbol{R}_{++}, \ldots, \boldsymbol{R}_{++})}^{\infty}$ be the efficient risk-sharing rule giving the solution to (1) and $u_\lambda \in \mathscr{U}_{\boldsymbol{R}_{++}}^{\infty}$ be its value function, that is, $u_\lambda = R\left(u_1, \ldots, u_I; \lambda\right)$. If $u_\lambda'(x)/\pi(x)$ does not depend on $x > 0$, then $\{\gamma_1, \ldots, \gamma_I\} = \{p_1, \ldots, p_N\}$. Moreover, for *each i, there exists a $\theta_i > 0$ such that $\sum_{\{i|\gamma_i = p_n\}} \theta_i = 1$ for every $n = 1, \ldots, N$ and*

$$f_{\lambda i}(x) = \theta_i c_n \left(\pi\left(x\right)\right)^{-p_n}$$

for every i and n with $\gamma_i = p_n$ and every $x > 0$.

The meaning of uniqueness of Proposition 4 can be explained as follows. Suppose that we are given a weighted sum of power functions with a constant term, just as (12). As Proposition 2 shows, there is an economy consisting of consumers exhibiting hyperbolic absolute risk aversion and giving rise to (12) as the representative consumer's marginal utility, or the state-pricing rule. The cautiousness of each consumer in this economy must then be one of the exponents p_n of (12), and each such exponent is equal to the cautiousness of at least one consumer in the economy. In other words, the exponents identify all the cautiousness present in the economy. The coefficients c_n of (12), on the other hand, identify the wealth shares, or the utility weights in the utilitarian welfare maximization problem (1), of the consumers. As can be seen from (12), the coefficients do not affect the elasticity of the individual consumers' consumption levels, in excess of the minimum subsistence levels, as a function of state prices, but they affect their slopes. In fact, if two consumers have the same cautiousness, their consumption levels are linear functions of each other, and the slopes of these linear functions are determined by the wealth shares between the two. Finally, we should point out that Proposition 4 also shows where a specification of the state-pricing rule (12) admits multiplicity in specifications of individual consumers' characteristics. First, we cannot pin down how the wealth is distributed across consumers of equal cautiousness. Second, we cannot pin down the individual consumers' minimum subsistence levels, except that they add up to the constant term of the given state-pricing rule.

Proof of Proposition 4 1. Let I, $(\gamma, \underline{d})$, and λ be as in the statement of this proposition. Then there exists a $\mu \in \boldsymbol{R}^I_{++}$ such that $\lambda_i u_i'(x_i) = \mu_i \left(x_i - \underline{d}_i\right)^{-1/\gamma_i}$ for every I and $x_i > \underline{d}_i$. Define $\hat{\pi} \in \mathscr{K}^I_{\sum_i \underline{d}_i}$ by

$$\hat{\pi}^{-1}(z) = \sum_{i=1}^{I} \mu_i^{\gamma_i} z^{-\gamma_i} + \sum_{i=1}^{I} \underline{d}_i$$

for every $z \in \boldsymbol{R}_{++}$, that is,

$$x = \sum_{i=1}^{I} \mu_i^{\gamma_i} \left(\hat{\pi}(x)\right)^{-\gamma_i} + \sum_{i=1}^{I} \underline{d}_i \tag{15}$$

for every $x > \sum_i \underline{d}_i$. Then define $f \in \mathscr{F}^{\infty}_{((\underline{d}_1,\infty),\ldots,(\underline{d}_I,\infty))}$ by

$$f_i(x) = \mu_i^{\gamma_i} \left(\hat{\pi}(x)\right)^{-\gamma_i} + \underline{d}_i$$

for every i and every $x > d$. It is easy to check that f in fact belongs to $\mathscr{F}^{\infty}_{((\underline{d}_1,\infty),\ldots,(\underline{d}_I,\infty))}$. In particular, Condition SUM follows from (15). Moreover,

$$\lambda_i u_i' \left(f_i(x)\right) = \mu_i \left(\mu_i^{\gamma_i} \left(\hat{\pi}(x)\right)^{-\gamma_i}\right)^{-1/\gamma_i} = \hat{\pi}(x)$$

for every i and $x > d$. Since the far right hand side does not depend on i, this implies that f gives the solution to (1), that is, $f = f_\lambda$. By the envelope theorem,

$$u_\lambda'(x) = \lambda_i u_i' \left(f_{\lambda i}(x)\right) = \lambda_i u_i' \left(f_i(x)\right) = \hat{\pi}(x).$$

Since $u_\lambda'(x)/\pi(x)$ does not depend on x, this means that $\pi(x)/\hat{\pi}(x)$ does not depend on x either. Denote this constant value by α. Then $\pi^{-1}(z) = \hat{\pi}^{-1}\left(\alpha^{-1}z\right)$ for every $z \in \boldsymbol{R}_{++}$. Note now that

$$\begin{aligned} \pi^{-1}(z) &= \sum_{n=1}^{N} c_n z^{-p_n} + d, \\ \hat{\pi}^{-1}(\alpha^{-1}z) &= \sum_{i=1}^{I} \left(\alpha\mu_i\right)^{\gamma_i} z^{-\gamma_i} + \sum_{i=1}^{I} \underline{d}_i \end{aligned}$$

for every $z \in \boldsymbol{R}_{++}$. Since $d = \sum_{i=1}^{I} \underline{d}_i$, $\sum_{n=1}^{N} c_n z^{-p_n} = \sum_{i=1}^{I} \left(\alpha\mu_i\right)^{\gamma_i} z^{-\gamma_i}$ for every $z \in \boldsymbol{R}_{++}$. This means that $\{\gamma_1, \ldots, \gamma_I\} = \{p_1, \ldots, p_N\}$. Moreover, since $p_1, \ldots, p_N$ are all distinct,

$$\sum_{\{i|\gamma_i = p_n\}} \left(\alpha\mu_i\right)^{\gamma_i} = c_n$$

for every n. For each i, let n be the unique n such that $\gamma_i = p_n$, and then define $\theta_i = \left(\alpha\mu_i\right)^{\gamma_i} c_n^{-1}$.

Then $\theta_i > 0$ and $\sum_{\{i|\gamma_i=p_n\}} \theta_i = 1$ for every n. Moreover,

$$\begin{aligned} f_{\lambda i}(x) &= \mu_i^{\gamma_i} \left(\hat{\pi}(x)\right)^{-\gamma_i} + \underline{d}_i \\ &= \mu_i^{\gamma_i} \left(\alpha^{-1}\pi(x)\right)^{-\gamma_i} + \underline{d}_i \\ &= (\alpha\mu_i)^{\gamma_i} \left(\pi(x)\right)^{-\gamma_i} + \underline{d}_i \\ &= \theta_i c_n \left(\pi(x)\right)^{-p_n} + \underline{d}_i \end{aligned}$$

for every i and $x > d$.

2. This part can be shown by replacing d and $\underline{d}_i$ by 0 in the proof of part 1. ///

The main results of this section can now be summarised as follows. First, the utility functions of which the inverse functions of marginal utilities is a weighted sum of power functions, possibly with a constant term, provides a larger class of utility functions exhibiting hyperbolic absolute risk aversion and yet closed under aggregation (Theorem 4). Second, any utility function in this class can be the representative consumer's utility function of an economy comprising consumers of hyperbolic absolute risk aversion (Proposition 2). Third, once we know the representative consumer's utility function, we can recover the individual consumers' cautiousness and their wealth shares in the economy (Proposition 4). As for this last point, we should note the stark contrast between the case of hyperbolic absolute risk aversion and the general case of Theorem 2. While knowing the representative consumer's utility function in no way restricts the risk-sharing rules in the general case, it allows us to essentially identify the risk-sharing rules in the case of hyperbolic absolute risk aversion.

6 Examples

As an implication of Corollaries 1 and 2, we give two examples of the representative and individual consumers' utility functions and risk-sharing rules.

6.1 Linear Risk-Sharing Rules and Possibly Non-Constant Relative Risk Aversion

Our first example involves linear risk-sharing rules. We shows in a series of claims that the linearity of risk-sharing rules places no restriction on the representative and individual consumers' risk attitudes.

Let $D_i = \boldsymbol{R}_{++}$ and $\theta_i \in \boldsymbol{R}_{++}$ for every i with $\sum_i \theta_i = 1$. Define $f \in \mathscr{F}^{\infty}_{\boldsymbol{R}_{++}\times\cdots\times\boldsymbol{R}_{++}}$ by letting $f_i(x) = \theta_i x$ for every i and $x \in \boldsymbol{R}_{++}$. Let $u \in \mathscr{U}^2_{\boldsymbol{R}_{++}}$ and denote its relative risk aversion by $b : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$. Then, for each i, define a $u_i \in \mathscr{U}^2_{\boldsymbol{R}_{++}}$ so that its relative risk aversion $b_i : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ satisfies $b_i(f_i(x)) = b(x)$ for every $x \in \boldsymbol{R}_{++}$.[2] It follows from equality (2) of Lemma 1 of HHK (which follows from Wilson (1968)) that there exists a $(\lambda_1, \dots, \lambda_I) \in \boldsymbol{R}^I_{++}$ (and appropriate scalar additions to the u_i) such that $u = R(u_1, \dots, u_I; \lambda)$.

By the envelope theorem,

$$u'(x) = \lambda_i u_i'(\theta_i x) \tag{16}$$

for every i and $x \in \boldsymbol{R}_{++}$. By differentiating both sides with respect to x, we obtain

$$u''(x) = \lambda_i u_i''(\theta_i x)\theta_i \tag{17}$$

Divide both sides of (17) by their counterparts of (16) and then multiply $-x$, then we establish the following equality.

Claim $b(x) = b_i(\theta_i x)$ for every i and $x \in \boldsymbol{R}_{++}$.

We know from this equality that u exhibits constant relative risk aversion if and only if every u_i exhibits constant relative risk aversion. The same can be said of strictly decreasing relative risk aversion, nonincreasing relative risk aversion, nondecreasing relative risk aversion, and strictly increasing relative risk aversion. In short, the linearity of risk-sharing rules does not significantly restrict the representative and individual consumers' risk attitudes.

We can also note that if u exhibits constant relative risk aversion, then all the b_i are identical, and the u_i are affine transformations of one another. But this sort of identify cannot be obtained if u exhibits strictly decreasing or strictly increasing relative risk aversion. To see this point,

[2] It can be shown that since u satisfies the Inada condition, any u_i defined in this way also satisfies the Inada condition.

suppose that u exhibits strictly decreasing relative risk aversion and $\theta_i > \theta_j$; that is, consumer i has a larger share of the aggregate consumption than consumer j. Then, for each individual consumption level $z \in \boldsymbol{R}_{++}$, we have $\theta_i^{-1}z < \theta_i^{-1}z$ and hence $b_i(z) = b(\theta_i^{-1}z) > b(\theta_j^{-1}z) = b_j(z)$. Thus the richer consumer is strictly more risk averse than the poorer consumer. In contrast, if u exhibits strictly increasing relative risk aversion, then the richer consumer is less risk averse than the poorer consumer.

In fact, more can be said of the relationship between the representative consumer's risk attitudes and the equality of individual consumers' risk attitudes.

Claim If b is either nonincreasing or nondecreasing, $\theta_i \neq \theta_j$ for some i and j, and $b_1 = \cdots = b_I$, then $b_1, \ldots, b_I$, and b are constant functions taking the same value.

In other words, if the representative consumer has nonincreasing or nondecreasing relative risk aversion, and if all individual consumers have the same risk attitudes, then the representative consumer and all the individual consumers must exhibit constant relative risk aversion, sharing the same value for it, unless all consumers have the equal share of the aggregate consumption.

Proof of the Claim We shall prove that if b is nonincreasing and nonconstant, then the b_i cannot be identical. This means that if b is nonincreasing and the b_i are identical, then b must be constant, which implies, together with the linearity of f, that all the b_i are constant and equal. The case of a nondecreasing b can also be analogously proven.

So let's assume that b is nonincreasing and nonconstant. Then there exists an $x \in \boldsymbol{R}_{++}$ such that $b(x) > b(y)$ for every $y > x$. Since the θ_i are not all equal, there exist an i and a j such that $\theta_i > \theta_j$. Let $z = \theta_i x$, then

$$b_i(z) = b\left(\frac{z}{\theta_i}\right) = b(x) > b\left(\frac{\theta_i}{\theta_j}x\right) = b\left(\frac{z}{\theta_j}\right) = b_j(z)$$

This shows that b_i and b_j are not the same. ///

In the above claim, the assumption that b is nonincreasing and nonconstant is indispensable. In other words, if b is allowed to oscillate, then b need not exhibit constant relative risk aversion even when all the b_i are identical. The following proposition substantiates this assertion.

Proposition 5 *For every* $(\theta_1, \ldots, \theta_I) \in \boldsymbol{R}^I_{++}$ *with* $\sum_i \theta_i = 1$*, every* $(\overline{\beta}, \underline{\beta}) \in \boldsymbol{R}_{++} \times \boldsymbol{R}_{++}$ *with* $\overline{\beta} \geq \underline{\beta}$*, and every* $\varepsilon > 0$*, there exist a* $(\hat{\theta}_1, \ldots, \hat{\theta}_I) \in \boldsymbol{R}^I_{++}$ *with* $\sum_i \hat{\theta}_i = 1$*, a* $\hat{u} \in \mathscr{U}^\infty_{\boldsymbol{R}_{++}}$*, with its relative risk aversion* $\hat{b} : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$*, and a* $\lambda \in \boldsymbol{R}^I_{++}$ *having the following properties:*

1. $\max \hat{b}(\boldsymbol{R}_{++})$ *exists and equals* $\overline{\beta}$*, and* $\min \hat{b}(\boldsymbol{R}_{++})$ *exists and equals* $\underline{\beta}$*.*
2. $|\hat{\theta}_i - \theta_i| < \varepsilon$ *for every* i*.*
3. *Define* $\hat{f} \in \mathscr{F}^\infty_{\boldsymbol{R}_{++} \times \cdots \times \boldsymbol{R}_{++}}$ *by letting* $\hat{f}_i(x) = \hat{\theta}_i x$ *for every* i *and* $x \in \boldsymbol{R}_{++}$*. Then* $\hat{f}$ *gives the solution of (1) where each* u_i *is replaced by* $\hat{u}$*.*

In other words, for every linear risk-sharing rule, we can find another linear risk-sharing rule arbitrarily close to it that is generated from a collection of consumers having the identical risk attitudes. The need to use another, yet arbitrarily close, risk-sharing rule arises from an integer problem of the oscillation of relative risk aversion, to be made clear in the proof. The first part shows that the linearity of risk-sharing rules in no way restricts the range within which the relative risk aversion oscillates.

Proof of Proposition 5 Let $(\theta_1, \ldots, \theta_I) \in \boldsymbol{R}^I_{++}$ with $\sum_i \theta_i = 1$. For each $n \in \boldsymbol{Z}_{++}$ and i, define $k_i^n \in \boldsymbol{Z}$ so that

$$\exp \frac{2k_i^n \pi}{n} \leq \frac{\theta_i}{\theta_1} < \exp \frac{2(k_i^n + 1)\pi}{n}.$$

It is then easy to show that

$$\exp \frac{2k_i^n \pi}{n} \to \frac{\theta_i}{\theta_1}$$

as $n \to \infty$. Define

$$\tau^n = \sum_{i=1}^{I} \exp \frac{2k_i^n \pi}{n} \in \boldsymbol{R}_{++}.$$

Then $\tau^n \to 1/\theta_1$ as $n \to \infty$. For each n and i, define

$$\theta_i^n = \frac{\exp \dfrac{2k_i^n \pi}{n}}{\tau^n} \in \boldsymbol{R}_{++},$$

then $\sum_i \theta_i^n = 1$ for every n. Moreover,

$$\theta_i^n \to \frac{\theta_i/\theta_1}{1/\theta_1} = \theta_i$$

as $n \to \infty$. Define $f^n \in \mathscr{F}^\infty_{\boldsymbol{R}_{++}\times\cdots\times\boldsymbol{R}_{++}}$ by letting $f^n_i(x) = \theta^n_i x$ for every i and x. We will later put $\hat{\theta}_i = \theta^n_i$ and $\hat{f} = f^n$ for a sufficiently large n such that $|\theta^n_i - \theta_i| < \varepsilon$ for every i.

Let $(\overline{\beta}, \underline{\beta})$ be as stated in this proposition, and define $b^n : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by

$$b^n(x) = \frac{\overline{\beta} + \underline{\beta}}{2} + \frac{\overline{\beta} - \underline{\beta}}{2} \sin(n \log x)$$

for every $x \in \boldsymbol{R}_{++}$. Then, for each i, define $b^n_i : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by

$$b^n_i(x_i) = b^n \left((f^n_i)^{-1}(x_i) \right) = b^n \left(\frac{x_i}{\theta^n_i} \right)$$

for every x_i. By this construction, for every $u^n_i \in \mathscr{U}^2_{\boldsymbol{R}_{++}}$, if b^n_i is the relative risk aversion of u^n_i, then there exists a $\lambda \in \boldsymbol{R}^I_{++}$ such that f^n gives the solution of (1) where each u_i is replaced by u^n_i. We now show that $b^n_1 = \cdots = b^n_I$. Indeed, for every i and $z \in \boldsymbol{R}_{++}$,

$$\begin{aligned}
b^n_i(z) &= b^n \left(\tau^n z \exp\left(-\frac{2k^n_i \pi}{n} \right) \right) \\
&= \frac{\overline{\beta} + \underline{\beta}}{2} + \frac{\overline{\beta} - \underline{\beta}}{2} \sin\left(n \log \tau^n + n \log z - 2k^n_i \pi \right) \\
&= \frac{\overline{\beta} + \underline{\beta}}{2} + \frac{\overline{\beta} - \underline{\beta}}{2} \sin\left(n \log \tau^n + n \log z \right).
\end{aligned}$$

Since the right-hand side does not depend on i, this proves that $b^n_1 = \cdots = b^n_I$. It also shows that $\max b^n_i(\boldsymbol{R}_{++})$ exists and equals $\overline{\beta}$, and $\min b^n_i(\boldsymbol{R}_{++})$ exists and equals $\underline{\beta}$.

The proof can now be completed by letting, for a sufficiently large n, $\hat{\theta}_i = \theta^n_i$, $\hat{f} = f^n$, and $\hat{u}$ be a utility function of which the relative risk aversion equals b^n_i. ///

The importance of the example of this subsection should not be underestimated. To see it, note first that the linearity of risk-sharing rules implies that if the consumers attain the corresponding consumption allocation via asset transactions, then the composition of their portfolios must be equal; that is, the proportion of wealth invested into each asset must be common across them. In the well known paper of Friend and Blume (1975) and subsequent empirical studies, researchers looked into whether the proportion of wealth invested into risky assets depend on consumers' (households') wealth levels, and whenever they found it does not, they concluded that the consumers exhibit constant relative risk aversion. The argument of this subsection shows that this line of reasoning is quite mistaken. First, we saw that they can exhibit increasing or decreasing relative risk aversion if their risk attitudes are heterogeneous. Second, Proposition 5 showed that even if they have identical risk attitudes, they can have nonconstant relative risk aversion; specifically, their relative risk aversion can oscillate between any two levels of relative risk aversion.

6.2 Nonlinear Risk-Sharing Rules and Constant Relative Risk Aversion

Our second example involves the individual utility functions derived from *correctly observed* risk-sharing rules and an *erroneously postulated* utility function for the representative consumer.

Let $I = 2$ and $D_1 = D_2 = \boldsymbol{R}_{++}$. Define $\hat{u}_1 : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by $\hat{u}_1(x) = \log x$ and $\hat{u}_2 : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by $\hat{u}_2(x) = 2x^{1/2}$. Thus consumer 1 exhibits constant relative risk aversion 1 and consumer 2 exhibits constant relative risk aversion 1/2. This is the special case that Wang (1996) studied. Let $\lambda = (\lambda_1, \lambda_2) = (1, 1)$. Then let $f_\lambda = (f_{\lambda 1}, f_{\lambda 2}) : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++} \times \boldsymbol{R}_{++}$ be the risk-sharing rule and $u_\lambda : \boldsymbol{R}_{++} \to \boldsymbol{R}$ be the representative consumer's utility function obtained by solving (1). By the first-order condition, $f_{\lambda 2}(x) = (f_{\lambda 1}(x))^2$ and, from $f_{\lambda 1}(x) + (f_{\lambda 1}(x))^2 = x$, that

$$f_{\lambda 1}(x) = \left(x + \frac{1}{4}\right)^{1/2} - \frac{1}{2} \text{ and } f_{\lambda 2}(x) = x + \frac{1}{2} - \left(x + \frac{1}{4}\right)^{1/2}$$

It is easy to confirm that $f \in \mathscr{F}^\infty_{\boldsymbol{R}_{++}}$. It is also easy to show that $f''_{\lambda 1}(x) < 0 < f''_{\lambda 2}(x)$ for every $x \in \boldsymbol{R}_{++}$. This means that $f_{\lambda 1}$ is strictly concave everywhere and $f_{\lambda 2}$ is strictly convex everywhere. Furthermore,

$$u_\lambda(x) = \hat{u}_1\left(f_{\lambda 1}(x)\right) + \hat{u}_2\left(f_{\lambda 2}(x)\right) = \log\left(\left(x + \frac{1}{4}\right)^{1/2} - \frac{1}{2}\right) + (4x+1)^{1/2} - 1$$

for every $x \in \boldsymbol{R}_{++}$, where the last equality follows from

$$\left(x + \frac{1}{2} - \left(x + \frac{1}{4}\right)^{1/2}\right)^{1/2} = \left(x + \frac{1}{4}\right)^{1/2} - \frac{1}{2}.$$

Define $b_\lambda : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $b_\lambda(x) = -u''_\lambda(x)x/u'_\lambda(x)$ for every $x \in \boldsymbol{R}_{++}$, then b_λ is the representative consumer's relative risk aversion. Then, according to part 2 of Corollary 8 of HHK, $b'_\lambda(x) < 0$ for every $x \in \boldsymbol{R}_{++}$. That is, the representative consumer's relative risk aversion is strictly decreasing everywhere.

Let's now imagine that we have erroneously postulated that the representative agent exhibits constant relative risk aversion 3/4, which is the arithmetic average of the individual consumers' true relative risk aversion, 1 and 1/2. Mathematically, define $u : \boldsymbol{R}_{++} \to \boldsymbol{R}$ by $u(x) = 4x^{1/4}$ for every $x \in \boldsymbol{R}_{++}$, and let $u_1 : \boldsymbol{R}_{++} \to \boldsymbol{R}$ and $u_2 : \boldsymbol{R}_{++} \to \boldsymbol{R}$ be the *inferred* utility functions, which are defined as in the proof of Theorem 2. Since $f_{\lambda 1}^{-1}(x_1) = x_1 + x_1^2$, (3) can be rewritten as

$$u_1'(x_1) = \left(x_1 + x_1^2\right)^{-3/4}.$$

Define $b_1 : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $b_1(x) = -u_1''(x)x/u_1'(x)$, then it is the relative risk aversion for the inferred utility function u_1. Then

$$b_1(x_1) = \frac{3}{4}\left(2 - \frac{1}{x_1 + 1}\right). \tag{18}$$

Thus, b_1 is strictly increasing, rather than constant, from 3/4 to 3/2. Define $b_2 : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ by $b_2(x) = -u_2''(x)x/u_2'(x)$, then it is the relative risk aversion for the inferred utility function u_2. We can analogously show that

$$b_2(x_2) = \frac{3}{4}\left(1 - \frac{1}{2x_2^{1/2} + 2}\right), \tag{19}$$

which is strictly increasing, rather than constant, from 3/8 to 3/4.

What we have done so far can be summarised as follows. First, we take up an economy consisting of two consumers, both of whom exhibit constant relative risk aversion, albeit at two different levels. Then the more risky consumer has a strictly convex risk-sharing rule and the less risky one has a strictly concave risk-sharing rule. Moreover, the representative consumer exhibits strictly decreasing relative risk aversion. Although the strictly decreasing relative risk aversion is a distinctive characteristic of the representative consumer's risk attitudes, we then move on to assume, contrary to this, that the representative consumer exhibits constant relative risk aversion, as is often postulated in the asset pricing literature. An implication of Theorem 2 (and Corollary 2) is that this assumption is consistent with the efficient risk-sharing rules that we have derived. The first consumer's utility function that is consistent with the representative consumer's constant relative risk aversion is still unambiguously more risk averse than the second consumer's counterpart, but both of them now exhibit strictly increasing relative risk aversion.

From a somewhat broader perspective, we can explain what we have done the emergence of the utility functions exhibiting strictly increasing relative risk aversion as follows. First, recall that the reciprocal of the absolute risk aversion is called the *absolute risk tolerance* and its first derivative of the reciprocal of the absolute risk aversion is called the *cautiousness.* Then the cautiousness of the true utility functions $\hat{u}_i$ are constantly equal to the reciprocals of the (constant) relative risk aversion. The cautiousness is therefore 1 for the first consumer and 2 for the second. According to Proposition 4 of HHK, this implies that f_1 is strictly concave and f_2 is strictly convex.

The inferred utility functions u_i have been constructed so that f_1 and f_2 remain to be efficient risk-sharing rules. Let $t_i : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$ be the risk tolerance for the inferred utility function u_i, defined by $t_i(x_i) = -u_i'(x_i)/u_i''(x_i)$ for every $x_i \in \boldsymbol{R}_{++}$. The corresponding cautiousness is its first derivative $t_i' : \boldsymbol{R}_{++} \to \boldsymbol{R}_{++}$. An application of Proposition 4 of HHK in the opposite direction establishes $t_1(f_{\lambda 1}(x)) < t_2(f_{\lambda 2}(x))$, that is, the cautiousness is lower for the first consumer's inferred utility function than for the second when evaluated at their own consumption levels implied by the risk-sharing rule f. We can in fact show, by direct calculation, that

$$
\begin{aligned}
t_1'(x_1) &= \frac{2}{3}\left(1 + \frac{1}{(2x_1+1)^2}\right), \\
t_2'(x_2) &= \frac{4}{3} + \frac{1}{3x_2^{1/2}}\left(1 - \frac{1}{\left(2x_2^{1/2}+1\right)^2}\right)
\end{aligned}
$$

for every $x_1 \in \boldsymbol{R}_{++}$ and $x_2 \in \boldsymbol{R}_{++}$. Since $t_1'(x_1) < 4/3 < t_2'(x_2)$ for every $x_1 \in \boldsymbol{R}_{++}$ and $x_2 \in \boldsymbol{R}_{++}$, the cautiousness is lower for the first consumer than for the second, not only at the consumption levels implied by the risk-sharing rules, but also at any two arbitrarily chosen consumption levels.

The relative risk aversion is not exactly equal to the reciprocal of the cautiousness for the u_i unless their relative risk aversion are constant. Rather, the former is equal to the latter multiplied by the elasticity of the absolute risk tolerance. By L'Hôpital's rule, the elasticity must converge to one as the consumption levels tends to zero or infinity. In our example, it so happens that this elasticity is always close to one, even at intermediate consumption levels. We can thus expect that we obtain a similar unambiguous ranking as regards to the relative risk aversion. In fact, we have $b_1(x_1) > 3/4 > b_2(x_2)$ for every $x_1 \in \boldsymbol{R}_{++}$ and every $x_2 \in \boldsymbol{R}_{++}$, that is, the relative risk aversion is higher for the first consumer than for the second, regardless of the two consumption levels at which the relative risk aversion are evaluated.

Since the representative consumer is postulated to exhibit constant relative risk aversion (and hence constant cautiousness), the nonlinearity of the risk-sharing rules implies that the two consumers must necessarily have heterogeneous risk attitudes. According to Theorem 5 and Proposition 7 of HHK, the presence of this type of heterogeneity makes it more likely for the representative consumer tends to exhibit strictly increasing cautiousness and strictly decreasing relative risk aversion. Given the constant relative risk aversion 3/4 (which is equivalent to constant cautiousness 4/3) for the representative consumer, to generate such a risk attitude for the representative consumer, the individual consumers must exhibit relative risk aversion that decreases at a sufficiently high rate as the consumption level increases. This is exactly what is happening, as we can see from (18) and (19).

7 Conclusion

In this paper we have shown that the efficiency of risk allocation in no way restricts the nature of the risk-sharing rules beyond comonotonicity, or the nature of the state-pricing rule beyond positivity and antimonotonicity. We have also explored implications of this result and investigated additional restrictions on the risk-sharing and state-pricing rules when the individual consumers exhibit hyperbolic absolute risk aversion.

There are a couple of unsolved problems. One is to identify the restrictions on the risk-sharing rules when the individual consumers have the same utility function but differing wealth levels. Such a case is of considerable interest because it is in line with the most commonly used macroeconomic setting, which consists of ex ante homogeneous but ex post heterogeneous consumers, as in Weil (1992). The other is to explore possible implications of heterogeneous probabilistic beliefs. Strictly speaking, the probabilistic beliefs are assumed to be homogeneous across consumers in this paper, but if the degree of heterogeneity in a given state depends only on the aggregate consumption level in the state, then the heterogeneity in beliefs may be equivalent to heterogeneity in risk attitudes, and we may be able to use the present framework to analyze the implication of belief heterogeneity on the risk-sharing and state-pricing rules.

References

[1] Karl Borch, 1962, Equilibrium in a reinsurance market, Econometrica 30, 424–444.

[2] Rose-Anne Dana and I. Meilijson, 2003, Modelling agents' preferences in complete markets by second order stochastic dominance, manuscript.

[3] Irwin Friend and Marshall E. Blume, 1975, The demand for risky assets, American Economic Review 65, 900–922.

[4] Chiaki Hara, James Huang, and Christoph Kuzmics, 2005, Efficient risk-sharing rules and the representative consumer's risk aversion, manuscript.

[5] Miki Kohara, Fumio Ohtake, and Makoto Saito, 2002, A test of the full insurance hypothesis: The Case of Japan, Journal of Japanese and International Economics 16, 335–352.

[6] Narayana R. Kocherlakota, 1996, The equity premium: It's still a puzzle, Journal of Economic Literature 34, 42–71.

[7] David M. Kreps, 1981, Arbitrage and equilibrium in economies with infinitely many commodities, Journal of Mathematical Economics 8, 15–35.

[8] David M. Kreps, 1990, A Course in Microeconomic Theory, Princeton University Press, Princeton.

[9] Barbara J. Mace, 1991, Full insurance in the presence of aggregate uncertainty, Journal of Political Economy 99, 928–956.

[10] Rajnish Mehra, and Edward C. Prescott, 1985, The Equity premium: A puzzle, Journal of Monetary Economics 15, 145–61.

[11] Masao Ogaki and Qiang Zhang, 2001, Decreasing relative risk aversion and tests of risk sharing, Econometrica 69, 515–526.

[12] Robert Townsend, 1994, Risk and insurance in village India, Econometrica 62, 539–591.

[13] Jiang Wang, 1996, The term structure of interest rates in a pure exchange economy with heterogeneous investors, Journal of Financial Economics 41, 75–110.

[14] Philippe Weil, 1992, Equilibrium asset prices with undiversifiable labor income risk, Journal of Economic Dynamics and Control 16, 769–790.

[15] Robert Wilson, 1968, The theory of syndicates, Econometrica 36, 119–132.