Condorcet Jury Theorem or Rational Ignorance*

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Abstract

We analyze a symmetric model of an election in which voters are uncertain about which of two alternatives is desirable for them. For each voter, an only information resource about the alternatives is a noisy signal, and she must pay some cost to reduce the noise. Although there are many possibilities of the signal structure, by focusing on *unbiased* voting strategies, we find that the problem can be reduced to one parameter: the probability that the desirable alternative in reality looks more plausible to the signal receiver. Another problem is a possibility of multiple equilibria. We show, however, that for any sequence of unbiased voting equilibria, the probability of electing the desirable alternative converges to the same value as the number of voters grows. Combining our results with the result of Martinelli (2005), we show that if the second order derivative of the information cost function is zero at no information then this probability converges to one, that is, the Condorcet Jury Theorem is valid, and otherwise converges to some value less than one, that is, the "rational ignorance" hypothesis is valid.

KEYWORDS: Condorcet jury theorem, rational ignorance, elections, strategic voting, information aggregation, costly information acquisition.

1 Introduction

Can elections aggregate private information in a large society? This is a central question concerning elections. Apropos of this question, the Condorcet Jury Theorem (hereafter CJT) gave a powerful justification for elections under the majority voting rule. CJT asserts that the probability of making an appropriate decision will converge to one as the number of voters grows. There is much of literature advocating CJT[1], nevertheless CJT may not valid if each voter voluntarily acquires costly information. Increasing the number of voters reduces incentives for information acquisition since the probability of affecting the outcome becomes small.

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[1]See, e.g., Austen-Smith and Banks (1986), Feddersen and Pesendorfer (1997), Berend and Paroush (1998)).

Recently, in his pioneering work, Martinelli (2005) showed that under certain conditions CJT is valid although information acquisition is costly. The main point of his paper is that "rational ignorant" voters are consistent with a well-informed electorate. Formally, Martinelli (2005) considered the sequence of symmetric equilibria and showed that CJT is valid along this sequence if and only if the second derivative of the information cost function at no information is zero.

Martinelli (2005), however, employed a quite restrictive framework to lead to this conclusion. Considering applications and the robustness of his results, there remain two problems to be solved: (i) there are only two signals in his model, and (ii) there still remains the possibility of asymmetric equilibria. We answer the following question in this paper. Does his result hold under general situations?

We study a symmetric model of an election in which voters are uncertain about which of two alternatives is desirable for them. For each voter, an only information resource about the alternatives is a noisy signal, and she must pay some cost to reduce the noise. Most of previous papers typically deal with simple signal structures, e.g., there are only two signals, while we consider general symmetric signal structures. We allow a wide variety of signal structures such that signals perturbed by a noise term distributed continuously.

Our first result is that if all voters adopt *unbiased* voting strategies such that each vote is symmetric between two alternatives, then the problem can be reduced to one parameter: the probability that the desirable alternative in reality looks more plausible to the signal receiver (Theorem 1). Due to this result, a symmetric signal structure can be degenerated to the simple one in which each voter can predict the desirable alternative with her own probability q. Thus, employing the simple signal structure is justified.

Our second result concerns the asymptotic property of the election outcome. Strategic voting models generally possess the multiplicity of equilibria. We show, however, that for any sequence of unbiased voting equilibria, the probability of electing the desirable alternative converges to the same value as the number of voters grows (Theorem 2). As a corollary of this result combined with Theorem 2 in Martinelli (2005), we can calculate the asymptotic probability of the electoral outcome. Consequently, we show that if the second order derivative of the information cost function is zero at no information then for any sequence of unbiased-voting equilibria, the probability of electing the desirable alternative converges to 1, that is, CJT is valid, otherwise this probability converges to $p < 1$, that is, CJT is not valid.

Although a large number of studies have been done on CJT, most of them assumed that each voter's competence is given exogenously. This assumption ignores salient behavior of voters: for the purpose of voting for a desirable alternative, each voter would be willing to acquire the information about the alternatives before the election. Taking such behavior into consideration, we suppose costly information acquisition. More precisely, for each voter an only information resource about the alternatives is a noisy signal, and she must pay some cost to reduce the noise. Such endogenous information acquisition exerts a serious influence on electoral outcomes (See Mukhopadhaya, 2003 and Martinelli, 2005).

Related to information aggregation under costly information acquisition, another significant thought is the "rational ignorance" hypothesis asserted by

Downs (1957). That is, voters will not have an incentive to acquire information about the alternatives before voting since it is extremely rare that each vote becomes pivotal in a large election. Therefore, it seems, at a glance, that elections lead to a poorer decision in a large society, namely, CJT is not valid.

This inference, however, is not necessarily true. Even if the amount of information acquired by each voter is small, the amount of aggregated information can be large enough to the correct decision. Martinelli (2005) showed this, confining to the sequence of symmetric equilibria. We show that his result is true under any general signal structure for any sequence of equilibria.

The rest of the paper is organized as follows. Section 2 examines the model. Section 3 derives two main results. Section 4 states conclusions. All formal proofs are in the Appendix.

2 The Model

We analyze a symmetric election with two alternatives, L and R. Either one of the alternatives is assumed to be commonly desirable. There are $2n+1$ voters indexed by i. A voter's payoff depends on the chosen alternative $d \in \{L, R\}$, the state $z \in \{z_L, z_R\}$ and the quality of information acquired by herself before the election $x \in [0, \bar{x}] = X$. Acquiring information of quality x has a cost given by $C(x)$, so the payoff of a voter can be written as

$$u(d, z) - C(x).$$

We assume that $x = 0$ is equivalent to acquiring no information, and that acquired information becomes more precise as x increases. We also assume that $C(0) = 0$, $C(\bar{x}) = \infty$ and $C(\cdot)$ is increasing. L (R) is the desirable alternative in state z_L (z_R):

$$u(d, z) = \begin{cases} 1 & \text{if } (d, z) = (L, z_L) \text{ or } (R, z_R), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

At the beginning of time, nature selects the state with equal probability: $\Pr(z_L) = \Pr(z_R) = 1/2$. Voters are uncertain about the realization of the state and they can only observe a noisy signal. After the realization of the state, each voter i must decide the quality of her information x_i. After deciding on x_i, i receives a private signal $s \in S$, which is independent among voters. The distribution of signals depends on both the quality of information x and the state z. Formally, let S be the signal space, $\mathcal{S}$ be the σ-algebra of subsets of S and $\{\mu_z^x\}$ be the family of probability measures on $\mathcal{S}$. Given (x, z_d), the distribution of signals follows μ_d^x. We restrict our attentions to symmetric signal structures.

Assumption 1
There exists a family of one-to-one transformations $\{\tau_x\}_{x \in X}$ on S such that for any $s \in S$,

$$\tau_x(\tau_x(s)) = s, \tag{2}$$

and for any $S' \in \mathcal{S}$,

$$\mu_L^x(S') = \mu_R^x(\tau(S')). \tag{3}$$

We call $\tau_x(s)$ the conjugate signal of s w.r.t. x. Obviously, the conjugate signal of $\tau_x(s)$ w.r.t. x is s. Intuitively, the probability of receiving s under z_L is equal to that of receiving $\tau(x)$ under z_R. Note that under Assumption 1 we obtain $\Pr\left(z_L \middle| s\right) = \Pr\left(z_R \middle| \tau(s)\right)$ by Bayes' rule, that is, if z_L is plausible under s then z_R is plausible under $\tau(s)$. We give two examples of symmetric signal structures.

Example 1 (2-signals: Martinelli, 2005)
$S = \{s_L, s_R\}$, $X = [0, 1/2]$ and the probability of receiving signal s_d in state z_d is given by $1/2 + x$. $\tau_x(s_L) = s_R$ and $\tau_x(s_R) = s_L$ for all x. We can regard s_d as the correct signal in the state z_d. This signal structure is the simplest one that describes the situation supposed by Condorcet. ‖

Example 2 (Normal noise: Kitahara and Sekiguchi, 2005)
$S = (-\infty, \infty)$, $X = [0, \infty]$ and

$$s(x) = \begin{cases} -x + \epsilon & \text{if } z = z_L, \text{ and} \\ x + \epsilon & \text{if } z = z_R, \end{cases} \tag{4}$$

where $\epsilon \sim N(0, 1)$. Obviously, $\tau_x(s) = -s$ for all x. ‖

The election takes place after voters receive their signals. A voter can either vote for L or vote for R. (That is, there is no abstention.) The alternative with most votes is chosen.

A strategy for a voter i is a tuple (x_i, v_i) where x_i specifies a quality of information and a measurable function $v_i : S \to [0, 1]$ specifies a probability of voting for L after receiving a signal. Since we assume no abstention, a probability of voting for R given s is $1 - v_i(s)$.

A strategy profile $(\mathbf{x}, \mathbf{v})$ is a *voting equilibrium* if it is a Nash equilibrium. We restrict our attention to some subset of voting equilibria in which each vote is unbiased between two alternatives. We define an *unbiased strategy* as follows.

Definition 1 *A strategy (x_i, v_i) is unbiased if $v_i(s) = 1 - v(\tau_{x_i}(s))$ for all $s \in S$.*

Namely, for any signal the probability of voting for L given this signal is equal to the probability of voting for R given the conjugate signal of this signal. An *unbiased voting equilibrium* is a voting equilibrium in which all voters adopt unbiased strategies. We restrict our attention to unbiased voting equilibria for the purpose of excluding trivial equilibria such that, for example, all voters vote for L regardless of their signals. Take notice of following two points. First, we do not assume that strategies are symmetric among voters, that is, voters may adopt different unbiased strategies. Second, we do not exclude deviation to a biased strategy, e.g., $v_i(s) = 1$ for all s.

Remarks:
We say our election model is symmetric in the following sense.

(i) Each alternative is desirable with equal probability ex ante: $\Pr(z_L) = \Pr(z_R)$.

(ii) The Payoff from the desirable alternative is independent of the names of the alternatives: Eq (1).

(iii) The signal structure is symmetric: Assumption 1.

(iv) In equilibria, each vote is symmetric between the alternatives: Definition 1.

3 Main Results

3.1 Degeneration to 2-signals

In this subsection, we show that if all voters adopt unbiased strategies, any symmetric signal structure can be degenerated to the two-signals case as in Example 1. We begin with some preliminary definitions.

At first, we define a family of partitions of S w.r.t. x as follows:

$$\begin{aligned} S^L(x) &= \{s \in S | \Pr(z_L|s,x) > \Pr(z_R|s,x)\}, \\ S^M(x) &= \{s \in S | \Pr(z_L|s,x) = \Pr(z_R|s,x)\}, \\ S^R(x) &= \{s \in S | \Pr(z_L|s,x) < \Pr(z_R|s,x)\}, \end{aligned}$$

where $x \in X$. Obviously, for any $x \in X$, $\{S^L(x), S^M(x), S^R(x)\}$ constitutes a partition of S and each element is measurable. $S^L(x)$ ($S^R(x)$) is the set of signals under which z_L (z_R) is more plausible when the quality of information is x, and $S^M(x)$ is the set of signals under which both states are similarly plausible. Notice that, by Assumption 1 and Bayes' rule, $s \in S^L(x)$ if and only if $\tau(s) \in S^R(x)$, and $s \in S^M(x)$ if and only if $\tau(s) \in S^M(x)$. As we will show below, in any unbiased voting equilibria each voter who chose x and received $s \in S^L(x)$ ($S^R(x)$) votes for L (R).

Next, define an accuracy of information $q : X \to [1/2, 1]$ as:

$$\begin{aligned} q(x) &= \mu_L^x\big(S^L(x)\big) + \frac{1}{2}\mu_L^x\big(S^M(x)\big) \\ &= \mu_R^x\big(S^R(x)\big) + \frac{1}{2}\mu_R^x\big(S^M(x)\big). \end{aligned}$$

By Assumption 1, $q(x)$ is well-defined. The accuracy $q(x)$ is, in other words, the probability that the desirable alternative in reality looks more plausible for a voter with her quality of information x. Thus, if a voter with x votes sincerely, i.e., as if her vote alone determines the outcome, then the probability that she votes for the desirable alternative is equal to $q(x)$. Note that $q(0) = 1/2$ since we assumed that there is no information when $x = 0$. For Example 1, $q(x) = x + 1/2$. For Example 2, $q(x) = \Phi(x)$, where Φ is the cumulative density function of $N(0,1)$. After this $q(x)$ plays a key part of our analysis. For simplicity, we exclude a redundant x, namely, no two qualities have same q. And we also assume that any $q(x) \in [1/2, 1]$ can be attained by appropriately choosing x.

Assumption 2
$q(\cdot)$ is a bijection, and hence $\tilde{C} = C \circ q^{-1}$ is well-defined.

$\tilde{C}(q)$ is the cost of information when an accuracy is q. In the rest of the paper, we consider $\tilde{C}$ instead of C. For a voter, by Assumption 2, selecting

$x \in X$ is equivalent to selecting $q \in [1/2, 1]$. Note that $\tilde{C}(1/2) = C(0) = 0$. For Examples 1 and 2, $\tilde{C}$ is well-defined.

To simplify notations, let y_i for $i = 1, ..., 2n+1$ be the random variable such that

$$y_i = \begin{cases} 1 & \text{if } i \text{ votes for the desirable alternative,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The only time i can influence the outcome of the election is if i's vote is pivotal, i.e., $\sum_{j \neq i} y_j = n$. A voter will votes for L if the expected payoff from voting for L, conditional on her vote being pivotal, is greater than the expected payoff from voting for R.

If voter i adopts an unbiased strategy, then $\Pr(y_i = 1|z_L) = \Pr(y_i = 1|z_R)$. Hence, in unbiased voting equilibria the probability of i's vote being pivotal is the same as in both states, i.e., $\Pr(\sum_{j \neq i} y_j = n|z_L) = \Pr(\sum_{j \neq i} y_j = n|z_R)$, and others' votes have no information. Therefore, each voter votes informatively, that is, she votes for the alternative, which is more likely to be desirable given her signal. Thus we obtain the following Lemma.

Lemma 1 *Suppose that* $(\mathbf{x}^*, \mathbf{v}^*)$ *is an unbiased voting equilibrium. Then* v_i^* *satisfies the followings for all* i,

$$v_i^*(s) = \begin{cases} 1 & \text{if } s \in S^L(x_i^*) \quad (a.s.), \\ 0 & \text{if } s \in S^R(x_i^*) \quad (a.s.), \end{cases} \tag{6}$$

$$E\Big[v_i^*(s)\Big| s \in S^M(x_i^*)\Big] = \frac{1}{2}. \tag{7}$$

Moreover, $\Pr\Big(y_i = 1 \Big| (x_i^*, v_i^*)\Big) = q(x_i^*)$.

Eq (7) is immediately obtained from the definition of the unbiased strategy. This Lemma shows that in unbiased voting equilibria all voters vote sincerely, and that the probability that i votes for the desirable alternative is exactly equal to $q(x_i^*)$. Hence equilibrium conditions are described by the accuracy of information $q(x_i^*)$. Before we state this formally, let $y_i(q)$ be the random variable which is independently distributed as

$$y_i(q) = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } 1 - q. \end{cases} \tag{8}$$

Theorem 1
For any unbiased voting equilibrium $(\mathbf{x}^*, \mathbf{v}^*)$, *for all* i,

$$q(x_i^*) \in \arg\max_{q \in [1/2,1]} \Big\{ q \Pr\Big(\sum_{j \neq i} y_j\big(q(x_j^*)\big) = n \Big) - \tilde{C}(q) \Big\}. \tag{9}$$

Conversely, if x_i^* *satisfies Eq (9) for all* i*, then there exists an unbiased voting equilibrium, in which each voter* i *chooses* x_i^**. The probability of electing the desirable alternative is equal to* $\Pr(\sum_i y_i(q(x_i^*)) > n)$.

This Theorem reveals that for any symmetric signal structure, the optimization problem for voters is equivalent to the maximization problem w.r.t. $q(x_i)$.

Therefore, any symmetric signal structure can be degenerated to the two-signals case as in Example 1.

Next we consider the existence of symmetric unbiased voting equilibria. To guarantee the existence, we suppose the following regular assumption in the remainder of the paper.

Assumption 3
$\tilde{C}$ is strictly increasing, strictly convex, and twice continuously differentiable.

For Examples 1 and 2, if C is strictly increasing, strictly convex, and twice continuously differentiable, then $\tilde{C}$ does too ($\tilde{C}$ are given by $C(q-1/2)$ and $C(\Phi^{-1}(q))$, respectively).

Corollary 1 *Assume $\tilde{C}'(1/2) = 0$. Then for arbitrary n there exists a symmetric unbiased voting equilibrium, in which all voters choose the same quality $\bar{x}_n$ which solves*

$$\binom{2n}{n}(q_n(1-q_n))^n = \tilde{C}'(q_n) \tag{10}$$

with $q_n = q(\bar{x}_n)$.

For the two-signals case, Martinelli also proved this statement (Theorem 1 in Martinelli, 2005).

3.2 Asymptotic Properties

We take an interest in the asymptotic property w.r.t. n. That is, whether or not the probability of the desirable alternative being chosen by the election converges to 1 as n goes to infinity. Even for the two-signals case, the convergence property has been clarified only for the sequence of symmetric equilibria (See Martinelli, 2005). Nevertheless, of course, there still remains the possibility of asymmetric equilibria. In strategic voting models the multiplicity of equilibria is generally observed. Can there be different performances by deliberately choosing asymmetric equilibria? The following theorem answers for this question.

Theorem 2
Assume $\tilde{C}'(0) = 0$. Then, for any sequence of unbiased voting equilibria $\{(\mathbf{x}^n, \mathbf{v}^n)\}$,

(i) if $\tilde{C}''(1/2) > 0$, then for sufficiently large n, $q(x_i^n) = q_n$ for all i, where q_n is the unique solution of Eq (10);

(ii) if $\tilde{C}''(1/2) = 0$, then

$$\Pr\Big(\sum_i y_i\big(q(x_i^n)\big) > n\Big) \to 1 \text{ as } n \to \infty. \tag{11}$$

In other words, for any sequence of unbiased equilibria the probability of the desirable alternative winning the election converges to that of the symmetric equilibria sequence defined in Corollary 1. Combining this Theorem with Martinelli's result, the convergence property of *all* equilibria sequences is clarified.

Theorem 2 in Martinelli (2005) asserted that if $\tilde{C}'(1/2) = 0$, and q_n satisfies (10) for all n, then $\Pr(\sum_i y_i(q_n) > n)$ approaches $\Phi(\delta)$, where δ solves

$$\frac{\phi(\delta)}{\delta} = \frac{1}{4}\tilde{C}''(1/2), \tag{12}$$

where ϕ is the density function of $N(0,1)$. Hence the following holds as a corollary of Theorem 2.

Corollary 2 *Assume $\tilde{C}'(0) = 0$. Then, for any sequence of unbiased voting equilibria, the probability of electing the desirable alternative converges to $\Phi(\delta)$, where δ is the unique solution of Eq (12).*

We can summarize the results obtained so far as the following remarks.

Remarks:

(i) If $\tilde{C}''(1/2) = 0$ then the probability of electing the desirable alternative approaches to 1, that is, CJT is valid.

(ii) If $\tilde{C}''(1/2) \neq 0$ then the probability of electing the desirable alternative approaches to $\Phi(\delta) < 1$, that is, rational ignorance hypothesis is valid.

4 Conclusions

In a symmetric two-alternatives election with costly information acquisition, general symmetric signal structures can be degenerated to the two-signals case if we assume that each vote is unbiased between the alternatives in equilibria.

For any unbiased voting equilibria sequence, the probability of electing the desirable alternative converges to the same value as the number of voters increases. If the second order derivative of the information cost function is zero at no information then this probability converges to 1, that is, CJT is valid. Otherwise, it converges to $\Phi(\delta) < 1$, that is, "rational ignorance" hypothesis is valid.

We believe that our results are useful to various applications. For instance, Kitahara and Sekiguchi (2005) considered the case in which signals are perturbed by normal noise as in Example 2 and showed that if C' is concave, then the election leads to poorer decision than the case of delegating the choice to one of voters. On the other hand, if C' is convex, then the election leads to better decision if and only if the value of choosing the desirable alternative is sufficiently large.

Appendix

The Proof of Lemma 1
Suppose that $(\mathbf{x}^*, \mathbf{v}^*)$ is an unbiased voting equilibrium. Then,

$$\begin{aligned}
\Pr\Big(y_i = 1\Big|z_L, (x_i^*, v_i^*)\Big) &= E\big[v_i^*(s)\big|z_L\big] \\
&= \int_S v_i^*(s)\mu_L^{x_i^*}(ds) \\
&= \int_S (1 - v_i^*(s))\mu_R^{x_i^*}(ds) \\
&= E\big[1 - v_i^*(s)\big|z_R\big] \\
&= \Pr\Big(y_i = 1\Big|z_R, (x_i^*, v_i^*)\Big).
\end{aligned}$$

Let piv^i be the event that i's vote is pivotal. Since $C(\bar{x}) = \infty$, $q(x_i^*) < 1$ for all i. Then, this event occurs with positive probability in equilibria. Since i's vote becomes pivotal if $\sum_{j \neq i} y_j^* = n$, we have

$$\Pr\Big(piv^i\Big|z_L, (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\Big) = \Pr\Big(piv^i\Big|z_R, (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\Big) > 0. \tag{13}$$

Thus, in unbiased voting equilibria the probability of i's vote being pivotal is the same as in both states. The probability distribution over states conditional on i receiving s and piv^i is computed by Bayes' rule. This is given by

$$\begin{aligned}
\Pr\Big(z\Big|piv^i, s, (\mathbf{x}^*, \mathbf{v}^*)\Big) &= \frac{\Pr\big(z\big|s, x_i^*\big)\Pr\big(piv^i\big|z, (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\big)}{\sum_{z' \in \{z_L, z_R\}} \Pr\big(z'\big|s, x_i^*\big)\Pr\big(piv^i\big|z', (\mathbf{x}_{-i}^*, \mathbf{v}_{-i}^*)\big)} \\
&= \Pr\Big(z\Big|s, x_i^*\Big).
\end{aligned}$$

The last equality is given by Eq (13). Therefore, when a voter receives the signal s, the difference between the expected payoff from voting for L and for R is

$$\Pr\big(z_L\big|s, x_i^*\big) - \Pr\big(z_R\big|s, x_i^*\big).$$

Consequently, i votes for L (R) if she receives $s \in S^L(x_i^*)$ ($S^R(x_i^*)$) almost surely. Eq (7) is immediately obtained from the definition of the unbiased strategy and obviously we have $\Pr\big(y_i = 1\big|(x_i^*, v_i^*)\big) = q(x_i^*)$. □

The Proof of Theorem 1
A voter i's expected payoff can be written as

$$\begin{aligned}
&U_i(x_i, y_i|\mathbf{x}_{-i}, \mathbf{v}_{-i}) \\
&\quad= \Pr\Big(\sum_{j \neq i} y_j > n\Big|(\mathbf{x}_{-i}, \mathbf{v}_{-i})\Big) + \Pr\Big(\sum_{j \neq i} y_j = n, y_i = 1\Big|(\mathbf{x}, \mathbf{v})\Big) - C(x_i) \\
&\quad= \Pr\Big(\sum_{j \neq i} y_j = n\Big|(\mathbf{x}_{-i}, \mathbf{v}_{-i})\Big)\Pr\Big(y_i = 1\Big|(x_i, v_i)\Big) - C(x_i) + \text{Const.}
\end{aligned}$$

Due to Lemma 1, we obtain

$$\begin{aligned}
&(x_i^*, v_i^*) \in \arg\max_{(x_i, v_i)} U_i(x_i, v_i|\mathbf{x}_{-i}, \mathbf{v}_{-i}),\ \forall i \\
&\qquad \iff \quad q(x_i^*) \in \arg\max_{q \in [1/2, 1]} \Big\{q \Pr\Big(\sum_{j \neq i} y_j\big(q(x_j^*)\big) = n\Big) - \tilde{C}(q)\Big\},\ \forall i.
\end{aligned}$$

Obviously, the probability of winning the desirable alternative is equal to $\Pr\Big(\sum_i y_i\big(q(x_i^*)\big) > n\Big)$. $\square$

The Proof of Corollary 1

Suppose that $(\mathbf{x}^*, \mathbf{v}^*)$ is a strategy profile such that $x_i^* = \bar{x}_n$ for all i and $\mathbf{v}_i^*$ satisfies Eq (6) (7). Due to Theorem 1, $(\mathbf{x}^*, \mathbf{v}^*)$ is an unbiased voting equilibrium if $\bar{x}_n$ solves Eq (10). If $\tilde{C}'(1/2) = 0$, then such an $\bar{x}_n$ uniquely exists for all n. $\square$

The Proof of Theorem 2

Suppose that $\{\mathbf{x}^n, \mathbf{v}^n\}$ is a sequence of unbiased voting equilibria.

(i) Suppose that $\tilde{C}''(1/2) > 0$. In this case, It saffices to show that $q(x_i^n) = q(x_j^n)$ for all i, j if n is sufficiently large.

Suppose, on the contrary, that $q(x_i^n) \neq q(x_j^n)$. By the F.O.C.,

$$\begin{aligned}
\tilde{C}'(q(x_i^n)) &= \Pr\Big(\sum_{k\neq i} y_k\big(q(x_k^n)\big) = n\Big) \\
&= \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n-1\Big)\cdot q(x_j^n) \\
&\quad + \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n\Big)\cdot \big(1-q(x_j^n)\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\tilde{C}'\big(q(x_i^n)\big) - \tilde{C}'\big(q(x_j^n)\big)}{q(x_i^n) - q(x_j^n)} \\
&\quad = \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n\Big) - \Pr\Big(\sum_{k\neq i,j} y_k\big(q(x_k^n)\big) = n-1\Big).
\end{aligned}$$

Since $\tilde{C}''$ is continuous then the LHS is bounded away from 0. On the other hand, the RHS converges to 0, it contradicts that $q(x_i^n) \neq q(x_j^n)$.

(ii) Suppose that $\tilde{C}''(1/2) = 0$. If $q(x_i^n) = q$ for all i then the probability that i's vote becomes pivotal is

$$f_n(q) = \tbinom{2n}{n}\big(q(1-q)\big)^n.$$

It reaches a maximum at $q = 1/2$ and it is decreasing. Define $\overline{q}_n$ and $\underline{q}_n$ such that

$$\begin{aligned}
\tilde{C}'(\overline{q}_n) &= r f_n\left(\frac{1}{2}\right), \\
\tilde{C}'(\underline{q}_n) &= r f_n\left(\overline{q}_n\right).
\end{aligned}$$

Since $\tilde{C}'(q(x_i^n)) = \Pr\Big(\sum_{j\neq i} y_j\big(q(x_j^n)\big) = n\Big)$ by the F.O.C. and $\Pr\Big(\sum_{j\neq i} y_j\big(q(x_j^n)\big) = n\Big)$ is decreasing in $q(x_j^n)$, $q(x_i^n) \in [\underline{q}_n, \overline{q}_n]$ for all i and n.

Suppose that $\tilde{C}'(q) = \alpha(q - 1/2)^2$. In such a case, those bounds satisfy

$$\left(\overline{q}_n - \frac{1}{2}\right) \simeq \left(\frac{r}{\alpha}\right)^{\frac{1}{2}} \left(\frac{1}{n\pi}\right)^{\frac{1}{4}}, \tag{14}$$

$$\left(\underline{q}_n - \frac{1}{2}\right) \simeq \left(\overline{q}_n - \frac{1}{2}\right) \exp\left(-\frac{r}{\alpha\sqrt{\pi}}\right). \tag{15}$$

for sufficiently large n (by Starling's Formula).

By Chebychev's inequality,

$$\sqrt{n}\left(q_n - \frac{1}{2}\right) \to \infty \Rightarrow \Pr\left(\sum_i y_i(q_n) > n\right) \to 1.$$

Obviously, $\sqrt{n}(q_n - 1/2) \to \infty$ if $\underline{q}_n$ satisfies Eq (14) (15) with equality. Then, even if all voters adopt worst quality $\underline{q}_n$, $\Pr\left(\sum_j y_j\left(\underline{q}_n\right) > n\right) \to 1$.

Consider a general $\tilde{C}'$ which satisfies $\tilde{C}''(1/2) = 0$. If α is sufficiently large, $\tilde{C}'(1/2 + \epsilon) \leq \alpha\epsilon^2$ for sufficiently small ϵ. Therefore each voter's equilibrium quality is not less than $\underline{q}_n$ in the case of $\tilde{C}'(q) = \alpha(q - 1/2)^2$ for sufficiently large n. Since $\Pr(\sum_i y_i(q(x_i^n)) > n)$ is increasing in $q(x_i^n)$, CJT is valid for any sequence of equilibria if $\tilde{C}''(1/2) = 0$. □

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