Love-for-Variety

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Abstract

We study how love-for-variety, -- productivity (or utility) gains from increasing variety of differentiated inputs (or consumer goods) --, depends on the underlying demand structure. Under general symmetric homothetic demand systems, love-for-variety and the substitutability across different varieties can be both expressed as functions of the mass of available varieties V only. Since the homotheticity alone imposes little restrictions on the properties of these two functions, we turn to three classes of homothetic demand systems, H.S.A., HDIA, and HIIA, which are pairwise disjoint with the sole exception of CES. For each of these three classes, we establish the three main results. First, the substitutability is increasing in V, if and only if Marshall's 2^{nd} law of demand (the price elasticity of demand for each variety is increasing in its price) holds. Second, increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety, but the converse is not true. Third, love-for-variety is constant, if and only if substitutability is constant, which occurs only under CES within these three classes. These three classes thus offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different varieties are more substitutable when more varieties are available.

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1. Introduction

Love-for-variety[1] captures the idea that producers (or consumers) can achieve higher level of productivity (or utility) when they have access to a wider variety of differentiated inputs (or consumer goods). It is a natural consequence of the convexity of the production technologies (or preferences). It represents productivity (or utility) gains from increasing variety of differentiated inputs (or consumer goods) and hence forms the basis for willingness to pay for new inputs (or goods); e.g., Dixit and Stiglitz (1977), Krugman (1980), Ethier (1982), and Romer (1987). As such, love-for-variety plays a central role in many fields of economics, as Matsuyama (1995) pointed out, but most prominently in economic growth (Grossman and Helpman 1993, Gancia and Zilibotti 2005, and Acemoglu 2008), international trade (Helpman and Krugman 1985), and economic geography (Fujita, Krugman, and Venables 1999).

However, little is known about how love-for-variety depends on the underlying production (or utility) function, and, in particular, how love-for-variety changes as more and more varieties become available. In a standard treatment, e.g., Matsuyama (1995, Sec.3A), the analytical expression for love-for-variety is obtained under the assumption of CES.[2] It is equal to $1/(\sigma - 1)$, where $\sigma > 1$ represents both the (constant) elasticity of substitution across different varieties and the (constant) price elasticity of demand for each variety. Even though this expression exhibits the appealing property that love-for-variety is smaller when different varieties are more substitutable and the price elasticity of demand for each variety is higher (i.e., a larger σ), it also exhibits the property many find less appealing; that is, love-for-variety is constant. It seems implausible to think that productivity gains enjoyed by producers from additional variety of inputs are independent of how many varieties they have already access to.[3] Of course, constant love-for-variety may be a peculiar artifact of the CES demand system. But, the question is then: under which non-CES demand systems should we expect love-for-variety to, say, decline as more and more varieties become available? This is the question we address in this paper.

In Section 2, we first recall some general properties of symmetric CRS production functions and symmetric homothetic demand systems for input varieties that they generate. Then, we show that both love-for-variety and the substitutability across differentiated input varieties can be defined as functions of the mass of

[1]Different authors called this concept differently; e.g., "the desirability of variety" (Dixit and Stiglitz 1977), "love of variety" (Helpman and Krugman 1985, sec. 6.2), "taste for variety" (Benassy 1996), etc. In the context of input variety, Ethier (1982) called it "gains from an increased division of labor" and Romer (1987) "increasing returns due to specialization." We call "love-for-variety," as in Parenti et.al. (2017) and Thisse and Ushchev (2020), since it seems most common in the recent literature.

[2]CES is also assumed in most empirical assessment of love-for-variety; see e.g., Feenstra (1994), Bils and Klenow (2001), and Broda and Weinstein (2006). A few exceptions include Feenstra and Weinstein (2017), which use translog.

[3]Perhaps due to this unappealing feature of love-for-variety under CES, some authors prefer "the ideal variety approach," e.g., Helpman and Krugman (1985, sec. 6.3), in which consumers are heterogenous in taste, and each consumer buys the only variety closest to his/her ideal variety. Despite each consumer buys only one variety, increasing variety is beneficial in that each consumer finds a variety closer to the ideal variety on average as more varieties becomes available, and yet the benefit of adding variety is diminishing, as the product space becomes congested.

available varieties V only, as $\mathcal{L}(V)$ and $\sigma(V)$, respectively. It turns out that the properties of these two functions, particularly their relation to each other, can be quite complex under general symmetric homothetic demand systems. Moreover, whether Marshall's 2nd law of demand holds or not (the price elasticity of demand for each input is increasing in its own price or not) tells us little about the properties of these two functions. Since the homothetic restriction alone is not strong enough to offer much insight, we turn to three classes of CRS production functions: H.S.A. (Section 3), HDIA (Section 4), and HIIA (Section 5), which are pairwise disjoint with the sole exception of CES, as shown in Figure. [4] For each of these three classes, we establish the three main results. First, the substitutability, $\sigma(V)$, is increasing in the mass of available varieties, V, if and only if Marshall's 2nd law of demand holds. Second, increasing (decreasing) substitutability $\sigma'(V) > (<) 0$, implies diminishing (increasing) love-for-variety $\mathcal{L}'(V) < (>) 0$, but the converse is not true. Third, love-for-variety is constant, $\mathcal{L}'(V) = 0$, if and only if substitutability is constant, $\sigma'(V) = 0$, which occurs only under CES within these three classes. These three classes thus offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different varieties are more substitutable when more varieties are available.[5]

We conclude in Section 6. All technical materials, including the proofs of lemmas, are in appendices.

2. General symmetric homothetic demand system

In what follows, we discuss a general symmetric homothetic demand system in the context of the producer's demand for a variety of differentiated inputs. Thus, consider a monotone, strictly quasi-concave, symmetric CRS production function, $X = X(\mathbf{x})$. Here, $\mathbf{x} = \{x_\omega; \omega \in \overline{\Omega}\}$ is the input quantity vector, defined over $\overline{\Omega}$, a continuum of all potential input varieties, which is divided into the set of available varieties, $\Omega \subset \overline{\Omega}$, and the set of unavailable varieties, $\overline{\Omega}\backslash\Omega$. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega}\backslash\Omega$. We denote the mass of available varieties by $V \equiv |\Omega|$. Our goal is to study the effect of changing V on productivity. To this end, it is necessary to assume that each input is inessential. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega}\backslash\Omega$ does *not* imply $X(\mathbf{x}) = 0$, so that it is possible to produce a positive output, even when some potential varieties are unavailable.

2.1 Duality Theory: A Refresher

Let us first recall some key results from the duality theory; see, e.g., Mas-Colell et al. (1995), and Jehle and Reny (2012). Let $\mathbf{p} = \{p_\omega; \omega \in \overline{\Omega}\}$ denote the input price vector, such that $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$ and $p_\omega < \infty$ for $\omega \in \Omega$. The non-essentiality of inputs ensures that the unit cost function corresponding to this production function, obtained by the following cost minimization problem,

[4] H.S.A., HDIA, and HIIA stand for *Homothetic with a Single Aggregator*, *Homothetic Direct Implicit Additivity*, and *Homothetic Indirect Implicit Additivity*. Matsuyama (2023) discusses the relation between these three and other classes of non-CES demand systems in detail.
[5] As such, they can be valuable alternatives to those who find "the ideal variety approach" more appealing than the love-for-love approach under CES despite that the former is less tractable. We thank Jim Markusen for pointing this out.

$$P(\mathbf{p}) \equiv \min_{\{x_\omega\}} \int_\Omega p_\omega x_\omega d\omega \quad s.t. \quad X(\mathbf{x}) \geq 1, \tag{1}$$

is well-defined, even though $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$. Furthermore, it also satisfies the monotonicity, strict quasi-concavity, linear homogeneity, and symmetry. From the first-order condition of this cost minimization problem, we obtain the inverse demand curve:

$$p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega}; \quad \omega \in \Omega, \tag{2}$$

from which we obtain the budget share of input ω as a function of $\mathbf{x}$ as follows:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega;\, \mathbf{x}); \;\; \omega \in \Omega. \tag{3}$$

Likewise, the production function can be recovered from the unit cost function as follows:

$$X(\mathbf{x}) \equiv \min_{\{p_\omega\}} \int_\Omega p_\omega x_\omega d\omega \quad s.t. \quad P(\mathbf{p}) \geq 1. \tag{4}$$

From the first-order condition of this minimization problem, we obtain the demand curve:

$$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}); \quad \omega \in \Omega. \tag{5}$$

from which we obtain the budget share of input ω as a function of $\mathbf{p}$ as follows:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega; \mathbf{p}); \; \omega \in \Omega. \tag{6}$$

We may thus use either $X(\mathbf{x})$ or $P(\mathbf{p})$ as the primitive of the CRS production technology.

Moreover, from either eq.(2) and eq.(5), and by using the Euler's theorem on linear homogeneity functions,

$$\mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega = P(\mathbf{p}) \int_\Omega \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega = X(\mathbf{x}) \int_\Omega p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} d\omega = P(\mathbf{p})X(\mathbf{x}).$$

This identity means that the total cost of inputs is equal to the total value of output.

2.2 Love-for-Variety

Let us denote symmetric quantity and price patterns among all the available varieties by:

$$\mathbf{x} = x\mathbf{1}_\Omega, \qquad \mathbf{p} = p\mathbf{1}_\Omega^{-1}$$

where $x > 0$ and $p > 0$ are scalars, while the unit quantity vector, $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \overline{\Omega}\}$ and the unit price vector, $\mathbf{1}_\Omega^{-1} \equiv \left\{\left(1_\Omega^{-1}\right)_\omega; \omega \in \overline{\Omega}\right\}$, are defined as follows:

$$(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases}; \qquad \left(1_\Omega^{-1}\right)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases},$$

which satisfies $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega \left(1_\Omega^{-1}\right)_\omega d\omega = |\Omega| \equiv V$.

Under symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, $P(\mathbf{p}) = pP(\mathbf{1}_\Omega^{-1})$, where $P(\mathbf{1}_\Omega^{-1})$ depends only on V, so that it can be rewritten as:

$$P(\mathbf{p}) = pP(\mathbf{1}_\Omega^{-1}) = \frac{p}{z(V)}.$$

The monotonicity of $P(\mathbf{p})$ implies that $z(V)$ is increasing in V. We measure love-for-variety by the rate of decline in the unit cost caused by a proportional increase in V, holding the price of each input $p > 0$ constant, that is,

$$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} > 0.$$

Likewise, under symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, $X(\mathbf{x}) = xX(\mathbf{1}_\Omega)$, where $X(\mathbf{1}_\Omega)$ depends only on V, so that it can be written as:

$$X(\mathbf{x}) = xX(\mathbf{1}_\Omega) = \frac{x}{\mathcal{y}(V)} = \frac{xV}{V\mathcal{y}(V)}.$$

The strict quasi-concavity of $X(\mathbf{x})$ implies that, holding the total amounts of inputs, xV, fixed, $X(\mathbf{x})$ is increasing in V. In other words, $V\mathcal{y}(V)$ is strictly decreasing in V. Thus, if we measure love-for-variety by the rate of increase in output caused by a proportional increase in V, holding xV fixed,

$$\mathcal{L}(V) \equiv -\frac{d \ln V\mathcal{y}(V)}{d \ln V} = -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1.$$

This is essentially the same with the definition proposed by Benassy (1996, eq.(2)) for what he called "taste for variety," even though he applied it only for CES demand systems with externalities.

These two measures of love-for-variety are indeed identical. To see this, inserting $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$ into the identity, $\mathbf{px} = P(\mathbf{p})X(\mathbf{x})$, yields

$$pxV = \frac{p}{z(V)}\frac{x}{\mathcal{y}(V)}$$

and hence

$$z(V) = \frac{1}{V\mathcal{y}(V)} \Longrightarrow \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1 > 0.$$

Hence, we use them interchangeably as the love-for-variety measure.

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1 > 0. \tag{7}$$

2.3 Price Elasticity of Demand for each variety and Marshall's 2nd Law of Demand

Next, using the expressions for the budget share given in eq.(3) and eq.(6), the price elasticity of demand for ω can be written as:

$$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega;\mathbf{p}) \equiv 1 - \frac{\partial \ln s(p_\omega;\mathbf{p})}{\partial \ln p_\omega} = \zeta^*(x_\omega;\mathbf{x}) \equiv \left[1 - \frac{\partial \ln s^*(x_\omega;\,\mathbf{x})}{\partial \ln x_\omega}\right]^{-1}.$$

Marshall's 2nd law states that the price elasticity for each input goes up as its price goes up (and its demand goes down) along its demand curve.

Definition: *Marshall's 2nd Law* holds if and only if

$$\partial\zeta(p_\omega;\mathbf{p})/\partial p_\omega > 0 \Leftrightarrow \partial\zeta^*(x_\omega;\mathbf{x})/\partial x_\omega < 0.$$

2.4 Substitutability Across Different Varieties

Since $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$ are both homogenous of degree zero, the price elasticity at symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ or $\mathbf{x} = x\mathbf{1}_\Omega$, is $\zeta(1;\mathbf{1}_\Omega^{-1}) = \zeta^*(1;\mathbf{1}_\Omega)$, which is independent of p or x, and can be written as a function of V only, as $\sigma(V)$. Moreover, Appendix A shows that $\sigma(V)$ is equal to the Allen-Uzawa elasticity of substitution between every pair of input varieties at the symmetric patterns. Thus, we use the following definition for the substitutability of inputs in the presence of mass V of available input varieties:

Definition: *The substitutability measure* is defined by

$$\sigma(V) \equiv \zeta(1;\mathbf{1}_\Omega^{-1}) = \zeta^*(1;\mathbf{1}_\Omega) > 1. \tag{8}$$

In the special case of CES with gross substitutes,

$$X = X(\mathbf{x}) = Z\left[\int_\Omega x_\omega^{1-\frac{1}{\sigma}}d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P = P(\mathbf{p}) = \frac{1}{Z}\left[\int_\Omega p_\omega^{1-\sigma}d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$ is the (constant) elasticity of substitution parameter and Z is the TFP parameter, it is easy to verify:

$$\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x}) = \sigma > 1$$

$$\sigma(V) = \sigma > 1;$$

$$\mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$$

Thus, under CES, the price elasticity of demand for each variety is *everywhere* constant and equal to σ. Obviously, this implies that our substitutability measure, which is always equivalent to the price elasticity evaluated at the symmetric patterns, $\sigma(V)$, is also equal to σ, and independent of V. Moreover, the love-for-variety measure, $\mathcal{L}(V)$, is also independent of V, and depends solely on the single parameter, σ, with a one-to-one inverse relation between the two.[6] Perhaps for these reasons, some authors have claimed that $\sigma(V)$ is constant

[6]Benassy (1996) proposed to break the tight relation between $\sigma(V)$ and $\mathcal{L}(V)$ under CES, by making TFP a function of V as $Z(V)$ with some sorts of externalities, through which V directly affects TFP. By driving the wedge between social and private demand for additional variety, such modified CES yields love-for-variety from the planner's viewpoint to be equal to $\mathcal{L}(V) = \partial \ln Z(V)/\partial \ln V + 1/(\sigma - 1)$. Moreover, he assumed that $\partial \ln Z(V)/\partial \ln V = \nu - 1/(\sigma - 1)$, so that $\mathcal{L}(V) = \nu$, which can be chosen independently from $\sigma(V) = \sigma$. If we assume instead $\partial \ln Z(V)/\partial \ln V$ is another parameter independent of $\sigma(V) = \sigma$, $\mathcal{L}(V)$ is still inversely related to $\sigma(V) = \sigma$. However,

only under CES, and/or that $\sigma(V)$ is the inverse measure of love-for-variety, even under general homothetic demand systems.

It should be pointed out, however, that the relation between the price elasticity, $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$, the substitutability, $\sigma(V)$, and love-for-variety, $\mathcal{L}(V)$ can be quite complex under general homothetic demand systems. First, whether Marshall's 2nd law holds or not in general tells us little about the sign of the derivative of $\sigma(V)$; This should not be surprising because the former is about how $\zeta\left(p_\omega; p\mathbf{1}_\Omega^{-1}\right)$ or $\zeta^*(x_\omega; x\mathbf{1}_\Omega)$ responds to a change in p_ω or x_ω, while the latter is about how $\zeta\left(p; p\mathbf{1}_\Omega^{-1}\right) = \zeta\left(1; \mathbf{1}_\Omega^{-1}\right)$ or $\zeta^*(x; x\mathbf{1}_\Omega) = \zeta^*(1; \mathbf{1}_\Omega)$ responds to a change in V through its effect on $\mathbf{1}_\Omega^{-1}$ or $\mathbf{1}_\Omega$. Second, as shown in Appendix B, there exists a parametric family of homothetic non-CES demand systems in which $\sigma(V)$ and $\mathcal{L}(V)$ are both independent of V, and they move in the same direction as one of the parameters changes. More generally, with both $\sigma(V)$ and $\mathcal{L}(V)$ being functions of V, one cannot expect $\sigma(V)$ and $\mathcal{L}(V)$ to always move in the opposite direction as V varies.

Nevertheless, it is intuitive to think that, when input varieties are more substitutable, the price elasticity of demand for each variety should be larger, and the love-for-variety measure should be smaller. One need to find some additional restrictions to capture this intuition, since the homothetic restriction alone is not enough. In what follows, we investigate the properties of the substitutability and love-for-variety measures, $\sigma(V)$ and $\mathcal{L}(V)$, in three classes of symmetric CRS production functions, H.S.A.(Section 3), HDIA (Section 4), and HIIA (Section 5). These three classes were initially proposed by Matsuyama and Ushchev (2017) and later extended to allow for a continuum of endogenous range of available inputs in the case of gross substitutes by Matsuyama and Ushchev (2020a). To our purpose, these three classes are useful for two reasons. First, they are pairwise disjoint with the sole exception of CES, as seen in Figure. Thus, they offer three alternative ways of departing from CES, while keeping CES as a special case. Second, each of the three classes generates the demand system with the property that the price elasticity of demand for each input can be expressed as $\zeta(p_\omega;\mathbf{p}) = \zeta(p_\omega/\mathcal{A}(\mathbf{p}))$ and $\zeta^*(x_\omega;\mathbf{x}) = \zeta^*(x_\omega/\mathcal{A}^*(\mathbf{x}))$. That is, the price elasticity is a function of a single variable, $p_\omega/\mathcal{A}(\mathbf{p})$ or $x_\omega/\mathcal{A}^*(\mathbf{x})$, where $\mathcal{A}(\mathbf{p})$ or $\mathcal{A}^*(\mathbf{x})$ is the linear homogeneous aggregator in $\mathbf{p}$ or in $\mathbf{x}$, whose value serves as a sufficient statistic that captures the interdependence of price elasticities across varieties. Thus, in these three classes, the price elasticity responds to an increase in p_ω and to a decline in $\mathcal{A}(\mathbf{p})$ in the same way, and hence also to an increase in V in the symmetric price patterns. Or equivalently, the price elasticity responds to a decline in x_ω and to an increase in $\mathcal{A}^*(\mathbf{x})$ in the same way, and hence also to an increase in V at the symmetric quantity patterns. This feature enables us to establish the following three results for each of the three classes. First, Marshall's 2nd law is equivalent to increasing substitutability, $\sigma'(V) > 0$. Second, increasing (decreasing) substitutability $\sigma'(V) > (<)0$ is sufficient

adding such externalities does not affect the willingness to pay for additional variety by atomistic producers and consumers. Furthermore, it does not address our question; that is, how $\mathcal{L}(V)$ depends on V under homothetic non-CES demand systems.

but not necessary for diminishing (increasing) love-for-variety, $\mathcal{L}'(V) < (>)0$. Third, $\mathcal{L}(V)$ is constant if and only if it is CES.

3. The HSA class.

We call a symmetric CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, H.S.A. (*Homothetic with a Single Aggregator*) if the budget share of ω, as a function of $\mathbf{p}$, can be written as:

$$s_\omega = \frac{\partial \ln P(\boldsymbol{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\boldsymbol{p})}\right). \tag{9}$$

Here, $s: \mathbb{R}_{++} \to \mathbb{R}_+$ is the *budget share function,* which is C^2 and strictly decreasing as long as $s(z) > 0$, with $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$, which can be finite or infinite, and $A(\mathbf{p})$ is linear homogenous in $\mathbf{p}$, defined implicitly and uniquely by the adding-up constraint,

$$\int_\Omega s\left(\frac{p_\omega}{A(\boldsymbol{p})}\right) d\omega \equiv 1, \tag{10}$$

which ensures, by construction, that the budget shares of all inputs are added up to one.

CES with gross substitutes is a special case where $s(z) = \gamma z^{1-\sigma}$ $(\sigma > 1)$. Translog unit cost function is another special case, where $s(z) = \gamma \max\{-\ln(z/\bar{z}), 0\}$, where $\bar{z} < \infty$.[7] The CoPaTh family[8] of H.S.A. is given by $s(z) = \gamma \max\left\{\left[\sigma - (\sigma - 1)z^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}, 0\right\}$, where $0 < \rho < 1$, with $\bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}} \to \infty$ and $s(z) \to \gamma z^{1-\sigma}$, as $\rho \nearrow 1$. Symmetric H.S.A. have been recently applied to a variety of monopolistic competition models.[9]

Eqs.(9)-(10) state that the budget share of an input is decreasing in its *normalized price,* $z_\omega \equiv p_\omega/A(\mathbf{p})$, which is defined as its own price, p_ω, divided by the *common price aggregator,* $A(\mathbf{p})$. Notice that $A(\mathbf{p})$ is independent of ω. Thus, it captures "the average price" against which the prices of *all* inputs are measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p})$, which is the key feature of H.S.A.[10] The monotonicity of $s(\cdot)$, combined with the assumptions,

[7] For $s: \mathbb{R}_{++} \to \mathbb{R}_+$, satisfying the above conditions, a class of the budget share functions, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$, generate the same demand system with the same common price aggregator. This can be seen by reindexing the varieties, as $\omega' = \gamma\omega$, so that $\int_\Omega s_\gamma(p_\omega/A(\mathbf{p}))d\omega = \int_\Omega s(p_{\omega'}/A(\mathbf{p}))d\omega' = 1$. In this sense, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$ are all equivalent. Note also that a class of the budget share functions, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$, generate the same demand system, with $A_\lambda(\mathbf{p}) = \lambda A(\mathbf{p})$, because $s_\lambda(p_\omega/A_\lambda(\mathbf{p})) = s(\lambda p_\omega/A_\lambda(\mathbf{p})) = s(p_\omega/A(\mathbf{p}))$. In this sense, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$ are all equivalent.

[8] CoPaTh stands for Constant Pass-Through; it is so named, since, when a monopolistic competitive firm faces the demand curve generated by this family, its pricing behavior features a constant pass-through rate, $0 < \rho < 1$, and it converges to CES, as $\rho \nearrow 1$. Matsuyama and Ushchev (2020b) developed the CoPaTh family of demand systems within H.S.A., HDIA, and HIIA.

[9] See, e.g., Baqaee, Farhi, and Sangani (2023), Fujiwara and Matsuyama (2022), Grossman, Helpman, and Luillier (2023), Matsuyama and Ushchev (2020a, 2020b, 2022a, 2022b). A large literature on monopolistic competition models under translog demand systems, which follows Feenstra (2003), may be also added to this list, because a symmetric translog unit cost function is a special case of symmetric H.S.A. with gross substitutes.

[10] In contrast, that $s(\cdot)$ is independent of ω is not a defining feature of H.S.A., but due to the assumption that the underlying production function is symmetric. Generally, the H.S.A. class of the production functions is defined by the property that the budget share of ω is given by $s_\omega(p_\omega/A(\mathbf{p}))$, where $A(\mathbf{p})$ is the unique solution to $\int_\Omega s_\omega(p_\omega/A(\mathbf{p}))d\omega = 1$. Note that $s_\omega(\cdot)$ may depend on ω but $A(\cdot)$ does not.

$\lim_{z\to 0}s(z) = \infty$ and $\lim_{z\to\bar{z}}s(z) = 0$, ensures that $A(\mathbf{p})$ is defined uniquely by eq.(10) for any $V \equiv |\Omega| > 0$. Note also that we allow for the possibility of $\bar{z} < \infty$, in which case $\bar{z}A(\mathbf{p})$ is the choke price, at which demand for a variety goes to zero. If $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector.

The price elasticity of each input can be written as a function of a single variable, $z_\omega \equiv p_\omega/A(\mathbf{p})$:

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta(z_\omega) > 1,$$

where $\zeta: (0,\bar{z}) \to (1,\infty)$ is continuously differentiable for $z \in (0,\bar{z})$, and $\lim_{z\to\bar{z}} \zeta(z) = \infty$ if $\bar{z} < \infty$. Note that the assumption that $s(\cdot)$ is strictly decreasing in $z < \bar{z}$ ensures $\zeta(z_\omega) > 1$. That is, inputs are *gross substitutes.*[11] In general, $\zeta(\cdot)$ can be nonmonotonic. Under CES, it is constant, $\zeta'(\cdot) = 0$. Marshall's 2nd law, $\partial\zeta(p_\omega; \mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta'(\cdot) > 0$, the condition satisfied both by translog with $\zeta(z_\omega) = 1 - \frac{1}{\ln(z_\omega/\bar{z})}$ and by CoPaTh with $\zeta(z_\omega) = \frac{\sigma}{\sigma-(\sigma-1)z_\omega{}^{(1-\rho)/\rho}} = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$.

The unit cost function, $P(\mathbf{p})$, can be obtained by integrating eq.(9), which yields

$$\frac{A(\boldsymbol{p})}{P(\boldsymbol{p})} = c\, exp\left[\int_\Omega \left[\int_{p_\omega/A(\boldsymbol{p})}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi\right] d\omega\right] \equiv c\, exp\left[\int_\Omega s\left(\frac{p_\omega}{A(\boldsymbol{p})}\right)\Phi\left(\frac{p_\omega}{A(\boldsymbol{p})}\right) d\omega\right] \tag{11}$$

where

$$\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0$$

and c is a positive constant, which is proportional to TFP.[12] The unit cost function, $P(\mathbf{p})$, satisfies the linear homogeneity, monotonicity, and strict quasi-concavity, and so does the corresponding production function, $X(\mathbf{x})$. This follows from Matsuyama and Ushchev (2017; Proposition 1-i)) and guarantees the integrability (in the sense of Hurwicz and Uzawa 1971) of the H.S.A. production function. It is important to note that, with the sole exception of CES, $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$.[13] This can be verified by differentiating eq.(10) to yield

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'})z_{\omega'}d\omega'} = \frac{[\zeta(z_\omega) - 1]s(z_\omega)}{\int_\Omega [\zeta(z_{\omega'}) - 1]s(z_{\omega'})d\omega'}$$

[11]Conversely, from any continuously differentiable $\zeta: (0,\bar{z}) \to (1+\varepsilon,\infty)$, satisfying $\lim_{z\to\bar{z}} \zeta(z) = \infty$ if $\bar{z} < \infty$, one could reverse-engineer as $s(z) = \gamma \exp\left[\int_{z_0}^{z}[1-\zeta(\xi)]\,d\xi/\xi\right] > 0$; $z_0, z \in (0,\bar{z})$, where $\gamma = s(z_0)$ is a positive constant. Thus, we could also use $\zeta(\cdot)$ instead of $s(\cdot)$, as a primitive of symmetric H.S.A. with gross substitutes.

[12]This constant term in eq.(11), which appears by integrating eq.(9), cannot be pinned down. First, $A(\mathbf{p})$, the "average input price", depends on the unit of measurement of inputs, but not on the unit of measurement of the final good. In contrast, $P(\mathbf{p})$ is the cost of producing one unit of the final good, when the input prices are given by $\mathbf{p}$. Hence, it depends not only on the unit of measurement of inputs but also on that of the final good. Second, a change in TFP, though it affects $P(\mathbf{p})$, leaves the budget share of each input, and hence $A(\mathbf{p})$, unaffected.

[13]This holds more generally, that is, for asymmetric H.S.A., as well as H.S.A. with gross complements, as shown in Matsuyama and Ushchev (2017; Proposition 1-iii).

which differs from

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta(z)$ is independent of z or $s(z) = \gamma z^{1-\sigma}$with $\zeta(z) = \sigma > 1$. This should not come as a surprise. After all, $A(\mathbf{p})$ is the "average input price", which captures the *cross-price effects* in the demand system, while $P(\mathbf{p})$ is the inverse measure of TFP, which captures the *productivity (or welfare) effects* of price changes. There is no reason to think *a priori* that they should move together.

We are now ready to derive the love-for-variety measure under H.S.A. For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$**,** eq.(10) is simplified to

$$s\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right)V = 1 \Longrightarrow z_\omega = \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right)$$

Hence, from eq.(11),

$$z(V) \equiv \frac{1}{P(\mathbf{1}_\Omega^{-1})} = \frac{1}{A(\mathbf{1}_\Omega^{-1})}\frac{A(\mathbf{1}_\Omega^{-1})}{P(\mathbf{1}_\Omega^{-1})} = cs^{-1}\left(\frac{1}{V}\right)\exp\left[\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)\right],$$

from which the love-for-variety measure under H.S.A. is given by

$$\mathcal{L}(V) \equiv \frac{d\ln z(V)}{d\ln V} = \frac{d\ln z(V)}{d\ln s^{-1}(1/V)}\cdot\frac{d\ln s^{-1}(1/V)}{d\ln V} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \ln\left(\frac{A(\mathbf{1}_\Omega^{-1})}{cP(\mathbf{1}_\Omega^{-1})}\right). \tag{12}$$

In contrast, the substitutability measure is given by:

$$\sigma(V) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right). \tag{13}$$

Since $s^{-1}(1/V)$ is increasing in V, eq.(13) implies

$$\zeta'(\cdot) \gtreqless 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0.$$

In particular, Marshall's 2nd law, $\zeta'(\cdot) > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under H.S.A.

The next lemma shows the relation between the following two functions:

$$\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1 \quad \text{and} \quad \Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}\mathrm{d}\xi > 0.$$

Lemma 1:

$$\zeta'(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \ \Longrightarrow \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}).$$

Furthermore,

$$\zeta'(z) = 0 \ \Leftrightarrow \Phi'(z) = 0 \ \Leftrightarrow \text{CES}.$$

The proof of Lemma 1 is in Appendix D. By combining Lemma 1, eq. (12) and eq.(13), we have:

Proposition 1

$$\zeta'(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/s(z_0), \infty)$$

$$\Longrightarrow$$

$$\Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/s(z_0), \infty).$$

Furthermore,

$$\zeta'(z) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \Phi'(z) = 0 \Leftrightarrow \mathcal{L}'(V) = 0 \Leftrightarrow \text{CES}.$$

In particular, Marshall's 2nd Law, $\zeta'(\cdot) > 0$ for all $z < \bar{z}$, or equivalently, increasing substitutability, $\sigma'(\cdot) > 0$ for all V, implies diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all V.[14] The converse is not true. Diminishing love-for-variety for all V does not imply increasing substitutability or Marshall's 2nd Law globally. However, constant love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all V, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all V, and constant price elasticity $\zeta'(\cdot) = 0$ for all $z < \bar{z} = \infty$.

Before proceeding, it should be pointed out that there exists an alternative (but equivalent) definition of H.S.A. For the sake of completeness, we discuss this alternative in Appendix C.

4. The HDIA class

We call a symmetric CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, HDIA (*Homothetic Direct Implicit Additivity*) with gross substitutes if $X = X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ can be defined implicitly by:

$$\int_{\Omega} \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) d\omega = \int_{\Omega} \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1, \tag{14}$$

where $\phi(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is C^3, with $\phi'(y) > 0$, $\phi''(y) < 0$, $-y\phi''(y)/\phi'(y) < 1$ for $\forall y \in (0, \infty)$ and $\phi(0) = 0$ and $\phi(\infty) = \infty$, and independent of $Z > 0$, TFP. Note that, unlike eq.(10), the adding-up constraint of the H.S.A, eq.(14) defines the production function $X(\mathbf{x})$ directly.[15] CES with gross substitutes is a special case where $\phi(y) = (y)^{1-1/\sigma}$ $(\sigma > 1)$. The CoPaTh family of HDIA is given by $\phi(y) = \int_0^y \left(1 + \frac{1}{\sigma-1}(\xi)^{\frac{1-\rho}{\rho}}\right)^{\frac{\rho}{\rho-1}} d\xi$, $0 < \rho < 1$, which converges to CES with $\rho \nearrow 1$. Symmetric HDIA defined as above may be viewed as an extension of the Kimball (1995) aggregator in that the set of available varieties Ω is not fixed, and in particular, its mass, $V \equiv |\Omega|$, is a variable.

From the cost minimization problem, eq.(1), subject to eq.(14), we obtain the inverse demand curve,

$$p_\omega = B(\mathbf{p})\phi'\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) = B(\mathbf{p})\phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right), \tag{15}$$

where $B(\mathbf{p})$ is defined by:

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[14] This generalizes the case of translog, $s(z) = \gamma \max\{-\ln(z/\bar{z}), 0\}$, where $\sigma(V) = 1 + \gamma V$ and $\mathcal{L}(V) = 1/(2\gamma V)$.

[15] This means that, unlike H.S.A. but similar to HIIA defined in the next section, we do not need to worry about the integrability of HDIA. Note also that $\hat{X}(\mathbf{x}) = X(\mathbf{x})/Z$ defined by eq.(14) is invariant of TFP, $Z > 0$, by construction. Thus, an increase in Z causes a proportionate increase in $X(\mathbf{x})$. This allows us to examine the effect of TFP without shifting $\phi(\cdot)$. Alternatively, we could have defined $X(\mathbf{x})$ by $\int_{\Omega} \phi(x_\omega/X(\mathbf{x}))d\omega = 1$, as in Matsuyama and Ushchev (2017). Though mathematically equivalent, this definition requires that $\phi(\cdot)$ would no longer be independent of TFP, which would make it harder to show that $\sigma(V)$ and $\mathcal{L}(V)$ are independent of TFP.

$$\int_{\Omega} \phi\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

This shows that the choke price is equal to $B(\mathbf{p})\phi'(0)$ if $\phi'(0) < \infty$, and that there is no choke price if $\phi'(0) = \infty$. The unit cost function is:

$$P(\mathbf{p}) = \frac{\hat{P}(\mathbf{p})}{Z} \equiv \frac{1}{Z}\int_{\Omega} p_\omega (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega,$$

Clearly, both $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$ are linear homogenous in $\mathbf{p}$, and independent of $Z > 0$. Hence, an increase in TFP, Z, causes a proportional decline in the unit cost function, $P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$.

The budget share is

$$\frac{p_\omega x_\omega}{P(\boldsymbol{p})X(\boldsymbol{x})} = \frac{p_\omega}{\hat{P}(\boldsymbol{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\boldsymbol{p})}\right) = \frac{x_\omega}{C^*(\boldsymbol{x})}\phi'\left(\frac{x_\omega}{\hat{X}(\boldsymbol{x})}\right), \tag{16}$$

where

$$C^*(\mathbf{x}) \equiv \int_{\Omega} x_\omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega$$

is linear homogenous in $\mathbf{x}$ and independent of $Z > 0$, and satisfies the identity

$$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \int_{\Omega} \frac{p_\omega}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega = \int_{\Omega} \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})}. \tag{17}$$

Eqs. (16)-(17) show that the budget share under HDIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/B(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/C^*(\mathbf{x})$, unless $\hat{P}(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/\hat{X}(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HDIA and H.S.A. do not overlap with the sole exception of CES.[16]

From the inverse demand curve, eq.(15), the price elasticity of demand can be written as a function of a single variable, $y_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$ or $p_\omega/B(\mathbf{p}) = \phi'(y_\omega)$, as:

$$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = -\frac{\phi'(y_\omega)}{y\phi''(y_\omega)} \equiv \zeta^D(y_\omega) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) = \zeta(p_\omega; \mathbf{p}) > 1,$$

where $\zeta^D(y) > 1$ ensures gross substitutability. Under CES, $\zeta^{D'}(\cdot) = 0$. Marshall's 2nd law, $\partial\zeta^*(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{D'}(\cdot) < 0$, the condition satisfied by CoPaTh, with $\zeta^D(y) = 1 + (\sigma - 1)(y)^{\frac{\rho-1}{\rho}}$.

We are now ready to derive the measure of love-for-variety under HDIA. For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(14) is simplified to

[16] This statement is a special case of Proposition 2-(ii) in Matsuyama and Ushchev (2017).

$$\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)V = 1 \implies y(V) \equiv \frac{1}{\hat{X}(\mathbf{1}_\Omega)} = \phi^{-1}\left(\frac{1}{V}\right)$$

Hence, the measure of love-for-variety under HDIA is given by:

$$\mathcal{L}(V) \equiv -\frac{d\ln y(V)}{d\ln V} - 1 = \frac{1}{\mathcal{E}_\phi\left(\phi^{-1}(1/V)\right)} - 1 > 0 \tag{18}$$

where

$$0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$$ [17]

In contrast, the substitutability measure is given by:

$$\sigma(V) = \zeta^D\left(\phi^{-1}(1/V)\right). \tag{19}$$

Since $\phi^{-1}(1/V)$ is decreasing in V, eq.(19) implies

$$\zeta^{D\prime}(\cdot) \lesseqgtr 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0.$$

In particular, Marshall's 2nd law, $\zeta^{D\prime}(\cdot) < 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under HDIA.

The next lemma shows the relation between the following two functions:

$$\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1 \quad \text{and} \quad 0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$$

Lemma 2:

$$\zeta^{D'}(y) \lesseqgtr 0, \forall y \in (0, y_0) \implies \mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

Furthermore,

$$\zeta^{D'}(y) = 0 \iff \mathcal{E}'_\phi(y) = 0 \iff \text{CES}.$$

The proof of Lemma 2 is in Appendix D. By combining Lemma 2, eq.(18), and eq.(19),

Proposition 2:

$$\zeta^{D'}(y) \lesseqgtr 0\ \forall y \in (0, y_0) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/\phi(y_0), \infty)$$

$$\implies$$

$$\mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/\phi(y_0), \infty).$$

Furthermore,

$$\zeta^{D'}(y) = 0 \iff \sigma'(V) = 0 \Leftrightarrow \mathcal{E}'_\phi(y) = 0 \iff \mathcal{L}'(V) = 0 \iff \text{CES}.$$

In particular, Marshall's 2nd Law, $\zeta^{D\prime}(\cdot) < 0$ for all $y > 0$, or equivalently, increasing substitutability, $\sigma'(\cdot) > 0$ for all V, implies diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all V. The converse is not true. Diminishing love-for-variety for all V does not imply increasing substitutability or Marshall's 2nd Law globally. However, constant

[17] Moreover, by evaluating eq.(17) at the symmetric price and quantity patterns,

$$\frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{B(\mathbf{1}_\Omega^{-1})} = \frac{C^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)d\omega = \mathcal{E}_\phi\left(\phi^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \frac{B(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{C^*(\mathbf{1}_\Omega)} = \mathcal{L}(V) + 1.$$

love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all V, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all V, and constant price elasticity $\zeta^{D\prime}(\cdot) = 0$ for all $y > 0$.

5. The HIIA class.

We call a symmetric CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, HIIA (*Homothetic Indirect Implicit Additivity*) with gross substitutes if $P = P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, can be defined implicitly by:

$$\int_{\Omega} \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega = \int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega = 1, \tag{20}$$

where $\theta: \mathbb{R}_{++} \to \mathbb{R}_{+}$ is C^3, with $\theta'(z) < 0$, $\theta''(z) > 0$, and $-z\theta''(z)/\theta'(z) > 1$, for $\theta(z) > 0$ with $\lim_{z\to 0} \theta(z) = \infty$ and $\lim_{z\to \bar{z}} \theta(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | \theta(z) = 0\}$, and it is independent of $Z > 0$. If $\bar{z} < \infty$, the choke price is equal to $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$, and $\lim_{z\to\bar{z}} \theta'(z) = 0$. If $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector. Note that, unlike eq.(10), the adding-up constraint of the H.S.A, eq.(20) defines the unit cost function $P(\mathbf{p})$ directly.[18] CES with gross substitutes is a special case where $\theta(z) = (z)^{1-\sigma}$ $(\sigma > 1)$. The CoPaTh family of HIIA is given by $\theta(z) = \sigma^{\frac{\rho}{1-\rho}} \int_{z/\bar{z}}^{1} \left((\xi)^{\frac{\rho-1}{\rho}} - 1\right)^{\frac{\rho}{1-\rho}} d\xi$ for $z < \bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}$; $0 < \rho < 1$, which converges to CES as $\rho \nearrow 1$.

The minimization problem, eq.(4), subject to eq.(20) leads to the demand curve

$$x_\omega = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) > 0 \tag{21}$$

where $B^*(\mathbf{x}) > 0$ is defined by

$$\int_{\Omega} \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) d\omega \equiv 1.$$

This indeed shows that the choke price is $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$, if $\bar{z} < \infty$. The production function is

$$X = X(\mathbf{x}) = Z\hat{X}(\mathbf{x}) \equiv Z \int_{\Omega} (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) x_\omega d\omega.$$

Clearly, both $B^*(\mathbf{x})$ and $\hat{X}(\mathbf{x})$ are linear homogeneous in $\mathbf{x}$ and independent of $Z > 0$, by construction. Thus, an increase in TFP, Z, causes a proportional increase in $X(\mathbf{x}) = Z\hat{X}(\mathbf{x})$.

The budget share is

[18]This means that, unlike H.S.A. but similar to HDIA defined in the previous section, we do not need to worry about the integrability of HIIA. Note also that $\hat{P}(\mathbf{p}) = ZP(\mathbf{p})$ defined by eq.(20) is invariant of TFP, $Z > 0$, by construction. Thus, an increase in Z causes a proportionate decline in $P(\mathbf{p})$. This allows us to examine the effect of TFP without shifting $\theta(\cdot)$. Alternatively, we could define $P(\mathbf{p})$ by $\int_{\Omega} \theta(p_\omega/P(\mathbf{p}))d\omega = 1$, as in Matsuyama and Ushchev (2017). Though mathematically equivalent, this definition requires that $\theta(\cdot)$ would no longer be independent of TFP, which would make it harder to show that $\sigma(V)$ and $\mathcal{L}(V)$ are independent of TFP.

$$\frac{p_\omega x_\omega}{P(\boldsymbol{p})X(\boldsymbol{x})} = -\theta'\left(\frac{p_\omega}{\hat{P}(\boldsymbol{p})}\right)\frac{p_\omega}{C(\boldsymbol{p})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\boldsymbol{x})}\right)\frac{x_\omega}{\hat{X}(\boldsymbol{x})}, \tag{22}$$

where

$$C(\mathbf{p}) \equiv -\int_\Omega \theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)p_\omega d\omega > 0$$

is linear homogenous in $\mathbf{p}$, and independent of $Z > 0$ and satisfies the identity,

$$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right]d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})}d\omega = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})}. \tag{23}$$

Eqs. (22)-(23) show that the budget share under HIIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/\hat{P}(\mathbf{p}) = \hat{X}(\mathbf{x})/B^*(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HIIA and H.S.A. do not overlap with the sole exception of CES.[19] Furthermore, by comparing the expressions for the budget share under HDIA and the budget share under HIIA, one could see that HDIA and HIIA do not overlap with the sole exception of CES.[20]

From the demand curve, eq.(21), the price elasticity can be expressed as a function of a single variable, $z_\omega \equiv p_\omega/P(\mathbf{p})$ or or $x_\omega/B^*(\mathbf{x}) = -\theta'(z_\omega)$, as:

$$\zeta_\omega = \zeta(p_\omega;\mathbf{p}) = -\frac{z_\omega\theta''(z_\omega)}{\theta'(z_\omega)} \equiv \zeta^I(z_\omega) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) = \zeta^*(x_\omega;\,\mathbf{x}) > 1,$$

where $\zeta^I(z) > 1$ ensures gross substitutability. Under CES, $\zeta^{I\prime}(\cdot) = 0$. Marshall's 2nd law, $\partial\zeta(p_\omega;\mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta^{I\prime}(\cdot) > 0$, the condition satisfied by CoPaTh with $\zeta^I(z_\omega) = \frac{\sigma}{\sigma-(\sigma-1)(z_\omega)^{(1-\rho)/\rho}} = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$.

We are now ready to derive the measure of love-for-variety under HIIA. For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, eq.(20) is simplified to

$$\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)V = 1 \implies z(V) \equiv \frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \theta^{-1}(1/V).$$

Hence, the measure of love-for-variety under HIIA is given by:

$$\mathcal{L}(V) \equiv \frac{d\ln z(V)}{d\ln V} = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)} > 0. \tag{24}$$

where

[19] This statement is a special case of Proposition 3-(ii) in Matsuyama and Ushchev (2017).
[20] This statement is a special case of Proposition 4-(iii) in Matsuyama and Ushchev (2017).

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$ [21]

In contrast, the substitutability measure is given by:

$$\sigma(V) = \zeta^I\big(\theta^{-1}(1/V)\big). \tag{25}$$

Since $\theta^{-1}(1/V)$ is increasing in V, eq.(25) implies

$$\zeta^{I\prime}(\cdot) \gtreqless 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0.$$

In particular, Marshall's 2nd law, $\zeta^{I\prime}(\cdot) > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under HIIA.

The next lemma shows the relation between the following two functions:

$$\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1 \quad \text{and} \quad \mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$

Lemma 3:

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \quad \Rightarrow \quad \mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \overline{z}).$$

Furthermore,

$$\zeta^{I'}(z) = 0 \quad \Leftrightarrow \quad \mathcal{E}'_\theta(z) = 0 \Leftrightarrow \text{CES}.$$

The proof of Lemma 3 is in Appendix D. By combining Lemma 3, eq.(24), and eq.(25),

Proposition 3:

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/\theta(z_0), \infty)$$

$$\Rightarrow$$

$$\mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/\theta(z_0), \infty).$$

Furthermore,

$$\zeta^{I'}(z) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \mathcal{E}'_\theta(z) = 0 \Leftrightarrow \mathcal{L}'(V) = 0 \Leftrightarrow \text{CES}.$$

In particular, Marshall's 2nd Law, $\zeta^{I\prime}(\cdot) < 0$ for all $z < \overline{z}$, or equivalently, increasing substitutability, $\sigma'(\cdot) > 0$ for all V, implies diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all V. The converse is not true. Diminishing love-for-variety for all V does not imply increasing substitutability or Marshall's 2nd Law globally. However, constant love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all V, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all V, and constant price elasticity $\zeta^{I\prime}(\cdot) = 0$ for all $z < \overline{z} = \infty$.

6. Concluding Remarks

In this paper, we study how love-for-variety is determined by the underlying demand structure. Under general symmetric homothetic demand systems, both love-for-variety and the substitutability across different

[21]Moreover, by evaluating eq.(23) at the symmetric price and quantity patterns,

$$\frac{C(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{B^*(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)d\omega = \mathcal{E}_\theta\left(\theta^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \mathcal{L}(V) = \frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{C(\mathbf{1}_\Omega^{-1})} = \frac{B^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)}.$$

varieties can be expressed as functions of the mass of available varieties V only, as $\mathcal{L}(V)$ and $\sigma(V)$. Since the homotheticity alone imposes little restrictions on their properties, we turn to three classes of homothetic demand systems, H.S.A., HDIA, and HIIA, which are pairwise disjoint with the sole exception of CES. For each of these three classes, we establish the three main results. First, the substitutability is increasing in V, if and only if Marshall's 2nd law of demand holds. Second, increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety, but the converse is not true. Third, love-for-variety is constant, if and only if substitutability is constant, which occurs only under CES within these three classes. These three classes thus offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different varieties are more substitutable when more varieties are available.

We would like to point out that this paper is all about the demand side for expanding variety. As such, the results are relevant regardless of what we assume on the supply side, that is the technology and market structure of innovation, which could be modelled as monopoly, oligopolistic competition, and monopolistic competition with or without heterogeneous firms and with or without multi-product firms. We hope that these three classes of demand systems will become a useful building block in many models across many different fields, where gains from endogenous variety are of central importance.

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Appendix A: Allen-Uzawa elasticity of substation at the symmetric patterns under general symmetric homothetic demand systems.

The Allen-Uzawa elasticity of substitution between two varieties, $\omega, \omega' \in \Omega$, are given by:

$$AES(p_\omega, p_{\omega'}, \mathbf{p}) = \frac{P(\mathbf{p})P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})}{x(p_\omega, \mathbf{p})x(p_{\omega'}, \mathbf{p})},$$

where $x(p_\omega, \mathbf{p})$ is the demand for input ω per unit of output, while the functions $P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})$ are the ``second cross-derivatives'' of $P(\mathbf{p})$. The second-order Taylor approximation of $P(\mathbf{p})$ is

$$P(\mathbf{p} + \alpha\mathbf{h}) = P(\mathbf{p}) + \alpha \int_\Omega x(p_\omega, \mathbf{p})h_\omega d\omega + \frac{\alpha^2}{2}\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} h_\omega^2 d\omega + \frac{\alpha^2}{2}\int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})h_\omega h_{\omega'}\, d\omega d\omega' + o(\alpha^2),$$

where $\mathbf{h}$ is a function over Ω, and α is a scalar. The linear homogeneity of $P(\mathbf{p})$ implies the following identity:

$$\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} p_\omega^2 d\omega + \int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})p_\omega p_{\omega'}\, d\omega d\omega' = 0.$$

By setting $(p_\omega, \mathbf{p}) = (1, \mathbf{1}_\Omega^{-1})$ and $(p_\omega, p_{\omega'}, \mathbf{p}) = (1,1, \mathbf{1}_\Omega^{-1})$ in the identity, we obtain:

$$\left[\frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\right]\Bigg|_{p_\omega=1} \underbrace{\left[\int_\Omega d\omega\right]}_{=V} + P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1}) \left[\underbrace{\int_\Omega \int_\Omega d\omega d\omega'}_{=V^2}\right] = 0.$$

Using the definition of $\sigma(V)$,

$$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = -\left[\frac{\partial \ln x(p_\omega, p\mathbf{1}_\Omega^{-1})}{\partial \ln p_\omega}\right]\Bigg|_{p_\omega=p} \quad\Rightarrow\quad \frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\Bigg|_{p_\omega=1} = -\sigma(V)x(1, \mathbf{1}_\Omega^{-1}),$$

the above identity can be further rewritten as:

$$P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1}) = \frac{\sigma(V)}{V} x(1, \mathbf{1}_\Omega^{-1}).$$

Moreover, by setting $\mathbf{p} = P(\mathbf{1}_\Omega^{-1})$ in $P(\mathbf{p}) = \int_\Omega x(p_\omega, \mathbf{p})p_\omega d\omega$,

$$P(\mathbf{1}_\Omega^{-1}) = Vx(1, \mathbf{1}_\Omega^{-1}).$$

Thus, the Allen-Uzawa elasticity of substitution evaluated at a symmetric outcome:

$$AES_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1}) = \frac{P(\mathbf{1}_\Omega^{-1})P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1})}{[x(1, \mathbf{1}_\Omega^{-1})]^2} = \frac{Vx(1, \mathbf{1}_\Omega^{-1})\frac{\sigma(V)}{V}x(1, \mathbf{1}_\Omega^{-1})}{[x(1, \mathbf{1}_\Omega^{-1})]^2} = \sigma(V).$$

Appendix B: $\boldsymbol{\sigma(V)}$ and $\boldsymbol{\mathcal{L}(V)}$ under Geometric Means of CES

This appendix shows that there exists a class of homothetic non-CES demand systems in which $\sigma(V)$ and $\mathcal{L}(V)$ are independent of V. Moreover, within this class, they exist a parametric family in which that $\sigma(V)$ and $\mathcal{L}(V)$ move in the same direction as one of the parameters changes.

Consider the symmetric CRS production function, $X(\mathbf{x})$, defined by a weighted geometric mean of symmetric CES production functions with different $\sigma \in (1, \infty)$:

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x}; \sigma)\, dF(\sigma),$$

where

$$[X(\mathbf{x}; \sigma)]^{1-\frac{1}{\sigma}} \equiv \int_\Omega \ {x_\omega}^{1-\frac{1}{\sigma}} d\omega$$

and $F(\cdot)$ is a c.d.f. of $\sigma \in (1, \infty)$,

$$\int_1^\infty dF(\sigma) = 1.$$

Proposition B: *Consider the homothetic demand system generated by a weighted geometric mean of symmetric CES production functions. Then,*

i) *The substitutability measure, $\sigma(V)$, is independent of V and given by:*

$$\sigma(V) = \frac{1}{E_F(1/\sigma)} > 1;$$

ii) *The love-for-variety measure, $\mathcal{L}(V)$, is independent of V and given by*

$$\mathcal{L}(V) = E_F\left(\frac{1}{\sigma - 1}\right) > 0;$$

iii) *The range of $\sigma(V)$ and $\mathcal{L}(V)$ is given by:*

$$0 < \frac{1}{\sigma(V) - 1} \le \mathcal{L}(V) < \infty,$$

where the equality holds if and only if F is degenerate.

Proof. The inverse demand for variety $\omega \in \Omega$ is

$$p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega} = P(\mathbf{p}) X(\mathbf{x}) \int_1^\infty \frac{{x_\omega}^{-\frac{1}{\sigma}}}{[X(\mathbf{x}; \sigma)]^{1-\frac{1}{\sigma}}} dF(\sigma) = P(\mathbf{p}) X(\mathbf{x}) \int_1^\infty {x_\omega}^{-\frac{1}{\sigma}} dF^*(\mathbf{x}; \sigma),$$

where $dF^*(\mathbf{x}; \sigma) \equiv dF(\sigma)/[X(\mathbf{x}; \sigma)]^{1-\frac{1}{\sigma}}$. Thus, the price elasticity of demand, $\zeta^*(x_\omega; \mathbf{x})$, as a function of the quantities, satisfies

$$\frac{1}{\zeta^*(x_\omega;\mathbf{x})} \equiv -\frac{\partial \ln p_\omega}{\partial \ln x_\omega} = \frac{\int_1^\infty \frac{1}{\sigma} x_\omega{}^{-\frac{1}{\sigma}} dF^*(\mathbf{x};\sigma)}{\int_1^\infty x_\omega{}^{-\frac{1}{\sigma}} dF^*(\mathbf{x};\sigma)} < 1.$$

By evaluating this at the symmetric quantity patterns $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\frac{1}{\sigma(V)} \equiv \frac{1}{\zeta^*(x;x\mathbf{1}_\Omega)} \equiv \frac{\int_1^\infty \frac{1}{\sigma} x^{-\frac{1}{\sigma}} \frac{dF(\sigma)}{[X(x\mathbf{1}_\Omega;\sigma)]^{1-\frac{1}{\sigma}}}}{\int_1^\infty x^{-\frac{1}{\sigma}} \frac{dF(\sigma)}{[X(x\mathbf{1}_\Omega;\sigma)]^{1-\frac{1}{\sigma}}}} = \frac{\int_1^\infty \frac{x^{-\frac{1}{\sigma}}}{Vx^{1-\frac{1}{\sigma}}} \frac{dF(\sigma)}{\sigma}}{\int_1^\infty \frac{x^{-\frac{1}{\sigma}}}{Vx^{1-\frac{1}{\sigma}}} dF(\sigma)} = \int_1^\infty \frac{dF(\sigma)}{\sigma} = E_F\left(\frac{1}{\sigma}\right) < 1.$$

This proves i).

Next, from $\ln X(\mathbf{1}_\Omega) \equiv \int_1^\infty \ln X(\mathbf{1}_\Omega;\sigma)\, dF(\sigma) = \int_1^\infty \ln V \frac{\sigma}{\sigma-1} dF(\sigma) = E_F\left(\frac{\sigma}{\sigma-1}\right) \ln V$,

$$\mathcal{L}(V) \equiv -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1 = \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 = E_F\left(\frac{\sigma}{\sigma-1}\right) - 1 = E_F\left(\frac{1}{\sigma-1}\right) > 0.$$

This proves ii).

For iii), Jensen's inequality implies

$$\mathcal{L}(V) = E_F\left(\frac{1/\sigma}{1-1/\sigma}\right) \geq \frac{E_F(1/\sigma)}{1-E_F(1/\sigma)} = \frac{1/\sigma(V)}{1-1/\sigma(V)} = \frac{1}{\sigma(V)-1},$$

where the lower bound is reached if and only if F is degenerate. Next, consider the Pareto distribution of σ:

$$F(\sigma) = 1 - \left(\frac{\sigma_{min}}{\sigma}\right)^\alpha, \qquad \sigma \geq \sigma_{min} \equiv \frac{\alpha\sigma_0}{\alpha+1} > 1,$$

where $\sigma_0 > 1$ and $\alpha > 1/(\sigma_0 - 1)$. The distribution and density of $x = 1/\sigma$ are given by:

$$G(x) = (\sigma_{min} x)^\alpha;\ g(x) = \alpha(\sigma_{min})^\alpha x^{\alpha-1}, \qquad x \in \left(0, \frac{1}{\sigma_{min}}\right).$$

Thus,

$$\frac{1}{\sigma(V)} = \mathbb{E}_F\left(\frac{1}{\sigma}\right) = \mathbb{E}_G(x) = \alpha(\sigma_{min})^\alpha \int_0^{1/\sigma_{min}} x^\alpha\, dx = \frac{1}{\sigma_{min}} \frac{\alpha}{\alpha+1} = \frac{1}{\sigma_0} < 1;$$

$$\mathcal{L}(V) = \mathbb{E}_F\left(\frac{1}{\sigma-1}\right) = \mathbb{E}_G\left(\frac{x}{1-x}\right) = \mathbb{E}_G\left(\sum_{k=1}^\infty x^k\right) = \sum_{k=1}^\infty \mathbb{E}_G\left(x^k\right) = \sum_{k=1}^\infty \alpha(\sigma_{min})^\alpha \int_0^{1/\sigma_{min}} x^{\alpha+k-1}\, dx$$

$$= \sum_{k=1}^\infty \frac{\alpha}{\alpha+k}\left(\frac{1}{\sigma_{min}}\right)^k = \sum_{k=1}^\infty \frac{\alpha}{\alpha+k}\left(\frac{\alpha+1}{\alpha}\right)^k \left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^\infty \frac{(1+1/\alpha)^k}{1+k/\alpha}\left(\frac{1}{\sigma_0}\right)^k$$

Holding $\sigma(V) = \sigma_0$ constant, $\mathcal{L}(V)$ is monotonically decreasing in α because

$$\frac{d\ln\left[\frac{(1+1/\alpha)^k}{1+k/\alpha}\right]}{d\ln\alpha} = -\frac{\alpha k(k-1)}{(\alpha+1)(\alpha+k)} \leq 0.$$

Moreover,

$$\lim_{\alpha\to\infty}\sum_{k=1}^{\infty}\frac{(1+1/\alpha)^k}{1+k/\alpha}\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\left(\frac{1}{\sigma_0}\right)^k = \frac{1}{\sigma_0-1} = \frac{1}{\sigma(V)-1};$$

$$\lim_{\alpha\to 1/(\sigma_0-1)}\sum_{k=1}^{\infty}\frac{(1+1/\alpha)^k}{1+k/\alpha}\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\frac{1}{1+(\sigma_0-1)k} > \int_1^{\infty}\frac{dz}{1+(\sigma_0-1)z} = \frac{\ln[1+(\sigma_0-1)z]}{(\sigma_0-1)}\Bigg]_1^{\infty} = \infty,$$

from which

$$\frac{1}{\sigma(V)-1} \leq \mathcal{L}(V) < \infty.$$

This completes the proof. ■

We are now ready to construct a parametric family of the distribution, F_α, in which $\sigma(V)$ and $\mathcal{L}(V)$ are independent of V and move in the same direction as α varies.

$$F_\alpha(\sigma) = 1-\left(\frac{\sigma_{min}}{\sigma}\right)^\alpha, \qquad \frac{1}{\sigma} \leq \frac{1}{\sigma_{min}} \equiv \frac{1+\alpha}{\alpha}h(\alpha) < 1.$$

Then, following the same step in the proof of Part iii) of Proposition,

$$\frac{1}{\sigma(V)} = \mathbb{E}_{F_\alpha}\left(\frac{1}{\sigma}\right) = \frac{1}{\sigma_{min}}\frac{\alpha}{\alpha+1} = h(\alpha);\ \mathcal{L}(V) = \mathbb{E}_{F_\alpha}\left(\frac{1}{\sigma-1}\right) = \sum_{k=1}^{\infty}\frac{(1+1/\alpha)^k}{1+k/\alpha}\big(h(\alpha)\big)^k$$

with

$$\frac{d\ln\left[\frac{(1+1/\alpha)^k}{1+k/\alpha}\big(h(\alpha)\big)^k\right]}{d\ln\alpha} = \alpha k\left[\frac{h'(\alpha)}{h(\alpha)} - \frac{(k-1)}{(\alpha+1)(\alpha+k)}\right].$$

Fix $h_0(\alpha)$, which is increasing in α, and satisfies $0 < h_0(\alpha) < 1$, and whose derivative is bounded. Then, for $0 < c < 1$, consider $h(\alpha,\varepsilon) \equiv (1-\varepsilon)c + \varepsilon h_0(\alpha)$. When $\varepsilon = 0, \sigma(V)$ is constant, while $\mathcal{L}(V)$ is decreasing in α. But, for a positive but sufficiently small ε, $\sigma(V)$ is decreasing, while $\mathcal{L}(V)$ continues to be decreasing in α by continuity.

Appendix C. An Alternative (and Equivalent) specification of the HSA class.

There exists an alternative (but equivalent) definition of H.S.A.. That is, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$ is called *homothetic with a single aggregator* (H.S.A.) if the budget share of input ω, as a function of $\mathbf{x}$, can be written as:

$$s_\omega = \frac{\partial \ln X(\boldsymbol{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\boldsymbol{x})}\right) \tag{26}$$

Here, $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$ is the *budget share function,* which is C^2 with $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1,\ s^*(0) = 0$ and $s^*(\infty) = \infty$, and $A^*(\mathbf{x})$ is linear homogenous in $\mathbf{x}$, defined implicitly and uniquely by

$$\int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1 \tag{27}$$

which ensures that the budget shares of all inputs are added up to one. Thus, the budget share of input ω is a function of its *normalized quantity*, $y_\omega \equiv x_\omega / A^*(\mathbf{x})$, defined as its own quantity x_ω divided by the *common quantity aggregator* $A^*(\mathbf{x})$.

Furthermore, the price elasticity of each input can be written as a function of a single variable, $y_\omega \equiv x_\omega / A^*(\mathbf{x})$:

$$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = \left[1 - \frac{d \ln s^*(y_\omega)}{d \ln y_\omega}\right]^{-1} \equiv \zeta^*(y_\omega) > 1,$$

where $\zeta^*: (0, \infty) \to (1, \infty)$ is continuously differentiable. Note that the assumption, $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1$ ensures $\zeta^*(y_\omega) > 1$. That is, inputs are *gross substitutes.*[22] In general, $\zeta^*(\cdot)$ can be nonmonotonic. Under CES, given by $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$, it is constant, $\zeta^*(y) = \sigma > 1$. Marshall's 2nd law, $\partial\zeta(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{*\prime}(\cdot) < 0$.

Note also that the choke price exists if and only if $\lim_{y\to 0} s^{*\prime}(y) < \infty$, which implies $\lim_{y\to 0} \frac{d \ln s^*(y)}{d \ln y} = 1$ and hence $\lim_{y\to 0} \zeta^*(y) = \infty$. For example, translog corresponds to $s^*(y)$, defined implicitly by $s^* \exp(s^*/\gamma) \equiv \bar{z}y$, for $\bar{z} < \infty$.

The production function, $X(\mathbf{x})$, can be obtained by integrating eq.(26), which yields

$$\frac{X(\boldsymbol{x})}{A^*(\boldsymbol{x})} = c^* \exp\left[\int_\Omega \left[\int_0^{x_\omega/A^*(x)} \frac{s^*(\xi^*)}{\xi^*} d\xi^*\right] d\omega\right] = c^* \exp\left[\int_\Omega s^*\left(\frac{x_\omega}{A^*(\boldsymbol{x})}\right) \Phi^*\left(\frac{x_\omega}{A^*(\boldsymbol{x})}\right) d\omega\right] \tag{28}$$

where c^* is a positive constant, which is proportional to TFP and

[22] Conversely, from any continuously differentiable $\zeta^*: (0, \infty) \to (1, \infty)$, one could reverse-engineer as $s^*(y) = \gamma^* \exp\left[\int_{y_0}^{y}\left[1 - \frac{1}{\zeta^*(\xi^*)}\right]\frac{d\xi^*}{\xi^*}\right] > 0$, where $\gamma^* = s^*(y_0)$ is a positive constant. Thus, we could also use $\zeta^*(\cdot)$ instead of $s^*(\cdot)$ as a primitive of symmetric H.S.A. with gross substitutes.

$$\Phi^*(y) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* = \frac{\int_0^y [s^*(\xi^*)/\xi^*]\,d\xi^*}{\int_0^y [s^*(y)/y]\,d\xi^*} > 1,$$

where the inequality follows from $\mathcal{E}_{s^*}(y) \equiv \frac{d\ln s^*(y)}{d\ln y} < 1$ implying that $s^*(y)/y$ is decreasing in y.

It is important to note that $X(\mathbf{x})/A^*(\mathbf{x})$ is constant, if and only if it is CES. To see this, differentiating eq.(27) yields,

$$\frac{\partial \ln A^*(\mathbf{x})}{\partial \ln x_\omega} = \frac{y_\omega s^{*\prime}(y_\omega)}{\int_\Omega s^{*\prime}(y_{\omega'})y_{\omega'}d\omega'} = \frac{\left[1-\frac{1}{\zeta^*(y_\omega)}\right]s^*(y_\omega)}{\int_\Omega \left[1-\frac{1}{\zeta^*(y_{\omega'})}\right]s^*(y_{\omega'})d\omega'},$$

which differs from $\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*(y_\omega)$, unless $\zeta^*(y_\omega)$ is constant.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(27) is simplified to

$$s^*\left(\frac{1}{A^*(\mathbf{1}_\Omega)}\right)V = 1 \Rightarrow y_\omega \equiv \frac{1}{A^*(\mathbf{1}_\Omega)} = s^{*-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(28),

$$\frac{X(\mathbf{x})}{A^*(\mathbf{x})} = c^* \exp\left\{\Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)\right\}.$$

from which

$$\mathcal{Y}(V) \equiv \frac{1}{X(\mathbf{1}_\Omega)} = \frac{1}{A^*(\mathbf{1}_\Omega)}\frac{A^*(\mathbf{1}_\Omega)}{X(\mathbf{1}_\Omega)} = \frac{1}{c^*}s^{*-1}\left(\frac{1}{V}\right)\exp\left\{-\Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)\right\},$$

and hence the love-for-variety measure is given by:

$$\mathcal{L}(V) \equiv -\frac{d\ln \mathcal{Y}(V)}{d\ln V} - 1 = \frac{d\ln \mathcal{Y}(V)}{d\ln s^{*-1}(1/V)}\left[-\frac{d\ln s^{*-1}(1/V)}{d\ln V}\right] - 1 = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1 \tag{29}$$

$$= \ln\left(\frac{X(\mathbf{1}_\Omega)}{c^* A^*(\mathbf{1}_\Omega)}\right) - 1$$

In contrast, the substitutability measure is given by:

$$\sigma(V) = \zeta^*\left(s^{*-1}\left(\frac{1}{V}\right)\right). \tag{30}$$

Since $s^{*-1}(1/V)$ is decreasing in V, eq.(30) implies

$$\zeta^{*\prime}(\cdot) \lesseqgtr 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0.$$

In particular, Marshall's 2nd law, $\zeta^{*\prime}(\cdot) < 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$.

The next lemma shows the relation between the following two functions:

$$\zeta^*(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1} > 1 \text{ and } \Phi^*(y) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* > 1.$$

Lemma 1*

$$\zeta^{*'}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Rightarrow \Phi^{*'}(y) \gtreqless 0, \forall y \in (0, y_0).$$

Furthermore,

$$\zeta^{*\prime}(y) = 0 \iff \Phi^{*'}(y) = 0 \iff \text{CES}.$$

The proof of Lemma 1* is in Appendix D. By combining Lemma 1*, eq.(29), and eq.(30),

Proposition 1*

$$\zeta^{*'}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/s^*(y_0), \infty)$$

$$\Rightarrow$$

$$\Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/s^*(y_0), \infty)$$

and

$$\zeta^{*\prime}(y) = 0 \iff \sigma'(V) = 0 \Leftrightarrow \Phi^{*\prime}(y) = 0 \Leftrightarrow \mathcal{L}'(V) = 0 \Leftrightarrow \text{CES}.$$

In particular, Marshall's 2nd Law, $\zeta^{*\prime}(\cdot) < 0$ for all $y > 0$, or equivalently, increasing substitutability, $\sigma'(\cdot) > 0$ for all V, implies diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all V. The converse is not true. Diminishing love-for-variety for all V does not imply increasing substitutability or Marshall's 2nd Law globally. However, constant love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all V, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all V, and constant price elasticity $\zeta^{*\prime}(\cdot) = 0$ for all $y > 0$.

Indeed, these two alternative definitions of H.S.A. are equivalent.[23] The isomorphism between the two is given by the one-to-one mapping between $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

Differentiating either of these two equalities yields the identity,

$$\zeta^*(y) \equiv \left[1 - \frac{d \ln s^*(y)}{d \ln y}\right]^{-1} = \zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} > 1,$$

which shows that the condition, $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1$, is equivalent to $\mathcal{E}_s(z) \equiv \frac{d \ln s(z)}{d \ln z} < 0$. Furthermore, the normalized quantity, $y_\omega \equiv x_\omega / A^*(\mathbf{x})$, and the normalized price, $z_\omega \equiv p_\omega / A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega} \Leftrightarrow y_\omega = \frac{s(z_\omega)}{z_\omega},$$

$$\frac{dy_\omega}{y_\omega} = -\zeta(z_\omega)\frac{dz_\omega}{z_\omega} \Leftrightarrow \frac{dz_\omega}{z_\omega} = -\frac{1}{\zeta^*(y_\omega)}\frac{dy_\omega}{y_\omega}$$

and

[23] This isomorphism has been shown for the broader class of H.S.A., which allows for asymmetry as well as gross complements; see Matsuyama and Ushchev (2017, sec. 3, Remark 3).

$$\frac{z_\omega \zeta'(z_\omega)}{y_\omega {\zeta^*}'(y_\omega)} = -\zeta(z_\omega) = -\zeta^*(y_\omega) < 0.$$

In addition, if $\lim_{y\to 0} s^{*\prime}(y) < \infty$, then $\lim_{y\to 0} \zeta^*(y) = \infty$ and

$$\lim_{y\to 0} \frac{s^*(y)}{y} = \lim_{y\to 0} s^{*\prime}(y) = \bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$$

is the (normalized) choke price.

Moreover,

$$\frac{p_\omega x_\omega}{A(\mathbf{p})A^*(\mathbf{x})} = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})}$$

hence we have the identity,

$$c \exp\left[\int_\Omega s(z_\omega)\Phi(z_\omega)d\omega\right] = \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})} = c^* \exp\left[\int_\Omega s^*(y_\omega)\Phi^*(y_\omega)\, d\omega\right]$$

which is a positive constant if and only if it is CES.

Furthermore, using

$$s(\xi) = s^*(\xi^*) = \xi\xi^* \to \frac{d\xi^*}{\xi^*} = \left[\frac{\xi s'(\xi)}{s(\xi)} - 1\right]\frac{d\xi}{\xi} \to s^*(\xi^*)\frac{d\xi^*}{\xi^*} = \left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi$$

$$\xi = z \leftrightarrow \xi^* = y;\ \xi = \bar{z} \leftrightarrow \xi^* = 0,$$

we have

$$\Phi^*(y) - \Phi(z) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* - \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}d\xi = \frac{1}{s(z)}\int_{\bar{z}}^{z}\left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi - \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}d\xi = 1.$$

Since $w(\xi) \equiv \frac{s(\xi)/\xi}{\int_z^{\bar{z}}[s(\xi')/\xi']d\xi'} \Leftrightarrow s(z)\Phi(z)w(\xi) = \frac{s(\xi)}{\xi}$ and $w^*(\xi^*) \equiv \frac{s^*(\xi^*)/\xi^*}{\int_0^y\left[s^*\left({\xi^*}'\right)/{\xi^*}'\right]d{\xi^*}'} \Leftrightarrow s^*(y)\Phi^*(y)w^*(\xi^*) = \frac{s^*(\xi^*)}{\xi^*}$, this implies

$$\frac{\xi w(\xi)}{\xi^* w^*(\xi^*)} = \frac{\Phi^*(y)}{\Phi(z)} = 1 + \frac{1}{\Phi(z)} = \frac{\Phi^*(y)}{\Phi^*(y) - 1},$$

$$\frac{c}{c^*} = \exp\left[\int_\Omega [s^*(y_\omega)\Phi^*(y_\omega) - s(z_\omega)\Phi(z_\omega)]d\omega\right] = \exp\left[\int_\Omega s(z_\omega)d\omega\right] = e.$$

and

$$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) = \Phi^*\left(s^{*-1}(1/V)\right) - 1.$$

Appendix D. Proofs of Lemmas

Proof of Lemma 1: Log-differentiating $s(z)\Phi(z) \equiv \int_z^{\overline{z}} \frac{s(\xi)}{\xi} d\xi$ yields

$$\frac{zs'(z)}{s(z)} + \frac{z\Phi'(z)}{\Phi(z)} = \frac{-s(z)}{\int_z^{\overline{z}}[s(\xi)/\xi]\, d\xi} = \frac{s(\overline{z}) - s(z)}{\int_z^{\overline{z}}[s(\xi)/\xi]\, d\xi} = \frac{\int_z^{\overline{z}} s'(\xi)\, d\xi}{\int_z^{\overline{z}}[s(\xi)/\xi]\, d\xi} = \frac{\int_z^{\overline{z}} \left[\frac{\xi s'(\xi)}{s(\xi)}\right]\left[\frac{s(\xi)}{\xi}\right] d\xi}{\int_z^{\overline{z}}[s(\xi)/\xi]\, d\xi}$$
$$= \int_z^{\overline{z}} [1 - \zeta(\xi)] w(\xi)\, d\xi = 1 - \int_z^{\overline{z}} \zeta(\xi) w(\xi)\, d\xi$$

where $w(\xi) \equiv \frac{s(\xi)/\xi}{\int_z^{\overline{z}}[s(\xi')/\xi']d\xi'}$ satisfies $\int_z^{\overline{z}} w(\xi)\, d\xi = 1$. Thus, $\zeta'(\cdot) \gtreqless 0, \forall z \in (z_0, \overline{z})$ implies

$$\frac{z\Phi'(z)}{\Phi(z)} = 1 - \int_z^{\overline{z}} \zeta(\xi) w(\xi)\, d\xi - \frac{zs'(z)}{s(z)} = \zeta(z) - \int_z^{\overline{z}} \zeta(\xi) w(\xi)\, d\xi \lesseqgtr 0.$$

Moreover, $\Phi'(z) = 0, \forall z \in (0, \infty)$ implies $\zeta(z) = \int_z^{\overline{z}} \zeta(\xi) w(\xi)\, d\xi,\ \forall z \in (0, \infty)$, which implies $\zeta'(z) = 0$, $\forall z \in (0, \infty)$. This completes the proof. ■

Proof of Lemma 2. First, from $\phi(0) = 0$ and $0 < y\phi'(y) < \phi(y)$ for all $y > 0$, $\lim_{y \to 0} y\phi'(y) = 0$. Thus, $\zeta^{D'}(y) \gtreqless 0, \forall y \in (0, y_0)$ implies

$$\left[1 - \frac{1}{\zeta^D(y)}\right] \phi(y) = \left[1 - \frac{1}{\zeta^D(y)}\right] \int_0^{y} \phi'(\xi) d\xi \gtreqless \int_0^{y} \left[1 - \frac{1}{\zeta^D(\xi)}\right] \phi'(\xi) d\xi = \int_0^{y} [\phi'(\xi) + \xi\phi''(\xi)] d\xi$$
$$= \int_0^{y} d[\xi\phi'(\xi)] = y\phi'(y).$$

which in turn implies

$$\frac{y\mathcal{E}'_\phi(y)}{\mathcal{E}_\phi(y)} = \frac{\phi(y)}{\phi'(y)} \frac{d}{dy}\left[\frac{y\phi'(y)}{\phi(y)}\right] = \frac{[\phi'(y) + y\phi''(y)]\phi(y) - y[\phi'(y)]^2}{\phi'(y)\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) \gtreqless 0.$$

Furthermore, from the above equation, $\mathcal{E}'_\phi(\cdot) = 0$ implies

$$1 - \frac{1}{\zeta^D(y)} = \mathcal{E}_\phi(y),$$

which is constant and hence $\zeta^{D'}(\cdot) = 0$. This completes the proof. ■

Proof of Lemma 3: First, we show $\overline{z}\theta'(\overline{z}) = 0$. For $\overline{z} < \infty$, this follows from $\theta'(\overline{z}) = 0$. For $\overline{z} = \infty$,

$$\theta(z) = -\int_z^\infty \theta'(\xi)d\xi = -\int_z^\infty \frac{\xi\theta'(\xi)}{\xi}d\xi = -\lim_{x\to\infty}\int_z^x \frac{\xi\theta'(\xi)}{\xi}d\xi$$

Suppose that there is $z_0 > 0$ such that, for all $z > z_0$, $-z\theta'(z) > c > 0$. Then,

$$\theta(z_0) = -\lim_{x\to\infty}\int_{z_0}^x \frac{\xi\theta'(\xi)}{\xi}d\xi > \lim_{x\to\infty}\int_{z_0}^x \frac{c}{\xi}d\xi = \infty,$$

a contradiction. Hence, $\overline{z}\theta'(\overline{z}) = \lim_{z\to\infty} z\theta'(z) = 0$. Then, $\zeta^{I'}(\cdot) \gtreqless 0, \forall z \in (z_0, \overline{z})$ implies

$$[\zeta^I(z) - 1]\theta(z) = [\zeta^I(z) - 1]\left[-\int_z^{\overline{z}} \theta'(\xi)d\xi\right] = \int_z^{\overline{z}} [\zeta^I(z) - 1](-\theta')(\xi)d\xi \lesseqgtr$$

$$\int_z^{\overline{z}} [\zeta^I(\xi) - 1](-\theta')(\xi)d\xi = \int_z^{\overline{z}} [\xi\theta''(\xi) + \theta'(\xi)]d\xi = \int_z^{\overline{z}} d[\xi\theta'(\xi)] = -z\theta'(z),$$

which in turn implies

$$\frac{z\mathcal{E}'_\theta(z)}{\mathcal{E}_\theta(z)} = -\frac{\theta(z)}{\theta'(z)}\frac{d}{dz}\left[-\frac{z\theta'(z)}{\theta(z)}\right] = \frac{[z\theta''(z) + \theta'(z)]\theta(z) - z[\theta'(z)]^2}{\theta'(z)\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) \gtreqless 0.$$

Furthermore, from the above equation, $\mathcal{E}'_\theta(\cdot) = 0$ implies $\zeta^I(z) - 1 = \mathcal{E}_\theta(z)$, which is constant and hence $\zeta^{I'}(\cdot) = 0$. This completes the proof. ■

Proof of Lemma 1*: Log-differentiating $s^*(y)\Phi^*(y) \equiv \int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^*$ yields

$$\frac{ys^{*\prime}(y)}{s^*(y)} + \frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{s^*(y)}{\int_0^y [s^*(\xi^*)/\xi^*]\,d\xi^*} = \frac{\int_0^y s^{*\prime}(\xi^*)\,d\xi^*}{\int_0^y [s^*(\xi^*)/\xi^*]\,d\xi^*} = \frac{\int_0^y \left[\frac{\xi s^{*\prime}(\xi^*)}{s^*(\xi^*)}\right]\left[\frac{s^*(\xi^*)}{\xi^*}\right]d\xi^*}{\int_0^y [s^*(\xi^*)/\xi^*]\,d\xi^*}$$

$$= \int_0^y \left[1 - \frac{1}{\zeta^*(\xi^*)}\right] w^*(\xi^*)\,d\xi^*,$$

where $w^*(\xi^*) \equiv \frac{s^*(\xi^*)/\xi^*}{\int_0^y \left[s^*\left(\xi^{*\prime}\right)/\xi^{*\prime}\right]d\xi^{*\prime}}$ satisfies $\int_0^y w^*(\xi^*)d\xi^* = 1$. Thus, $\zeta^{*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0)$ implies

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \int_0^y \left[1 - \frac{1}{\zeta^*(\xi^*)}\right] w^*(\xi^*)\,d\xi^* - \frac{ys^{*\prime}(y)}{s^*(y)} = \int_0^y \left[\frac{1}{\zeta^*(y)} - \frac{1}{\zeta^*(\xi^*)}\right] w^*(\xi^*)\,d\xi^* \gtreqless 0.$$

Moreover, $\Phi^{*\prime}(y) = 0, \forall y \in (0, \infty)$ implies $\frac{1}{\zeta^*(y)} = \int_0^y \frac{w^*(\xi^*)}{\zeta^*(\xi^*)}d\xi^*, \forall y \in (0, \infty)$,which implies $\zeta^{*'}(y) = 0, , \forall y \in (0, \infty)$. This completes the proof. ■

Figure: Three Classes of Homothetic Symmetric Demand Systems with Gross Substitutes.