MULTIPLE-BELIEF RATIONAL-EXPECTATIONS EQUILIBRIA IN OLG MODELS WITH AMBIGUITY*

by

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May 11, 2005

Abstract

In this paper, we extend the concept of rational-expectations equilibrium, from a traditional single-belief framework to a multiple-belief one. In traditional framework of single belief, agents are supposed to know the equilibrium price "correctly." We relax this requirement in the framework of multiple beliefs. While agents do not have to know the equilibrium exactly, they must be correct in that the equilibrium price must be always contained in the support of each probability distribution they think possible. We call this equilibrium concept a multiple-belief rational-expectations equilibrium. We then show that the indeterminacy and complexity could arise even when the degree of risk aversion is moderate.

*The work reported here is partially supported by a grant from the Economic and Social Research Insitute, the Cabinet Office, the Government of Japan.

1. Introduction

It is well-known that in overlapping-generations models with money, the indeterminacy of equilibria arises only when the degree of risk aversion of each generation is extremely high. (See, for example, Woodford, 1984.) In this paper, we show that such indeterminacy could arise regardless of the degree of risk aversion if each generation faces *Knightian uncertainty* or *ambiguity* with respect to equilibrium prices. Here, the ambiguity refers to the situation where information about uncertain prospects is too imprecise to be summarized into a single probability distribution, and hence, the belief is represented only by a *set* of distributions, rather than a single distribution. In addition, we suppose that each generation is *uncertainty-averse* in the sense that they use the "worst" probability distribution in solving the utility maximization problem. Such a behavior is axiomatized by Gilboa and Schmeidler (1989).

To highlight the importance of ambiguity, we analyze an overlapping-generation model in which there is no uncertainty in endowments and preferences. Thus, uncertainty and ambiguity are present only in generations' subjective expectations about market prices. We assume each generation has only vague confidence about the "true" equilibrium prices and that confidence is represented by a set of some plausible prices rather than a single price.

In this model, we extend the concept of rational-expectations equilibrium, or in other words perfect-foresight equilibrium in our setting, in which agents in the model are supposed to know the equilibrium price "correctly." Thus, there is no surprise in this rational-expectations equilibrium. We relax this requirement to the one that while agents do not know the equilibrium price exactly, they must not be surprised by the realization of the equlibrium price. That is, agents' multiple-belief expectations must be "correct" in that the equilibrium price is always contained in the support of each probability distribution they think possible. We call this equilibrium concept multiple-belief rational-expectations equilibrium. At this equilibrium, despite of the agents' unawareness of the precise equilibrium price, their utility-maximization problems are well-defined through the assumption of uncertainty aversion. Furthermore, the realization of the price which clears the market never disappoints agents' expectations since they assign a positive (but possibly less than unity) probability to the occurrence of that price. Thus, their expectations are "realized." Obviously, single-belief rational-expectations equilibrium where agents' expectations are singleton sets is ordinary rational-expectations equilibrium.

The rational-expectations equilibrium is typically characterized by a solution to a difference equation. In contrast, the multiple-belief rational-expectations equilibrium we consider is shown to be characterized by a solution to a difference *inclusion*. Since a difference inclusion has multiple solutions in general, the multiple-belief rational-expectations equilibrium is quite easy to be indeterminate. In fact, we construct an overlapping-generations model in which each generation has a moderate degree of risk aversion so that there exists a unique rational-expectations equilibrium, but that there are uncountably many multiple-belief rational-expectations equilibria, some of which may be very complex.

Furthermore, we show that when each generation has a linear utility function, there exists a multiple-belief rational-expectations equilibrium in which equilibrium price sequences are steadily *decreasing* over time. This example is a sharp contrast with the traditional rational-expectations equilibrium in which the equilibrium price sequence is typically constant or increasing over time. Our result shows that there could be sustained *de*flation in the economy when the agents have only ambiguous beliefs about uncertain prospects of the future.

2. The Framework

2.1. Preliminaries

Although the model presented in this paper is fairly simple and standard, we have to put it into a framework of Knightian uncertainty or ambiguity literature. This necessitates some mathematical formulations and definitions, which are proven to be important in the following analysis.

When X is a compact subset of $\mathbb{R}$, we denote by $\mathcal{M}(X)$ the set of all probability measures on $(X, \mathcal{B}_X)$, where $\mathcal{B}_X$ is the Borel σ-algebra on X. For a probability measure $P \in \mathcal{M}(X)$, its *support*, denoted $\text{supp}(P)$, is given by

$$\text{supp}(P) \equiv X \backslash N$$

where $N \subseteq X$ is defined by

$$N = \cup \{ U \subseteq X \mid U \text{ is open and } P(U) = 0 \}.$$

It turns out that N is open and such that $P(N) = 0$,[1] and hence, that $\mathrm{supp}(P)$ is closed and such that $P(\mathrm{supp}(P)) = 1$. We denote by $\mathcal{M}^\circ(X)$ the set of all probability measures on $(X, \mathcal{B}_X)$ whose supports are equal to X. That is, $P \in \mathcal{M}^\circ(X)$ if and only if $\mathrm{supp}(P) = X$. Clearly, it holds that $\mathcal{M}^\circ(X) \subseteq \mathcal{M}(X)$.

2.2. An Economy with Overlapping Generations

We assume a fairly simple and standard structure. The economy consists of a countably infinite number of generations (the 0-th generation, the 1-st generation, the 2-nd generation, and so on). Each generation lives for two periods. In the t-th period ($t = 1, 2, \dots$), the old of the $(t-1)$-th generation and the youth of the t-th generation coexist. (As usual, we assume that only the youth of the 0-th generation lives in the 0-th period.)

The preference of the t-th generation is defined over its consumptions both in the t-th and $(t+1)$-th periods and is represented by a time-separable utility function: $u(c_t^y) + v(c_t^o)$. Here c_t^y is the t-th generation's consumption in the t-th period, c_t^o is its consumption in the $(t+1)$-th period, and $u : \mathbb{R}_+ \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ are the felicity function. We assume that u and v are common for all generations. Each generation earns one unit of consumption good in its youth and then is retired when it is old, earning no income.

The consumption good is perishable and cannot be carried over into the next period. Thus, the old generation could not consume if there were no means of exchange. Here, the money comes into the model. We assume that the government provides the 0-th generation in the 1-st period with $M\ (> 0)$ units of fiat money. Each generation believes that the money circulates.

It should be noted that in our model, there is no exogenous, unpredictable shocks to the economy. *The "real structure" of the economy is certain for all generations.* Thus, only relevant uncertainty comes from each generation's subjective evaluation about the state of the economy.[2]

[1] That N is open is immediate. To see that $P(N) = 0$, assume on the contrary that $P(N) > 0$. Then, by the regularity of a probability measure on a compact set (Ash, 1972, p.180, Theorem 4.3.8), there exists a compact set $K \subseteq N$ such that $P(K) > 0$. By the compactness of K and the fact that $K \subset N$, there exists a finite family of open sets, $U_1, \dots, U_k$, such that $K \subseteq \cup_{i=1}^k U_i$ and $(\forall i)\ P(U_i) = 0$. Therefore, $0 < P(K) \leq \sum_{i=1}^k P(U_i) = 0$, which is a contradiction.

[2] In this regard, our model is similar to the one with sunspot equilibria.

Let Q be a set of prices that contains the supports of all possible price distributions: $Q \equiv [\underline{q}, \overline{q}]$ where $0 \le \underline{q} < \overline{q} < +\infty$. (Thus, we implicitly assume away infinite prices.) After observing the price $q_t \in Q$ in the t-th period, the t-th generation forms a belief on the price q_{t+1} which will be realized in the $(t+1)$-th. We assume that the information the t-th generation possesses is too imprecise to be summarized as a single probability measure. Instead, we assume that the t-th generation's belief is represented by a nonempty subset of $\mathcal{M}(Q)$, $\mathcal{P}_t(q_t)$. Here, $\mathcal{P}_t : Q \twoheadrightarrow \mathcal{M}(Q)$ is a correspondence which maps the price observed in the current period into a belief, which itself is a set of distributions, on the price in the next period and may be called a *belief correspondence*.

2.3. Generations' Utility-Maximization

Suppose that the t-th generation's belief correspondence is given by $\mathcal{P}_t$. We assume that each generation is *uncertainty-averse* in accordance with the preference axiomatized by Gilboa and Schmeidler (1989). Then, after observing $q_t \in Q$, the t-th $(t = 1, 2, \dots)$ generation chooses a current consumption, $c_t^y \in \mathbb{R}_+$; a consumption plan for the next period, $c_t^o : \mathbb{R}_+ \to \mathbb{R}_+$, which is contingent upon the realization of q_{t+1}; and the money amount, $M_t \in \mathbb{R}_+$, which is carried over to the next period at the sacrifice of some of the current consumption, in order to maximize

$$\inf \left\{ u(c_t^y) + \int_Q v(c_t^o(q_{t+1}))\, P(dq_{t+1}) \,\middle|\, P \in \mathcal{P}_t(q_t) \right\} \tag{1}$$

$$\text{subject to} \quad M_t = q_t\,(1 - c_t^y) \quad \text{and} \quad (\forall q_{t+1} \in Q)\;\; q_{t+1} c_t^o(q_{t+1}) = M_t\,. \tag{2}$$

Note that each integral is taken with respect to the next-period price q_{t+1}. We denote the solution to this problem, when it exists, by $c_t^{y*}(q_t)$, $c_t^{o*}(q_t; \cdot)$ and $M_t^*(q_t)$.

Similarly, the 0-th generation chooses a current consumption, $c_0^y \in \mathbb{R}_+$ and a consumption plan for the next period, $c_0^o : \mathbb{R}_+ \to \mathbb{R}_+$, in order to maximize

$$\inf \left\{ u(c_0^y) + \int_Q v(c_0^o(q_1))\, P(dq_1) \,\middle|\, P \in \mathcal{P}_0 \right\}$$

$$\text{subject to} \quad c_0^y = 1 \quad \text{and} \quad (\forall q_1 \in Q)\;\; q_1 c_0^o(q_1) = M\,,$$

where $\mathcal{P}_0 \subseteq \mathcal{M}(Q)$ represents the 0-th generation's belief on q_1, and M is the money amount given to the 0-th generation by the government. The solution, when it exists, is denoted by c_0^{y*} and $c_0^{o*}(\cdot)$.

3. Three Concepts of Equilibria

We consider three concepts of equilibria in this model economy. The first one corresponds to ordinary Perfect Foresight Equilibrium, which can be called *single-belief rational-expectations equilibrium*. Economic agents expect that one non-stochastic price sequence is a prevailing market price sequence, and their expectations are realized.

The second equilibrium concept is a natural extension of the Perfect Foresight Equilibrium to multi-belief cases, and thus called *multiple-belief rational-expectations equilibrium*. Here economic agents expect that the true market price sequence is among several stochastic and non-stochastic price sequences, and their expectations are realized: the market price sequence is in fact among the supports of expected price sequences.

The third one is a hybrid of these two. It assumes an additional structure on multiple beliefs. In particular, economic agents hold multiple beliefs about market price sequence, but they also assume the true market price sequence is not far from the perfect-foresight sequence. In particular, they are assumed to believe that the deviation should be no more than γ. In a sense, they are imprecise in forming their expectations by γ. Rational expectation equilibrium in this case is naturally defined as the state that their expectations are fulfilled: the actual market price is in fact within γ of the perfect foresight price sequence. This equilibrium concept can be called γ*-imprecise rational-expectations equilibrium*.

3.1. Single-Belief Rational-Expectations Equilibrium

Let us now start from Single-Belief Rational-Expectations Equilibrium. In this case, the belief correspondence $\mathcal{P}_t(\cdot)$ is singleton-valued and independent of the current price.

To make notation simple, we furthermore assume that there exists $\langle q'_t \rangle_{t=1}^{\infty} \subseteq \mathbb{R}_+^{\infty}$ such that

$$(\forall t \geq 0)(\forall q_t) \quad \mathcal{P}_t(q_t) = \{\delta_{q'_{t+1}}\}\,.$$

Here, $\delta_{q'_{t+1}}$ denotes a point mass concentrated at q'_{t+1}. Then, each generation's objective function (1) is simplified to

$$u(c_t^y) + v(c_t^o(q'_{t+1}))\,.$$

A quadruplet of sequences of real numbers,

$$\big(\langle \hat{c}_t^y \rangle_{t=1}^{\infty}, \langle \hat{c}_t^o \rangle_{t=0}^{\infty}, \langle \hat{M}_t \rangle_{t=1}^{\infty}, \langle q'_t \rangle_{t=1}^{\infty}\big)\,,$$

is a *single-belief rational-expectations equilibrium* of this economy if the following holds:

$$(\forall t \geq 1) \quad \hat{c}_t^y = c_t^{y*}(q'_t)\,,\ \hat{c}_{t-1}^o = c_{t-1}^{o*}(q'_{t-1}; q'_t)\,,\ \hat{M}_t = M_t^*(q'_t)\,,$$
$$\text{and} \quad (\forall t \geq 1) \quad \hat{c}_t^y + \hat{c}_{t-1}^o = 1\,,\ \hat{M}_t = M\,.$$

Alternatively, it is easily shown that $\langle q'_t \rangle_{t=1}^{\infty} \subseteq \mathbb{R}_+^{\infty}$ is single-belief rational-expectations equilibrium if the following holds:

$$(\forall t \geq 1) \quad c_t^{y*}(q'_t) + c_{t-1}^{o*}(q'_{t-1}; q'_t) = 1 \ \text{ and } \ M_t^*(q'_t) = M\,.$$

3.2. Multiple-Belief Rational-Expectations Equilibrium

A quadruplet of sequences of real numbers and correspondences,

$$\big(\langle \hat{c}_t^y \rangle_{t=1}^{\infty}, \langle \hat{c}_t^o \rangle_{t=0}^{\infty}, \langle \hat{M}_t \rangle_{t=1}^{\infty}, \langle \hat{\mathcal{P}}_t(\cdot) \rangle_{t=0}^{\infty}\big)\,,$$

is a *multiple-belief rational-expectations equilibrium* of this economy if there exists a *non*stochastic price process, $\langle \hat{q}_t \rangle_{t=1}^{\infty}$, such that

$$(\forall t \geq 1) \quad \hat{c}_t^y = c_t^{y*}(\hat{q}_t)\,,\ \hat{c}_{t-1}^o = c_{t-1}^{o*}(\hat{q}_{t-1}; \hat{q}_t)\,,\ \hat{M}_t = M_t^*(\hat{q}_t)\,, \tag{3}$$

$$(\forall t \geq 1) \quad \hat{c}_t^y + \hat{c}_{t-1}^o = 1\,,\ \hat{M}_t = M\,, \tag{4}$$

$$\text{and} \quad (\forall t \geq 0) \quad (\forall P \in \hat{\mathcal{P}}_t(\hat{q}_t)) \ \ \hat{q}_{t+1} \in \operatorname{supp}(P)\,. \tag{5}$$

The equations (3) require that the consumption allocation and the money holdings should be optimal along the equilibrium price path. (Here, we set $c_0^{o*}(q_0; q_1) = c_0^{o*}(q_1)$.) The equations (4) are market-clearance conditions both for good and money markets in each period.

Unlike these standard conditions, the last condition (5) is new. This equation requires that the actual price realized in each period should be consistent with each generation's belief in the sense that it is in the supports of all the distributions that each generation considers plausible. (Here, we set $\mathcal{P}_0(q_0) = \mathcal{P}_0$.) Thus the old generation is never surprised in observing the price since all probability distributions in $\hat{\mathcal{P}}_t(\hat{q}_t)$, which was formed by each generation when it was young, have assigned a "positive" probability to that price. (In our nonstochastic framework, this is equivalent to say that the price is not a surprise to the old, since the price is among the old's price expectations $\hat{\mathcal{P}}_t(\hat{q}_t)$.) Each generation's expectations are thus realized.

We make two observations about the multiple-belief rational-expectations equilibrium. First, a sequence of belief correspondences constitutes a part of the equilibrium. This means that the belief itself is determined endogenously within the equilibrium. This property is analogous to the property that a prior distribution over the nodes in the same information set (or an "assessment") constitutes a part of the "sequential equilibrium" of a game.

Second, each generation in the model need not be assumed to know the equilibrium "correctly." This fact differentiates the multiple-belief rational-expectations equilibrium largely from the single-belief rational-expectations equilibrium, where the t-th generation should correctly anticipate the equilibrium price $\hat{q}_{t+1}$ when that generation is young (that is, in the t-th period).

Perfect foresight assumes both the agents in the model and the outside observer (economist) share the common knowledge about the model's structure. This assumption is admittedly strong and unrealistic. The assumption of multiple beliefs is a natural alternative to this straight jacket assumption. Although beliefs are multiple, the utility-maximization problem is well-defined through the assumption of uncertainty aversion. Moreover, the equilibrium concept itself is well-defined through the condition (5), even though we do not assume that each generation shares the correct knowledge about the model's structure with the outside observer.

3.3. γ-imprecise Rational-Expectations Equilibrium

Finally, we introduce a special kind of multiple-belief rational-expectations equilibrium, which can be viewed as a small deviation from single-belief rational-expectations equilibrium.

Let $\langle q'_t \rangle_{t=1}^{\infty} \subseteq \mathbb{R}_+^{\infty}$ be the single-belief rational-expectations equilibrium price sequence, or equivalently, the perfect foresight equilibrium price sequence.[3] We define $\mathcal{P}_t(\cdot)$ by

$$(\forall t \geq 0)(\forall q_t) \quad \mathcal{P}_t(q_t) = \mathcal{M}^{\circ}([q'_{t+1} - \gamma, q'_{t+1} + \gamma])$$

Then, a quadruplet of sequences of real numbers,

$$\left(\langle \hat{c}_t^y \rangle_{t=1}^{\infty}, \langle \hat{c}_t^o \rangle_{t=0}^{\infty}, \langle \hat{M}_t \rangle_{t=1}^{\infty}, \langle \hat{q}_t \rangle_{t=1}^{\infty} \right),$$

is a γ*-imprecise rational-expectations equilibrium* if the following holds:

$$(\forall t \geq 1) \quad \hat{c}_t^y = c_t^{y*}(\hat{q}_t)\,,\ \hat{c}_{t-1}^o = c_{t-1}^{o*}(\hat{q}_{t-1}; \hat{q}_t)\,,\ \hat{M}_t = M_t^*(\hat{q}_t)\,, \tag{6}$$

$$(\forall t \geq 1) \quad \hat{c}_t^y + \hat{c}_{t-1}^o = 1\,,\ \hat{M}_t = M\,, \tag{7}$$

$$\text{and} \quad (\forall t \geq 0) \quad \hat{q}_{t+1} \in [q'_{t+1} - \gamma, q'_{t+1} + \gamma]\,. \tag{8}$$

Obviously, when $\gamma = 0$, γ*-imprecise rational-expectations equilibrium* turns out to be single-belief rational-expectations equilibrium or equivalently, perfect foresight equilibrium.

4. Existence, Multiplicity and Complexity

4.1. Difference Inclusion

Throughout this section, we assume that the felicity functions of each generation, u and v, satisfy (i) u and v are twicely continuously differentiable; (ii) u and v are strictly increasing and strictly concave; and (iii) $\lim_{c^y \to 0} u'(c^y) = \lim_{c^o \to 0} v'(c^o) = +\infty$ and $\lim_{c^y \to \infty} u'(c^y) = \lim_{c^o \to \infty} v'(c^o) = 0$.

In this section, we employ a constructive approach to show the existence of multiple-belief rational-expectations equilibrium and to characterize it. There the concept of difference inclusion plays a crucial role.

First, given $q_t, q_{t+1} \in Q$, consider an optimization problem given by

$$\max_{0 \leq M_t \leq q_t} u\left(1 - \frac{M_t}{q_t}\right) + v\left(\frac{M_t}{q_{t+1}}\right). \tag{9}$$

[3] Theoretically, this “focal” price sequence is not necessarily the perfect foresight sequence. It can be any sequence, including the perfect foresight one. However, since the most natural focal-price-sequence in our setting is the perfect foresight one, we adopt the definition of the text.

Under the stated assumptions on u, there exists a unique solution to this problem, and we denote it by $M^*(q_t, q_{t+1})$. Second, consider the equation:

$$M^*(q_t, q_{t+1}) = M \,, \tag{10}$$

where M is the total money supply of this economy (see Section 2.2). We can show the following lemma:

Lemma 1. *The equation (10) can be uniquely solved for q_t as a function of q_{t+1}. That is, we may write as $q_t = f(q_{t+1})$, where f is some continuously differentiable function which maps $(0, +\infty)$ into $(M, +\infty)$.*

By Lemma 1, it holds that

$$(\forall q_{t+1} \in \mathbb{R}_{++}) \quad M^*(f(q_{t+1}), q_{t+1}) = M \,. \tag{11}$$

Let us now consider a (backward) *difference inclusion* defined by

$$(\forall t \geq 1) \quad f(q_{t+1}) \in \left[\, \underline{q}, q_t \right] \,. \tag{12}$$

A solution to (12) is defined as any sequence, $\langle q_t^* \rangle_{t=1}^{\infty}$, which satisfies (12). Obviously, this generalizes a (backward) difference *equation*:

$$(\forall t \geq 1) \quad f(q_{t+1}) = q_t \tag{13}$$

by replacing the equality with the inclusion. Clearly, any solution to (13) is also a solution to (12).

4.2. Description of Equilibria

This section characterizes multiple-belief rational-expectations equilibria of this economy. Proofs are relegated to Section 6.

Theorem 1 (Existence). *Suppose that there exists a solution $\langle q_t^* \rangle_{t=1}^{\infty}$ to the difference inclusion (12). Then, the quadruplet of sequences, $\left(\langle \hat{c}_t^y \rangle, \langle \hat{c}_t^o \rangle, \langle \hat{M}_t \rangle, \langle \hat{\mathcal{P}}_t(\cdot) \rangle\right)$, each component of which is defined by*

$$(\forall t \geq 1) \quad \hat{c}_t^y = 1 - M/\hat{q}_t \,,\ \hat{c}_{t-1}^o = M/\hat{q}_t \,,\ \hat{M}_t = M \tag{14}$$

$$\textit{and} \quad (\forall t \geq 0)(\forall q_t) \quad \hat{\mathcal{P}}_t(q_t) = \hat{\mathcal{P}}_t = \mathcal{M}^\circ\left(\left[\underline{q}, q^*_{t+1}\right]\right), \tag{15}$$

is a multiple-belief rational-expectations equilibrium with the supporting price process $\langle \hat{q}_t \rangle_{t=1}^{\infty}$ *given by*

$$(\forall t) \quad \hat{q}_t = f(q^*_{t+1}). \tag{16}$$

The following theorem provides a sufficient condition for the existence of uncountably many solutions to the difference inclusion (12), which we will use to show multiplicity of multiple-belief rational-expectations equilibrium. (We hereafter use a "supporting price process to equilibrium" and an "equilibrium price process" interchangeably so long as there is no confusion.)

Theorem 2 (Multiplicity of Solutions to Difference Inclusion). *Suppose that there exists* $\hat{q} \in (M, \overline{q})$ *such that* $f(\hat{q}) = \hat{q}$*. Also, assume that* $f'(\hat{q}) < 1$*. Then, there exist* $\tilde{q} \in (\hat{q}, \overline{q})$ *such that for any* $q^* \in (\hat{q}, \tilde{q})$*, the sequence* $\langle q^*_t \rangle_{t=1}^{\infty}$ *defined by* $(\forall t)$ $q^*_t = q^*$ *solves the difference inclusion (12).*

The condition that $f'(\hat{q}) < 1$ in this theorem is equivalent to the condition:

$$u'\left(1 - \frac{M}{\hat{q}}\right) - u''\left(1 - \frac{M}{\hat{q}}\right) \cdot \left(\frac{M}{\hat{q}}\right) > v'\left(\frac{M}{\hat{q}}\right) + v''\left(\frac{M}{\hat{q}}\right) \cdot \left(\frac{M}{\hat{q}}\right)$$

since

$$\begin{aligned} f'(\hat{q}) &= -\frac{M^*_2(\hat{q}, \hat{q})}{M^*_1(\hat{q}, \hat{q})} \\ &= \left[v'\left(\frac{M}{\hat{q}}\right) + v''\left(\frac{M}{\hat{q}}\right) \cdot \left(\frac{M}{\hat{q}}\right)\right] \Big/ \left[u'\left(1 - \frac{M}{\hat{q}}\right) - u''\left(1 - \frac{M}{\hat{q}}\right) \cdot \left(\frac{M}{\hat{q}}\right)\right], \end{aligned}$$

where M^*_i denotes the partial derivative of M^* with respect to the i-th argument. This condition is easily verified in specific examples and we will show in the next section an example satisfying all the assumptions of Theorem 2, by using this condition.

Under the assumptions of Theorem 2, (14) and (15) characterize a multiple-belief rational-expectations equilibrium by Theorem 1. If we assume additionally that $f'(\hat{q}) \neq 0$ also holds, f takes on uncountably many values on $(\hat{q}, \tilde{q})$ by its continuity. Therefore, Theorems 1 and 2 yield the following corollary.

Corollary 1 (Multiplicity). *Suppose that all the assumptions of Theorem 2 hold. Then, (14) and (15) characterize a multiple-belief rational-expectations equilibrium with* $\langle q_t^* \rangle_{t=1}^{\infty}$ *defined in Theorem 2. Furthermore, if* $f'(\hat{q}) \neq 0$ *also holds, there exists uncountably many supporting prices to this equilibrium.*

Define the degree of the relative risk aversion R_v of the felicity function v by

$$(\forall c \in \mathbb{R}_+) \quad R_v(c) = -\frac{v''(c)c}{v'(c)} .$$

Lemma 2. *Assume that* $(\forall c)\ R_v(c) < 1$*. Then, the function* f *is strictly increasing.*

In Theorem 2 there are multiple solutions, but they are "simple" in the sense that they are constant over time. In contrast, the next theorem shows that under the assumption that the degree of relative risk aversion is mild, a very complex price process can be a solution to the difference inclusion.

Theorem 3 (Complexity of Solutions to Difference Inclusion). *Suppose that all the assumptions of Theorem 2 hold. Further assume that* $(\forall c)\ R_v(c) < 1$*. Then, there exists a non-degenerate interval* $[q_*, q^*]$ *such that any sequence* $\langle q_t^* \rangle_{t=1}^{\infty}$ *satisfying* $(\forall t)\ q_t^* \in [q_*, q^*]$ *solves the difference inclusion (12).*

The next corollary is immediate by the strictly increasing property of f (Lemma 2). In contrast to Corollary 2 where equilibrium prices are constant over time, this corollary implies that equilibrium price processes may be very complex. In particular, it implies that there exists a multiple-belief rational-expectations equilibrium with declining supporting prices.

Corollary 2 (Complexity). *Suppose that all the assumptions of Theorem 3 hold. Then, (14) and (15) characterize a multiple-belief equilibrium with* $\langle q_t^* \rangle_{t=1}^{\infty}$ *defined in Theorem 3. Furthermore, any sequence* $\langle \hat{q}_t \rangle_{t=1}^{\infty}$ *satisfying* $(\forall t)\ \hat{q}_t \in [f(q_*), f(q^*)]$ *is a supporting price to this equilibrium. In particular, there exists a supporting price which is steadily decreasing.*

Finally, the next theorem clarifies sufficient conditions for the existence of γ-imprecise rational-expectations equilibrium.

Theorem 4 (Existence of γ-imprecise Rational-Expectations Equilibrium). *Suppose that there exists $q' \in (M, \overline{q})$ such that $f(q') = q'$ (i.e., perfect foresight equilibrium). Also, assume that $f'(q') < 1$ and that $(\forall c)\ R_v(c) < 1$. Then, there exists $\gamma > 0$ such that there exists γ-imprecise rational-expectations equilibrium around $(q', q', q', \dots)$.*

In particular, it can be shown that any γ satisfying

$$\frac{f(q'+\gamma) - f(q')}{\gamma} < 1 \tag{17}$$

can be γ in Theorem 4.

5. Parametric Examples

5.1. CRRA Utility

In this subsection, we specify the preference of each generation by a time-separable utility function with a constant relative risk aversion (CRRA). That is, we assume

$$(\forall c^y, c^o) \quad u(c^y) + v(c^o) = v(c^y) + v(c^o)$$

where $v : \mathbb{R}_+ \to \mathbb{R}$ is a function defined by

$$v(c) = \frac{c^{1-\theta}}{1-\theta}.$$

Here, $\theta > 0$ is a degree of relative risk aversion, which we suppose to be moderate by assuming that $\theta \in (0, 1)$. Throughout this section, we set $\underline{q} = 0$, and assume that $\overline{q} > 2M$.

5.1.1. Single-Belief Rational-Expectations Equilibrium

We compute the single-belief rational-expectations equilibrium of this economy. Although this is reduced to the well-known perfect foresight equilibrium, it is worthwhile to explain it in our multi-belief setting for discussions that follow. To this end, we suppose that each generation's belief correspondence satisfies

$$(\forall t \geq 0)(\forall q_t) \quad \mathcal{P}_t(q_t) = \{\delta_{q'_{t+1}}\}, \tag{18}$$

where $\langle q'_t \rangle_{t=1}^{\infty}$ is an equilibrium price process and $\delta_{q'_{t+1}}$ is a point mass concentrated at q'_{t+1}. The equation (18) reflects the assumption that along the equilibrium path, each generation expects the next period's price perfectly.

Under (18), it can be shown that the equilibrium price process $\langle q'_t \rangle_{t=1}^{\infty}$ satisfies the *forward* difference equation defined by[4]

$$(\forall t \geq 1) \quad q_{t+1} = g(q_t) \equiv q_t \left(\frac{q_t}{M} - 1 \right)^{\frac{\theta}{1-\theta}} . \tag{19}$$

It can be easily verified that (i) $g(M) = 0$; (ii) g is strictly increasing as far as $q_t \geq M$; (iii) aside from $q_t = 0$, g has a unique fixed point, $q_t = 2M$; and (iv)

$$g'(2M) = \frac{1+\theta}{1-\theta} > 1 . \tag{20}$$

By (ii), we know that (19) can be converted into the *backward* difference equation as (13) with $f = g^{-1}$. (See Figure 1.)

We now show that the single-belief rational-expectations equilibrium price process is unique. First, suppose that $q_1 < 2M$. Then, (20) shows that there exists $t' > 1$ such that $q_{t'} < M$, which contradicts the nonnegativity of $c^y_{t'}$. Second, suppose that $q_1 > 2M$. Then, (20) shows that $\lim_{t \to \infty} q_t = +\infty$, which contradicts the assumption that $(\forall t)\ q_t \leq \overline{q} < +\infty$. We thus conclude that $\langle \hat{q}_t \rangle_{t=1}^{\infty}$ such that $(\forall t)\ \hat{q}_t = 2M$ is the unique single-belief rational-expectations equilibrium price process, or the perfect foresight equilibrium price process.

5.1.2. Multiple-Belief Rational-Expectations Equilibrium

We now show that there are uncountably many multiple-belief equilibrium prices in the sense defined in Section 4.2. Let q^* be any price such that $q^* > 2M$. Then, $\langle \hat{q}_t \rangle_{t=1}^{\infty}$ defined by $(\forall t)\ \hat{q}_t = f(q^*)$ is an equilibrium price.

To see this, define a price process $\langle q^*_t \rangle_{t=1}^{\infty}$ by $(\forall t)\ q^*_t = q^*$. We note that (20) implies that $\langle q^*_t \rangle_{t=1}^{\infty}$ solves (12) since f is the inverse function of g defined by (19). Therefore, we may invoke Theorem 1 to conclude that $\langle \hat{q}_t \rangle_{t=1}^{\infty}$ defined by $(\forall t)\ \hat{q}_t = f(q^*)$ is an equilibrium price.

[4]This follows by solving the first-order condition for each generation's problem:

$$\left(1 - \frac{M_t}{q_t}\right)^{-\theta} \left(\frac{1}{q_t}\right) = \left(\frac{M_t}{q_{t+1}}\right)^{-\theta} \left(\frac{1}{q_{t+1}}\right)$$

for q_{t+1} as a function of q_t, and then by setting $M_t = M$.

Furthermore, since q^* was arbitrary as long as it is greater than $2M$ and since f is strictly increasing, there are uncountably many multiple-belief equilibrium prices.

It should be noted that those multi-belief rational expectations equilibrium are simple constant-price ones. There are other multi-belief rational-expectations in which equilibrium price processes are complex.

(We need a simple numerical example of complex equilibrium price processes here, which is a simple application of Corollary 2.)

5.1.3. γ-imprecise Rational-Expectations Equilibrium

Let $\gamma \in [0, \overline{q} - 2M]$. Then, it is immediate from (20) to see that γ satisfies (17), where $q' = 2M$. By the proof of Theorem 4 (see Section 6), we know that $(\forall t)\ \hat{q}_t = f(2M + \gamma)$ is the equilibrium: in particular, (8) is satisfied.

5.2. Linear Uitility

Throughout this section, we assume that utility function is given by a linear form[5], that is,

$$(\forall c_t^y, c_t^o) \quad u(c_t^y) + v(c_t^o) = c_t^y + c_t^o\,.$$

We observe that any perfect foresight equilibrium price process must be constant over time. To see this, first assume that an equilibrium price process $\langle q_t' \rangle$ is such that $(\exists t)\ q_t' < q_{t+1}'$. Then, it follows that $M_t^* = 0$, which contradicts the money-market equilibrium condition that $M_t^* = M > 0$. Second, assume that $(\exists t)\ q_t' > q_{t+1}'$. Then, it follows that $M_t^* = q_t'$. Hence, the money-market equilibrium condition implies that $M_{t+1}^* = M = M_t^* = q_t' > q_{t+1}'$, the inequality

[5] When there are time preferences, there exists no perfect foresight price process. To see this, let

$$u(c_t^y) + v(c_t^o) = c_t^y + \beta c_t^o$$

and $\beta \neq 1$. Note that by a similar argument to the one in the text, it turns out that any perfect foresight price process $\langle q_t' \rangle$ must satisfy

$$(\forall t) \quad q_{t+1}' = \beta^t q_1'\,.$$

When $\beta \in (0,1)$, there exists some t' such that $q_{t'}' < M$, which contradicts the nonnegativity of $c_{t'}^y$. When $\beta > 1$, $\lim_{t\to\infty} q_t' = +\infty$, which is also a contradiction.

in which contradicts the fact that c^y_{t+1} is nonnegative. Actually, it can easily be seen that any *constant* price process $\langle q'_t \rangle$ is a perfect foresight equilibrium price process as long as $(\forall t)\ q'_t \geq M$.

The previous paragraph shows that there are uncountably many perfect foresight equilibria when the utility function of each generation is linear. This indeterminacy of equilibria cannot happen when each generation is risk-averse with a moderate degree of relative risk aversion as we saw in the previous section. However, only possible perfect foresight equilibrium price processes under the linear preference are those which are constant over time. In contrast, under the linear preference, any *decreasing* price process will be a multiple-belief rational expectation equilibrium price sequence as the following proposition shows.

Proposition 1. *Assume that* $\underline{q} \leq M < \overline{q}$ *and let* $\langle \hat{q}_t \rangle_{t=1}^{\infty}$ *be any price process such that* $(\forall t)\ M \leq \hat{q}_{t+1} \leq \hat{q}_t \leq \overline{q}$. *Then, the quadruplet of sequences,* $\left(\langle \hat{c}^y_t \rangle, \langle \hat{c}^o_t \rangle, \langle \hat{M}_t \rangle, \langle \hat{\mathcal{P}}_t(\cdot) \rangle \right)$, *each component of which is defined by*

$$(\forall t \geq 1) \quad \hat{c}^y_t = 1 - M/\hat{q}_t\,,\ \hat{c}^o_{t-1} = M/\hat{q}_t\,,\ \hat{M}_t = M \tag{21}$$

$$\text{and} \quad (\forall t \geq 0)(\forall q_t) \quad \hat{\mathcal{P}}_t(q_t) = \hat{\mathcal{P}}(q_t) = \mathcal{M}^{\circ}\left(\left[\underline{q}, q_t \right] \right)\,, \tag{22}$$

is a multiple-belief rational-expectations equilibrium with $\langle \hat{q}_t \rangle_{t=1}^{\infty}$ *being the supporting price.*

6. Proofs

Proof of Lemma 1. From the first-order condition of the problem (9), it follows that

$$\frac{1}{q_t} u' \left(1 - \frac{M^*}{q_t} \right) = \frac{1}{q_{t+1}} v' \left(\frac{M^*}{q_{t+1}} \right)\,,$$

where M^* is the unique solution to this problem. Then, by (10), it holds that

$$\frac{1}{q_t} u' \left(1 - \frac{M}{q_t} \right) = \frac{1}{q_{t+1}} v' \left(\frac{M}{q_{t+1}} \right)\,.$$

Define a function $\varphi : (M, +\infty) \to (+\infty, 0)$ by

$$(\forall x) \quad \varphi(x) = \frac{1}{x} u' \left(1 - \frac{M}{x} \right)\,.$$

Then, φ is strictly decreasing and hence invertible. Therefore, we have

$$q_t = f(q_{t+1}) \equiv \varphi^{-1} \circ \frac{1}{q_{t+1}} v' \left(\frac{M}{q_{t+1}} \right) ,$$

which completes the proof. ■

Proof of Theorem 1. Let $\langle q_t^* \rangle_{t=1}^{\infty}$ be any solution to the difference inclusion (12), let $\langle \hat{q}_t \rangle_t$ be a price process defined by (16), and suppose that each generation's belief is given by $\hat{\mathcal{P}}_t = \mathcal{M}^\circ \left(\left[\underline{q}, q_{t+1}^* \right] \right)$ as in (15). This paragraph proves that the objective function (1) of each generation is simplified as

$$\inf \left\{ u(c_t^y) + \int_Q v(c_t^o(q_{t+1}))\, P(dq_{t+1}) \,\middle|\, P \in \mathcal{M}^\circ \left(\left[\underline{q}, q_{t+1}^* \right] \right) \right\} = u\left(1 - \frac{M_t}{\hat{q}_t}\right) + v\left(\frac{M_t}{q_{t+1}^*}\right) .$$

To do this, define a sequence of probability measures, $\langle P^n \rangle_{n=1}^{\infty}$, by

$$(\forall n \geq 1) \quad P^n = \frac{1}{n} P^0 + \left(1 - \frac{1}{n}\right) \delta_{q_{t+1}^*} ,$$

where P^0 is the uniform distribution over $\left[\underline{q}, q_{t+1}^* \right]$ and $\delta_{q_{t+1}^*}$ is a point mass concentrated at q_{t+1}^*. Note that for any n, $P^n \in \mathcal{M}^\circ \left(\left[\underline{q}, q_{t+1}^* \right] \right)$ and that $P^n \to \delta_{q_{t+1}^*}$ as $n \to +\infty$ in the weak topology. Then,

$$\begin{aligned}
& \inf \left\{ u(c_t^y) + \int_Q v(c_t^o(q_{t+1}))\, P(dq_{t+1}) \,\middle|\, P \in \mathcal{M}^\circ \left(\left[\underline{q}, q_{t+1}^* \right] \right) \right\} \\
= \; & \inf \left\{ u\left(1 - \frac{M_t}{\hat{q}_t}\right) + \int_Q v\left(\frac{M_t}{q_{t+1}}\right) P(dq_{t+1}) \,\middle|\, P \in \mathcal{M}^\circ \left(\left[\underline{q}, q_{t+1}^* \right] \right) \right\} \\
\leq \; & \lim_{n \to +\infty} \left[u\left(1 - \frac{M_t}{\hat{q}_t}\right) + \int_Q v\left(\frac{M_t}{q_{t+1}}\right) P^n(dq_{t+1}) \right] \\
= \; & u\left(1 - \frac{M_t}{\hat{q}_t}\right) + v\left(\frac{M_t}{q_{t+1}^*}\right) ,
\end{aligned}$$

where the last equality holds by the continuity of v and the fact that $P^n \to \delta_{q_{t+1}^*}$ as $n \to +\infty$ in the weak topology. Since the inequality of the other direction clearly holds by the strict increase of v, the claim has been proved.

Therefore, the solution $M_t^*(\hat{q}_t)$ to each generation's problem as is defined in Section 2.3 exists uniquely and it satisfies that

$$M_t^*(\hat{q}_t) = M^*(\hat{q}_t, q_{t+1}^*) ,$$

where the right-hand side is the solution to the problem (9) with q_t now being $\hat{q}_t$. Furthermore, this, the definitions of $\hat{q}_t$ and (11) in turn imply that

$$M_t^*(\hat{q}_t) = M^*(\hat{q}_t, q_{t+1}^*) = M^*(f(q_{t+1}^*), q_{t+1}^*) = M \,,$$

which shows that $\hat{M}_t \equiv M$ satisfies the last equation in the condition (3). Also, it follows from this and the budget constraint (2) that

$$c_t^{y*}(\hat{q}_t) = 1 - M_t^*(\hat{q}_t)/\hat{q}_t = 1 - M/\hat{q}_t \quad \text{and} \quad c_{t-1}^{o*}(\hat{q}_{t-1};\hat{q}_t) = M_{t-1}^*(\hat{q}_{t-1})/\hat{q}_t = M/\hat{q}_t \,,$$

which shows that the first two equations in the condition (3) are also satisfied. It is immediate from the definitions of $\hat{c}_t^y$, $\hat{c}_{t-1}^o$ and $\hat{M}_t$ in the theorem that the condition (4) is also met.

Finally, we turn to the condition (5). We observe that for any $P \in \hat{\mathcal{P}}_t = \mathcal{M}^\circ\left(\left[\underline{q}, q_{t+1}^*\right]\right)$, it holds that $\text{supp}(P) = \left[\underline{q}, q_{t+1}^*\right]$ by the definition of $\mathcal{M}^\circ$. Therefore, the condition (5) is now met since it holds that

$$\hat{q}_{t+1} = f(q_{t+2}^*) \in \left[\underline{q}, q_{t+1}^*\right]$$

by the assumption that $\langle q_t^* \rangle$ solves the difference inclusion (12). ■

Proof of Theorem 2. By the assumption that $f'(\hat{q}) < 1$ and the continuity of f, there exists $\tilde{q}$ such that if $q \in (\hat{q}, \tilde{q})$, then $f(q) < q$. Let $q^* \in (\hat{q}, \tilde{q})$ and let $\langle q_t^* \rangle_{t=1}^\infty$ be a sequence such that $(\forall t)\ q_t^* = q^*$. Then, $(\forall t)\ f(q_{t+1}^*) = f(q^*) < q^* = q_t^*$, and hence, $\langle q_t^* \rangle_{t=1}^\infty$ solves the difference inclusion (12). ■

Proof of Lemma 2. By the proof of Lemma 1, it suffices to show that a function defined by

$$q \mapsto \frac{1}{q} v'\left(\frac{M}{q}\right)$$

is strictly decreasing. But, this follows from the assumption that $(\forall c)\ R_v(c) < 1$ since

$$\begin{aligned} \frac{d}{dq}\left[\frac{1}{q} v'\left(\frac{M}{q}\right)\right] &= -\frac{1}{q^2}\left[v'\left(\frac{M}{q}\right) + \frac{M}{q} v''\left(\frac{M}{q}\right)\right] \\ &= -\frac{1}{q^2} v'\left(\frac{M}{q}\right)\left[1 - R_v\left(\frac{M}{q}\right)\right] \\ &< 0\,. \end{aligned}$$

■

Proof of Theorem 3. By Lemma 2, there exists an inverse function of f, $f^{-1} : (M, +\infty) \to (0, +\infty)$, which is strictly increasing. We denote it by g. By the assumptions of the theorem, it holds that $g(\hat{q}) = \hat{q}$ and $g'(\hat{q}) > 1$. Then, there exist q_* and q^* such that $\hat{q} < q_* < q^*$ and $g(q_*) = q^*$. Let $\langle q_t^* \rangle_{t=1}^{\infty}$ be any sequence satisfying $(\forall t)\ q_t^* \in [q_*, q^*]$. We show that $(\forall t)\ q_{t+1}^* \leq g(q_t^*)$, which completes the proof by the strict increase of f. But, this holds because $q_{t+1}^* \leq q^* = g(q_*) \leq g(q_t^*)$, where the last inequality holds by the strict increase of g. ■

Proof of Theorem 4 Let $\langle q_t' \rangle$ and $\langle \hat{q}_t \rangle$ be such that $(\forall t \geq 1)\ q_t' = q'$ and $\hat{q}_t = f(q_{t+1}' + \gamma)$. By a similar argument to the proof of Theorem 1, the objective function (1) of each generation is simplified to

$$u\left(1 - \frac{M_t}{\hat{q}_t}\right) + v\left(\frac{M_t}{q_{t+1}' + \gamma}\right)$$

under $\mathcal{P}_t(q_t) = \mathcal{M}^\circ([q_{t+1}' - \gamma, q_{t+1}' + \gamma])$. Therefore, it follows that

$$M_t^*(\hat{q}_t) = M^*(\hat{q}_t, q_{t+1}' + \gamma) = M^*(f(q_{t+1}' + \gamma), q_{t+1}' + \gamma) = M\,,$$

where the first equality holds by the definition of M^*, the second equality holds by the definition of $\hat{q}_t$ and the last equality holds by the definition of f. This shows that $\hat{M}_t \equiv M$ satisfies the last equation in the condition (6). Also, it follows from this and the budget constraint (2) that

$$c_t^{y*}(\hat{q}_t) = 1 - M_t^*(\hat{q}_t)/\hat{q}_t = 1 - M/\hat{q}_t \quad \text{and} \quad c_{t-1}^{o*}(\hat{q}_{t-1}; \hat{q}_t) = M_{t-1}^*(\hat{q}_{t-1})/\hat{q}_t = M/\hat{q}_t\,,$$

which shows that the first two equations in the condition (6) are also satisfied with $\hat{c}_t^y = 1 - M/\hat{q}_t$ and $\hat{c}_{t-1}^o = M/\hat{q}_t$. It is immediate that the condition (7) is also met.

In order to show (8), let γ be such that

$$\frac{f(q' + \gamma) - f(q')}{\gamma} < 1\,. \tag{23}$$

Since $f'(q') < 1$, such a γ exists. Then, $\hat{q}_{t+1} \geq q_{t+1}' - \gamma$ holds since $\hat{q}_{t+1} = f(q_{t+2}' + \gamma) \geq f(q_{t+2}') = q_{t+2}' = q_{t+1}' \geq q_{t+1}' - \gamma$, where the first inequality holds by the increase of f. Also, $\hat{q}_{t+1} \leq q_{t+1}' + \gamma$ holds since $\hat{q}_{t+1} = f(q_{t+2}' + \gamma) \leq f(q_{t+2}') + \gamma = q_{t+2}' + \gamma = q_{t+1}' + \gamma$, where the first inequality holds by (23). ■

Proof of Proposition 1. Let $\langle \hat{q}_t \rangle_t$ be any price process such that $(\forall t)$ $M \leq \hat{q}_{t+1} \leq \hat{q}_t \leq \overline{q}$ and suppose that each generation's belief is given by $\hat{\mathcal{P}}_t(q_t) = \mathcal{M}^\circ\left(\left[\underline{q}, q_t\right]\right)$ as in (22). We then observe that the objective function (1) of each generation's problem is reduced to

$$\begin{aligned}
& \inf\left\{ c_t^y + \int_Q \left(c_t^o(q_{t+1})\right) P(dq_{t+1}) \,\middle|\, P \in \mathcal{M}^\circ\left(\left[\underline{q}, \hat{q}_t\right]\right) \right\} \\
= \; & \inf\left\{ \left(1 - \frac{M_t}{\hat{q}_t}\right) + \int_Q \left(\frac{M_t}{q_{t+1}}\right) P(dq_{t+1}) \,\middle|\, P \in \mathcal{M}^\circ\left(\left[\underline{q}, \hat{q}_t\right]\right) \right\} \\
= \; & 1 - \frac{M_t}{\hat{q}_t} + \frac{M_t}{\hat{q}_t} = 1
\end{aligned}$$

by a similar reasoning to the one made in the proof of Theorem 1. Therefore, $\hat{M}_t \equiv M$ is certainly a solution to each generation's problems since $M \leq \hat{q}_t$. (Actually, any $\hat{M}_t$ is a solution as long as it satisfies $\hat{M}_t \leq \hat{q}_t$.) This shows that the condition (3) is met. Clearly, the condition (4) is also met.

Finally, we turn to the condition (5). We observe that for any $P \in \hat{\mathcal{P}}_t(\hat{q}_t) = \mathcal{M}^\circ\left(\left[\underline{q}, \hat{q}_t\right]\right)$, it holds that $\text{supp}(P) = \left[\underline{q}, \hat{q}_t\right]$ by the definition of $\mathcal{M}^\circ$. Therefore, the condition (5) is now met since it holds that $\hat{q}_{t+1} \leq \hat{q}_t$ by the assumption. ■

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