Allocations most realizable through strategic manipulation

Yuji Fujinaka *and Toyotaka Sakai †‡

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Abstract

This paper studies the consequences of the strategic manipulation of resource allocation rules in the problem of assigning an indivisible object with monetary transfers. We introduce the notion of most realizable allocations, which are allocations chosen as ε-Nash equilibria for arbitrary small $\varepsilon > 0$ in the direct revelation game of any rule. We show that for every rule satisfying certain normative requirements, the set of most realizable allocations is non-empty and in fact coincides with the set of efficient and envy-free allocations. Thus possible strategic manipulation achieves efficiency and envy-freeness, while any other distributional objective cannot be attained. This result much contrasts with the well-known non-existence of efficient and strategy-proof rules in the literature. Our result also suggests a way not to depend on the Groves rules, which violate budget balance.

Keywords: Mechanism design, Implementation, Direct revelation game, ε-Nash equilibrium, Manipulability, Strategy-proofness, Groves mechanism, Indivisible good, Fair allocation.

JEL codes: C72, D63, D61, C78, D71.

*Graduate School of Economics, Kobe University, 2-1 Rokkodaicho, Nada, Kobe 657-8501; 011d258e@y01.kobe-u.ac.jp

†Division of Economics and Business Administration, Yokohama City University, 22-2 Seto, Kanazawa, Yokohama 236-0027, Japan; toyotaka_sakai@yahoo.co.jp; http://www.geocities.jp/toyotaka_sakai/index.html

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1 Introduction

Designing allocation rules that are robust to strategic misrepresentation of preferences is the main topic in the literature of mechanism design and implementation. However, this requirement is so demanding in general that no reasonable allocation rule satisfies it in many economic models. This motivates us to investigate if possible manipulations are really serious. In this paper, we study this issue in the public decision problem of assigning one indivisible object with monetary transfers. [1]

An *allocation* is a pair of an assignment function that determines who receives the object and a vector of feasible monetary transfers. A *rule* is a function that maps each preference profile to an allocation. We shall analyze which allocations are realizable in the direct revelation game of any rule satisfying certain normative requirements. Nash equilibrium does not exist in many cases, thus we slightly relax the equilibrium notion. An allocation is *most realizable* if it can be chosen by an ε-Nash equilibrium for arbitrary small $\varepsilon > 0$. Our main theorem fully characterizes the set of most realizable allocations for a broad class of rules. In fact they have exactly the same non-empty set. We show that for arbitrary rule satisfying certain efficiency, fairness, and continuity conditions, the set of most realizable allocations is non-empty and in fact coincides with the set of efficient and envy-free allocations. Thus strategic manipulation leads efficiency and envy-freeness, while any other distributional objective cannot be attained.

It is known that the Groves rules (Groves, 1973) are the only decision efficient and strategy-proof rules. [2] However, the Groves rules do not balance budget among agents. Though this is admissible in the auction context at which a third party is supposed to receive money from the accepter, it is a serious drawback in public decision at which monetary transfers are supposed to be closed among the agents. Our theorem suggests that this problem can be resolved by using any rule satisfying certain conditions instead of the Groves rules. Examples of such rules are Shapley solutions and all envy-free rules that are continuous in money.

A closely related study by Tadenuma and Thomson (1995a) also analyzes the manipulability of rules in the same environment. [3] They show that if a (possibly) multi-valued rule satisfies envy-freeness and a technical require-

[1]This problem has a lot of applications such as the assignment of a task, location, or right. Sakai (2005a,b) interprets the object as a garbage incineration facility and analyze fair compensation to the siting district in generalized models. See, Thomson (2005, Ch. 10) for a survey.

[2]This result is due to Green and Laffont (1977) and Holmström (1979).

[3]Thomson (1984, 1988) investigates the same topic in exchange economies. In the papers, he offers surveys of the literature.

ment called "non-discrimination", then the set of certain equilibrium allocations coincides with the set of envy-free allocations. [4] Non-discrimination is a condition that almost immediately implies that all envy-free allocations are supported by equilibria. This condition is hard to be met for single-valued rules. Indeed, all standard rules satisfying non-discrimination in this context are multi-valued. Thus Tadenuma and Thomson's result is strongly oriented to understand the manipulability of multi-valued rules. On the other hand, we focus on single-valued rules. This enables us to compare our results with the results on strategy-proofness in the literature, since strategy-proofness is well-defined only for single-valued rules. Also, it enables us to interpret that our study is to find outcome functions that implement certain desirable allocations in direct revelation games, since every single-valued rule can be seen as the outcome function in the corresponding direct revelation game.

This paper is organized as follows: Section 2 defines the model. Section 3 explains the equilibrium notion and provides the main theorem. Section 4 concludes the discussion. The proof of the theorem is relegated to the Appendix.

2 Model

2.1 Basic notion

Let $I = \{1, 2, \ldots, n\}$ be a finite set of *agents*. There is one indivisible object α to be assigned to agents. For convenience, we consider that if an agent does not receive α, she instead receives a "null" object ν. We assume that monetary transfers are possible. Each $i \in I$ has a valuation over the object, $v_i \in \mathbb{R}$. Agent i's quasi-linear preference over $\{\alpha, \nu\} \times \mathbb{R}$ is then represented by $u(\,\cdot\,; v_i) : \{\alpha, \nu\} \times \mathbb{R} \to \mathbb{R}$ such that for each $m_i \in \mathbb{R}$,

$$u(\alpha, m_i; v_i) = v_i + m_i,$$
$$u(\nu, m_i; v_i) = m_i.$$

Let $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^I$ be a profile of valuations.

An *assignment function* is a function $\sigma : I \to \{\alpha, \nu\}$ such that $|\sigma^{-1}(\alpha)| = 1$. Given $i \in I$, $\sigma(i) = \alpha$ means that i receives the object and $\sigma(i) = \nu$ means that i does not receive it. A *monetary transfer vector* is $m = (m_1, m_2, \ldots, m_n) \in \mathbb{R}^I$ such that $\sum_{i \in I} m_i \leq 0$. Given $i \in I$, $m_i \geq 0$ (resp. $m_i < 0$) is the amount of money he is paid (resp. pays). An *allocation* is a pair of an assignment function and a monetary transfer vector, $x = (\sigma, m)$. We write $(x_i)_{i \in I} = (\sigma(i), m_i)_{i \in I}$. Let X be the set of all allocations.

[4] Generalizations of this result are obtained by Fujinaka and Sakai (2005).

A *rule* is a function ψ from $\mathbb{R}^I$ to X, which associates with each valuation profile $v \in \mathbb{R}^I$ an allocation, $\psi(v) \in X$.

2.2 Axioms

We first introduce efficiency conditions. They respectively state that: no unanimous welfare improvement is possible: an agent whose valuation to the object is highest should receives it; no money should be wasted.

Efficiency: An allocation $x \in X$ is *efficient* at v if there exists no $y \in X$ such that for each $i \in I$, $u(y_i; v_i) \geq u(x_i; v_i)$, with strict inequality holding for at least one agent. A rule ψ is *efficient* if for each $v \in \mathbb{R}^I$, $\psi(v)$ is *efficient* at v.

Decision efficiency: An allocation $x = (\sigma, m) \in X$ is *decision efficient* at v if $v_{\sigma^{-1}(\alpha)} = \max_{i \in I} v_i$. A rule ψ is *decision efficient* if for each $v \in \mathbb{R}^I$, $\psi(v)$ is *decision efficient* at v.

Budget balancedness: An allocation $x = (\sigma, m) \in X$ is *budget balanced* if $\sum_{i \in I} m_i = 0$. A rule ψ is *budget balanced* if for each $v \in \mathbb{R}^I$, $\psi(v)$ is *budget balanced.*

We next introduce fairness requirements. The next notion states that every agent weakly prefers his/her own consumption to anyone else's (Foley, 1967).

Envy-freeness: An allocation $x \in X$ is *envy-free* at v if for each $i, j \in I$, $u(x_i; v_i) \geq u(x_j; v_i)$. A rule ψ is *envy-free* if for each $v \in \mathbb{R}^I$, $\psi(v)$ is *envy-free* at v.

Given $i \in I$ and $v_i \in \mathbb{R}$, imagine the hypothetical situation where all agents have the same valuation as i. Here, all agents are identical, hence it is fair to treat everyone equally. At any such allocation, the identical agents enjoy the equal utility level of $\frac{v_i}{n}$. This is the reference utility level in the hypothetical situation. The following notion states that everyone should be weakly better off than this situation (Moulin, 1990). Alternately, it says that everyone should benefit from the diversity of preferences.

Identical preferences lower bound: An allocation $x \in X$ meets the *identical preferences lower bound* at v if for each $i \in N$, $u(x_i; v_i) \geq \frac{v_i}{n}$. A solution ψ satisfies the *identical preferences lower bound* if for each $v \in \mathbb{R}^I$, $\psi(v)$ meets the *identical preferences lower bound* at v.

The next symmetry condition states that every identical two agents should be treated equally.

Equal treatment of equals: For each $v \in \mathbb{R}^I$ and each $i, j \in I$ with $v_i = v_j$,

$u(\psi_i(v); v_i) = u(\psi_j(v); v_i)$.

The next proposition summarizes relations among the properties defined so far.

Proposition 1. The following relations hold:
(i) *Efficiency* is equivalent to the pair of *decision efficiency* and *budget balancedness*;
(ii) *Envy-freeness* implies *decision efficiency* and *equal treatment of equals*;
(iii) *Envy-freeness* and *budget balancedness* together imply *efficiency* and the *identical preferences lower bound*;
(iv) Given $v \in \mathbb{R}^I$, $x = (\sigma, m) \in X$ is *efficient* and *envy-free* at v if and only if, letting $(\sigma, m) \equiv \psi(v)$ and $j \equiv \sigma^{-1}(\alpha)$,

$$v_j = \max_{i \in I} v_i, \quad \sum_{i \in I} m_i = 0$$
$$\frac{\max_{i \neq j} v_i}{n} \leq m_i = m_k \leq \frac{v_j}{n} \quad \text{for each } i, k \in N \setminus \{j\}$$

Proof. (i) immediately follows from quasi-linearity of preferences. Svensson (1983) shows that *envy-freeness* and *budget balancedness* together imply *efficiency*. A proof similar to this and the trivial fact that *envy-freeness* implies *equal treatment of equals* show (ii). Moulin (1990, p152) shows that *envy-freeness* and *budget balancedness* together imply the *identical preferences lower bound*. [5] Thus (iii) holds.

General versions of (iv) can be found in Tadenuma and Thomson (1995a, Lemma 2) and Bochet and Sakai (2005, Lemma 2). We only show that any *efficient* and *envy-free* allocation is as stated in the statement. Let $v \in \mathbb{R}^I$ and x be such that x is *efficient* and *envy-free* at v. Let $(\sigma, m) \equiv x$ and $j \equiv \sigma^{-1}(\alpha)$. By (i), x is *decision efficient* and *budget balanced*. Since all non-accepters do not envy each other, for each $i, k \neq j$, $m_i = m_k$. Since j does not envy any other agent, and $m_j = -(n-1)m_i$, $v_j + m_j = v_j - (n-1)m_i \geq m_i$. So, $m_i \leq \frac{v_j}{n}$. Since any $i \neq j$ does not envy j, $m_i \geq v_i - (n-1)m_i$. So, $\frac{\max_{i \neq j} v_i}{n} \leq m_j$. Thus (iv) holds. □

The next incentive condition states that no one can gain by preference misrepresentation.

Strategy-proofness: For each $v \in \mathbb{R}^I$, each $i \in I$, and each $v'_i \in \mathbb{R}$, $u(\psi_i(v); v_i) \geq u(\psi_i(v'_i, v_{-i}); v_i)$.

[5] Bevia (1996, Proposition 1) generalizes this result to multi objects cases.

A general result by Homström (1979) implies that the Groves rules (Groves, 1973) are the only *decision efficient* and *strategy-proof* rules. [6] However, it is known that none of them satisfies *budget balancedness.* [7] Thus we have:

Proposition 2. There exists no *efficient* and *strategy-proof* rule.

In this sense all *efficient* rules are manipulable. As explained in Introduction, this motivates us to analyze if manipulation is serious.

We introduce a continuity property, which guarantees the robustness of social decision to small changes or misspecification of preferences. [8] In this economy with an indivisible object, we simply define the condition by continuous choice of monetary transfers when the accepter is unchanged.

Continuity: Let $\{v^t\}_{t\in\mathbb{N}}$ be a sequence in $\mathbb{R}^I$ that converges to some $v^0 \in \mathbb{R}^I$ such that

$$\sigma^t = \sigma^0 \text{ for each } t \in \mathbb{N},$$

where $(\sigma^t, m^t) \equiv \psi(v^t)$ for each $t \in \mathbb{N} \cup \{0\}$. Then $\{m^t\}_{t\in\mathbb{N}}$ converges to m^0.

2.3 Examples of rules

We present a few examples of rules satisfying the axioms. We say that rules ψ, ϕ are *equivalent in welfare* if for each $v \in \mathbb{R}^I$ and each $i \in I$, $u(\psi_i(v); v_i) = u(\phi_i(v); v_i)$.

A rule ψ is an *equal welfare rule* if for each $v \in \mathbb{R}^I$, whenever $(\sigma, m) \equiv \psi(v)$ and $j \equiv \sigma^{-1}(\alpha)$, $v_j = \max_{i\in I} v_i$, $m_j = -\frac{n-1}{n} v_j$ and $m_i = \frac{v_j}{n}$ for each $i \in I \setminus \{j\}$ at such an allocation (Tadenuma and Thomson, 1993). This rule is referred to as "equal utility", since $u(\psi_i(v); v_i) = \frac{v_j}{n}$ for each $i \in I$. All equal welfare rules are equivalent in welfare and in fact coincide with each other when there is only one agent whose valuation is highest. Equal welfare rules satisfy all the properties defined so far except for strategy-proofness. Other rules satisfying all the properties can be found in Tadenuma and Thomson (1995b).

The next rules are based on the Shapley value (Shapley, 1953) in transferable utility games. We first explain them by an example. Let $I = \{1, 2, 3\}$ and

[6] Holmström's result is based on Green and Laffont (1977).

[7] See, Ohseto (2000) and Schummer (2000) for discussion on Holmström's result in this environment. Other studies of *strategy-proof* rules are Miyagawa (2001), Svensson and Larsson (2002), Ohseto (1999, 2004, 2006), and Bochet and Sakai (2005).

[8] Here each rule plays the role of an outcome function in the corresponding direct revelation game. Since we analyze direct revelation games, arguments on the desirability of continuous outcome functions in mechanism design directly apply to rules (see, Postlewaite and Wettstein, 1989).

$v_1 < v_2 < v_3$. Consider the allocation $x = (\sigma, m)$ such that $\sigma(3) = \alpha$ and

$$m_1 = \frac{v_1}{3},\ m_2 = \frac{v_1}{3} + \frac{v_2 - v_1}{2}, \quad \text{and } m_3 = -v_3 + \frac{v_1}{3} + \frac{v_2 - v_1}{2} + \frac{v_3 - v_2}{1}.$$

Then the utility levels of agents at the allocation are

$$u(x_1; v_1) = \frac{v_1}{3},\ u(x_2; v_2) = \frac{v_1}{3} + \frac{v_2 - v_1}{2}, \quad \text{and } u(x_3; v_3) = \frac{v_1}{3} + \frac{v_2 - v_1}{2} + \frac{v_3 - v_2}{1},$$

which are the Shapley values when the worth of each coalition $S \subset I$ is defined by $\max_{i \in S} v_i$.[9] Formally, a rule ψ is a *Shapley rule* if for each $v \in \mathbb{R}^I$, whenever $(\sigma, m) \equiv \psi(v)$ and $j \equiv \sigma^{-1}(\alpha)$,

$$\begin{aligned}
&v_j = v_{[n]},\\
&m_j = -v_j + \frac{v_{[1]}}{|I(v_{[1]})|} + \frac{v_{[2]} - v_{[1]}}{|I(v_{[2]})|} + \cdots + \frac{v_{[n]} - v_{[n-1]}}{|I(v_{[n]})|},\\
&m_i = \frac{v_{[1]}}{|I(v_{[1]})|} + \frac{v_{[2]} - v_{[1]}}{|I(v_{[2]})|} + \cdots + \frac{v_{[\ell]} - v_{[\ell-1]}}{|I(v_{[\ell]})|} \text{ for each } i \neq j \text{ with } v_i = v_{[\ell]},
\end{aligned}$$

where $v_{[\ell]}$ is the ℓ-th lowest valuation among $(v_1, v_2, \ldots, v_n)$ [10] and $I(v_{[\ell]}) \equiv \{k \in I : v_k \geq v_{[\ell]}\}$ for each ℓ. All Shapley rules are equivalent in welfare and in fact coincide with each other when there is only one agent whose valuation is highest. They satisfy the properties defined so far except for *envy-freeness* and *strategy-proofness*.

3 Allocations realizable through manipulation

To study which allocations are realizable as consequences of strategic manipulation, we analyze the direct revelation game of a rule ψ (henceforth, the game of ψ). Given a true profile $v \in \mathbb{R}^I$, a reported profile $b \in \mathbb{R}^I$ is a *Nash equilibrium* in the game of ψ at v if for each $i \in I$ and each $b'_i \in \mathbb{R}$,

$$u(\psi_i(b); v_i) \geq u(\psi_i(b'_i, b_{-i}); v_i).$$

Let $\mathcal{N}(\psi, v)$ be the set of Nash equilibria in the direct revelation game of ψ at v. Then the set

$$\psi(\mathcal{N}(\psi, v)) \equiv \{x \in X : \exists\, b \in \mathcal{N}(\psi, v) \text{ such that } x = \psi(b)\}$$

[9] This argument is based on the simple expression of the Shapley value by Littlechild and Owen (1973). Moulin (1992) analyzes the Shapley value in economies with indivisible objects and money.

[10] For example, whenever $v_1 \leq v_2 \leq \ldots \leq v_n$, $v_{[\ell]} = v_\ell$ for each ℓ.

is the set of Nash equilibrium allocations in the game of ψ at v.

We are interested in which types of allocations belong to $\psi(\mathcal{N}(\psi, v))$. However, this set is often empty due to the non-existence of Nash equilibria. For example, consider the case $I = \{1, 2\}$ and the equal welfare rule ψ that breaks a tie by giving priority to agent 1, i.e., for each $v \in \mathbb{R}^I$ with $v_1 = v_2$, agent 1 is chosen to be the $\psi(v)$-accepter. [11]

Consider any $v \in \mathbb{R}^I$ with $v_1 < v_2$. We shall show that no Nash equilibrium exists here. Let $b \in \mathbb{R}^I$ and $j \in I$ be the $\psi(b)$-accepter. Let $i \neq j$.

Case 1: $\boldsymbol{v_j < b_j}$. Since

$$u(\psi_j(v_j, b_i); v_j) \geq \frac{v_j}{2} > v_j - \frac{b_j}{2} = u(\psi_j(b); v_j),$$

agent j can gain by switching from b_j to v_j.

Case 2: $\boldsymbol{b_j \leq v_j}$ and $\boldsymbol{j = 1}$. Then $b_2 \leq b_1 \leq v_1 < v_2$. Since

$$u(\psi_2(b_1, v_2); v_2) = \frac{v_2}{2} > \frac{b_1}{2} = u(\psi_2(b); v_2),$$

agent 2 can gain by switching from b_2 to v_2.

Case 3: $\boldsymbol{b_j \leq v_j}$ and $\boldsymbol{j = 2}$. Then $b_1 \leq b_2 \leq v_2$. Since ψ breaks a tie in favor of agent 1, $b_1 < b_2$. For every $\varepsilon > 0$ such that $b_1 < b_2 - \varepsilon$,

$$u(\psi_2(b_1, b_2 - \varepsilon); v_2) = v_2 - \frac{b_2 - \varepsilon}{2} > v_2 - \frac{b_2}{2} = u(\psi_2(b); v_2).$$

Hence agent 2 can gain by switching from b_2 to such $b_2 - \varepsilon$. Overall, we observed that $\mathcal{N}(\psi, v) = \emptyset$ in all cases.

The key reason for the non-existence can be found in Case 3. Here, agent 2 is the $\psi(b)$-accepter who pays $\frac{b_2}{2}$. However, if she reports $b_2 - \varepsilon$ and if ε is sufficiently small so as not to change the order of reported valuations, agent 2 can reduce her payment to $\frac{b_2 - \varepsilon}{2}$ with accepting the object. However, she cannot find the most profitable ε due to the lack of the smallest number in the open interval (b_1, b_2).

However, it is natural to consider that agent 2 does not care very small gains. In such a case, whenever the interval (b_1, b_2) is sufficiently small, agent 2 does not switch from b_2 to $b_2 - \varepsilon$ with $b_1 < b_2 - \varepsilon$, since the gain from the switch is only $\frac{\varepsilon}{2}$, which is smaller than $\frac{b_2 - b_1}{2}$. This argument motivates us to slightly relax the equilibrium notion so that agents do not care very small gains.

[11] We can proceed the same discussion for the case that agent 2 is given the priority by changing the role of the agents.

Given $\varepsilon > 0$ and a true profile $v \in \mathbb{R}^I$, a reported profile $b \in \mathbb{R}^I$ is an *ε-Nash equilibrium* in the direct revelation game of ψ at v if for each $i \in I$ and each $b'_i \in \mathbb{R}$,

$$u(\psi_i(b); v_i) \geq u(\psi_i(b'_i, b_{-i}); v_i) - \varepsilon.$$

Let $\mathcal{N}^\varepsilon(\psi, v)$ be the set of ε-Nash equilibria in the direct revelation game of ψ at v. Then the set

$$\psi(\mathcal{N}^\varepsilon(\psi, v)) \equiv \{x \in X : \exists\, b \in \mathcal{N}^\varepsilon(\psi, v) \text{ such that } x = \psi(b)\}$$

is the set of ε-Nash equilibrium allocations in the game of ψ at v.

One can easily check that $\mathcal{N}(\psi, v) = \cap_{\varepsilon>0}\mathcal{N}^\varepsilon(\psi, v)$, and $\mathcal{N}(\psi, v) \subseteq \mathcal{N}^\varepsilon(\psi, v)$ for each $\varepsilon > 0$. This implies that

$$\psi(\bigcap_{\varepsilon>0} \mathcal{N}^\varepsilon(\psi, v)) = \psi(\mathcal{N}(\psi, v)) \subseteq \bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(\psi, v)).$$

Recall that our purpose is to understand which allocations are realized through strategic manipulation. Most realizable allocations here should be such that they can be selected by ε-Nash equilibria for arbitrary small $\varepsilon > 0$, i.e., allocations belonging to $\psi(\mathcal{N}^\varepsilon(\psi, v))$ for each $\varepsilon > 0$. Hence, we shall analyze which allocations belong to the set

$$\bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(\psi, v)).$$

Allocations in this set are said to be *most realizable* for ψ at v.

The equal welfare rule in the above discussion has a non-empty set of most realizable allocations. Indeed, consider the v in the discussion and any *efficient* and *envy-free* allocation (σ, m) at v. Proposition 1 implies that $\sigma(2) = \alpha$ and $\frac{v_1}{2} \leq m_1 \leq \frac{v_2}{2}$. Then for each $\varepsilon > 0$, let $b_1 \equiv 2m_1 - \varepsilon$ and $b_2 \equiv 2m_1$. Then agent 1 cannot gain by any deviation and agent 2 cannot gain more than ε by any deviation. Thus such (σ, m) is a most realizable allocation for ψ at v. This implies that all *efficient* and *envy-free* allocations are most realizable.

By a logic similar to Case 3 in the previous discussion, one can conversely show that all most realizable allocations are *efficient* and *envy-free*. Thus the set of most realizable allocations coincides with the set of *efficient* and *envy-free* allocations here. Our theorem implies that this is not an accidental coincidence for the particular rule. It establishes the same equivalence for a quite large class of rules. For each $v \in \mathbb{R}^I$, we denote by $PF(v)$ the set of all *efficient* and *envy-free* allocations for v. Note that Proposition 1 ensures the non-emptiness of this set. [12]

[12] Svensson (1983) and Alkan, Demange, and Gale (1991) prove the non-emptiness in more general situations.

Theorem. Let ψ be an arbitrary rule satisfying *efficiency*, the *identical preferences lower bound*, *equal treatment of equals*, and *continuity*. Then for each $v \in \mathbb{R}^I$,

$$\emptyset \neq \bigcap_{\varepsilon > 0} \psi(\mathcal{N}^{\varepsilon}(\psi, v)) = PF(v).$$

Furthermore, in order to only obtain $\bigcap_{\varepsilon>0} \psi(\mathcal{N}^{\varepsilon}(\psi, v)) \subseteq PF(v)$, *equal treatment of equals* and *continuity* can be dropped from the list.

Proof. See the Appendix. □

From this theorem, we can predict that, for all rules satisfying the four conditions, by letting agents selfishly report their preferences, *efficiency* and *envy-freeness* are achieved through strategic manipulation. In this sense, strategic manipulation does not matter. This implication drastically contrasts with the absence of *efficient* and *strategy-proof* rules. However, the theorem also suggests that all rules satisfying the four conditions coincide with the *efficient* and *envy-free* correspondence through strategic manipulation. This means that any other distributional objective other than *efficiency* and *envy-freeness* cannot be achieved.

4 Concluding remarks

We established the equivalence of the set of most realizable allocations and the set of efficient and envy-free allocations for a broad class of rules. The notion of most realizable allocations was introduced by this study to analyze the consequences of strategic manipulation. In the analysis, this notion helped a lot, since the existence of Nash equilibria in direct revelation games is a difficult requirement here. The non-existence problem is common in other economic environments such as exchange economies with divisible goods. Using this notion to understand the consequences of strategic manipulation in such environments is an interesting future work.

Our theorem clarified the distributional properties of allocations that can be selected as ε-Nash equilibria for every $\varepsilon > 0$. Thus a next step to be done is to analyze the distributional properties of the set of ε-Nash equilibrium allocations for each $\varepsilon > 0$. Our theorem implies that, when ψ satisfies the four conditions in Theorem, the set at least contains all efficient and envy-free allocations. We conjecture that the rest allocations are nearly efficient and envy-free in some sense. This issue is also left to the future research.

Appendix: Proof of Theorem

We hereafter fix a rule ψ that satisfies *efficiency*, the *identical preferences lower bound*, *equal treatment of equals*, and *continuity*. We also fix a true valuation profile $v \in \mathbb{R}^I$. To simplify the discussion, we assume that there is only one agents whose true valuation is highest. This is just to simplify the proof. One can easily extend the proof so as to deal with tie cases. Hence, without loss of generality, we assume that $v_1 \leq v_2 \leq \cdots \leq v_{n-1} < v_n$.

Our purpose is to show $\bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(\psi, v)) = PF(v)$. To prove this, we divide the proof by two parts. The first part shows $\bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(\psi, v)) \subseteq PF(v)$, and the second part proves the converse, $\bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(\psi, v)) \supseteq PF(v)$.

Part I. Most realizable allocations are efficient and envy-free

We show the inclusion relation $\bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(v)) \subseteq PF(v)$. We do not use *equal treatment of equals* and *continuity* in this part.

Lemma 1. For each $x \in \cap_{\varepsilon>0}\psi(\mathcal{N}^\varepsilon(\psi, v))$ and each $i \in I$, $u(x_i; v_i) \geq \frac{v_i}{n}$.

Proof. Let $x \in \cap_{\varepsilon>0}\psi(\mathcal{N}^\varepsilon(\psi, v))$ and $i \in I$. For each $\varepsilon > 0$, there exists b^ε such that $\psi(b^\varepsilon) = x$ and $b^\varepsilon \in \mathcal{N}^\varepsilon(\psi, v)$. By the *identical preferences lower bound*,

$$u(\psi_i(v_i, b^\varepsilon_{-i}); v_i) \geq \frac{v_i}{n}. \tag{1}$$

Since $\psi(b^\varepsilon) = x$ and $b^\varepsilon \in \mathcal{N}^\varepsilon(\psi, v)$, by (1),

$$u(x_i; v_i) = u(\psi_i(b^\varepsilon); v_i) \geq u(\psi_i(v_i, b^\varepsilon_{-i}); v_i) - \varepsilon \geq \frac{v_i}{n} - \varepsilon.$$

Hence, $u(x_i; v_i) \geq \frac{v_i}{n}$. □

Let

$$C(v) \equiv \{(\sigma, m) \in X : m_i = -\frac{m_{\sigma^{-1}(\alpha)}}{n-1} \text{ for each } i \in I \setminus \{\sigma^{-1}(\alpha)\}\}$$

be the set of allocations at which monetary transfers of all non-accepters are the same. Obviously $PF(v) \subseteq C(v)$.

Lemma 2. $\bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(\psi, v)) \subseteq C(v)$.

Proof. Let $x = (\sigma, m) \in \cap_{\varepsilon>0}\psi(\mathcal{N}^\varepsilon(\psi, v))$. Let $j \in I$ be the x-accepter.

Let $\varepsilon > 0$. Let $b^\varepsilon \in \mathcal{N}^\varepsilon(\psi, v)$ be such that $\psi(b^\varepsilon) = x$. Let $\bar{b}^\varepsilon \equiv \max_{i\in I} b^\varepsilon_i$. By *efficiency*, $b^\varepsilon_j = \bar{b}^\varepsilon$ and $\sum_{i\in I} m_i = 0$.

For each $i \neq j$, and each $b'_i < \bar{b}^\varepsilon$, by *efficiency*, i is not the $\psi(b'_i, b^\varepsilon_{-i})$-accepter, so by the *identical preferences lower bound*,

$$u(\psi_i(b'_i, b^\varepsilon_{-i}); v_i) \geq \frac{b'_i}{n}.$$

Since $b^\varepsilon \in \mathcal{N}^\varepsilon(\psi, v)$,

$$u(x_i; v_i) = u(\psi_i(b^\varepsilon); v_i) \geq u(\psi_i(b'_i, b^\varepsilon_{-i}); v_i) - \varepsilon \geq \frac{b'_i}{n} - \varepsilon. \tag{2}$$

For each $i \in I \setminus \{j\}$, since $u(x_i; v_i) = m_i$, by (2),

$$m_i \geq \frac{b'_i}{n} - \varepsilon \quad \text{for each } b'_i < \bar{b}^\varepsilon.$$

Thus,

$$m_i \geq \frac{\bar{b}^\varepsilon}{n} - \varepsilon \quad \text{for each } i \neq j, \text{ and each } \varepsilon > 0. \tag{3}$$

Since $\sum_{i \in I} m_i = 0$, by (3),

$$-m_j = \sum_{i \in I \setminus \{j\}} m_i \geq \frac{n-1}{n}\bar{b}^\varepsilon - (n-1)\varepsilon. \tag{4}$$

Since $b^\varepsilon_j = \bar{b}^\varepsilon$, by the *identical preferences lower bound*,

$$\bar{b}^\varepsilon + m_j \geq \frac{\bar{b}^\varepsilon}{n}. \tag{5}$$

By (4) and (5),

$$-\frac{n-1}{n}\bar{b}^\varepsilon + (n-1)\varepsilon \geq m_j \geq -\frac{n-1}{n}\bar{b}^\varepsilon > -\frac{n-1}{n}\bar{b}^\varepsilon - (n-1)\varepsilon.$$

Thus,

$$\left| m_j - \left(-\frac{n-1}{n}\bar{b}^\varepsilon \right) \right| \leq (n-1)\varepsilon \quad \text{for each } \varepsilon > 0.$$

This implies that

$$\lim_{\varepsilon \to 0} -\frac{n-1}{n}\bar{b}^\varepsilon = m_j,$$

so

$$\lim_{\varepsilon \to 0} \frac{\bar{b}^\varepsilon}{n} = -\frac{m_j}{n-1}. \tag{6}$$

By (3) and (6),

$$m_i = u(x_i; v_i) \geq -\frac{m_j}{n-1} \text{ for each } i \in I \setminus \{j\}. \tag{7}$$

Since $\sum_{i \in I \setminus \{j\}} m_i = -m_j$, (7) implies that

$$m_i = u(x_i; v_i) = -\frac{m_j}{n-1} \text{ for each } i \in I \setminus \{j\}.$$

Hence $x \in C(v)$. □

The next lemma establishes the claim of Part I.

Lemma 3. $\bigcap_{\varepsilon>0} \psi(\mathcal{N}^{\varepsilon}(\psi, v)) \subseteq PF(v)$.

Proof. Let $x = (\sigma, m) \in \cap_{\varepsilon>0}\psi(\mathcal{N}^{\varepsilon}(\psi, v))$. Note that $\sum_{i \in I} m_i = 0$. By Lemma 2, $x \in C(v)$. Let $m_\nu \equiv m_i$ for $i \in I$ with $\sigma(i) = \nu$.

Case 1: $\boldsymbol{\sigma(n) = \alpha}$. By Lemma 1, $v_n + m_n = u(x_n; v_n) \geq \frac{v_n}{n}$. Hence, by $-(n-1)m_\nu = m_n$,

$$-(n-1)m_\nu \geq -\frac{n-1}{n} v_n. \tag{8}$$

By Lemma 1,

$$(n-1)m_\nu \geq \frac{n-1}{n} v_{n-1}. \tag{9}$$

By (8) and (9),

$$\frac{n-1}{n} v_n \geq (n-1)m_\nu \geq \frac{n-1}{n} v_{n-1}.$$

Thus by Proposition 1(iv), $x \in PF(v)$.

Case 2: $\boldsymbol{\sigma(n) = \nu}$. Let $j \neq n$ be the x-accepter. By Lemma 1, $m_\nu \geq \frac{v_n}{n}$. Hence $\sum_{i \in I \setminus \{j\}} u(x_i; v_i) \geq \frac{n-1}{n} v_n$. Since $\sum_{i \in I} u(x_i; v_i) = v_j < v_n$,

$$u(x_j; v_j) = v_j - \sum_{i \in I \setminus \{j\}} u(x_i; v_i) \leq v_j - \frac{n-1}{n} v_n < v_j - \frac{n-1}{n} v_j = \frac{v_j}{n}.$$

This contradicts Lemma 1. Thus this case does not occur. □

Part II. Efficient and envy-free allocations are most realizable

In this part we conversely show that $PF(v) \subseteq \bigcap_{\varepsilon>0} \psi(\mathcal{N}^{\varepsilon}(\psi, v))$.

Lemma 4. Let $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^I$ be such that for each $i, j \in I \setminus \{n\}$, $b_i = b_j < b_n$. Let $(\sigma, m) \equiv \psi(b)$. Then $\sigma(n) = \alpha$ and

$$\frac{b_{n-1}}{n} \leq m_i = m_j \leq \frac{b_n}{n} \text{ for each } i, j \in I \setminus \{n\}.$$

Proof. Let us write $b_\nu \equiv b_i$ for $i \in I \setminus \{n\}$. Since $b_\nu < b_n$, by *efficiency*, $\sigma(n) = \alpha$. Since all $i \in N \setminus \{n\}$ report the same valuation b_ν, by *equal treatment of equals*, they receive the same amount of money, m_ν. By the *identical preferences lower bound*,

$$m_n \geq -\frac{n-1}{n} b_n \text{ and } m_\nu \geq \frac{b_\nu}{n}.$$

Since $m_n = -(n-1)m_\nu$,

$$\frac{b_\nu}{n} \leq m_\nu \leq \frac{b_n}{n}.$$

□

We introduce a technical condition. If $x = (\sigma, m) \in X$ is such that, for each $\varepsilon > 0$, there exists $b \in \mathbb{R}^I$ for which

$$\psi(b) = x, \tag{10}$$
$$nm_\nu - \frac{\varepsilon}{n} \leq b_n \leq nm_\nu + \frac{\varepsilon}{n}, \tag{11}$$
$$b_n - \frac{\varepsilon}{n} \leq b_i < b_n \text{ for each } i \neq n, \tag{12}$$

then we say that *x is ψ-supportable.*

Lemma 5. Every $x \in PF(v)$ is ψ-supportable.

Proof. Let $x = (\sigma, m) \in PF(v)$. By Proposition 1, $\sigma(n) = \alpha$ and

$$\frac{v_{n-1}}{n} \leq m_i = m_j \leq \frac{v_n}{n} \text{ for each } i, j \in I \setminus \{n\}.$$

Let $m_\nu \equiv m_i$ for $i \in I \setminus \{n\}$.

Let $\varepsilon > 0$. Let $b^0 = (b_1^0, b_2^0, \ldots, b_n^0) \in \mathbb{R}^I$ be such that

$$b_n^0 = nm_\nu + \frac{\varepsilon}{2n},$$
$$b_i^0 = nm_\nu - \frac{\varepsilon}{2n} \text{ for each } i \in I \setminus \{n\}.$$

Let $(\sigma^0, m^0) = \psi(b^0)$. By Lemma 4,

$$\sigma^0(n) = \alpha,$$
$$m_\nu - \frac{\varepsilon}{2n^2} \leq m_i^0 = m_j^0 \leq m_\nu + \frac{\varepsilon}{2n^2} \text{ for each } i, j \in I \setminus \{n\}.$$

Let $m_\nu^0 \equiv m_i^0$ for $i \in I \setminus \{n\}$. Using b^0 and (σ^0, m^0), we shall find b that satisfies (10), (11), and (12). There are three cases to consider:

Case 1: $\boldsymbol{m_\nu^0 = m_\nu}$. Then, $\psi(b^0) = x$, and b^0 satisfies (10), (11) and (12).

Case 2: $\boldsymbol{m_\nu^0 > m_\nu}$. Define the function $f : \left[nm_\nu - \frac{\varepsilon}{4n}, nm_\nu + \frac{\varepsilon}{2n}\right] \to \mathbb{R}$ by

$$f(b_n') \equiv m_\nu' \text{ for each } b_n' \in \left[nm_\nu - \frac{\varepsilon}{4n}, nm_\nu + \frac{\varepsilon}{2n}\right],$$

where m_ν' is the amount of money all non-accepters commonly receive at $\psi(b_n', b_{-n}^0)$. This function is well-defined, since at any such $\psi(b_n', b_{-n}^0)$, *efficiency* implies that all agents other than n do not accept the object and *equal treatment of equals* implies that they receive the same amount of money. Furthermore, *continuity* implies that f is continuous.

By Lemma 4,

$$f(nm_\nu - \frac{\varepsilon}{4n}) \leq m_\nu - \frac{\varepsilon}{4n^2}. \tag{13}$$

Hence

$$f(nm_\nu - \frac{\varepsilon}{4n}) \leq m_\nu - \frac{\varepsilon}{4n^2} < m_\nu < m_\nu^0 = f(nm_\nu + \frac{\varepsilon}{2n}). \tag{14}$$

Since f is continuous, by the intermediate value theorem, there exists

$$\hat{b}_n \in \left[nm_\nu - \frac{\varepsilon}{4n}, nm_\nu + \frac{\varepsilon}{2n}\right]$$

such that $f(\hat{b}_n) = m_\nu$. This implies that $\psi(\hat{b}_n, b_{-n}^0) = x$. Thus $(\hat{b}_n, b_{-n}^0)$ satisfies (10), (11), and (12).

Case 3: $\boldsymbol{m_\nu^0 < m_\nu}$. Define the function $g : \left[nm_\nu - \frac{\varepsilon}{2n}, nm_\nu + \frac{\varepsilon}{4n}\right] \to \mathbb{R}$ by

$$g(b_\nu') \equiv m_\nu' \text{ for each } b_\nu' \in \left[nm_\nu - \frac{\varepsilon}{2n}, nm_\nu + \frac{\varepsilon}{4n}\right],$$

where m_ν' is the common amount of money all agents other than n receive at $\psi(b_\nu', \ldots, b_\nu', b_n^0)$. This function is well-defined, since at any such $\psi(b_\nu', \ldots, b_\nu', b_n^0)$, *efficiency* implies that all agents other than n do not accept the object and *equal treatment of equals* implies that they receive the same amount of money. Furthermore, *continuity* implies that g is continuous.

By Lemma 4,

$$m_\nu + \frac{\varepsilon}{4n^2} \leq g(nm_\nu + \frac{\varepsilon}{4n}).$$

Hence

$$g(nm_\nu - \frac{\varepsilon}{2n}) = m_\nu^0 < m_\nu < m_\nu + \frac{\varepsilon}{4n^2} \leq g(nm_\nu + \frac{\varepsilon}{4n}).$$

Since g is continuous, by the intermediate value theorem, there exists

$$\hat{b}_\nu \in \left[nm_\nu - \frac{\varepsilon}{2n}, nm_\nu + \frac{\varepsilon}{4n}\right]$$

such that $g(\hat{b}_\nu) = m_\nu$. This implies that $\psi(\hat{b}_\nu, \ldots, \hat{b}_\nu, b_n^0) = x$. Thus $(\hat{b}_\nu, \ldots, \hat{b}_\nu, b_n^0)$ satisfies (10), (11), and (12).

In either case, we established the existence of a valuation profile that satisfies (10), (11), and (12). Hence x is ψ-supportable. □

The next lemma establishes the claim of Part II.

Lemma 6. *$PF(v) \subseteq \bigcap_{\varepsilon>0} \psi(\mathcal{N}^\varepsilon(\psi, v))$.*

Proof. Let $x = (\sigma, m) \in PF(v)$. Note that n is the x-accepter. Let $m_\nu \equiv m_i$ for $i \in I \setminus \{n\}$.

Let $\varepsilon > 0$. By Lemma 5, x is ψ-supportable. Therefore, there exists $b \in \mathbb{R}^I$ satisfying (10), (11), and (12). It suffices to show that $b \in \mathcal{N}^\varepsilon(\psi, v)$. Let $j \in I$, $b'_j \neq b_j$, and $\psi(b'_j, b_{-j}) = (\sigma', m')$.

Case 1: $\boldsymbol{\sigma'(j) = \alpha}$. By the *identical preferences lower bound*, for each $i \neq j$,

$$u(\psi_i(b'_j, b_{-j}); v_i) = m'_i \geq \frac{b_i}{n} \geq \frac{b_n - \frac{\varepsilon}{n}}{n} \geq \frac{nm_\nu - \frac{\varepsilon}{n} - \frac{\varepsilon}{n}}{n} = m_\nu - \frac{2\varepsilon}{n^2}. \tag{15}$$

Since $\sum_{i \in I} u(\psi_i(b'_j, b_{-j}); v_i) = v_j$, by (15),

$$\begin{aligned}
u(\psi_j(b'_j, b_{-j}); v_j) &= v_j - \sum_{i \in I \setminus \{j\}} u(\psi_i(b'_j, b_{-j}); v_i) = v_j - \sum_{i \in I \setminus \{j\}} m'_i \\
&\leq v_j - (n-1)(m_\nu - \frac{2\varepsilon}{n^2}) = v_j + m_n + \frac{2(n-1)}{n^2}\varepsilon \\
&= u(x_n; v_j) + \frac{2(n-1)}{n^2}\varepsilon
\end{aligned} \tag{16}$$

Since $x \in F(v)$ and $\psi(b) = x$,

$$\begin{aligned}
u(x_n; v_j) + \frac{2(n-1)}{n^2}\varepsilon &\leq u(x_j; v_j) + \frac{2(n-1)}{n^2}\varepsilon \\
&= u(\psi_j(b); v_j) + \frac{2(n-1)}{n^2}\varepsilon.
\end{aligned} \tag{17}$$

By (16) and (17),

$$u(\psi_j(b'_j, b_{-j}); v_j) \le u(\psi_j(b); v_j) + \frac{2(n-1)}{n^2}\varepsilon < u(\psi_j(b); v_j) + \varepsilon.$$

Hence j cannot gain more than ε by this deviation.

Case 2: $\boldsymbol{\sigma'(j) = \nu}$. Let $k \neq j$ be such that $\sigma'(k) = \alpha$. By the *identical preferences lower bound*,

$$m'_i \ge \frac{b_i}{n} \ge \frac{b_n - \frac{\varepsilon}{n}}{n} \text{ for each } i \in I \setminus \{j, k\}, \tag{18}$$

$$m'_k \ge -\frac{n-1}{n} b_k \ge -\frac{n-1}{n} b_n. \tag{19}$$

Hence, since $\sum_{i \in I} u(\psi_i(b'_j, b_{-j}); v_i) = v_k$, by (18) and (19)

$$\begin{aligned} u(\psi_j(b'_j, b_{-j}); v_j) &= v_k - \sum_{i \in I \setminus \{j\}} u(\psi_i(b'_j, b_{-j}); v_i) \\ &= v_k - (v_k + m'_k) - \sum_{i \in I \setminus \{j,k\}} m'_i \\ &\le \frac{(n-1)b_n}{n} - (n-2)\left(\frac{b_n - \frac{\varepsilon}{n}}{n}\right) = \frac{b_n}{n} + \frac{n-2}{n^2}\varepsilon. \end{aligned} \tag{20}$$

By the definition of b_n,

$$\begin{aligned} \frac{b_n}{n} + \frac{n-2}{n^2}\varepsilon &\le \frac{n m_\nu + \frac{\varepsilon}{n}}{n} + \frac{n-2}{n^2}\varepsilon = m_\nu + \frac{n-1}{n^2}\varepsilon \\ &= u(x_{n-1}; v_j) + \frac{n-1}{n^2}\varepsilon. \end{aligned} \tag{21}$$

Since $x \in F(v)$ and $\psi(b) = x$,

$$u(x_{n-1}; v_j) + \frac{n-1}{n^2}\varepsilon \le u(x_j; v_j) + \frac{n-1}{n^2}\varepsilon = u(\psi_j(b); v_j) + \frac{n-1}{n^2}\varepsilon. \tag{22}$$

By (20), (21), and (22),

$$u(\psi_j(b'_j, b_{-j}); v_j) \le u(\psi_j(b); v_j) + \frac{n-1}{n^2}\varepsilon < u(\psi_j(b); v_j) + \varepsilon.$$

Hence j cannot gain more than ε by this deviation. □

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