Corporate Default and Investment Policies under Rollover Risk and Solvency Concern*

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Abstract

We consider the role of a market learning process in identifying the firm's solvency risk in its financing and investment decisions under endogenous interaction between rollover risk and solvency concern. We show that an increase in liquidity (solvency) uncertainty raises (reduces) the firm's incentives to default, but reduces (raises) the firm's incentives to invest, although unlike the standard real options model, an increase in liquidity (solvency) uncertainty reduces (raises) the volatility of the state variable defined as the posterior expectation of the drift of cash flows. We also find that the firm's incentive to default (invest) decreases (increases) as debt maturity increases. In addition, our model predicts that an increase in liquidity (solvency) uncertainty raises (reduces) the leverage ratio of the firm, while reducing (raising) credit spreads if debt maturity is sufficiently long.

JEL Classification Codes: D83, G31, G32, G33.

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1. Introduction

The 2007–2008 financial crisis shed new light on the importance of the interaction between rollover risk and solvency concern in financial markets: the deterioration of rollover risk caused severe financial difficulties for many firms and exacerbated their solvency concern, whereas the aggravated solvency concerns conversely increased rollover risk. In fact, because economic circumstances changed drastically during the financial crisis, market participants were forced to reconsider the solvency risk of firms by determining their current and future profitability.

The objective of this paper is to study the role of a market learning process in identifying the firm's solvency risk in its default, investment, and leverage decisions under the interaction between rollover risk and solvency concern. Rollover risk in this paper means not only the possibility of the firm's failure to roll over debt, but also the fluctuations in rollover gains/losses incurred by the firm.

We consider the situation in which a firm must continuously roll over maturing bonds. As in Diamond and He (2014), we assume that equity holders pay the principal back on maturing bonds by issuing new bonds with the same principal and maturity at market prices, which can be higher or lower than the principal of the maturing bonds. In addition, we introduce two sources of uncertainty in cash flows—liquidity and solvency uncertainties—and incorporate a learning process over time regarding the solvency uncertainty, following the learning models of Gryglewicz (2011), DeMarzo and Sannikov (2017), and He, Wei, Yu, and Gao (2017).[1,2] Then, liquidity is the ability of a firm to compensate for rollover losses at each point in time and is short-term in nature. Solvency is the ability of a firm to incur debt obligations over longer periods of time. More specifically, we assume that cash flows follow a Brownian motion with drift that is not directly observable. In this framework, liquidity uncertainty is represented by the Brownian motion, whereas solvency uncertainty is expressed by the uncertain drift. Investors observe noisy cash flows and learn about the drift through a Bayesian-type updating process.

In our model, both the debt rollover and the learning process generate the endogenous

[1] Chang, Dasgupta, Wang, and Yao (2014) show empirically the importance of decomposing corporate cash flows into temporary and permanent components to understand how firms allocate cash flows, and whether financial constraints matter in this allocation decision.

[2] However, DeMarzo and Sannikov (2017) and He, Wei, Yu, and Gao (2017) examine the learning process in a continuous-time agency model rather than a capital structure model.

interaction between rollover risk and solvency concern, which is characterized as a type of feedback effect. For example, via the learning process, persistent cash flow shocks affect solvency concern, which has an effect on the firm's default, investment, and leverage policies. The changes in these policies influence rollover gains/losses and cause fluctuations in rollover risk. More specifically, the firm's equity holders must absorb rollover gains/losses because the market prices of newly issued bonds can be higher or lower than the principal of the maturing bonds. This implies that the more severely the firm's fundamentals have deteriorated, the heavier are the rollover losses incurred by equity holders. This conflict of interest between equity and debt holders may induce equity holders to default optimally when they find it unprofitable to absorb further losses. Thus, rollover risk generates solvency concern. Consequently, variations in rollover risk affect solvency concern.

However, the learning process in our model enables us to isolate the effects of solvency uncertainty on the firm's policies from those of liquidity uncertainty. The reason is that the two uncertainties work oppositely in the learning process: greater liquidity uncertainty updates the posterior expectation of the drift of cash flows slower, whereas greater solvency uncertainty updates the posterior expectation of the drift of cash flows faster. Because greater liquidity (solvency) uncertainty makes cash flow signals less (more) informative, greater liquidity (solvency) uncertainty decreases (increases) the present value of the expected profit of the firm. Hence, the two sources of uncertainty have different implications for the informativeness of cash flow signals under the learning process and affect the firm's policies differently through the variations in the informativeness of cash flows.

In fact, this mechanism is very different from that of a real options model in which the firm's policies are affected through the option value of waiting to execute the irreversible decisions. In particular, in our model, the firm's decisions depend on the volatility of the posterior expectation of the drift of cash flows, whereas in real options models they depend on the volatility of cash flows (the state variable) involved in the option value of waiting. In our model, the posterior expectation of the drift of cash flows serves as the state variable, while cash flows work as a signal to make market participants form the posterior expectation.

However, it is complicated to analyze the market learning process of the firm's solvency risk under the endogenous interaction between rollover risk and solvency concern in the learning model because it is not straightforward to disentangle the effects on the default and investment decisions. To make the analysis tractable, we compare our baseline model with two benchmarks. The first benchmark is the "constant capital stock" model, in which neither investment nor depreciation occurs.[3] Only the effect on the default decision is investigated. The second benchmark is the "equity finance" model, in which all the required funds are financed by equity. In this case, no default decision is considered because there is no debt. Only the effect on the investment decision is examined.

The main results of our model are summarized as follows. We first discuss how liquidity and solvency uncertainties affect optimal default and investment policies. An increase in liquidity uncertainty raises the firm's incentives to default, whereas solvency uncertainty reduces them. However, an increase in liquidity uncertainty reduces the firm's incentives to invest, while solvency uncertainty raises them. The interesting implication of these results is that an increase in the volatility of cash flows hastens the default timing and delays the investment timing, even though it reduces the volatility of the posterior expectation of the drift as the state variable. By contrast, in standard real options models, such as in Dixit and Pindyck (1994), the default (investment) timing is delayed (hastened) if the volatility of the state variable decreases. Furthermore, our finding regarding the firm's investment policy also provides additional new results that the effect of solvency uncertainty on the investment policy—debt overhang—is opposite to that of liquidity uncertainty.

Intuitively, greater liquidity (solvency) uncertainty delays (hastens) the update of the posterior expectation of the drift of cash flows by decreasing (increasing) the volatility of the posterior expectation of the drift of cash flows. Thus, greater liquidity (solvency) uncertainty makes cash flow signals less (more) informative through the learning process. As a result, under our framework, the two sources of uncertainty affect the present value of the additional expected profit of the firm differently because of a change in the expectation of the firm's profitability and the market value of debt, thereby leading to the different effects on the default and investment policies. In particular, greater liquidity (solvency) uncertainty hastens (delays) the default timing and delays (hastens) the investment timing through a mechanism in which greater liquidity (solvency) uncertainty decreases (increases) the present value of the additional expected profit of the firm because of a change in the expectation of the firm's profitability by making cash flow signals less (more) informative.

Second, we examine the effect of debt maturity on optimal default and investment policies.

[3]Note that because of the presence of depreciation, the baseline model is not reduced to the constant capital stock model even if the investment cost is infinitely large.

Then, default is more likely to occur if debt maturity is shorter. In addition, default is more likely to arise in the baseline model than in the constant capital stock model. Intuitively, investment opportunities induce equity holders to be more likely to choose default earlier. However, unlike Diamond and He (2014), investment incentives for equity holders improve as debt maturity increases. This novel result implies that less debt overhang occurs for longer maturities. However, investment incentives are more aggravated in the baseline model than in the equity finance model. Intuitively, as in the ex post debt overhang model of Gertner and Scharfstein (1991), the market value of debt decreases with debt maturity because the future portion of debt is riskier. This leverage effect increases equity holders' incentive to invest when debt maturity is longer. In addition, debt overhang forces equity holders to invest later in the baseline model than in the equity finance model.

Third, our model predicts that an increase in liquidity (solvency) uncertainty raises (reduces) the leverage ratio of the firm, while reducing (raising) credit spreads if debt maturity is sufficiently long. These results again depend on the fact that the greater liquidity (solvency) uncertainty makes cash flow signals less (more) informative through the learning process.

This paper provides several empirical implications of the relationship between uncertainty and the firm's decisions on investment and capital structure. For investment, there remains a possibility that increasing uncertainty enhances firms' investment activities such as research and development (see Bloom, 2014, and Kraft, Schwartz, and Weiss, 2013). Given that the two sources of uncertainty have opposite effects on the firm's optimal investment policy in our model and that solvency uncertainty plays a more important role in the firm's research and development activity, we suggest that solvency uncertainty should be distinguished from liquidity uncertainty in the empirical analysis of investment. We also predict that increasing solvency uncertainty enhances the investment activities of firms in "new economy" industries, such as information technology and bioscience.

For capital structure, the existing empirical literature provides competing views about a relationship between cash flow volatility and capital structure (see Bradley, Jarrell, and Kim, 1984). A possible reason for the divergence in views may be because of a difference in identification of the cash flow volatility (see Keefe and Yaghoubi, 2016). However, our finding shows that greater solvency (liquidity) uncertainty decreases (increases) the leverage ratio. Consequently, our paper uncovers the importance of making a distinction between liquidity and solvency uncertainties in interpreting the mixed results in the existing empirical literature.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the model setting. Section 4 derives the debt and equity valuations. Section 5 investigates the optimal default, investment, and leverage policies of the firm. Section 6 clarifies the empirical implications of our results. Section 7 concludes. All proofs are in the Appendix.

2. Literature Review

This paper is related to several recent studies on default–liquidity interaction under debt rollover.[4] He and Xiong (2012a) consider the setting in which bond investors hit by liquidity shocks are forced to sell their holdings immediately at an exogenous transaction cost. Because such deterioration in liquidity causes a firm to suffer losses in rolling over its maturing debt, equity holders must absorb the losses when they do not choose to default.[5] However, when the equity value drops to zero, equity holders choose to default optimally. He and Milbradt (2014) endogenize secondary bond market liquidity (or the transaction cost) by modeling bond trading in a search-based secondary market, and examine debt valuations, equity valuations, and default policy. Unlike the authors of these two papers, Diamond and He (2014) neglect both the liquidity shock of bond investors and transaction costs in secondary bond markets. Instead, they incorporate investment opportunities, determine debt maturity endogenously, and discuss how debt maturity affects debt overhang. In contrast to these three papers, our paper distinguishes between liquidity and solvency uncertainties and incorporates a market learning process over time into the debt rollover model regarding the firm's solvency risk. Relative to models without such a learning process, we can show that liquidity uncertainty affects the investment decision through a mechanism that is different from the standard real

[4] The methodology of these papers extends the constant debt maturity structure models of Leland (1994, 1998) and Leland and Toft (1996). By ruling out Brownian cash flow shocks and assuming deterministically decreasing cash flows with a possible terminal upward jump, He and Milbradt (2016) consider how a firm chooses its debt maturity structure and default timing dynamically without any commitment to a policy of constant debt maturity structure. DeMarzo and He (2016) also study a model in which equity holders lack the ability to commit to their future leverage choices and can fully issue or buy back debt at the current market price at any time.

[5] He and Xiong (2012b) focus on rollover risk originated from coordination problems between debt holders in firms that cannot raise funds by issuing new equity. Cheng and Milbradt (2012) also discuss a similar coordination problem under the risk-shifting incentive of the manager. However, neither the investment decision nor the learning process is considered in these models.

options model. We can also clarify the effects of solvency uncertainty on the decisions of the firm about default, investment, and leverage policies, by separating these effects from the effects of liquidity uncertainty, under the endogenous interaction between rollover risk and solvency concern generated by both debt rollover and learning.

Our analysis is related to several existing studies on the effect of debt overhang first formalized by Myers (1977), which points out that outstanding debt may distort the firm's investment incentives downward. Myers (1977) suggests that the shorter the maturity of debt, the smaller the ex ante debt overhang. This is because the value of shorter-term debt is less sensitive to the value of the firm, and the debt value represents a smaller benefit from new investment taken after the debt is issued. Gertner and Scharfstein (1991) incorporate investment opportunities in a two-period model, conditional on ex post financial distress. Holding constant the promised payment to debt holders, they show that shorter-term debt imposes a stronger overhang ex post because making an early fixed-promised debt payment causes debt to be safer and raises the market value of the debt (the firm's market leverage), increases transfers to debt holders, and thus causes ex post debt overhang.[6] In the analysis of the effect of maturity on debt overhang, Diamond and He (2014) stress the timing of investment decisions and hold the initial market value of debt constant by varying the promised debt payment. They develop a dynamic model based on Leland (1994, 1998), in which firms have many investment opportunities in the present and future, and show that investment incentives first increase with debt maturity for very short maturities, but then decrease with debt maturity for longer maturities. By contrast, our paper indicates that investment incentives improve with debt maturity as debt maturity increases. The difference between the results of our paper and Diamond and He (2014) reflects the following: in our model, the possibility of debt rollover creates leverage effects through the changes not only in the debt value when rolling over debt, but also in the default loss for debt holders because the initial market value of debt per capital stock varies, while the promised payment to debt holders per unit of capital stock is held constant. Note that our leverage effect does not arise in the two-period framework of Gertner and Scharfstein (1991) because they do not consider the possibility of debt rollover so that there is no default loss in the date-1 portion of debt

[6] The quantitative study of Titman and Tsyplakov (2007) based on Leland (1998) also fixes the promised debt payment and focuses on leverage adjustments by incorporating a tax shield and physical costs of default and adjusting leverage. They suggest that the shorter-term debt improves investment incentives further but triggers the earlier default.

in their model.

Our paper also complements the recent study of Gryglewicz (2011) on solvency–liquidity interaction under a learning process.[7] In his model, the firm faces both liquidity and solvency uncertainties. In addition, at each point in time, the positive net earnings of the firm can be distributed as dividends or retained to increase cash holdings, although the firm cannot issue new equity. As a result, losses and dividends must be covered from cash reserves instead of new equity issues in his model, unlike He and Xiong (2012a), He and Milbradt (2014), Diamond and He (2014), DeMarzo and He (2016), and our model.[8] Using this framework, Gryglewicz (2011) focuses on the role of initial cash holdings under liquidity and solvency concerns by ruling out debt rollover and the investment decision of the firm, although he assumes that the firm has sufficient cash holdings to avoid liquidity default (liquidity concern). Hence, he cannot obtain any results regarding debt maturity or investment. Accordingly, the main difference between our paper and Gryglewicz (2011) is that we investigate a market learning process of the firm's solvency risk under the interaction between rollover risk and solvency concern when newly issued equity covers the net loss under debt rollover and when the choice of investment is considered. Again, we highlight that Gryglewicz (2011) assumes perpetual debt and rules out the investment decision problem. Therefore, we can derive the effect of liquidity uncertainty on the default, investment, and leverage policy decisions through a different mechanism from a standard real options model and also clarify the effect of solvency uncertainty on the default, investment, and leverage policy decisions of the firm under the endogenous interaction between rollover risk and solvency concern.

3. The Model

3.1. Outline of the model.—

We consider a firm that generates uncertain cash flows and selects investment and default policies at $t \geq 0$. We assume that all of the agents in the model are risk neutral and discount cash flows at a constant risk-free rate r, and that management acts in the interest of equity

[7] A different learning model is also developed using the continuous-time agency framework of DeMarzo and Sannikov (2017), in which the firm's expected cash flows are controlled through costly effort observed by the agent alone and the firm's expected profitability is learned over time.

[8] In fact, the benchmark model of Gryglewicz (2011) follows the framework of Leland (1994). As a result, losses and dividends can be covered from new equity issues in his benchmark model. However, the valuations of equity, debt, and the firm in his benchmark model are different from those derived in our model because he assumes perpetual debt and neglects the investment decision problem.

holders.

The firm's financing comes from a combination of equity and multiple debt issues. Because it is difficult to analyze dynamic models of multiple debt issues, we use a framework based on He and Xiong (2012a) and Diamond and He (2014) by extending it with the incorporation of capital stock accumulation.[9] In this framework, even though conditions change, the firm keeps constant the total amount promised to debt holders per capital stock at each refinancing, and does not adjust this amount in response to new conditions. To satisfy this requirement, we assume that the firm can always raise equity as needed whenever the value of equity is positive. Then, equity holders are willing and able to inject any funds necessary to cover investment costs or losses at refinancing. To focus on the effect of external market liquidity, we also assume that internal liquidity such as cash holdings and credit lines is unavailable.

The assumption on the financing of the firm allows us to eliminate the possibility of the firm defaulting because of illiquidity, that is, liquidity default. Instead, this assumption enables us to focus on solvency default, which is defined as a situation in which the firm voluntarily defaults if the value of equity falls below zero.

3.2. Earnings and learning.—

At each time t, a firm produces output by employing capital. The firm's capital stock K_t evolves according to

$$dK_t = (i_t - \delta)K_t dt, \tag{1}$$

where i_t is the firm's growth investment rate controlled by equity holders and $\delta \geq 0$ is the rate of depreciation. We assume that $i_t \in \{0, i\}$ takes a binary value, and that the investment cost is $\lambda i_t K_t$. We also assume that $r + \delta > i > \delta$. The firm generates a stochastic flow of earnings:

$$dX_t = \overline{\mu} K_t dt + \sigma K_t d\overline{Z}_t - \lambda i_t K_t, \tag{2}$$

where $\overline{\mu}$ is the true mean value of earnings per capital stock, σ is the constant volatility, and $\{\overline{Z}_t : 0 \leq t \leq \infty\}$ is a standard Brownian motion.

The firm faces two sources of uncertainty about the instantaneous flow of earnings. The

[9] He and Xiong (2012a) and Diamond and He (2014) use the framework of Leland (1994, 1998) and Leland and Toft (1996). The latter three papers take as fixed parameters both the frequency of refinancing and the total amount of promised repayments of debt.

first uncertainty arises from Brownian shocks $d\overline{Z}_t$, whereas the second uncertainty comes from the fact that the true $\overline{\mu}$ is ex ante unknown to all parties. We assume that $\overline{\mu}$ is a fixed parameter and can take either of the two values μ_L or μ_H, with $\mu_L < \mu_H$.

The information structure of our model is as follows. We assume that all parties have the same information at each time t. More specifically, at the initial time 0, all parties share a common prior expectation μ_0 about $\overline{\mu}$, with $\mu_0 \in (\mu_L, \mu_H)$. As time evolves, more information generated by X_t becomes available. Thus, all parties update their expectation of $\overline{\mu}$. Let $\mathcal{F}_t$ denote the current set of information generated by X_t. Then, the posterior expectation of the mean earnings based on information up to time t, μ_t, is given by $\mu_t = \boldsymbol{E}[\overline{\mu} \mid \mathcal{F}_t]$.

Let dZ_t denote the difference between the realized and expected earnings. Then, the dynamics of X_t in terms of observables are represented as follows:

$$dX_t = (\mu_t - \lambda i_t)K_t dt + \sigma K_t dZ_t. \tag{3}$$

Note that the process $\{Z_t : 0 \leq t \leq \infty\}$ is a Brownian motion adapted to filtration $\mathcal{F}_t$.

The posterior expectation of the mean earnings per capital stock, μ_t, evolves as (see Lipster and Shiryaev (2001))[10]

$$d\mu_t = \frac{1}{\sigma}(\mu_t - \mu_L)(\mu_H - \mu_t)dZ_t. \tag{4}$$

The key point of equation (4) is that expectations adjust more rapidly if σ is small, while learning slows down if μ_t is close to either μ_L or μ_H. This feature will help us understand our results regarding the differences in the effects of short-term liquidity and long-term solvency uncertainties in the subsequent analysis.

As discussed in Gryglewicz (2011), the specification written as (3) and (4) elucidates a close relation between cash flow shocks and solvency. Compared with equation (2), we find that in equation (3), short-term negative (positive) cash flow shocks $dZ_t < 0$ ($dZ_t > 0$) are

[10] DeMarzo and Sannikov (2017) and He, Wei, Yu, and Gao (2017) show that the evolution equation of the posterior expectation of the mean earnings can be represented independently of an unobservable effort level of the agent, even though the stochastic flow of cash earnings depends on the unobservable effort level. Similarly, our evolving equation of μ can be represented independently of the observable investment level, although the stochastic flow of cash earnings depends on the observable investment level. In addition, we may formalize the time-varying $\overline{\mu}$ like DeMarzo and Sannikov (2017). Then, the true value of $\overline{\mu}_t$ is never known with certainty, but all the agents believe at time 0 that $\overline{\mu}_0 \sim N(\mu_0, \varsigma_0)$. Even in this case, under certain conditions, μ_t evolves like equation (4). See Subsection 2.1 of DeMarzo and Sannikov (2017) for the detailed discussion.

more likely to occur if the firm is of low (high) expected long-term profitability $\overline{\mu}$. The reason is that $dZ_t = \frac{dX_t - (\mu_t - \lambda i_t) K_t dt}{\sigma K_t} < 0$ is more likely to arise if dX_t is more likely to fall below $(\mu_t - \lambda i_t) K_t dt$. Because (2) implies that dX_t is more likely to be small if the true $\overline{\mu}$ is low ($\overline{\mu} = \mu_L$), we show that $dZ_t < 0$ is more likely to occur if $\overline{\mu} = \mu_L$. Similarly, $dZ_t > 0$ is more likely if $\overline{\mu} = \mu_H$. Hence, this specification indicates that cash flow shocks and solvency are closely interrelated.

Suppose that equity holders always invest ($i_t = i$) and the firm does not default. Given the current value of the posterior expectation μ_t, the present value of the firm per capital stock, which is equal to the expected discounted future cash flows per capital stock, is represented by $\frac{\mu_t - \lambda i}{r - i + \delta}$. Because the present value of the firm per capital stock without investment is $\frac{\mu_t}{r+\delta}$, investment is always profitable for the firm if $\frac{\mu_t - \lambda i}{r - i + \delta} \geq \frac{\mu_t}{r+\delta}$. Hence, if $\mu_t \geq (r+\delta)\lambda$, investment at time t maximizes the total value of the firm. We also assume that investment can be undertaken only by equity holders, and that future investment policies are lost when debt holders take over the firm at default. Thus, if default occurs, the first-best policy that investment occurs at every instant when $\mu_t \geq (r+\delta)\lambda$ cannot be achieved.

3.3. Stationary debt structure and rolling over debt.—

According to Leland (1994, 1998) and Leland and Toft (1998), we assume that the firm has one unit of debt per capital stock with a constant aggregate principal face value of debt p and maintains a stationary debt structure per capital stock under a refinancing policy in which at each instant a constant fraction of debt per capital stock, $f dt$, becomes due and must be refinanced to keep the amount of total debt outstanding per capital stock constant.[11] Thus, given refinancing frequency f, the average debt maturity is $m \equiv \frac{1}{f}$. In addition, because each debt is retired exponentially, the firm's existing debt per capital stock is identical at any point in time.

Let $D(R_t, K_t)$ denote the market value of the firm's debt, where $R_t = \mu_t K_t$ is the posterior expectation of the mean earnings. In issuing new debt to replace maturing debt, the firm receives total proceeds $\frac{D(R_t, K_t)}{m} dt$ by issuing $\frac{K_t}{m} dt$ units of new debt and pays $\frac{p K_t}{m} dt$ to replace maturing debt. Because the market price of newly issued debt fluctuates with the posterior expectation of the mean earnings μ_t, the net payments to bond holders lead to rollover

[11] We assume that there is no coupon payment. Thus, debt in our model can be interpreted as zero-coupon debt.

gains/losses, which are represented by $\frac{1}{m}\left[D(R_t, K_t) - pK_t\right] dt$.[12] The rollover gains or losses are received or paid by equity holders. This implies that any gain will be paid out to equity holders immediately, whereas any loss will be paid off by issuing more equity at the market price. Thus, for μ_t, the expected net cash flow to equity holders is

$$(\mu_t - \lambda i_t)K_t dt + \frac{1}{m}\left[D(R_t, K_t) - pK_t\right] dt, \tag{5}$$

where the first term indicates the firm's expected net cash flows and the second term the rollover gains or losses.

When the firm issues additional equity to absorb rollover losses, the equity issuance dilutes the value of the existing shares. Hence, rollover losses affect the equity value. In fact, as investment can only be undertaken by equity holders, future investment opportunities are lost when debt holders take over the firm from bankruptcy. Thus, equity holders are willing and able to pay off rollover losses to keep the firm's operations running whenever the equity value is positive, that is, whenever the option value of keeping the firm alive justifies expected rollover losses. This means that insolvency default is triggered by equity holders when the equity value drops to zero.

4. Valuations of Debt and Equity for Different Debt Maturities

We now determine the values of claims held by debt and equity holders for different debt maturities. These values depend on the flows to the claimants and on the insolvency default and investment times chosen by equity holders. The insolvency default occurs when the posterior expectation μ_t drops to an endogenously determined threshold μ_B. In the Appendix, we assume that the total value of the firm increases in μ_t,[13] and can show that the optimal investment time is determined by an endogenous investment threshold μ_I.

4.1. Debt and equity values with evolution of capital stock.—

[12] As we assume zero-coupon debt, it follows from discounting that the firm always incurs rollover losses. However, whether rollover gains are possible or not is not essential to our analysis, as discussed in He and Xiong (2012a) and Diamond and He (2014).

[13] Our numerical calculation ensures that this assumption holds in our parameter set.

First, using (1) and (4), note that

$$dR_t = d(\mu_t K_t) = \mu_t(i_t - \delta)K_t dt + \frac{K_t}{\sigma}(\mu_t - \mu_L)(\mu_H - \mu_t)dZ_t. \tag{6}$$

Then, because equity holders use the investment threshold policy, the debt value before default satisfies the following ordinary differential equations:

$$rD(R,K) = \begin{cases} \frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2(K)^2 D_{RR}(R,K) + \mu(i-\delta)KD_R(R,K) \\ \quad +(i-\delta)KD_K(R,K) + \frac{1}{m}\left[pK - D(R,K)\right], \quad \text{if } \mu \geq \mu_I, \\ \frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2(K)^2 D_{RR}(R,K) - \mu\delta KD_R(R,K) - \delta KD_K(R,K) \\ \quad +\frac{1}{m}\left[pK - D(R,K)\right], \quad \text{if } \mu_I > \mu \geq \mu_B. \end{cases} \tag{7}$$

The first two terms (third term) on the right-hand side of (7) capture(s) the expected change in the debt value from a change in R in equation (6) (K in equation (1)), and the final term is the change in the debt value caused by rolling over debt.

Using the scale invariance of the firm's technology arising from the homogeneity assumption, we write $D(R,K) \equiv d(\frac{R}{K},1)K \equiv d(\mu)K$. Hence, we can reduce (7) to the following equations with a single state variable μ.

$$\left(r - i + \delta + \frac{1}{m}\right)d(\mu) = \frac{p}{m} + \frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2 d''(\mu), \quad \text{if } \mu \geq \mu_I, \tag{8a}$$

$$\left(r + \delta + \frac{1}{m}\right)d(\mu) = \frac{p}{m} + \frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2 d''(\mu), \quad \text{if } \mu_I > \mu \geq \mu_B. \tag{8b}$$

We need several boundary conditions to solve equation (8). Equity holders choose default at $\mu = \mu_B$. Then, we assume that the liquidation value per capital stock is equal to the value per capital stock of the all-equity firm at the moment of default, $\frac{\mu_B}{r+\delta}$. Furthermore, debt holders take over the firm with the value $\frac{\mu_B}{r+\delta}$ per capital stock without future investment.[14] This requirement is represented by the following value-matching condition:

$$d(\mu_B) = \frac{\mu_B}{r+\delta}. \tag{9}$$

[14]This assumption implies that debt holders can sell the firm to other investors without any liquidation costs in default. Gryglewicz (2011) imposes a similar assumption although he takes account of liquidation costs. However, even though liquidation costs are considered, our main results are unaffected.

At the investment boundary μ_I, the boundary conditions are also needed, that is,

$$\lim_{\mu \uparrow \mu_I} d(\mu) = \lim_{\mu \downarrow \mu_I} d(\mu), \tag{10}$$

$$\lim_{\mu \uparrow \mu_I} d'(\mu) = \lim_{\mu \downarrow \mu_I} d'(\mu). \tag{11}$$

Finally, if μ hits μ_H, we need the condition that $d(\mu_H)$ is bounded and is equal to the default-free debt value per capital stock, $\frac{p}{1+m(r-i+\delta)}$, as imposed in Gryglewicz (2011) and Diamond and He (2014). Thus,

$$d(\mu_H) = \frac{p}{1 + m(r - i + \delta)}. \tag{12}$$

This condition implies that μ_H is an absorbing state for μ.

We now consider the equity value. Using (1) and (6), the equity value must satisfy the following differential equation:

$$\begin{aligned} rE(R,K) = & \max_{i \in \{0,i\}} \frac{1}{2\sigma^2}(\mu - \mu_L)^2(\mu_H - \mu)^2(K)^2 E_{RR}(R,K) + \mu(i-\delta)KE_R(R,K) \\ & + (i-\delta)KE_K(R,K) + (\mu - \lambda i)K - \frac{1}{m}\left[pK - D(R,K)\right]. \end{aligned} \tag{13}$$

The first two terms (third term) on the right-hand side of (13) capture(s) the expected change in the equity value caused by a change in R in equation (6) (K in equation (1)), and the final term is the rollover gain/loss of equity holders.

Again, using the scale invariance of the firm's technology arising from the homogeneity assumption, we write $E(R,K) \equiv e(\frac{R}{K},1)K \equiv e(\mu)K$. Then,

$$re(\mu) = \max_{i \in \{0,i\}} \mu - \lambda i + (i-\delta)e(\mu) + \frac{1}{2\sigma^2}(\mu - \mu_L)^2(\mu_H - \mu)^2 e''(\mu) - \frac{1}{m}[p - d(\mu)]. \tag{14}$$

We specify boundary conditions to solve equation (14). In the Appendix, we can show that the threshold investment strategy is optimal. Thus, it follows from the maximization of the right-hand side of (14) with respect to i that

$$i(\mu) = \begin{cases} i, & \text{if } \mu \geq \mu_I, \\ 0, & \text{if } \mu_I > \mu \geq \mu_B, \end{cases}$$

and that the endogenous investment threshold μ_I chosen by equity holders must satisfy the following value-matching condition:[15]

$$e(\mu_I) = \lambda. \tag{15}$$

At the boundary μ_H, the debt value per capital stock is equal to the default-free debt value per capital stock $\frac{p}{1+m(r-i+\delta)}$. Thus, for consistency, we need to have

$$e(\mu_H) = \frac{\mu_H - \lambda i}{r - i + \delta} - \frac{p}{1 + m(r - i + \delta)}, \tag{16}$$

where the firm value per capital stock at $\mu = \mu_H$ is equal to the expected discounted future cash flows per capital stock that prevail if the firm always invests and does not default. To ensure this, we assume that $e(\mu_H) > \lambda$, that is, $\frac{\mu_H - \lambda(r+\delta)}{r-i+\delta} > \frac{p}{1+m(r-i+\delta)}$.

However, equity holders default at μ_B and receive zero under limited liability, which implies

$$e(\mu_B) = 0. \tag{17}$$

The endogenous default boundary also needs to satisfy the smooth-pasting condition:

$$e'(\mu_B) = 0. \tag{18}$$

To ensure the existence of μ_B ($\geq \mu_L$) and the immediate default for $\mu < \mu_B$, we assume that $e(\mu_L) < 0$, that is, $\frac{\mu_L}{r+\delta} < \frac{p}{1+m(r+\delta)}$. Finally, the following boundary conditions at the investment boundary are required:

$$\lim_{\mu \uparrow \mu_I} e(\mu) = \lim_{\mu \downarrow \mu_I} e(\mu), \tag{19}$$

$$\lim_{\mu \uparrow \mu_I} e'(\mu) = \lim_{\mu \downarrow \mu_I} e'(\mu). \tag{20}$$

Now, we provide the following proposition that clarifies the debt and equity values as

[15] In Diamond and He (2014), this condition is represented by the smooth-pasting condition. The difference depends on the difference between the formulation of the two models. More specifically, Diamond and He (2014) suppose that cash flows follow a geometric Brownian motion, whereas they do not consider capital accumulation. By contrast, we suppose that cash flows per capital stock follow an arithmetic Brownian motion by incorporating the capital accumulation process.

solutions to equations (8) and (14) together with boundary conditions (9)–(12) and (15)–(20).[16] We also verify the optimality of the threshold investment strategy in the Appendix by assuming that $v(\mu)$ is increasing in μ.

Proposition 1: *There exists a unique μ_I that satisfies (14). Thus, the optimal investment policy for each different debt maturity is given by the investment threshold policy: given μ_B and μ_I, equity holders invest as long as* the *posterior expectation μ exceeds a critical value μ_I:*

$$i(\mu) = \begin{cases} i, & \text{if } \mu \geq \mu_I, \\ 0, & \text{if } \mu_I > \mu \geq \mu_B. \end{cases}$$

The debt value is: if $\mu \geq \mu_I$,

$$d(\mu) = \frac{p}{1 + m(r - i + \delta)} + A_1(\mu - \mu_L)^{1-\beta_1}(\mu_H - \mu)^{\beta_1}; \tag{21a}$$

and if $\mu_I > \mu \geq \mu_B$,

$$d(\mu) = \frac{p}{1 + m(r + \delta)} + A_3(\mu - \mu_L)^{1-\beta_2}(\mu_H - \mu)^{\beta_2} + A_4(\mu - \mu_L)^{\beta_2}(\mu_H - \mu)^{1-\beta_2}. \tag{21b}$$

The equity value is: if $\mu \geq \mu_I$,

$$e(\mu) = \frac{\mu - \lambda i}{r - i + \delta} - \frac{p}{1 + m(r - i + \delta)} + B_1(\mu - \mu_L)^{1-\gamma_1}(\mu_H - \mu)^{\gamma_1} - A_1(\mu - \mu_L)^{1-\beta_1}(\mu_H - \mu)^{\beta_1}; \tag{22a}$$

and if $\mu_I > \mu \geq \mu_B$,

$$\begin{aligned} e(\mu) = {} & \frac{\mu}{r + \delta} - \frac{p}{1 + m(r + \delta)} + B_2(\mu - \mu_L)^{1-\gamma_2}(\mu_H - \mu)^{\gamma_2} + B_3(\mu - \mu_L)^{\gamma_2}(\mu_H - \mu)^{1-\gamma_2} \\ & - A_3(\mu - \mu_L)^{1-\beta_2}(\mu_H - \mu)^{\beta_2} - A_4(\mu - \mu_L)^{\beta_2}(\mu_H - \mu)^{1-\beta_2}. \end{aligned} \tag{22b}$$

The constants β_1, β_2, γ_1, γ_2, A_1, A_3, A_4, B_1, B_2, and B_3 are given by (A1), (A3), (A17), (A19), (A5)–(A7), and (A23)–(A25), respectively, in the Appendix. The endogenous boundaries μ_B and μ_I are also given by (A29) and (A30) in the Appendix. In particular, μ_B

[16] In the subsequent analysis, for brevity, we delete the term "per capital stock" from each variable unless confusion occurs.

satisfies $\frac{\mu_B}{r+\delta} < \frac{p}{1+m(r+\delta)}$.

The value functions of $d(\mu)$ and $e(\mu)$ are interpreted as follows. If μ exceeds the investment threshold μ_I, the debt value is equal to the value of default-free debt minus the impact of potential default (the present value of the expected potential loss for debt holders when defaulting at μ). However, outside the investment region $\mu_I > \mu \geq \mu_B$, the debt value includes an additional term that captures the adjustment for entering the investment region in the future. This additional term represents the loss of the debt value caused by the execution of investment. This loss arises when potential default occurs because of a change in the expectation of the firm's profitability, thereby moving the equity value close to zero. Regarding the equity value, if μ exceeds μ_I, it is comprised of four terms. The first and second terms are equal to the firm value at μ that would prevail if the firm always invested and did not default, minus the default-free debt value. The remaining two terms reflect the present value of the additional expected profit for equity holders generated by updating μ as well as the impact of potential default (the present value of the additional expected profit for equity holders when defaulting at μ). Outside the investment region $\mu_I > \mu \geq \mu_B$, the equity value now consists of six terms. The first and second terms are the firm value at μ without investment in the case of no default, minus the default-free debt value. The remaining four terms capture both the present value of the additional expected profit for equity holders generated by updating μ when the adjustment for entering the investment region in the future is included and the impact of potential default (the present value of the additional expected profit for equity holders in defaulting at μ when the adjustment for entering the investment region in the future is included).

Proposition 1 indicates that $\frac{\mu_B}{r+\delta} < \frac{p}{1+m(r+\delta)}$. This implies that on the date of default, there is a loss to debt holders.

4.2. Debt and equity values with default and investment boundaries under benchmark models.—

To facilitate discussion, we now solve two other cases that serve as benchmark models. We first consider a benchmark case in which neither investment nor depreciation occurs ($i = \delta = 0$) so that the firm's capital stock is constant ($K_t = const$ for any $t \geq 0$). This corresponds to the case in which no investment decision is considered. We call this the "constant capital stock" case, and indicate the corresponding solution by a superscript "c".

Then, ignoring μ_I and using $i = \delta = 0$, we rearrange Proposition 1 as follows.

Proposition 2: *Suppose that $i = \delta = 0$ and the capital stock is constant for any $t \geq 0$. Then, the debt value is:*

$$d^c(\mu) = \frac{p}{1+mr} - \left(\frac{p}{1+mr} - \frac{\mu_B^c}{r}\right)\left(\frac{\mu - \mu_L}{\mu_B^c - \mu_L}\right)^{1-\beta^c}\left(\frac{\mu_H - \mu}{\mu_H - \mu_B^c}\right)^{\beta^c}; \tag{23}$$

and the equity value is:

$$e^c(\mu) = \frac{\mu}{r} - \frac{p}{1+mr} + \left(\frac{p}{1+mr} - \frac{\mu_B^c}{r}\right)\left(\frac{\mu - \mu_L}{\mu_B^c - \mu_L}\right)^{1-\beta^c}\left(\frac{\mu_H - \mu}{\mu_H - \mu_B^c}\right)^{\beta^c}, \tag{24}$$

where $\beta^c > 1$ is the positive root of $(\beta^c)^2 - \beta^c - \frac{2(1+mr)\sigma^2}{m(\mu_H - \mu_L)^2} = 0$. The default threshold is given by

$$\mu_B^c = \frac{\mu_H\mu_L + [(\beta^c - 1)\mu_H - \beta^c\mu_L]\frac{rp}{1+mr}}{(1-\beta^c)\mu_L + \beta^c\mu_H - \frac{rp}{1+mr}}. \tag{25}$$

Note that μ_B^c satisfies $\frac{\mu_B^c}{r} < \frac{p}{1+mr}$.

Several remarks can be made. First, as we do not consider investment, we need not change the debt or equity value function according to whether investment occurs or not. In addition, except for the term regarding the value of default-free debt (the firm value at μ that would prevail if the firm always invested and did not default, minus the default-free debt value), the debt (equity) value consists of only the term involved in the present value of the expected potential loss (profit) for debt (equity) holders when defaulting at μ; in other words, the debt value does not include any terms involved in the adjustment for entering the investment region in the future, whereas the equity value does not include any terms involved in the present value of the additional expected profit for equity holders generated by updating μ. Second, the default threshold has the closed-form expression (25) in this case.

Gryglewicz (2011) derives the debt and equity value equations by assuming that new external financing is available only at the start of the project while the firm uses this initial financing to ensure cash holdings enable the firm to avoid liquidity default (see Proposition

4 in Gryglewicz (2011)). This assumption allows only solvency default by excluding liquidity default. Because he does not consider debt rollover or the investment decision of the firm, the debt and equity value equations in his model are essentially similar to (23) and (24) if we rule out debt rollover ($m \to \infty$) and make debt perpetual.[17] Note that there is no coupon payment in our model.

More formally, we provide the following corollary to clarify the relationship between our model and the benchmark model of Gryglewicz (2011, Proposition 1).

Corollary to Proposition 2: *Suppose that $i = \delta = 0$ and capital stock is constant for any $t \geq 0$. If we rule out debt rollover ($m \to \infty$), the debt value is:*

$$d^c(\mu) = \frac{\mu_B^c}{r}\left(\frac{\mu - \mu_L}{\mu_B^c - \mu_L}\right)^{1-\beta^c}\left(\frac{\mu_H - \mu}{\mu_H - \mu_B^c}\right)^{\beta^c}; \tag{23'}$$

and the equity value is:

$$e^c(\mu) = \frac{\mu}{r} - \frac{\mu_B^c}{r}\left(\frac{\mu - \mu_L}{\mu_B^c - \mu_L}\right)^{1-\beta^c}\left(\frac{\mu_H - \mu}{\mu_H - \mu_B^c}\right)^{\beta^c}, \tag{24'}$$

where $\beta^c > 1$ is the positive root of $(\beta^c)^2 - \beta^c - \frac{2r\sigma^2}{(\mu_H - \mu_L)^2} = 0$. The default threshold is given by

$$\mu_B^c = \frac{\mu_H \mu_L}{(1 - \beta^c)\mu_L + \beta^c \mu_H}. \tag{25'}$$

We next discuss the other benchmark case in which equity holders do not issue any debt ($d(\mu) = 0$ for any $t \geq 0$) and finance all the required funds by issuing equity. This corresponds to the case in which neither endogenous default decisions nor debt overhang problems are investigated. We denote this case as the "equity finance" case, and indicate the corresponding solution with a superscript "e". Then, ignoring μ_B and applying a proof procedure similar to that of Proposition 1, we obtain the following proposition.

Proposition 3: *Suppose that equity holders do not issue any debt ($d(\mu) = 0$ for any*

[17] Note that β^c in our model converges to β given in Gryglewicz (2011) as $m \to \infty$ (see the following corollary to Proposition 2).

$t \geq 0$). Then, the equity value is: if $\mu \geq \mu_I^e$,

$$e^e(\mu) = \frac{\mu - \lambda i}{r - i + \delta} + \left[\frac{\lambda(r+\delta) - \mu_I^e}{r - i + \delta}\right] \left(\frac{\mu - \mu_L}{\mu_I^e - \mu_L}\right)^{1-\gamma_1^e} \left(\frac{\mu_H - \mu}{\mu_H - \mu_I^e}\right)^{\gamma_1^e}, \tag{26a}$$

and if $\mu_I^e > \mu$,

$$e^e(\mu) = \frac{\mu}{r + \delta} + \left[\frac{\lambda(r+\delta) - \mu_I^e}{r + \delta}\right] \left(\frac{\mu - \mu_L}{\mu_I^e - \mu_L}\right)^{\gamma_2^e} \left(\frac{\mu_H - \mu}{\mu_H - \mu_I^e}\right)^{1-\gamma_2^e}, \tag{26b}$$

where $\gamma_1^e > 1$ and $\gamma_2^e > 1$ are the positive roots of $(\gamma_1^e)^2 - \gamma_1^e - \frac{2(r-i+\delta)\sigma^2}{(\mu_H-\mu_L)^2} = 0$ and $(\gamma_2^e)^2 - \gamma_2^e - \frac{2(r+\delta)\sigma^2}{(\mu_H-\mu_L)^2} = 0$, respectively. The investment threshold is given by

$$\mu_I^e = -\frac{\begin{array}{c}(r-i+\delta)\left[\mu_L\lambda(r+\delta) - \mu_H\mu_L + \gamma_2^e\lambda(r+\delta)(\mu_H - \mu_L)\right] \\ -(r+\delta)\left[(\mu_H - \gamma_1^e(\mu_H - \mu_L))\lambda(r+\delta) - \mu_H\mu_L\right]\end{array}}{(r-i+\delta)\left[\mu_H - \gamma_2^e(\mu_H-\mu_L) - \lambda(r+\delta)\right] - (r+\delta)\left[\mu_L + \gamma_1^e(\mu_H - \mu_L) - \lambda(r+\delta)\right]}. \tag{27}$$

We provide several comments. First, this benchmark model is different from a standard real options model of investment à la Dixit and Pindyck (1994), in that in our model the learning process is incorporated. Second, as equity holders do not need to consider future default in the absence of debt, there is no default effect in this case. Thus, the debt overhang problem discussed in the baseline model does not arise in this case. Indeed, it follows from the proof of this proposition that the second term on the right-hand side of (26a) ((26b)) corresponds to $B_1(\mu - \mu_L)^{1-\gamma_1}(\mu_H - \mu)^{\gamma_1}$ ($B_3(\mu - \mu_L)^{\gamma_2}(\mu_H - \mu)^{1-\gamma_2}$) on the right-hand side of (22a) ((22b)), which can be interpreted as the present value of the additional expected profit for equity holders generated by updating μ. Thus, by comparing the results of Propositions 1 and 3 ((22) and (26)), we see that the equity value in this case includes the terms involved in the present value of the additional expected profit for equity holders generated by updating μ, whereas it does not include any terms involved in the value of default-free debt or in the impact of potential default.

5. Optimal Default and Investment Policies

Comparing the baseline model of Proposition 1 with the other two benchmark models of Propositions 2 and 3, we now consider how changes in σ, $\mu_H - \mu_L$, and m affect the optimal default and investment policies. We also investigate the effects of σ and $\mu_H - \mu_L$ on the leverage ratio and credit spreads in the baseline model.

5.1. The effects of liquidity and solvency uncertainties on default and investment policies.—

We now discuss how the two sources of uncertainty affect optimal default and investment policies. In this model, liquidity uncertainty arises from the unpredictable immediate earnings. Thus, liquidity uncertainty is the liquidity shock generated by the Brownian motion. However, solvency uncertainty is the profitable uncertainty represented by the uncertain drift $\overline{\mu}$ that may cause the firm to undergo solvency distress. We analyze the effects of these two uncertainties on default and investment policies using the following set of basic parameters: $\mu_L = 0$, $\mu_H = 0.1$, $\sigma = 0.3$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$. However, the initial market value of debt varies with liquidity uncertainty σ or solvency uncertainty $\mu_H - \mu_L$ relative to $p = 0.8$. We also use the same basic parameter set in the discussions of the next two subsections.

Figure 1A indicates that in the baseline model, an increase in σ (from 0.2 to 0.4) leads to an increase in the default threshold μ_B. Thus, equity holders are more likely to default earlier as σ increases. This tendency is also observed in the case of constant capital stock (see also Figure 1A). In the case of constant capital stock, it follows from (24) that the equity value includes the term involved in the present value of the additional expected profit for equity holders when defaulting at μ, whereas it does not include any terms involved in the present value of the additional expected profit for equity holders generated by updating μ. In the baseline model case, it follows from (22b) that the equity value also includes the term involved in the present value of the additional expected profit for equity holders generated by updating μ, as well as the terms observed in the case of constant capital stock. This implies that incorporation of the present value of the additional expected profit for equity holders generated by updating μ does not modify the effect of σ on the default threshold. In addition, as argued in the corollary to Proposition 2, the case of constant capital stock when

$m = \infty$ corresponds to the benchmark case of Gryglewicz (2011). Figure 1B suggests that even for the case that corresponds to Gryglewicz (2011), the comparative static result with respect to liquidity uncertainty has the same tendency as in the other cases for sufficiently large $\mu_H - \mu_L$ ($\mu_H - \mu_L \geq 0.101$), although we have the corner solution $\mu_B = \mu_L$ in this case if $\mu_H - \mu_L$ is not sufficiently large ($\mu_H - \mu_L < 0.101$).

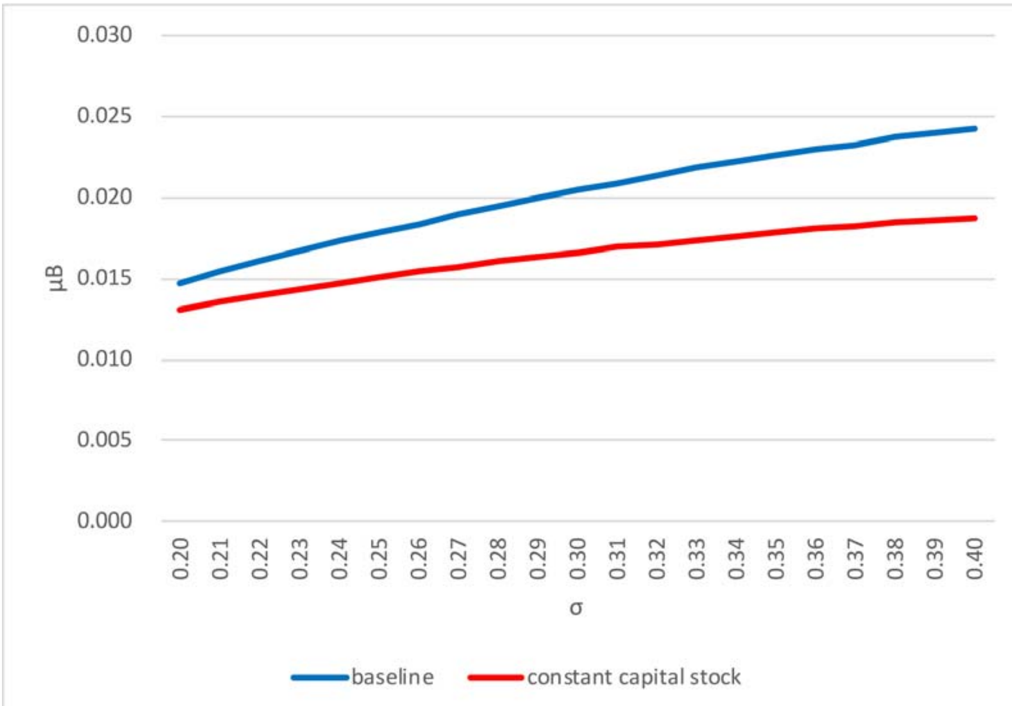

Figure 1A. The effect of a change in σ on the default threshold in the baseline model (blue line) and in the constant capital stock model (red line). Each solid line expresses the optimal default threshold relative to σ. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

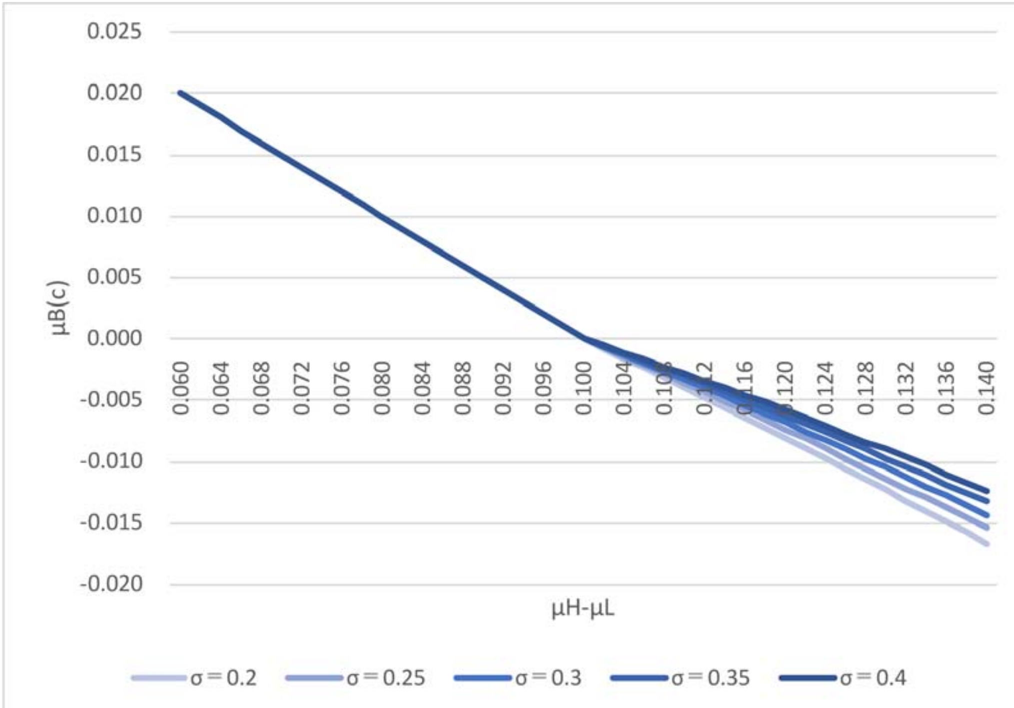

Figure 1B. The effects of changes in σ and $\mu_H - \mu_L$ on the default threshold in the Gryglewicz (2011) model. Each solid line expresses the optimal default threshold as a function of $\mu_H - \mu_L$ relative to σ. Parameters are $m = \infty$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

The intuition for these results is explained as follows. A higher σ makes instantaneous cash flows less informative about $\overline{\mu}$. Then, the effect of increasing σ lowers the present value of the additional expected profit for equity holders when defaulting at μ. This leads to hastening the decision on default to wait for new information in both the baseline model and the constant capital stock model. Hence, in these two models, the default threshold μ_B increases as σ increases.

Next, Figure 2 illustrates that in the baseline model, an increase in σ increases the investment threshold μ_I. Hence, the higher σ aggravates the investment incentives for equity holders and increases debt overhang. Similarly, in the equity finance model, Figure 2 also shows that an increase in σ increases the investment threshold μ_I^e. This result means that even though the present value of the additional expected profit for equity holders when defaulting μ is incorporated, increasing σ does not modify the effect of σ on the investment threshold.

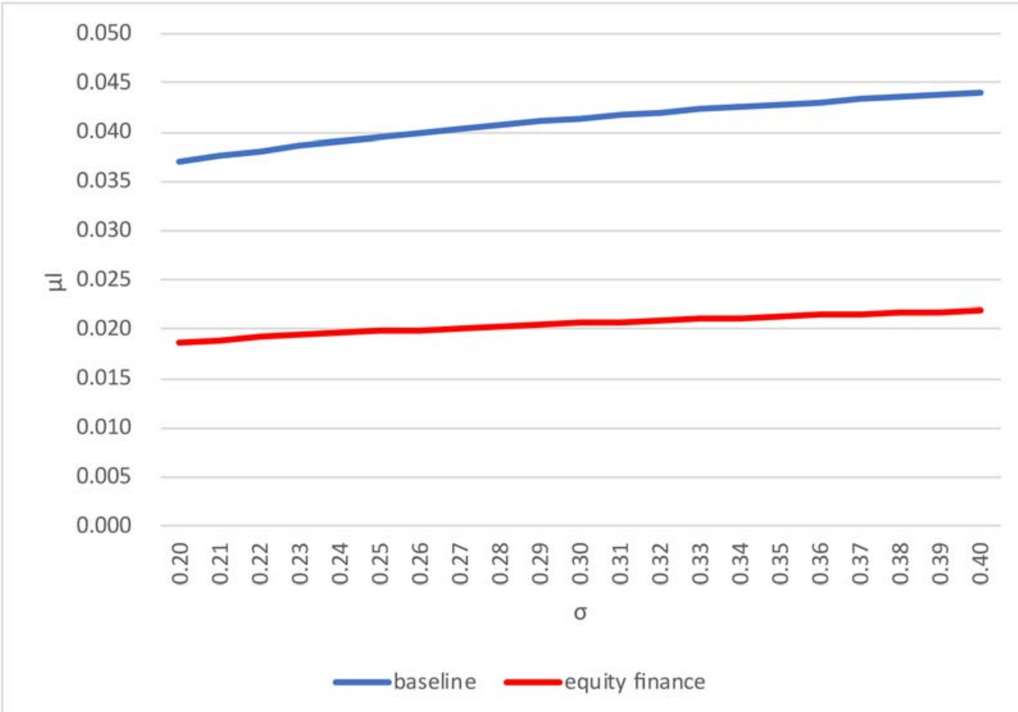

Figure 2. The effect of a change in σ on the investment threshold in the baseline model (blue line) and in the equity finance model (red line). Each solid line expresses the optimal investment threshold relative to σ. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

The logic behind these results is explained as follows. A direct effect of increasing σ causes the increased volatility of cash flows and the less-informative cash flow signals, thereby decreasing the present value of the additional expected profit for equity holders generated by updating μ.[18] This direct learning effect created by an increase in volatility σ induces the firm to choose the higher investment threshold. Furthermore, given that rollover gains/losses are covered from new equity issues, the lower informativeness of cash flows raises the market value of debt because debt is information-insensitive security. This indirect learning effect increases

[18] Because an increase in σ delays the adjustment of μ, the prediction of whether $\overline{\mu} = \mu_L$ or μ_H using additional information becomes more difficult. As a result, the present value of the additional expected payoff for equity holders generated by updating μ decreases.

the transfer to debt holders and decreases equity holders' incentives to invest. Another indirect learning effect of increasing μ_I arises from the above finding that μ_B increases in σ. This is because the increase in μ_B triggers earlier default and induces the firm to make do with smaller cash flows or to hedge negative liquidity shocks to a lesser extent at each point in time as a result of the greater likelihood of failing to receive cash flows when investing. In the equity finance model, the firm does not default because it does not issue any debt. Hence, it follows from (26a) and (26b) that the equity value includes the term involved in the present value of the additional expected profit for equity holders generated by updating μ, but does not include any terms involved in the present value of the additional expected profit for equity holders when defaulting. Then, in the equity finance model, increasing volatility σ involves only the first of these three learning effects because there is no debt. Thus, in this model, an increase in σ always increases the investment threshold μ_I^e. However, the baseline model also includes the remaining two indirect learning effects because the levered firm may default. In fact, even though the last indirect learning effect may have the opposite to the other two learning effects, it is dominated by these other learning effects. Consequently, consideration of debt does not modify the relation between μ_I and σ observed in the equity finance model.

We now consider the effect of a change in profitability uncertainty. Because we use the binomial distribution of $\overline{\mu}$, this uncertainty is measured by a mean preserving spread between the high value (μ_H) and low value (μ_L) realizations of mean earnings, as in Gryglewicz (2011). More specifically, we vary $\mu_H - \mu_L$ around the mean $\mu_0 = \frac{1}{2}(\mu_H + \mu_L) = 0.05$. Increasing $\mu_H - \mu_L$ around the mean μ_0 has two main direct effects. One effect is that increasing $\mu_H - \mu_L$ around the mean μ_0 directly increases the profit potential of the firm at success. The other effect is that the greater the spread of $\mu_H - \mu_L$ is, the more rapid are the learning dynamics in μ_t. The reason is that cash flow signals are then more informative about the realization of either μ_H or μ_L (see equation (4)) because μ_t is farther away from μ_L and μ_H on average. Thus, this effect increases the present value of waiting for new information.[19]

As illustrated in Figure 3, the default threshold μ_B decreases with $\mu_H - \mu_L$ in the baseline model. Thus, equity holders are more likely to default later as $\mu_H - \mu_L$ increases. This is also observed in the constant capital stock model (see Figure 3). Hence, consideration of the present value of the additional expected payoff for equity holders generated by updating μ

[19] DeMarzo and Sannikov (2017)

does not modify the effect of $\mu_H - \mu_L$ on the default threshold. Again, Figure 1B suggests that even for the case that corresponds to Gryglewicz (2011), the comparative static result with respect to solvency uncertainty has the same tendency as in the other cases for sufficiently large $\mu_H - \mu_L$ ($\mu_H - \mu_L \geq 0.101$), although we have the corner solution $\mu_B = \mu_L$ in this case if $\mu_H - \mu_L$ is not sufficiently large ($\mu_H - \mu_L < 0.101$).

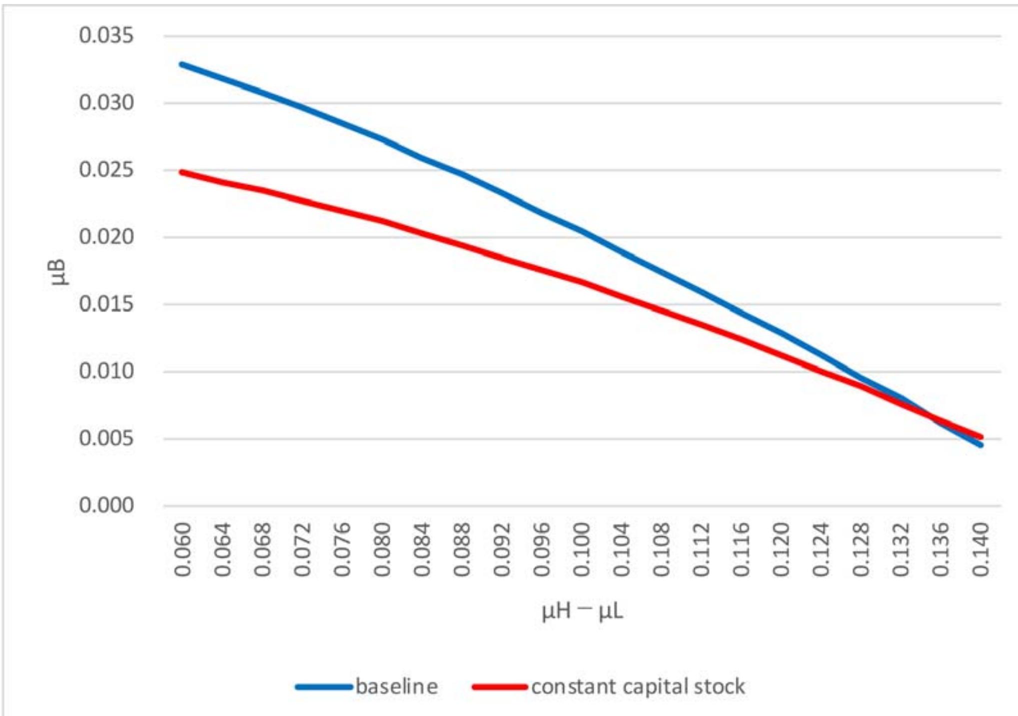

Figure 3. The effect of a change in $\mu_H - \mu_L$ on the default threshold in the baseline model (blue line) and in the constant capital stock model (red line). Each solid line expresses the optimal default threshold relative to $\mu_H - \mu_L$. Parameters are $\sigma = 0.3$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

The mechanism for these results is as follows. The two main direct effects of increasing $\mu_H - \mu_L$ discussed above increases the present value of the additional expected payoff for equity holders when defaulting at μ and increases the present value of waiting for new information in both the baseline model and the constant capital stock model. Hence, in these two models, the default threshold μ_B decreases for any m as $\mu_H - \mu_L$ increases.

Figure 4 shows that in the baseline model, the investment threshold μ_I decreases with $\mu_H - \mu_L$. In the equity finance model, Figure 4 also indicates that an increase in $\mu_H - \mu_L$

decreases the investment threshold μ_I^e. Thus, this result suggests that even though the present value of the additional expected profit for equity holders when defaulting at μ is incorporated, increasing $\mu_H - \mu_L$ still induces equity holders to invest earlier.

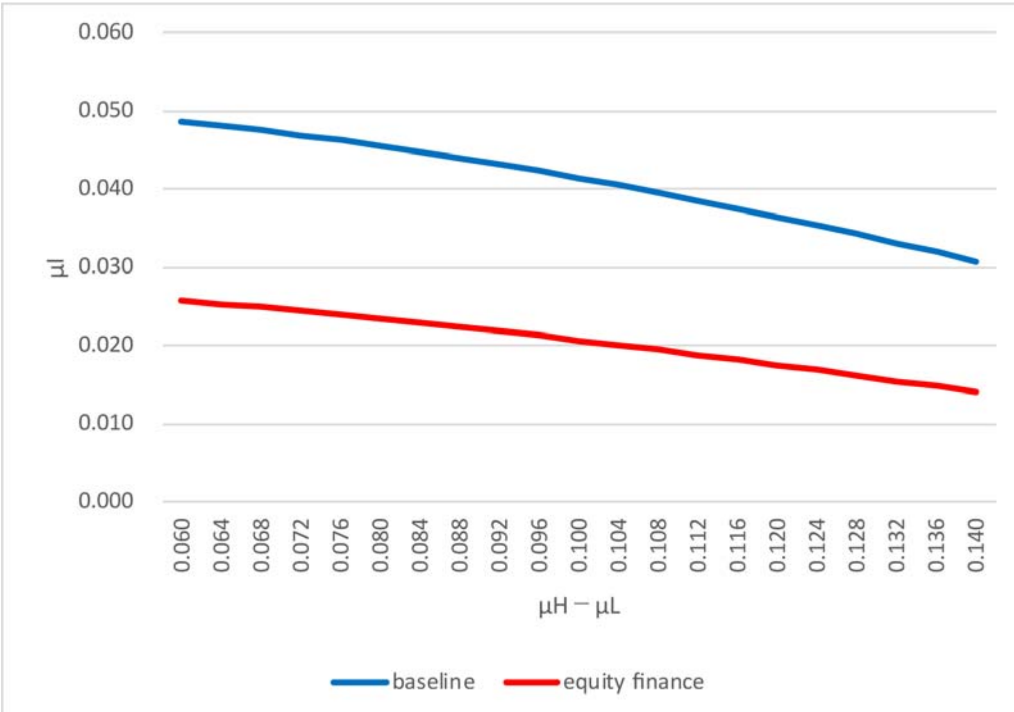

Figure 4. The effect of a change in $\mu_H - \mu_L$ on the investment threshold in the baseline model (blue line) and in the equity finance model (red line). Each solid line expresses the optimal investment threshold relative to $\mu_H - \mu_L$. Parameters are $\sigma = 0.3$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

Intuitively, the four main effects of increasing $\mu_H - \mu_L$ around μ_0 operate as follows. First, increasing $\mu_H - \mu_L$ directly increases the profit potential of the firm at success. Hence, increasing $\mu_H - \mu_L$ induces equity holders to invest earlier. Second, an increase in $\mu_H - \mu_L$ raises the speed of learning from cash flow shocks about expected profitability. Hence, this creates the higher present value of the additional expected profit for equity holders generated by updating μ. Thus, an increase in $\mu_H - \mu_L$ reduces the investment threshold. Third, given that rollover gains/losses are covered from new equity issues and that debt is information-insensitive security, the greater informativeness of cash flows reduces the market value of debt. This indirect learning effect decreases the transfer to debt holders and increases equity holders' incentive to invest. Fourth, the decrease in the default threshold μ_B because of an increase in $\mu_H - \mu_L$ triggers a later default and induces the firm to obtain larger cash flows or the greater need to hedge negative liquidity shocks at each point in time as a result of the greater likelihood of being able to receive cash flows when investing. Thus, this indirect effect also motivates the firm to choose the lower investment threshold. In the equity finance model, increasing profitability uncertainty $\mu_H - \mu_L$ involves only the first and second direct effects because there is no debt. Hence, in the equity finance model, an increase in $\mu_H - \mu_L$ always decreases the investment threshold μ_I^e, thereby inducing equity holders to start their investments earlier. In the baseline model, consideration of debt additionally involves the two indirect effects. However, as the two indirect effects also decrease the investment threshold, consideration of debt does not modify the tendency of the equity finance model. Consequently, increasing solvency uncertainty is more likely to induce equity holders to invest earlier in the baseline model.

The analysis in this section also shows that the effect on the default threshold of solvency uncertainty is opposite to that of liquidity uncertainty. Furthermore, the effect on the investment threshold (debt overhang) of solvency uncertainty is also opposite to that of liquidity uncertainty. Intuitively, this is because the greater degree of uncertainty about liquidity (solvency) makes cash flow signals less (more) informative through the learning process of profitability uncertainty. As a result, liquidity and solvency uncertainties affect the present value of the additional expected payoff for equity holders generated by updating μ, the firm's demand for cash flows, and the market value of debt at each point in time in different ways, thereby leading to different effects on the default and investment policies of the firm. These results have not been tested empirically.

Our investigation also indicates that the effect of increasing solvency uncertainty not only mitigates the incentives for equity holders to default, but also improves their incentives to invest. This result depends on the fact that increasing solvency uncertainty raises the profit potential of the firm at success and the present value of the additional expected payoff for equity holders generated by updating μ, while reducing the market value of debt, thereby decreasing (increasing) equity holders' incentives to default (invest).

Finally, we should notice that the mechanism through which two sources of uncertainty affect the firm's default and investment decisions is very different from that of the standard real options model. In the standard real options model, the firm's default and investment policies are affected through the variations in the option value of waiting to execute the irreversible decisions. As a result, the default (investment) timing is hastened (delayed) if the volatility of the state variable *increases*. By contrast, in our learning model, the default (investment) timing is hastened (delayed) if the volatility of the state variable *decreases*. Intuitively, in our learning model, the decrease in the volatility of the state variable defined as the posterior expectation of the drift of cash flows delays the adjustment of the state variable itself, thus making cash flows less informative through the learning process. Consequently, this effect reduces the present value of the additional expected profit for equity holders generated by updating the firm's profitability and raises the market value of debt, thereby hastening (delaying) the default (investment) timing.

5.2. The effect of debt maturity on default and investment policies.—

First studied by Myers (1977), debt overhang captures the idea that equity holders underinvest relative to the level that maximizes the total value of the firm because a part of investment benefits accrues to the firm's debt claims. In this subsection, we provide numerical examples to illustrate a new insight offered by our paper into the effect of debt maturity on default and debt overhang when debt maturity is exogenously determined.

As indicated at the beginning of Section 5.1, the initial market value of debt varies with debt maturity m relative to $p = 0.8$. By contrast, Diamond and He (2014) hold the initial market value of debt constant by varying the face value of debt. We do not follow the setting of Diamond and He (2014), not only because the default and investment policies of the firm depend strongly on the *leverage effect* caused by the effect of debt maturity on the market value of debt, but also because we need to investigate the *leverage effect* created by learning. In addition, because our *leverage* effect is derived from the possibility of debt rollover and learning, unlike Gertner and Scharfstein (1991), it would seem interesting to consider this effect fully under our framework.

Figure 5 illustrates the optimal default policies when varying debt maturity m for the two models: the baseline model and the constant capital stock ($i = \delta = 0$) model. In the constant capital stock case, Figure 5 indicates that the default threshold μ_B^c decreases with m: default is more likely to occur if debt maturity is shorter. In the baseline model case, Figure 5 shows that the default threshold μ_B in this case still decreases with m; and it is higher than μ_B^c for any m. Hence, consideration of the present value of the additional expected payoff for equity holders generated by updating μ is more likely to induce equity holders to default for any m.

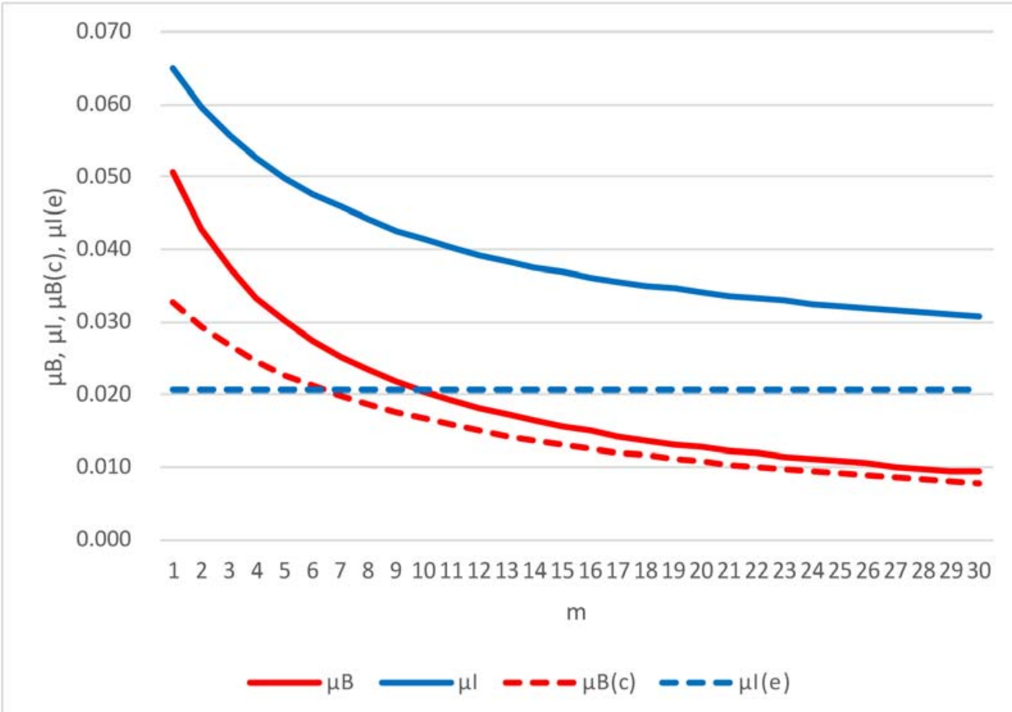

Figure 5. The effect of a change in m on the default and investment thresholds. The red (blue) solid line expresses the optimal default (investment) threshold in the baseline model. The red (blue) dashed line represents the optimal default (investment) threshold in the constant capital stock (equity finance) model. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $\sigma = 0.3$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

The reason why μ_B decreases with m is that equity holders default earlier to refuse to subsidize debt holders as m is shorter. This is because shorter-term debt requires equity holders to absorb greater rollover losses when the firm's future prospects deteriorate.[20] Such a relation between the default threshold and debt maturity is also obtained in He and Xiong (2012a) and Diamond and He (2014), although their model frameworks are different from

[20] In our model, reducing m raises equity holders' rollover losses, $\frac{1}{m}[p - d(m)]$, because the effect of the higher rollover frequency $\frac{1}{m}$ brought about by reducing m dominates the effect of the larger market value of debt caused by reducing m.

ours. Our analysis also indicates that for any m, the default threshold is higher in the baseline model than in the constant capital stock model. As the firm has an option to invest in the former case but not in the latter case, equity holders are more likely to choose default in the former case than in the latter case, when the firm's future prospects deteriorate. Consequently, for any m, the default threshold is higher in the baseline model than in the constant capital stock model.

Figure 5 also illustrates the optimal investment policies when varying debt maturity m for the two models: the baseline model and equity finance ($d = 0$ for any $t \geq 0$) model. In the equity finance model, as the firm does not issue any debt, the equity value does not depend on m. Consequently, the investment threshold μ_I^e does not depend on m. In the baseline model, the equity value includes the terms involved in the present value of the additional expected payoff for equity holders when defaulting. Then, Figure 5 indicates that the investment threshold μ_I is higher than μ_I^e for any m. Hence, consideration of debt is more likely to induce equity holders to delay investment for any m. Furthermore, Figure 5 also illustrates that the investment threshold μ_I in this case decreases with m. This implies that debt has less overhang when the debt maturity is long.

Intuitively, because of the possibility of default, consideration of debt forces the firm to consider the likelihood of failing to receive cash flows as a result of investment. Hence, it is not surprising that this consideration is more likely to induce equity holders to delay the investment timing for any m. However, the reason why μ_I in the baseline model decreases with m is explained as follows. As shorter-term debt induces more frequent repricing, shorter-term debt holders share less gains given good news. Hence, shorter-term debt may improve equity holders' incentives to invest. However, our model also incorporates the leverage effect caused by debt rollover, through which the market value of debt decreases with debt maturity (see Figure 6) because the future portion of debt is more risky due to the greater possibility of default. Thus, the leverage effect decreases the transfer to debt holders and increases equity holders' incentives to invest, as debt maturity is longer. In addition, the other effect by which longer-term debt increases equity holders' incentives to invest results from the above finding that μ_B decreases with debt maturity. This is because the decrease in μ_B is more likely to delay default and induces the firm to obtain larger cash flows or to hedge negative shocks to a greater extent at each point in time as a result of the greater likelihood of being able to receive cash flows when investing. As the second and third effects are stronger than the first effect, longer-term debt has less debt overhang.

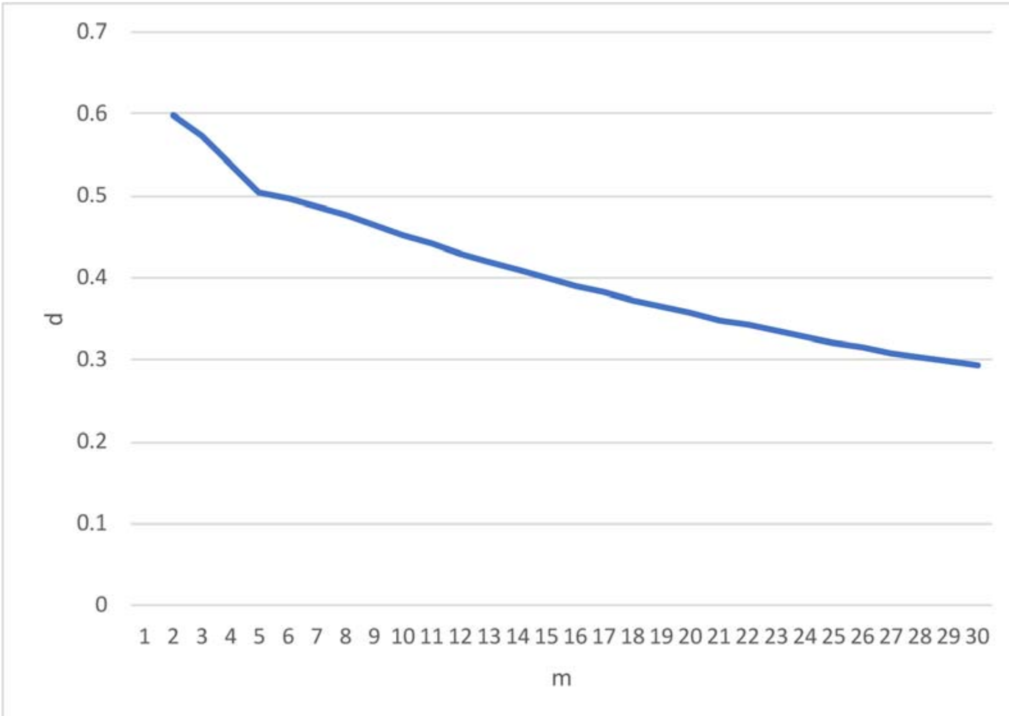

Figure 6. The initial debt value in the baseline model. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $\sigma = 0.3$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, $p = 0.8$, and $\mu_0 = 0.5$.

In contrast, holding the initial market value of debt constant by varying the promised debt payment, Diamond and He (2014) suggest that investment incentives first increase with debt maturity for very short maturities, then decrease with debt maturity as debt maturity increases. By varying the initial market value of debt, we consider the leverage effect caused by debt rollover, which is not derived by Gertner and Scharfstein (1991), although Diamond and He (2014) rule out this effect by holding the initial market value of debt constant. Thus, investment incentives increase with debt maturity. In fact, holding the promised debt payment constant by varying the initial market value of debt, we might choose a considerably higher or lower leverage ratio, which would affect the results of this paper. To check this problem, we report the default and investment thresholds, the debt value, and the firm value in Figures A1 and A2 using our basic parameter set. The results confirm that our choice of the value of the promised debt payment does not lead to extraordinary results (see the discussion at the end of the Appendix).

5.3. Leverage and credit spreads.—

In this subsection, we examine the effects of liquidity and solvency uncertainties on the leverage ratio (debt to firm value) and credit spreads in the baseline model.

Figure 7A displays the effects of the two sources of uncertainty on the leverage ratio.[21] The effect of increasing σ on the leverage ratio is positive for all m. As discussed in Section 5.1, higher volatility makes cash flow signals less informative about $\overline{\mu}$ and causes default to occur relatively early. Although the latter effect increases the cost of debt, the former learning effect induces equity holders to issue more debt because debt is information-insensitive security. As the latter learning effect dominates, the higher volatility increases the debt value and leverage ratio.

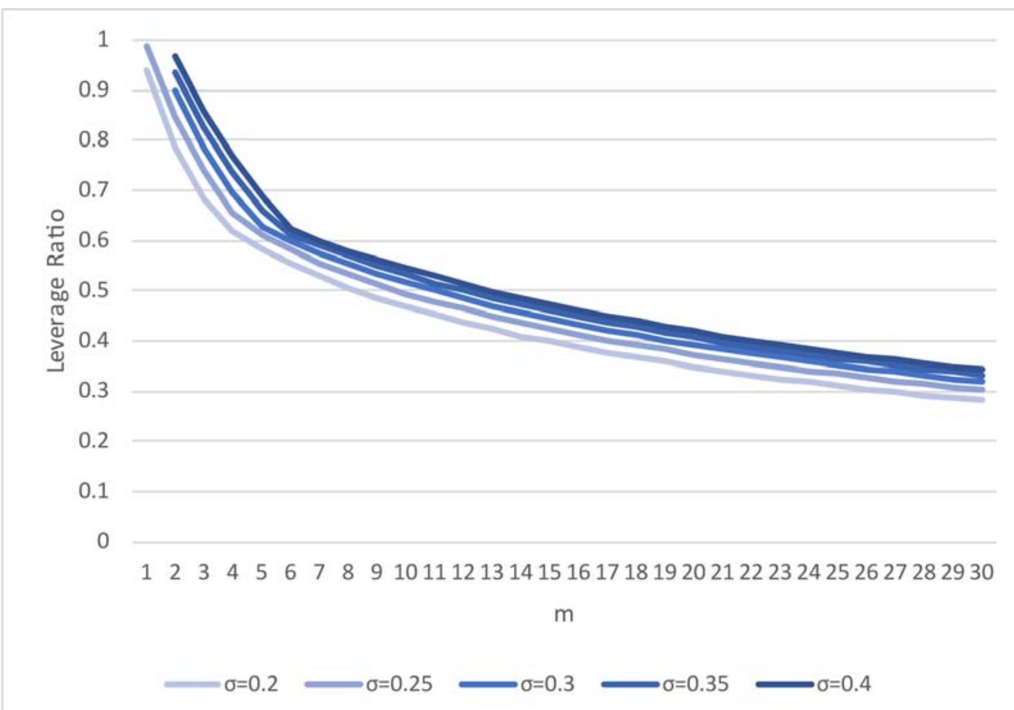

Figure 7A. The effect of a change in σ on the leverage ratio for different debt maturities. Each solid line expresses the leverage ratio relative to σ. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

[21] In Figures 7 and 8, note that in several cases of $m < 3$, we do not obtain any computation results because the firm does not operate in this case as a result of $\mu_0 \leq \mu_B$.

However, Figure 7B also shows that the effect of a mean preserving spread of $\mu_H - \mu_L$ around the mean is negative for all m. As argued in the preceding section, a higher spread $\mu_H - \mu_L$ increases the profit potential for the firm at success, brings out the higher informativeness of cash flows, and induces equity holders to default relatively late. In particular, because the second learning effect dominates and debt is information-insensitive security, the higher $\mu_H - \mu_L$ motivates equity holders to issue less debt and decreases the leverage ratio.

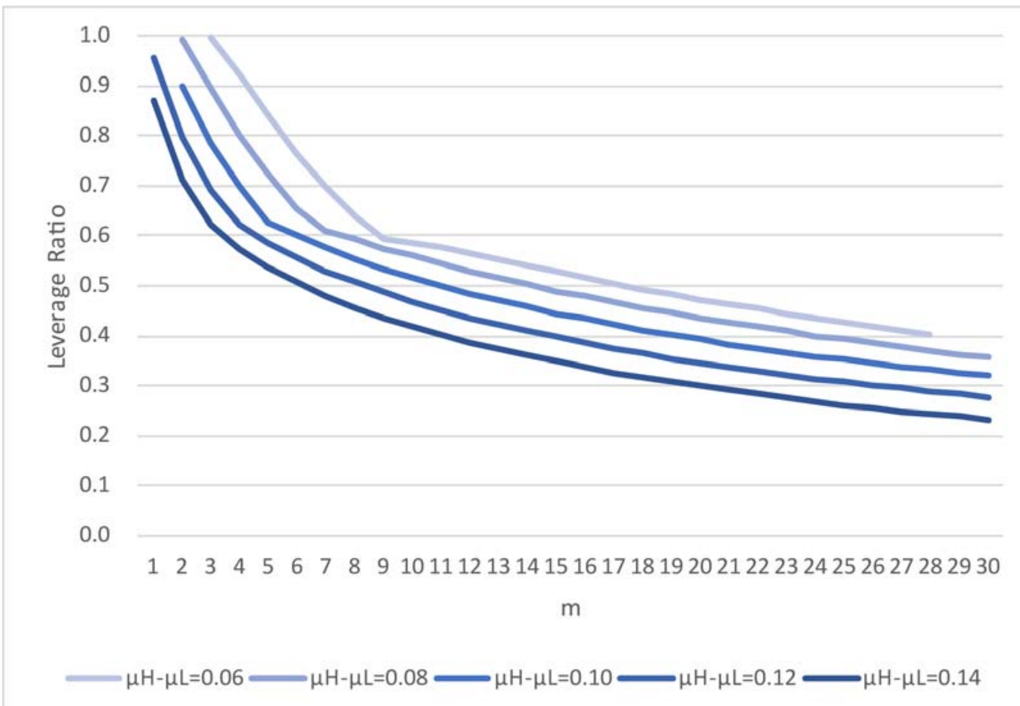

Figure 7B. The effect of a change in $\mu_H - \mu_L$ on the leverage ratio for different debt maturities. Each solid line expresses the leverage ratio relative to $\mu_H - \mu_L$. Parameters are $\sigma = 0.3$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

Gryglewicz (2011) suggests that the consideration of cash holdings in his model lessens the problem of the standard trade-off model of capital structure that the optimal leverage implied by the standard model exceeds the leverage ratio observed empirically. However, our numerical analysis also indicates that the leverage ratio is significantly lower as the debt maturity is longer. Hence, our model implies that one of the driving forces of the reduced leverage is the incorporation of debt maturity.[22]

Figure 8A illustrates the effects of liquidity and solvency uncertainties on credit spreads. The high volatility σ decreases credit spreads for all $m \geq 9$, but does not necessarily do so for shorter-maturity debt. The high volatility risk lowers the informativeness of cash flows and results in the greater possibility of default. Indeed, as the first learning effect dominates for $m \geq 9$ (but does not necessarily dominate for $m < 9$), credit spreads decrease for $m \geq 9$ (but does not necessarily decrease for $m < 9$).

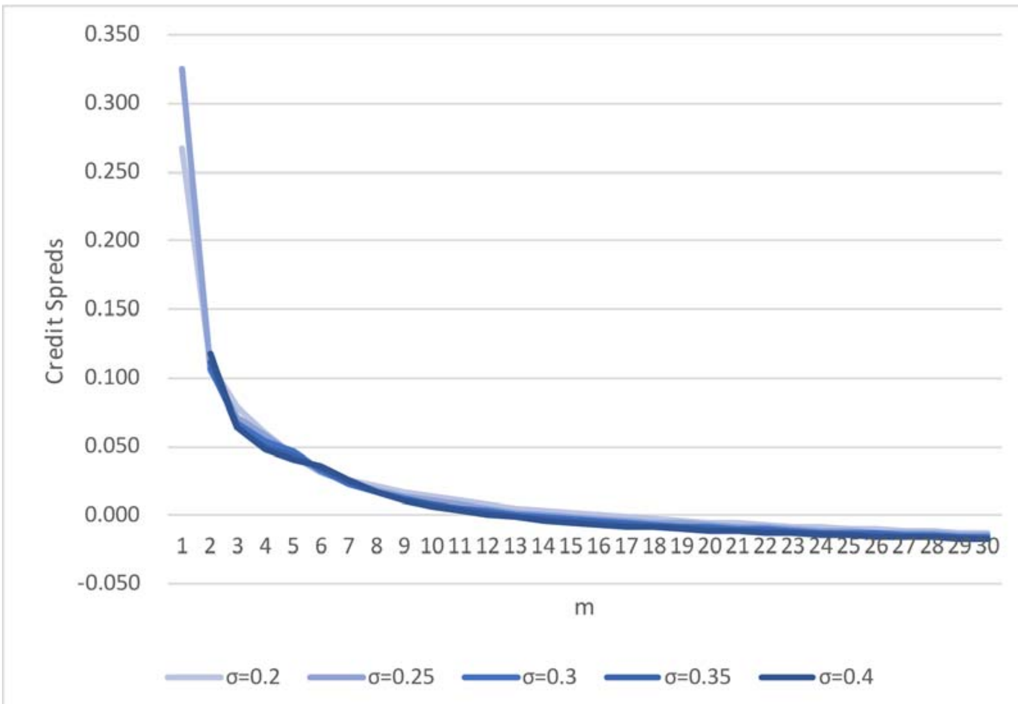

Figure 8A. The effect of a change in σ on credit spreads for different debt maturities. Each solid line expresses credit spreads relative to σ. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

However, Figure 8B indicates that the higher spread $\mu_H - \mu_L$ increases credit spreads for all $m \geq 12$, but does not necessarily do so for shorter-maturity debt. The higher spread creates the greater profit potential of the firm at success, the greater informativeness of cash flows, and the lower default threshold. Because the second learning effect dominates for m

[22] Note that in the model of Gryglewicz (2011), the debt maturity is infinite.

≥ 12 (but does not necessarily dominate for $m < 12$), credit spreads increase for $m \geq 12$ (but does not necessarily decrease for $m < 12$).

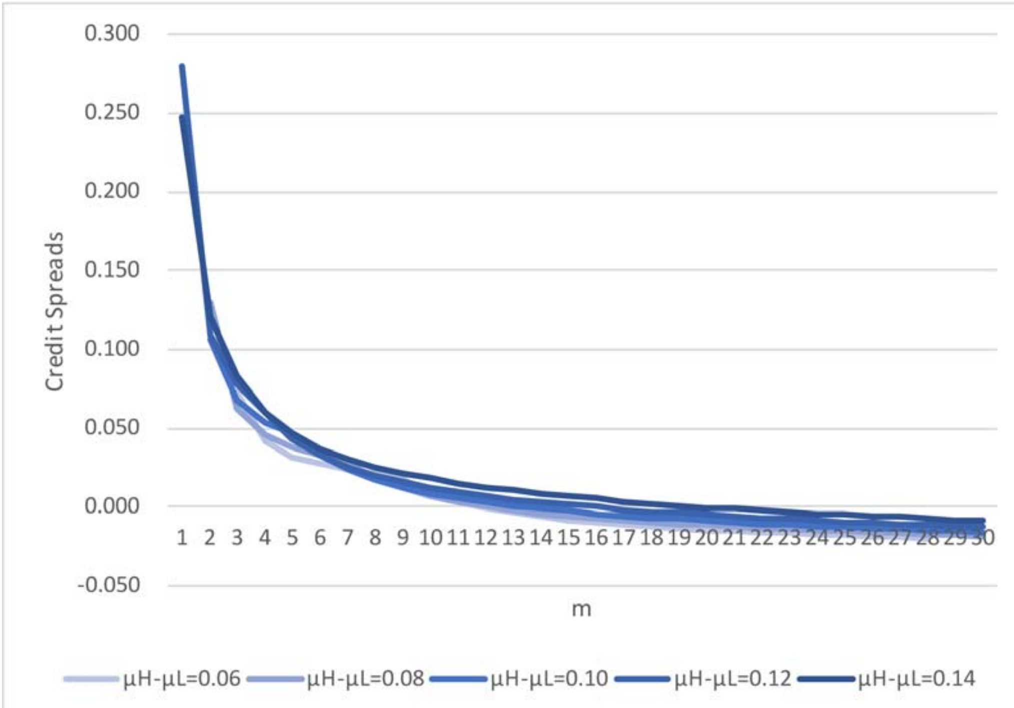

Figure 8B. The effect of a change in $\mu_H - \mu_L$ on credit spreads for different debt maturities. Each solid line expresses credit spreads relative to $\mu_H - \mu_L$. Parameters are $\sigma = 0.3$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $p = 0.8$.

When the maturity of debt is not sufficiently long, our results for credit spreads do not necessarily coincide with those of Gryglewicz (2011). Again, the incorporation of debt maturity affects the results significantly.

6. Empirical implications

6.1. Investment.—

Although many theoretical and empirical studies support the argument that uncertainty has a negative effect on firms' investments, there remains a possibility that increasing uncertainty enhances firms' investment activities such as research and development.[23] This is

[23] Bloom (2014) discusses this issue. Kraft, Schwartz, and Weiss (2013) indicate that R&D-intensive firms with higher uncertainty can have higher stock values.

because a "growth options effect" exists: increasing uncertainty extends the upper range of the revenues from a new project, whereas it also expands the lower range of the revenues from the new project but the expansion is bounded by limited liability. Our research in Section 5.1 shows that liquidity and solvency uncertainties have opposite effects on investment policy: increasing solvency (liquidity) uncertainty is more likely to induce equity holders to invest earlier (later). Because solvency (liquidity) uncertainty works in the long- (short-) term, the growth options effect depends on solvency uncertainty rather than liquidity uncertainty. Indeed, firms in some industries, for example, in "new economy" industries such as information technology and bioscience, are more likely to have higher R&D expenditures. Thus, our result suggests that increasing solvency uncertainty enhances the investment activities of firms in "new economy" industries such as information technology and bioscience by the growth options effect. Hence, our finding can be interpreted such that solvency uncertainty should be distinguished from liquidity uncertainty in the empirical literature on the investment activities of R&D-intensive firms.

6.2. Capital structure.—

The literature on corporate finance provides competing views on how cash flow volatility influences capital structure. For example, Bradley, Jarrell, and Kim (1984) show a negative relationship between volatility and leverage, while Kim and Sorensen (1986) find a positive one. In addition, Leary and Roberts (2005) find that volatility has no role in explaining capital structure. A possible reason for such divergence in views is because of a difference in identification of the cash flow volatility. Keefe and Yaghoubi (2016) investigate the effect of cash flow volatility on capital structure using several measures of a firm's cash flow volatility. They find that firms with more volatile cash flows use less debt. Furthermore, their result is robust regardless of the different measures of cash flow volatility and debt ratio. However, they do not consider the endogenous interaction between solvency and liquidity concerns in calculating the cash flow volatility measures.

In Section 5.3, taking debt maturity as given, we indicate that the larger solvency (liquidity) uncertainty decreases (increases) the leverage ratio for any debt maturity. Thus, our finding suggests that solvency uncertainty should be distinguished from liquidity uncertainty in interpreting the mixed results reported in the existing empirical literature.

7. Conclusion

We explore the roles of debt rollover and the market learning process in the firm's solvency risk in its default, investment, and leverage policy decisions under the interaction between rollover risk and solvency concern. We distinguish between liquidity uncertainty (cash flow shock) and solvency uncertainty (profitability uncertainty) and incorporate an assessment of the firm's solvency risk via the learning process over time. Under the learning model framework, the effects on the decisions of the firm about default, investment, and leverage policies resulting from solvency uncertainty are separated from those resulting from liquidity uncertainty. In addition, both liquidity and solvency uncertainties affect the firm's policies through a different mechanism from that of the real options model because an increase in liquidity (solvency) uncertainty makes cash flow signals less (more) informative by reducing (raising) the volatility of the state variable defined by the posterior expectation of the drift term. We consider how the two sources of uncertainty affect such decisions of the firm under the endogenous interaction between rollover risk and solvency concern, generated by both debt rollover and the learning process, when newly issued equity covers losses under debt rollover.

Our results show that an increase in liquidity (solvency) uncertainty raises (reduces) the firm's incentives to default and the leverage ratio, whereas an increase in liquidity (solvency) uncertainty reduces (raises) the firm's incentives to invest. The latter result regarding the firm's investment incentives implies that the effect of solvency uncertainty on the investment policy—debt overhang—is opposite to that of liquidity uncertainty. In addition, an increase in liquidity (solvency) uncertainty reduces (raises) credit spreads if debt maturity is sufficiently long. Our findings further indicate that default is more likely to occur if debt maturity is shorter. Our findings also show that as debt maturity becomes longer, less debt overhang subsequently occurs.

Appendix

Proof of Proposition 1: We begin by solving for the debt value function. Initially, we assume that equity holders use the investment threshold policy. Later, we will prove that the investment threshold policy is optimal for equity holders. Then, if $\mu \geq \mu_I$, ordinary differential equation (8) has a solution of the following general form:

$$d(\mu) = \frac{m}{2[1+m(r-i+\delta)]\sigma^2}\beta_1(\beta_1-1)(\mu_H-\mu_L)^2\left[A_1(\mu-\mu_L)^{1-\beta_1}(\mu_H-\mu)^{\beta_1}\right.$$
$$\left.+A_2(\mu-\mu_L)^{\beta_1}(\mu_H-\mu)^{1-\beta_1}\right]+\frac{p}{1+m(r-i+\delta)},$$

where $\beta_1 > 1$ is the positive root of

$$\beta_1^2-\beta_1-\frac{2[1+m(r-i+\delta)]\sigma^2}{m(\mu_H-\mu_L)^2}=0. \tag{A1}$$

Because $A_2(\mu-\mu_L)^{\beta_1}(\mu_H-\mu)^{1-\beta_1} \to \pm\infty$ as $\mu \to \mu_H$, boundary condition (12) implies that $A_2 = 0$. Thus, using (A1), we obtain

$$d(\mu) = A_1(\mu-\mu_L)^{1-\beta_1}(\mu_H-\mu)^{\beta_1}+\frac{p}{1+m(r-i+\delta)}, \qquad \text{if } \mu \geq \mu_I. \tag{A2}$$

If $\mu_I > \mu \geq \mu_B$, ordinary differential equation (8) still has a solution of the following general form:

$$d(\mu) = \frac{m}{2[1+m(r+\delta)]\sigma^2}\beta_2(\beta_2-1)(\mu_H-\mu_L)^2\left[A_3(\mu-\mu_L)^{1-\beta_2}(\mu_H-\mu)^{\beta_2}\right.$$
$$\left.+A_4(\mu-\mu_L)^{\beta_2}(\mu_H-\mu)^{1-\beta_2}\right]+\frac{p}{1+m(r+\delta)},$$

where $\beta_2 > 1$ is determined by

$$\beta_2^2-\beta_2-\frac{2[1+m(r+\delta)]\sigma^2}{m(\mu_H-\mu_L)^2}=0. \tag{A3}$$

Thus, it follows from (A3) that the above solution can be reduced to

$$d(\mu) = A_3(\mu-\mu_L)^{1-\beta_2}(\mu_H-\mu)^{\beta_2}+A_4(\mu-\mu_L)^{\beta_2}(\mu_H-\mu)^{1-\beta_2}+\frac{p}{1+m(r+\delta)}, \qquad \text{if } \mu_I > \mu \geq \mu_B. \tag{A4}$$

Solving the constants A_1, A_3, and A_4 using (9)–(11), (A2), and (A4), we obtain

$$\begin{aligned} A_1 = & \left[\frac{\mu_B}{r+\delta} - \frac{p}{1+m(r+\delta)}\right] \frac{1}{\mu_B-\mu_L} \left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{\beta_2} \left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_2-\beta_1} \\ & + \left[\left(\frac{\mu_I-\mu_L}{\mu_H-\mu_I}\right)^{\beta_1+\beta_2-1} - \left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{2\beta_2-1} \left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_2-\beta_1}\right] A_4 \\ & - \frac{\frac{pmi}{\mu_H-\mu_I}\left(\frac{\mu_I-\mu_L}{\mu_H-\mu_I}\right)^{\beta_1-1}}{[1+m(r+\delta)]\,[1+m(r-i+\delta)]}, \end{aligned} \tag{A5}$$

$$A_3 = \left[\frac{\mu_B}{r+\delta} - \frac{p}{1+m(r+\delta)}\right] \frac{1}{\mu_B-\mu_L} \left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{\beta_2} - \left(\frac{\mu_H-\mu_B}{\mu_B-\mu_L}\right)^{1-2\beta_2} A_4, \tag{A6}$$

$$\begin{aligned} A_4 = & \left\{\Psi_3 + \left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{2\beta_2-1} \Psi_2 \right. \\ & \left. + \left[\left(\frac{\mu_I-\mu_L}{\mu_H-\mu_I}\right)^{\beta_1+\beta_2-1} - \left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{2\beta_2-1} \left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_2-\beta_1}\right] \Psi_1 \right\}^{-1} \\ & \times \left\{ \left[\frac{\mu_B}{r+\delta} - \frac{p}{1+m(r+\delta)}\right] \frac{1}{\mu_B-\mu_L} \left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{\beta_2} \left[\Psi_2 - \left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_2-\beta_1} \Psi_1\right] \right. \\ & \left. + \frac{pmi\left(\frac{\beta_1}{\mu_H-\mu_I} + \frac{\beta_1-1}{\mu_I-\mu_L}\right)}{[1+m(r+\delta)]\,[1+m(r-i+\delta)]} \right\}, \end{aligned} \tag{A7}$$

where

$$\Psi_1 = \beta_1 \left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_1-1} + (\beta_1-1)\left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_1}, \tag{A8}$$

$$\Psi_2 = \beta_2 \left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_2-1} + (\beta_2-1)\left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\beta_2}, \tag{A9}$$

$$\Psi_3 = \beta_2 \left(\frac{\mu_I-\mu_L}{\mu_H-\mu_I}\right)^{\beta_2-1} + (\beta_2-1)\left(\frac{\mu_I-\mu_L}{\mu_H-\mu_I}\right)^{\beta_2}. \tag{A10}$$

Now, we move on to equity. We first show that the optimal investment policy is given by the investment threshold policy. Note that $\frac{\mu_H-\lambda(r+\delta)}{r-i+\delta} > \frac{p}{1+m(r-i+\delta)}$ is assumed,[24] which ensures that $e(\mu_H) > \lambda$. Hence, investment is optimal and default does not occur at $\mu = \mu_H$. Then, it follows from (12) and (14) that $e(\mu) \to \frac{\mu_H-\lambda i}{r-i+\delta} - \frac{p}{1+m(r-i+\delta)} > \lambda$ as $\mu \to \mu_H$. Given $e(\mu_B) = 0$ from (17), there must exist a solution $\mu_I \in (\mu_B,\, \mu_H)$ that satisfies $e(\mu_I) = \lambda$. Suppose that we have multiple solutions to $e(\mu_I) = \lambda$, and take the smallest one as μ_{I0}. To prove that the threshold strategy is optimal, we need to show that $e(\mu) > \lambda$ for any $\mu \in (\mu_{I0},\, \mu_H]$, where $e(\mu)$ solves the following ordinary differential equation for any $\mu \in (\mu_{I0},\, \mu_H]$:

$$(r-i+\delta)e(\mu) = \mu - \lambda i + \frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2 e''(\mu) - \frac{1}{m}[p-d(\mu)]. \tag{A11}$$

Suppose that there are at least two other solutions $\mu_{I1},\, \mu_{I2} \in (\mu_{I0},\, \mu_H]$ that satisfy $\mu_{I1} < \mu_{I2}$, $e(\mu_{Ii}) = \lambda$ for $i = 1, 2$, and $e'(\mu_{I1}) < 0 < e'(\mu_{I2})$. Then, there are intermediate points $\mu_I^\circ \in (\mu_{I0},\, \mu_{I1})$ and $\mu_I^{\circ\circ} \in (\mu_{I1},\, \mu_{I2})$ so that $e(\mu_I^\circ) > \lambda > e(\mu_I^{\circ\circ})$, $e'(\mu_I^\circ) = e'(\mu_I^{\circ\circ}) = 0$, $e''(\mu_I^\circ) < 0$, and $e''(\mu_I^{\circ\circ}) > 0$.[25,26] Now, evaluating (A11) at μ_I° and $\mu_I^{\circ\circ}$ and using $e(\mu_I^\circ) > \lambda > e(\mu_I^{\circ\circ})$, we obtain

$$\begin{aligned}&\mu_I^\circ - \lambda i + \frac{1}{2\sigma^2}(\mu_I^\circ-\mu_L)^2(\mu_H-\mu_I^\circ)^2 e''(\mu_I^\circ) - \frac{1}{m}[p-d(\mu_I^\circ)]\\ &> \frac{\lambda}{r-i+\delta} > \mu_I^{\circ\circ} - \lambda i + \frac{1}{2\sigma^2}(\mu_I^{\circ\circ}-\mu_L)^2(\mu_H-\mu_I^{\circ\circ})^2 e''(\mu_I^{\circ\circ}) - \frac{1}{m}[p-d(\mu_I^{\circ\circ})].\end{aligned}$$

Thus,

$$\mu_I^\circ - \mu_I^{\circ\circ} > \frac{1}{m}[d(\mu_I^{\circ\circ})-d(\mu_I^\circ)] + \frac{1}{2\sigma^2}(\mu_I^{\circ\circ}-\mu_L)^2(\mu_H-\mu_I^{\circ\circ})^2 e''(\mu_I^{\circ\circ}) - \frac{1}{2\sigma^2}(\mu_I^\circ-\mu_L)^2(\mu_H-\mu_I^\circ)^2 e''(\mu_I^\circ). \tag{A12}$$

Given $\mu_I^\circ \in (\mu_{I0},\, \mu_{I1})$, $\mu_I^{\circ\circ} \in (\mu_{I1},\, \mu_{I2})$, $e''(\mu_I^\circ) < 0$, and $e''(\mu_I^{\circ\circ}) > 0$, it follows from (A12) that $d(\mu_I^{\circ\circ}) < d(\mu_I^\circ)$. Because of $e(\mu_I^\circ) > \lambda > e(\mu_I^{\circ\circ})$, this implies that $v(\mu_I^\circ) > v(\mu_I^{\circ\circ})$. However, this contradicts the assumption that $v(\mu)$ is increasing in μ. Hence, the solution to $e(\mu_I) = \lambda$ is uniquely determined. Consequently, we verify that $e(\mu) > \lambda$ for any $\mu \in (\mu_I,\, \mu_H]$.

We next proceed to characterize the equity value function. As discussed in Diamond and

[24] Note that our parameter set satisfies this assumption.

[25] Even though $\mu_{I0} = \mu_{I1}$ ($\mu_{I1} = \mu_{I2}$) so that $e(\mu_{I1}) = \lambda$ and $e'(\mu_{I1}) = 0$ ($e(\mu_{I2}) = \lambda$ and $e'(\mu_{I2}) = 0$), we can set $\mu_I^\circ = \mu_{I1}$ ($\mu_I^{\circ\circ} = \mu_{I2}$). Then, the following argument still holds.

[26] μ_I° is a local maximum point of $e(\mu)$. Thus, $e(\mu)$ is flat and concave at μ_I°. In contrast, $\mu_I^{\circ\circ}$ is a local minimum point of $e(\mu)$. Thus, $e(\mu)$ is flat and convex at $\mu_I^{\circ\circ}$.

He (2014), the equity value can be indirectly derived as the difference between the total firm value and the debt value: $e(\mu) = v(\mu) - d(\mu)$. The total firm value $v(\mu)$ satisfies

$$v(\mu) = \begin{cases} \frac{\mu-\lambda i}{r-i+\delta} + B_1(\mu-\mu_L)^{1-\gamma_1}(\mu_H-\mu)^{\gamma_1}, & \text{if } \mu \geq \mu_I, \\ \frac{\mu}{r+\delta} + B_2(\mu-\mu_L)^{1-\gamma_2}(\mu_H-\mu)^{\gamma_2} + B_3(\mu-\mu_L)^{\gamma_2}(\mu_H-\mu)^{1-\gamma_2}, & \text{if } \mu_I > \mu \geq \mu_B. \end{cases} \tag{A13}$$

The function $v(\mu)$ can be interpreted as follows. If $\mu \geq \mu_I$, the first term is equal to the firm value at μ that would be realized if the firm always invested and did not default. The second term indicates the adjustment for stopping investment at least temporarily. If $\mu_I > \mu \geq \mu_B$, the first term is the firm value at μ without investment in the case of no default. The remaining terms reflect the adjustment for entering the investment region again in the future.

Thus, it follows from (A2), (A13), and $e(\mu) = v(\mu) - d(\mu)$ that if $\mu \geq \mu_I$, the equity value is given by

$$e(\mu) = \frac{\mu-\lambda i}{r-i+\delta} + B_1(\mu-\mu_L)^{1-\gamma_1}(\mu_H-\mu)^{\gamma_1} - A_1(\mu-\mu_L)^{1-\beta_1}(\mu_H-\mu)^{\beta_1} - \frac{p}{1+m(r-i+\delta)}. \tag{A14}$$

Then,

$$\begin{aligned} e'(\mu) &= v'(\mu) - d'(\mu) \\ &= \frac{1}{r-i+\delta} + (1-\gamma_1)B_1\left(\frac{\mu_H-\mu}{\mu-\mu_L}\right)^{\gamma_1} - \gamma_1 B_1\left(\frac{\mu-\mu_L}{\mu_H-\mu}\right)^{1-\gamma_1} \\ &\quad - \left[(1-\beta_1)A_1\left(\frac{\mu_H-\mu}{\mu-\mu_L}\right)^{\beta_1} - \beta_1 A_1\left(\frac{\mu-\mu_L}{\mu_H-\mu}\right)^{1-\beta_1}\right], \end{aligned} \tag{A15}$$

$$e''(\mu) = \frac{(\mu_H-\mu_L)^2}{(\mu-\mu_L)(\mu_H-\mu)^2}\left[\gamma_1(\gamma_1-1)B_1\left(\frac{\mu_H-\mu}{\mu-\mu_L}\right)^{\gamma_1} - \beta_1(\beta_1-1)A_1\left(\frac{\mu_H-\mu}{\mu-\mu_L}\right)^{\beta_1}\right]. \tag{A16}$$

Using (14) and $v(\mu) = d(\mu) + e(\mu)$, it is found from (A1), (A2), (A13), and (A16) that $\gamma_1 > 1$ is the positive root of

$$\gamma_1^2 - \gamma_1 - \frac{2\sigma^2(r-i+\delta)}{(\mu_H-\mu_L)^2} = 0. \tag{A17}$$

If $\mu_I > \mu \geq \mu_B$, we repeat the above argument with (14), (A3), (A4), (A13), and $e(\mu) = v(\mu) - d(\mu)$. Then, we can show that the equity value is given by

$$e(\mu) = \frac{\mu}{r+\delta} + B_2(\mu-\mu_L)^{1-\gamma_2}(\mu_H-\mu)^{\gamma_2} + B_3(\mu-\mu_L)^{\gamma_2}(\mu_H-\mu)^{1-\gamma_2} \\ - A_3(\mu-\mu_L)^{1-\beta_2}(\mu_H-\mu)^{\beta_2} - A_4(\mu-\mu_L)^{\beta_2}(\mu_H-\mu)^{1-\beta_2} - \frac{p}{1+m(r+\delta)}, \tag{A18}$$

where $\gamma_2 > 1$ is given by the positive root of

$$\gamma_2^2 - \gamma_2 - \frac{2\sigma^2(r+\delta)}{(\mu_H-\mu_L)^2} = 0. \tag{A19}$$

We now solve the constants B_1, B_2, and B_3. Combining (9)–(11), (17), (19), and (20) under $v(\mu) = e(\mu) + d(\mu)$, we have

$$v(\mu_B) = \frac{\mu_B}{r+\delta}, \tag{A20}$$

$$\lim_{\mu\uparrow\mu_I} v(\mu) = \lim_{\mu\downarrow\mu_I} v(\mu), \tag{A21}$$

$$\lim_{\mu\uparrow\mu_I} v'(\mu) = \lim_{\mu\downarrow\mu_I} v'(\mu). \tag{A22}$$

Then, it follows from (A13) and (A20)–(A22) that

$$B_1 = \frac{1}{\Psi_4}\left\{\frac{i}{(r-i+\delta)(r+\delta)} - \left[\left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{2\gamma_2-1}\Psi_5 + \Psi_6\right]B_3\right\}, \tag{A23}$$

$$B_2 = -\left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{2\gamma_2-1} B_3, \tag{A24}$$

$$B_3 = \left\{(\mu_I-\mu_L)\left(\frac{\mu_B-\mu_L}{\mu_H-\mu_B}\right)^{2\gamma_2-1}\left[\left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\gamma_1}\frac{\Psi_5}{\Psi_4} - \left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\gamma_2}\right]\right. \\ \left. +(\mu_H-\mu_I)\left(\frac{\mu_I-\mu_L}{\mu_H-\mu_I}\right)^{\gamma_2} + (\mu_I-\mu_L)\left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\gamma_1}\frac{\Psi_6}{\Psi_4}\right\}^{-1} \\ \times\left\{\frac{i}{(r-i+\delta)(r+\delta)}\left[\mu_I + \frac{(\mu_I-\mu_L)\left(\frac{\mu_H-\mu_I}{\mu_I-\mu_L}\right)^{\gamma_1}}{\Psi_4}\right] - \frac{\lambda i}{r-i+\delta}\right\}, \tag{A25}$$

where

$$\Psi_4 = \gamma_1 \left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\gamma_1 - 1} + (\gamma_1 - 1)\left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\gamma_1}, \tag{A26}$$

$$\Psi_5 = \gamma_2 \left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\gamma_2 - 1} + (\gamma_2 - 1)\left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\gamma_2}, \tag{A27}$$

$$\Psi_6 = \gamma_2 \left(\frac{\mu_I - \mu_L}{\mu_H - \mu_I}\right)^{\gamma_2 - 1} + (\gamma_2 - 1)\left(\frac{\mu_I - \mu_L}{\mu_H - \mu_I}\right)^{\gamma_2}. \tag{A28}$$

Now, it follows from (A14) and (A18) that the investment and default thresholds μ_I and μ_B are simultaneously determined by (15) and (18), that is,

$$\begin{aligned}
&\frac{\mu_I - \lambda i}{r - i + \delta} - \frac{p}{1 + m(r - i + \delta)} - \lambda \\
&+ (\mu_I - \mu_L)\left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\gamma_1} \frac{1}{\Psi_4}\left\{\frac{i}{(r - i + \delta)(r + \delta)} - \left[\left(\frac{\mu_B - \mu_L}{\mu_H - \mu_B}\right)^{2\gamma_2 - 1}\Psi_5 + \Psi_6\right] B_3\right\} \\
&- (\mu_I - \mu_L)\left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\beta_1}\left\{\left[\frac{\mu_B}{r + \delta} - \frac{p}{1 + m(r + \delta)}\right]\frac{1}{\mu_B - \mu_L}\left(\frac{\mu_B - \mu_L}{\mu_H - \mu_B}\right)^{\beta_2}\left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\beta_2 - \beta_1}\right. \\
&+ \left[\left(\frac{\mu_I - \mu_L}{\mu_H - \mu_I}\right)^{\beta_1 + \beta_2 - 1} - \left(\frac{\mu_B - \mu_L}{\mu_H - \mu_B}\right)^{2\beta_2 - 1}\left(\frac{\mu_H - \mu_I}{\mu_I - \mu_L}\right)^{\beta_2 - \beta_1}\right] A_4 \\
&\left. - \frac{\frac{pmi}{\mu_H - \mu_I}\left(\frac{\mu_I - \mu_L}{\mu_H - \mu_I}\right)^{\beta_1 - 1}}{[1 + m(r + \delta)]\,[1 + m(r - i + \delta)]}\right\} \\
&= 0,
\end{aligned} \tag{A29}$$

$$\begin{aligned}
&\frac{1}{r + \delta} + \left[\gamma_2\left(\frac{\mu_B - \mu_L}{\mu_H - \mu_B}\right)^{\gamma_2 - 1}\frac{\mu_H - \mu_L}{\mu_H - \mu_B} + (\gamma_2 - 1)\left(\frac{\mu_B - \mu_L}{\mu_H - \mu_B}\right)^{\gamma_2 - 1}\frac{\mu_H - \mu_L}{\mu_H - \mu_B}\right] B_3 \\
&+ \left[\beta_2\left(\frac{\mu_H - \mu_B}{\mu_B - \mu_L}\right)^{\beta_2 - 1} + (\beta_2 - 1)\left(\frac{\mu_H - \mu_B}{\mu_B - \mu_L}\right)^{\beta_2}\right] A_3 \\
&- \left[\beta_2\left(\frac{\mu_B - \mu_L}{\mu_H - \mu_B}\right)^{\beta_2 - 1} + (\beta_2 - 1)\left(\frac{\mu_B - \mu_L}{\mu_H - \mu_B}\right)^{\beta_2}\right] A_4 = 0.
\end{aligned} \tag{A30}$$

Finally, to prove $\frac{\mu_B}{r+\delta} < \frac{p}{1+m(r+\delta)}$, we assume that $\frac{\mu_B}{r+\delta} \geq \frac{p}{1+m(r+\delta)}$. Then, it follows from (9) that $d(\mu_B) \geq \frac{p}{1+m(r+\delta)}$, which means that the debt is riskless. Thus, with the option to default, equity holders must incur strictly negative expected cash flows at default, as discussed in Diamond and He (2014). However, because equity holders set $i_t = 0$ at μ_B, it follows from (5) that the expected net cash flow for equity at μ_B is at least equal to $\{\mu_B + \frac{1}{m}[d(\mu_B) - p]\}dt > \left(\mu_B - \frac{p(r+\delta)}{1+m(r+\delta)}\right) dt \geq 0$, which is a contradiction. ■

Proof of Proposition 2: Suppose that $i = \delta = 0$. Given that the investment threshold μ_I does not need to be considered, we can derive (23)–(25) by applying the same procedure as that of the proof of Proposition 1 with $i = \delta = 0$. Given boundary conditions (12) and (16), note that (23) and (24) are determined so that neither $d(\mu)$ nor $e(\mu)$ becomes infinite as $\mu \to \mu_H$. Again, note that μ_B^c must satisfy $\frac{\mu_B^c}{r} < \frac{p}{1+mr}$. ■

Proof of Proposition 3: In this case, the firm does not issue any debt. Then, the equity value is obtained as

$$e^e(\mu) = \begin{cases} \frac{\mu-\lambda i}{r-i+\delta} + B_1^e (\mu - \mu_L)^{1-\gamma_1^e} (\mu_H - \mu)^{\gamma_1^e}, & \text{if} \quad \mu \geq \mu_I^e, \\ \frac{\mu}{r+\delta} + B_3^e (\mu - \mu_L)^{\gamma_2^e} (\mu_H - \mu)^{1-\gamma_2^e}, & \text{if} \quad \mu_I^e > \mu, \end{cases} \tag{A31}$$

where $\gamma_1^e > 1$ and $\gamma_2^e > 1$ are the positive roots of $(\gamma_1^e)^2 - \gamma_1^e - \frac{2\sigma^2(r-i+\delta)}{(\mu_H-\mu_L)^2} = 0$ and $(\gamma_2^e)^2 - \gamma_2^e - \frac{2\sigma^2(r+\delta)}{(\mu_H-\mu_L)^2} = 0$, respectively. Note that the equity value is determined so that it does not become infinite as $\mu \to \mu_H$ or $\mu \to \mu_L$. It follows from (15) and (A31) that

$$B_1^e = \left[\frac{\lambda(r+\delta) - \mu_I^e}{r-i+\delta}\right] (\mu_I^e - \mu_L)^{-1+\gamma_1^e} (\mu_H - \mu_I^e)^{-\gamma_1^e}. \tag{A32}$$

It is also found from (15), (19), and (A31) that

$$B_3^e = \left[\frac{\lambda(r+\delta) - \mu_I^e}{r+\delta}\right] (\mu_I^e - \mu_L)^{-\gamma_2^e} (\mu_H - \mu_I^e)^{-1+\gamma_2^e}. \tag{A33}$$

Substituting (A32) and (A33) into (A31), we obtain (26a) and (26b). It also follows from (26a), (26b), and (20) that (27) is derived.

The effect of the promised debt payment p: To confirm that our choice of the value of the promised debt payment p does not lead to extraordinary results, we examine the effect of p on the default and investment thresholds, the debt value, and the firm value. We report the results in Figures A1 and A2. These results suggest that our choice of p does not cause any trouble. ■

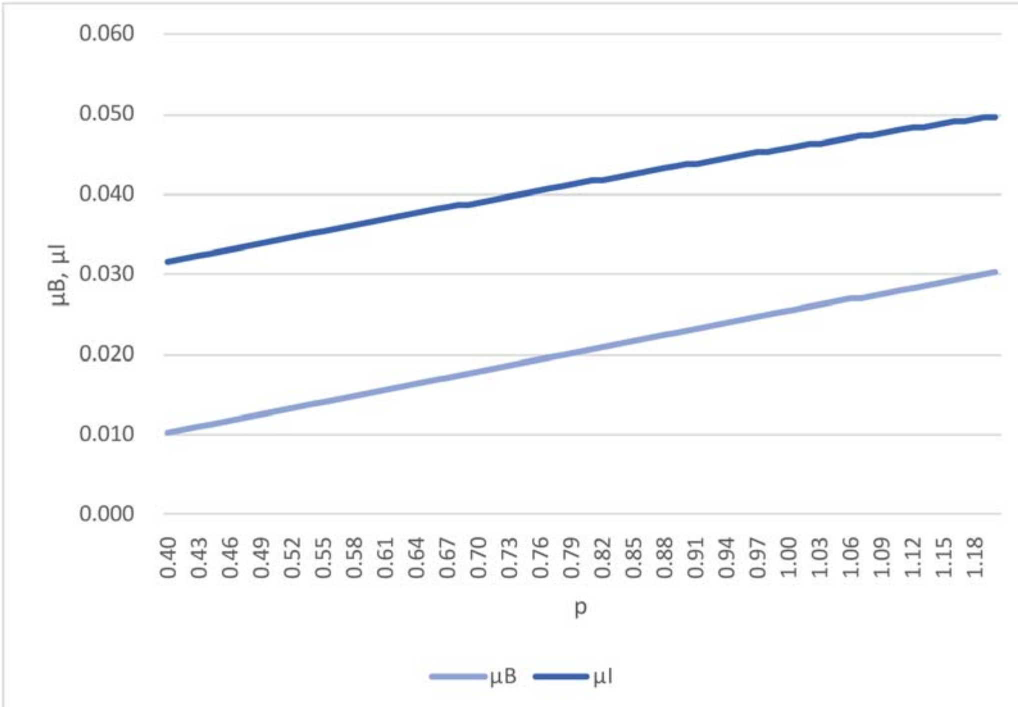

Figure A1. The effects of a change in p on the default and investment thresholds, μ_B and μ_I. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $\sigma = 0.3$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, and $\delta = 0.04$.

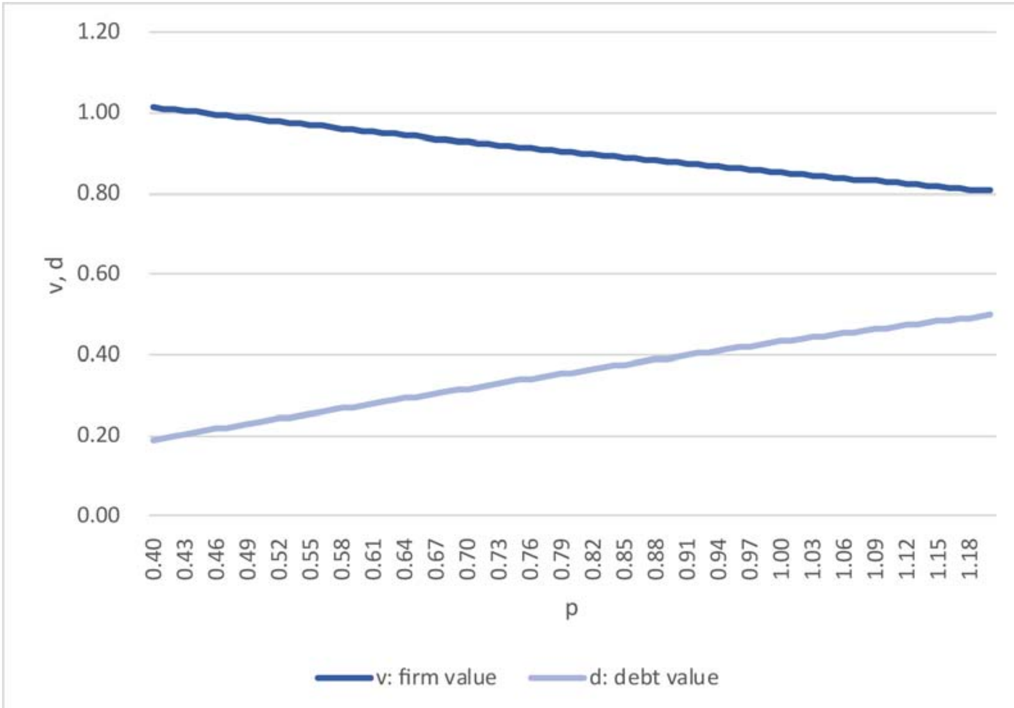

Figure A2. The effects of a change in p on the debt and firm values, d and v. Parameters are $\mu_L = 0$, $\mu_H = 0.1$, $\sigma = 0.3$, $m = 10$, $r = 0.05$, $\lambda = 0.3$, $i = 0.05$, $\delta = 0.04$, and $\mu_0 = 0.5$.

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