# The Risky Steady State and the Interest-Rate Lower Bound*

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**Abstract**

The possibility that the policy rate becomes constrained by its effective lower bound (ELB) in the future leads price-setters to choose lower prices than they would otherwise choose in the absence of such a possibility. As a consequence, inflation falls below its long-run target in the interest-rate feedback rule even when all exogenous shocks dissipate and the economy is at the risky steady state. In an empirically rich DSGE model calibrated to match key features of the U.S. economy, we find that inflation is 25 basis points below the inflation target at the risky steady state. Our model can explain why some central banks systematically undershot their inflation targets even before the policy rates became constrained by the ELB.

JEL: E32, E52, E61, E62, E63

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# 1 Introduction

This paper characterizes the risky steady state in an empirically rich sticky-price model with occasionally binding effective lower bound (ELB) constraints on nominal interest rates. The risky steady state is the "point where agents choose to stay at a given date if they expect future risk and if the realization of shocks is 0 at this date" (Coeurdacier, Rey, and Winant (2011)). The risky steady state is an important object in dynamic macroeconomic models: This is the point around which the economy fluctuates. This is the point where the economy eventually converges to when all headwinds and tailwinds dissipate.

We first use a stylized New Keynesian model to illustrate how, and why, the risky steady state differs from the deterministic steady state. We show that inflation and the policy rate are lower, and output is higher, at the risky steady state than at the deterministic steady state. The lower bound constraint on interest rates makes the distribution of inflation asymmetric; The decline in inflation due to a large negative shock is larger than the increase in inflation due to a positive shock of the same magnitude. As a result, the lower bound constraint reduces the expected inflation facing forward-looking firms, leading them to lower their prices even when the policy rate is not currently constrained. Reflecting the lower risky-steady-state inflation, the policy rate is lower at the risky steady state than that at the deterministic steady state. The decline in the policy rate due to the lower bound risk is larger than the decline in inflation, making the expected real rate slightly lower at the risky steady than at the deterministic steady state. Accordingly, the output gap is positive.

We then explore the quantitative importance of the wedge between the risky and deterministic steady states in an empirically rich DSGE model calibrated to match key features of the U.S. economy over the past two decades—roughly corresponding to a period when long-run inflation expectations were low and stable and the lower bound constraint on the policy rate was either a concern or a binding constraint to the Federal Reserve. We find the wedge between the deterministic and risky steady states is non-trivial in our calibrated empirical model. Inflation is about 25 basis points lower than the target inflation of 2 percent at the risky steady state. The policy rate and the output gap are about [30] basis point lower and [0.2] percent higher at the risky steady state than at the deterministic steady state. The wedge depends importantly on the frequency of hitting the ELB, which in turn depends importantly on the level of the long-run equilibrium policy rate. If the policy rate at the deterministic steady state is 50 basis points lower than our baseline of 3.75 percent, then the deflationary bias would increase to [60] basis points.

The observation that the risky steady-state inflation falls below the target in the policy rule is distinct from the well-known fact that the average inflation falls below the target in the model with the ZLB constraint. The decline in inflation arising from a negative demand shock is exacerbated at the ZLB, while the rise in inflation arising from a positive demand shock is always tempered by corresponding increase in the policy rate. As a result, the distribution of inflation has large tail risks on the downside and the average inflation fall below the target. This fact is intuitive and has been well known in the profession for a long time (Coenen, Orphanides, and Wieland (2004) and Reifschneider and Williams (2000)). The risky steady-state inflation is different from the average inflation; it is the rate of inflation that would prevail at the economy's steady state when agents are aware of risks. It is worth mentioning that the average inflation falls below the target even in perfect-foresight models or backward-looking models. On the other hand, for the risky steady-state inflation to fall below the inflation target, price-setters need to be forward looking and take uncertainty into account in their pricing decisions.

As discussed in section 7, some central banks had systematically undershot their inflation targets even before the policy rate became constrained by the ELB. For example, in the U.S., core PCE Price inflation averaged about 1.8 percent from mid 1990s to the fourth quarter of 2008, the last quarter in which the federal funds rate was on average above the ELB. The undershooting of the inflation target away from the ELB is also observed in other countries that faced the ELB during the recent global recession, including Canada, Euro Area, United Kingdom, Sweden, and Switzerland. Our model provides a potential explanation for this systematic undershooting of the inflation target in economies with low inflation and low interest rates.

The question of how the possibility of returning to the ELB affects the economy has remained largely unexplored. The majority of the literature adopts the assumption that the economy will eventually return to an absorbing state where the policy rate is permanently away from the ELB constraint and analyzes the dynamics of the economy, and the effects of various policies, when the policy rate is *at* the ELB (Eggertsson and Woodford (2003), Christiano, Eichenbaum, and Rebelo (2011)). While an increasing number of studies have recently departed from the assumption of an absorbing state, the focus of these studies remains mostly what happens at the ELB, instead of how the lower bound risk affects the economy away from the ELB.[1] With the federal funds rate widely expected to lift off from the ELB in the near future, the question of how the lower bound risk affects the economy is as relevant as ever.[2]

Our paper builds on the work by Adam and Billi (2007) and Nakov (2008) who first observed that the possibility of future ELB episodes has consequences for the economy away from the ELB. Our work differs from these papers in two substantive ways. First, they have pointed out the anticipation effect of the future ELB episode on the economy when the policy

[1]For example, Schmidt (2013) and Nakata (2013a) analyzes how the government should conduct fiscal policy at the ZLB. Gavin, Keen, Richter, and Throckmorton (2015) study how the anticipated monetary policy shock affect the dynamics of the economy at the ZLB. Gust, López-Salido, and Smith (2012) asks how much the ZLB constraint depressed output and inflation during the Great Recession.

[2]According to the September 2015 Primiary Dealer Survey, respondents on average attached 70 percent chance to the event that the federal funds rate will rise from the ELB by the end of 2015. According to the Summary of Economic Projections in September 2015, 13 out of 17 FOMC participants projected the federal funds rate will rise from the ELB by the end of 2015.

rate is *near* the ELB and the economy is away from the steady state. Our work shows that the possibility of returning to the ELB has consequences for the economy even when the policy rate is sufficiently above the ELB and the economy is at the steady state. Second, and more importantly, the existing papers have only studied the effects of the lower bound risk in a highly stylized model. Our key contribution is to quantify the magnitude of the effects of the lower bound risk in an empirically rich model.

Our work compliments a body of work that explores the deflationary steady state in various sticky price models. A seminal work of Benhabib, Schmitt-Grohe, and Uribe (2001) shows the existence of a deflationary steady state with the policy rate stuck at the lower bound in a standard sticky-price model with a Taylor rule.[3] Some have recently studied deflationary steady states with zero nominal interest rates in other interesting models with a nominal friction and a Taylor rule (Benigno and Fornaro (2015); Eggertsson and Mehrotra (2014); Schmitt-Grohe and Uribe (2013)). Bullard (2010) and Bullard (2015) have argued that these deflationary steady states are relevant in understanding the Japanese economy last two decades as well as what may happen in other advanced economies. The steady state we focus on is similar to the deflationary steady state analyzed in these papers in that both entail below-target inflation. However, unlike in the deflationary steady states studied in these papers, the nominal interest rate is above the ELB in our risky steady state.

Finally, this paper is related to recent papers that have emphasized the importance of the effect of risk on steady states in various nonlinear dynamic models. Gertler, Kiyotaki, and Queralto (2012) and de Groot (2014) discuss how the degree of risk affects the balance sheet conditions of financial intermediaries at the steady state. Coeurdacier, Rey, and Winant (2011), Devereux and Sutherland (2011) and Tille and van Wincoop (2010) study optimal portfolio choices at the risky steady state in open-economy models. While our work is similar to theirs in analyzing the effect of risk on the steady state, we differ from them in a fundamental way: While the wedge between the deterministic and risky steady states is driven by the nonlinearity of smooth differentiable functions in their models, the ELB constraint is the key nonlinearity in our model. Reflecting the difference in the types of nonlinearity involved, the solution methods employed are different as well. While they all solve the model by using local approximation methods that take advantage of differentiability of policy functions, we use a global method to solve the model.[4]

The rest of the paper is organized as follows. After a brief review of the concept of the risky steady state in section 2, section 3 analyzes the risky steady state in a stylized New Keynesian economy. Section 4 quantifies the wedge between the deterministic and risky steady states in an empirically rich DSGE model. Sections 5 and 6 examine how the wedge depends on the equilibrium real rate and monetary policy rules. After a brief discussion on

[3] Armenter (2014) and Nakata and Schmidt (2014) show that the deflationary steady state exists even when policy is conducted optimally, provided that the central bank does not have a commitment technology.

[4] See de Groot (2013) and Meyer-Gohde (2015) for recent progress in computing the risky steady state in nonlinear differentiable economies.

the policy relevance of our analysis in section 7, section 8 concludes.

## 2 The Risky Steady State: Definition

The risky steady state is defined generically as follows.[5] Let $\Gamma_t$ and $S_t$ denote a vector of endogenous variables and a vector of exogenous variables in the model under investigation. Let $S_{SS}$ denote the steady state of $S_t$. There is no distinction between deterministic and risky steady states for $S_t$ because $S_t$ is exogenous. Let $f(\cdot,\cdot)$ denote a vector of policy functions mapping the values of endogenous variables in the previous period and today's realizations of exogenous variables into the values of endogenous variables today.[6] Then, the risky steady state of the economy, $\Gamma_{RSS}$, is given by a vector satisfying the following condition.

$$\Gamma_{RSS} = f(\Gamma_{RSS}, S_{SS}) \tag{1}$$

That is, the risky steady state is where the economy will eventually converge to as the exogenous variables settle at their steady state. In this risky steady state, the agents are aware of the fact that shocks to the exogenous variables can occur, but realization of those shocks are zero. On the other hand, the deterministic steady state of the economy, $\Gamma_{DSS}$, is defined as follows:

$$\Gamma_{DSS} = f_{PF}(\Gamma_{DSS}, S_{SS}) \tag{2}$$

where $f_{PF}(\cdot,\cdot)$ denotes a vector of the policy functions obtained under the perfect-foresight assumption.

## 3 The Risky Steady State in a Stylized Model with the ELB

We start by characterizing the risky steady state in a stylized New Keynesian model. The model we use is identical to the one used in the previous work of one of the authors (Nakata (2013b)). Since the model is standard, we only present its equilibrium conditions here. The details of the model is explained in the Appendix for the sake of brevity.

$$C_t^{-\chi_c} = \beta\delta_t R_t \mathrm{E_t} C_{t+1}^{-\chi_c}\Pi_{t+1}^{-1} \tag{3}$$

$$w_t = N_t^{\chi_n} C_t^{\chi_c} \tag{4}$$

$$\frac{Y_t}{C_t^{\chi_c}}\left[\varphi\left(\frac{\Pi_t}{\bar{\Pi}}-1\right)\frac{\Pi_t}{\bar{\Pi}} - (1-\theta) - \theta(1-\tau)w_t\right] = \beta\delta_t \mathrm{E_t}\frac{Y_{t+1}}{C_{t+1}^{\chi_c}}\varphi\left(\frac{\Pi_{t+1}}{\bar{\Pi}}-1\right)\frac{\Pi_{t+1}}{\bar{\Pi}} \tag{5}$$

[5]Our definition is identical to those in the literature on the risky steady state, though our notations are somewhat uncommon.

[6]Note that the policy function does not need to depend on the entire set of the endogenous variables in the prior period. It may not depend on any endogenous variables in the prior period at all, as in the stylized model presented in the next section.

$$Y_t = C_t + \frac{\varphi}{2}\left[\frac{\Pi_t}{\bar{\Pi}} - 1\right]^2 Y_t \tag{6}$$

$$Y_t = N_t \tag{7}$$

$$R_t = \max\left[1, \quad \frac{\bar{\Pi}}{\beta}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}\right] \tag{8}$$

$$(\delta_t - 1) = \rho_\delta(\delta_{t-1} - 1) + \epsilon_t^\delta \tag{9}$$

$C$, $N$, $Y$, $w$, $\Pi$, and $R$ are consumption, labor supply, output, real wage, inflation, and the policy rate, respectively. $\delta$ is an exogenous shock to the household's discount rate, and follows an AR(1) process with mean one, as shown in equation 9. Equation 3 is the consumption Euler equation, Equation 4 is the intratemporal optimality condition of the household, Equation 5 is the optimal condition of the intermediate good producing firms (forward-looking Phillips Curve) relating today's inflation to real marginal cost today and expected inflation tomorrow, Equation 6 is the aggregate resource constraint capturing the resource cost of price adjustment, and Equation 7 is the aggregate production function. Equation 8 is the interest-rate feedback rule.

A recursive equilibrium of this stylized economy is a given by a set of policy functions for $\{C(\cdot), N(\cdot), Y(\cdot), w(\cdot), \Pi(\cdot), R(\cdot)\}$ satisfying the equilibrium conditions described above. The model is solved with a global solution method described in detail in the Appendix. Table 1 lists the parameter values used for this exercise.

Table 1: **Parameter Values for the Stylized Model**

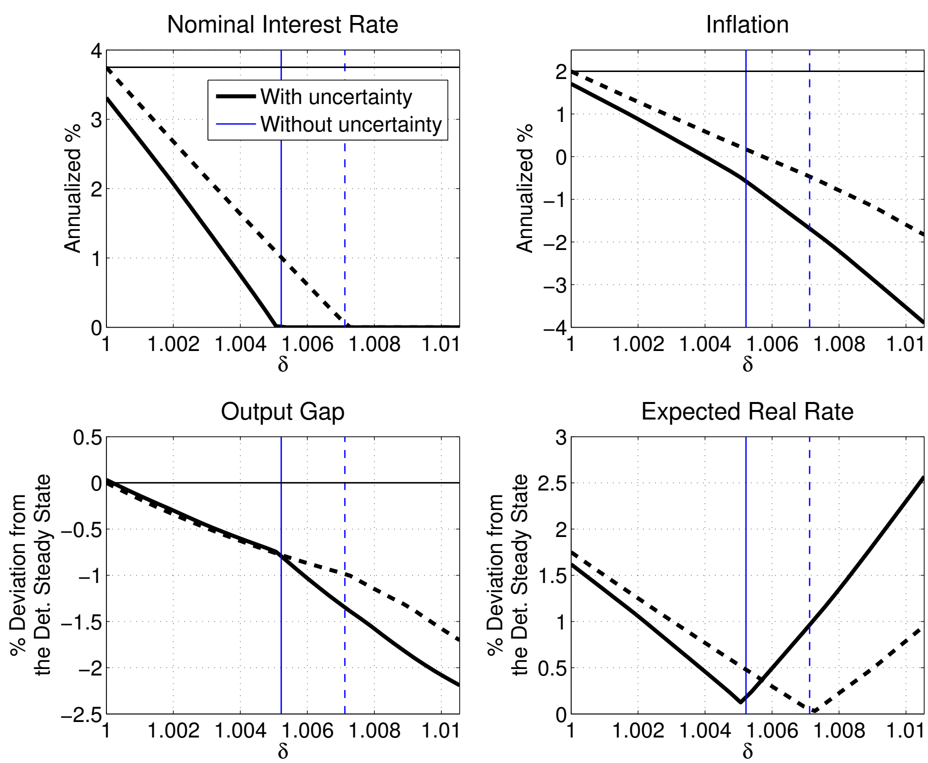
| Parameter | Description | Parameter Value |
|---|---|---|
| $\beta$ | Discount rate | $\frac{1}{1+0.004365}$ |
| $\chi_c$ | Inverse intertemporal elasticity of substitution for $C_t$ | 1.0 |
| $\chi_n$ | Inverse labor supply elasticity | 1 |
| $\theta$ | Elasticity of substitution among intermediate goods | 11 |
| $\varphi$ | Price adjustment cost | 200 |
| $400(\bar{\Pi} - 1)$ | (Annualized) target rate of inflation | 2.0 |
| $\phi_\pi$ | Coefficient on inflation in the Taylor rule | 1.5 |
| $\phi_y$ | Coefficient on the output gap in the Taylor rule | 0 |
| $\rho$ | AR(1) coefficient for the discount factor shock | 0.8 |
| $\sigma_\epsilon$ | The standard deviation of shocks to the discount factor | $\frac{0.32}{100}$ |
| | *The implied prob. that the policy rate is at the lower bound | 10% |

Black lines in Figure 1 show the policy functions for the policy rate, inflation, the output gap, and the expected real ineterest rate. Red lines show the policy function of the model obtained under the assumption of perfect foresight. Under the perfect-foresight case, the agents in the model attach zero probability to the event that the policy rate will be at the ELB when the policy rate is currently away from the effective lower bound.[7]

[7] Nakata (2013b) provides detailed accounts of the large differences between stochastic and perfect-foresight

In the stylized model of this section, the policy functions do not depend on any of the model's endogenous variables from the previous period. In other words, there is no endogenous *state* variable. Thus, the risky steady state is given by the vector of the policy functions evaluated at $\delta = 1$. According to the figure, the policy rate and inflation are lower at the risky steady state than those at the deterministic steady state, while the output gap is higher at the risky steady state. As shown in the first and second rows of Table 2, inflation and the policy rate are about [40] and [45] basis points lower at the risky steady state than at the deterministic steady state, respectively. The risky-steady-state output gap is slightly positive.

Figure 1: Policy Functions from the Stylized Model




*The dashed black lines ("Without uncertainty" case) show policy functions obtained under the perfect-foresight assumption (i.e., $\sigma_\epsilon = 0$).

Due to the lower bound constraint, the conditional distribution of inflation is asymmetric at any level of the discount rate shock, $\delta$. This asymmetry arises because the decline in inflation due to a large negative shock is larger than the increase in inflation due to a positive shock of the same magnitude. As a result, the lower bound constraint reduces the expected inflation facing forward-looking firms, leading them to lower their prices today at any level of the discount rate shock. The fact the the lower bound risk reduces inflation is referred

solutions while the policy rate is at the ZLB. The focus of this paper, in contrast, is to examine the differences between these two solutions when the policy rate is away from the ZLB constraint.

Table 2: **The Risky Steady State in the Stylized Model**

| | Inflation | Output gap | Policy rate |
|---|---|---|---|
| Deterministic steady state | 2 | 0 | 3.75 |
| Risky steady state | 1.70 | 0.03 | 3.31 |
| (Wedge) | (−0.30) | (0.03) | (−0.44) |
| Risky steady state w/o the ELB | 1.99 | −0.02 | 3.72 |
| (Wedge) | (−0.01) | (−0.02) | (−0.03) |

to as *deflationary bias* in the literature (Nakov (2008); Nakata and Schmidt (2014)). The deflationary bias is larger the closer the policy rate is to the ELB constraint because the degree of asymmetry is stronger when the policy rate is at or near the ELB than when the policy rate is comfortably above the ELB. This can be seen by the fact that the wedge between the correct (stochastic) and the perfect-foresight policy functions for inflation is larger when the discount rate is high. However, the effect of asymmetry remains nontrivial even when the discount rate shock is at its steady state level and the economy is at its risky steady state.

Reflecting the lower risky steady-state inflation, the policy rate is lower at the risky steady state than at the deterministic steady state. The decline in the policy rate due to the lower bound risk is larger than the decline in inflation and, as a result, the expected real rate is lower at the risky steady than at the deterministic steady state, as shown in the bottom-right panel of figure 1. Thus, the output gap is slightly higher at the risky steady state than at the deterministic steady state.[8]

In our model, the ELB constraint is not the only source of nonlinearity. Our specifications of the utility function and the price adjustment cost also make the model nonlinear, and thus explain some of the wedge between the deterministic and risky steady states. To understand the extent to which these other nonlinearities matter, the fourth line of Table 2 shows the risky steady state in the version of our model without the ELB constraint. Overall the differences between the deterministic and risky steady states are small. Inflation and the policy rate at the risky steady state are respectively [1] and [3] basis points below those at the deterministic steady states. The output gap at the risky steady state is about 2 basis points below that at the deterministic steady state. Thus, the majority of the overall wedge between the deterministic and risky steady states is accounted for by the nonlinearity induced by the ELB constraint, as opposed to other nonlinearities of the model.

The wedge between the deterministic and risky steady states depends importantly on the probability of being at the ELB. The blue line in the left panel of figure 2 illustrates this point for inflation. In this figure, we vary the standard deviation of the discount rate shock to induce changes in the probability of being at the ELB. According to the blue line, a higher

[8] Since the output gap today depends on the expected output gap in the next period, the output gap can be negative even with lower expected real rates. However, we find that the output gap is positive in all the calibrations we have examined. Using a two-state shock model in which the central bank is optimizing under discretion, Nakata and Schmidt (2014) analytically prove that the output gap is always positive in the risky steady state.

probabilty of being at the ELB is associated with a larger deflationary bias at the risky steady state. In this stylized model, the risky steady state increases from 25 basis points to 50 basis points when the probability of being at the ELB increases from 10 percent to 12 percent.

Figure 2: Conditional and Unconditional Averages of Inflation in the Stylized Model

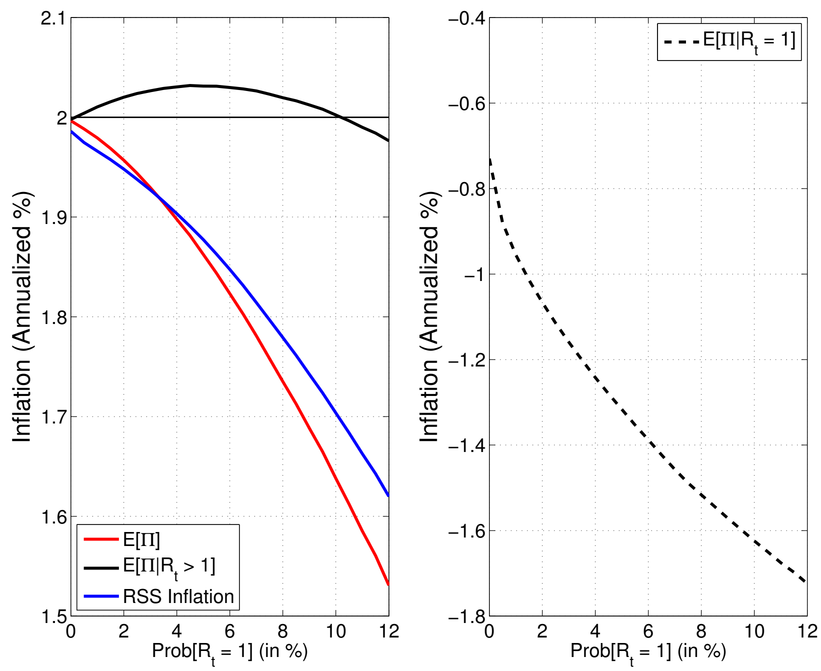



Figure 2 also plots how the conditional average of inflation away from the ELB (the black line in the left panel), the unconditinal average of inflation (the red line in the left panel), and the conditional average of inflation at the ELB (the dashed black line in the right panel). Not surprisingly, the conditional average of inflation away from the ELB is higher than the unconditional average, which in turn is higher than the conditional average of inflation at the ELB. These averages decline with the probability of being at the ELB, as the risky steady state inflation does. When the probability of being at the ZLB is sufficiently high, the risky steady state inflation is higher than the unconditional average of inflation, reflecting the fact that the ergodic distribution of inflation has a fat tail on the negative side. However, when the probability of being at the ZLB is not sufficiently high, the risky steady state inflation is lower than the unconditional average of inflation, reflecting the fact that the ergodic distribution of inflation has a slightly fat tail on the positive side due to other nonlinearities of the model.

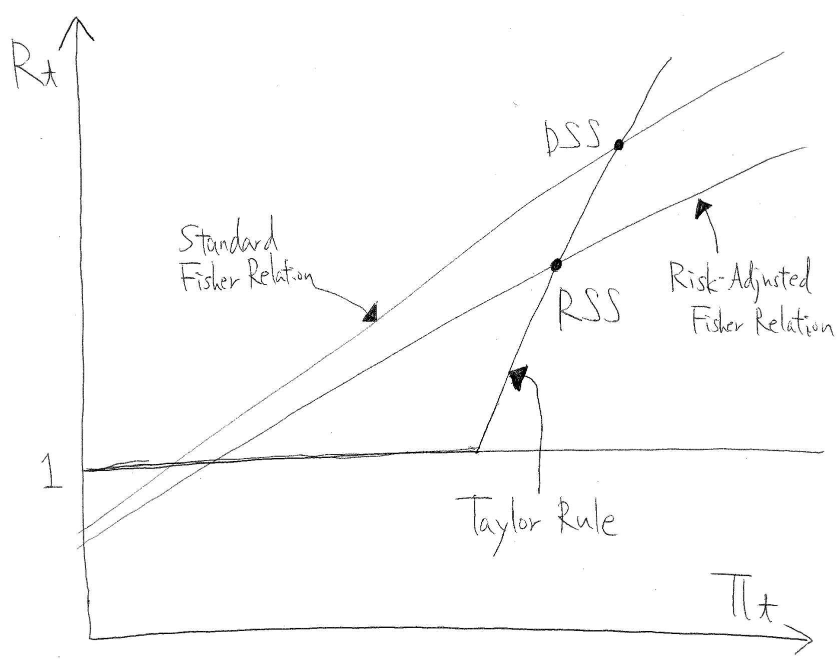
Note that the conditional average of inflation away from the ELB is above 2 percent when the lower bound risk is positive but not sufficiently high. This is because the conditional distribution of inflation away from the ZLB exclude the lower tail of the unconditional distribution which is centered around 2 percent. However, the conditional average of inflation away from the ELB becomes below the inflation target when the lower bound risk is sufficiently high. This happens because the unconditional distribution of inflation is centered around a point sufficiently below 2 percent. While the risky steady state inflation cannot be measured in the data, the conditional average of inflation away from the ELB can be. Thus, the importance of the lower bound risk in the data manifests itself in the extent to which the conditional average inflation away from the ELB falls below the target rate of inflation. We will later examine the average inflation away from the lower bound in several advanced economies in Section 7.

One way to understand the wedge between deterministic and risky steady states is to examine the effect of the lower bound risk on the Fisher relation. Let $R_{DSS}$ and $\Pi_{DSS}$ be the deterministic steady-state policy rate and inflation. In the deterministic environment, the consumption Euler equation evaluated at the steady state becomes

$$R_{DSS} = \frac{\Pi_{DSS}}{\beta} \tag{10}$$

after dropping the expectation operator from the consumption Euler equation and eliminating the deterministic steady-state consumption from both sides of the equation. This relation is often referred to as the Fisher relation.

Figure 3: The Risk-Adjusted Fisher Relation and the Taylor Rule




In the stochastic environment, the consumption Euler equation evaluated at the (risky) steady state can be written as

$$R_{RSS} = \frac{\Pi_{RSS}}{\beta} \cdot \frac{1}{\mathrm{E}_{RSS}\big[\Big(\frac{C_{RSS}}{C_{t+1}}\Big)^{\chi_c} \frac{\Pi_{RSS}}{\Pi_{t+1}}\big]} \tag{11}$$

where $R_{RSS}$, $\Pi_{RSS}$, and $C_{RSS}$ are the risky steady-state policy rate, inflation, and consumption. $\mathrm{E}_{RSS}[\cdot]$ is the conditional expectation operator when the economy is at the risky steady state today. In the stylized model with one shock and without any endogenous state variables, $\mathrm{E}_{RSS}[\cdot] := E_t[\cdot|\delta_t = 1]$. We will refer to equation 11 as the *risk-adjusted Fisher relation*. Relative to the standard Fisher relation, there is an adjustment term that reflects the discrepancy between the expected economic conditions and today's economic conditions. Notice that the adjustment term is less than one,

$$\frac{1}{\mathrm{E}_{RSS}\big[\Big(\frac{C_{RSS}}{C_{t+1}}\Big)^{\chi_c} \frac{\Pi_{RSS}}{\Pi_{t+1}}\big]} < 1, \tag{12}$$

because of the fat tails on the lower ends of the distributions of future inflation and consumption induced by the ELB constraint.

In equilibrium, the steady state is given by the intersection of the line representing the Fisher relation and the line representing the Taylor rule. Since the risk-adjustment term is less than one, the line representing the risk-adjusted Fisher relation crosses the line representing the Taylor rule at a point below the line for the standard Fisher relation crosses it, as shown in figure 3. Thus, inflation and the policy rate are lower at the risky steady state than at the deterministic steady state.[9]

# 4 The Risky Steady State in an Empirical Model with the ELB

We now quantify the magnitude of the wedge between the deterministic and risky steady states in an empirically rich model with multiple state variables that is calibrated to match key features of the U.S. economy.

## 4.1 Model

Our empirical model adds four additional features on top of the stylized New Keynesian model of the previous section. Four additional features are a non-stationary productivity process, consumption habits, sticky wages, and an interest-rate smoothing term in the interest-

[9] The other intersection of the risk-adjusted Fisher relation and the Taylor-rule equation indicates that, in a deflationary equilibrium, inflation is higher at the risky steady state than at the deterministic steady state. We have confirmed that this is indeed the case using a semi-loglinear model with a three-state discount rate shock. We will leave an in-depth examination of the risky steady state in a deflationary equilibrium to future research.

rate feedback rule. Since these features are standard, we relegate the detailed description of the model to the Appendix and only show the equilibrium conditions of the model here. Let $\tilde{Y}_t = \frac{Y_t}{A_t}$, $\tilde{C}_t = \frac{C_t}{A_t}$, $\tilde{w}_t = \frac{w_t}{A_t}$, and $\tilde{\lambda}_t = \frac{\lambda_t}{A_t^{-\chi_c}}$ be the stationary representations of output, consumption, real wage, and marginal utility of consumption respectively. The stationary equilibrium is characterized by the following system of equations:

$$\tilde{\lambda}_t = \frac{\beta}{a^{\chi_c}} \delta_t R_t \mathrm{E_t} \tilde{\lambda}_{t+1} \left( \Pi^p_{t+1} \right)^{-1} \tag{13}$$

$$\tilde{\lambda}_t = (\tilde{C}_t - \frac{\zeta}{a} \tilde{C}_{t-1})^{-\chi_c} \tag{14}$$

$$\frac{N_t \tilde{w}_t}{\tilde{\lambda}_t^{-1}} \left[ \varphi_w \left( \frac{\Pi^w_t}{\bar{\Pi}^w} - 1 \right) \frac{\Pi^w_t}{\bar{\Pi}^w} - (1 - \theta^w) - \theta^w \frac{N_t^{\chi_n}}{\tilde{\lambda}_t \tilde{w}_t} \right] = \frac{\beta \varphi_w}{a^{\chi_c - 1}} \delta_t \mathrm{E}_t \frac{N_{t+1} \tilde{w}_{t+1}}{\lambda_{t+1}^{-1}} \left( \frac{\Pi^w_{t+1}}{\bar{\Pi}^w} - 1 \right) \frac{\Pi^w_{t+1}}{\bar{\Pi}^w} \tag{15}$$

$$\Pi^w_t = \frac{\tilde{w}_t}{\tilde{w}_{t-1}} \Pi^p_t \tag{16}$$

$$\frac{\tilde{Y}_t}{\tilde{\lambda}_t^{-1}} \left[ \varphi_p \left( \frac{\Pi^p_t}{\bar{\Pi}^p} - 1 \right) \frac{\Pi^p_t}{\bar{\Pi}^p} - (1 - \theta^p) - \theta^p \tilde{w}_t \right] = \frac{\beta \varphi_p}{a^{\chi_c - 1}} \delta_t \mathrm{E}_t \frac{\tilde{Y}_{t+1}}{\tilde{\lambda}_{t+1}^{-1}} \left( \frac{\Pi^p_{t+1}}{\bar{\Pi}^p} - 1 \right) \frac{\Pi^p_{t+1}}{\bar{\Pi}^p} \tag{17}$$

$$\tilde{Y}_t = \tilde{C}_t + \frac{\varphi_p}{2} \left[ \frac{\Pi^p_t}{\bar{\Pi}^p} - 1 \right]^2 \tilde{Y}_t + \frac{\varphi_w}{2} \left[ \frac{\Pi^w_t}{\bar{\Pi}^w} - 1 \right]^2 \tilde{w}_t N_t \tag{18}$$

$$\tilde{Y}_t = N_t \tag{19}$$

and

$$R_t = \max \left[ 1, R^*_t \right] \tag{20}$$

where

$$\frac{R^*_t}{\bar{R}} = \left( \frac{R^*_{t-1}}{\bar{R}} \right)^{\rho_R} \left( \frac{\Pi^p_t}{\bar{\Pi}^p} \right)^{(1-\rho_r)\phi_\pi} \left( \frac{\tilde{Y}_t}{\bar{Y}} \right)^{(1-\rho_r)\phi_y} \tag{21}$$

and the following processes for the discount rate:

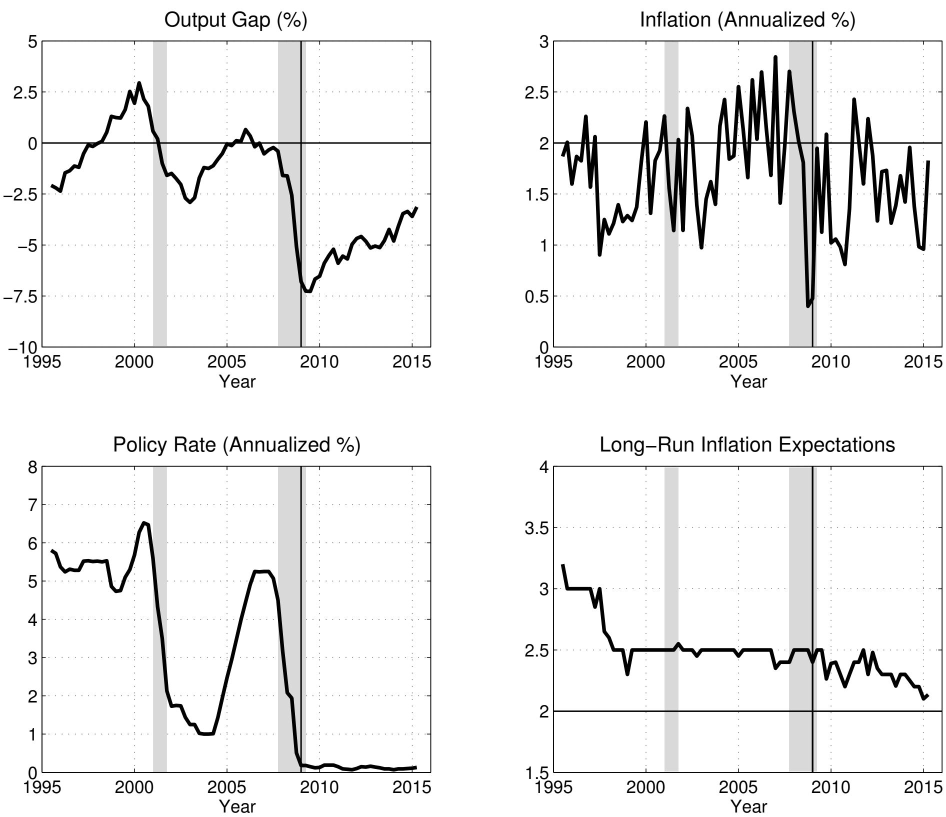
$$(\delta_t - 1) = \rho_\delta (\delta_{t-1} - 1) + \epsilon^\delta_t \tag{22}$$

$\zeta$ is a degree of consumption habits in the household's utility function and $a$ is the trend growth rate of productivity. $\varphi_p$ and $\varphi_w$ are the price and wage adjustment costs. $\rho_R$ is the weight on the lagged shadow policy rate in the truncated interest-rate feedback rule. $\bar{\Pi}^p$ and $\bar{\Pi}^w$ are price and wage inflation rates in the determistic steady state. $\bar{R}$ and $\bar{Y}$ are the policy rate and output (normalized by $A_t$) at the deterministic steady state and are functions of the structural parameters In the Appendix, we also study the risky steady state in several models which add more frictions and shocks to the empirical model of this section.

## 4.2 Calibration

We calibrate our model to match key features of the U.S. economy over the past two decades. We focus on this relatively recent past because long-run inflation expectations were low and stable and the ELB was either a concern or a binding constraint to the Federal Reserve during this period. As shown in the bottom-right panel of figure 4, the median of CPI inflation forecasts 5-10 years ahead in the Survey of Professional Forecasters, a commonly used measure of longer-run inflation expectations, declined to 2.5 percent in late 1990s and has been relatively stable since then.[10] Also, the concern for the ELB surged in the U.S. in the second half of 1990s when the Bank of Japan lowered the policy rate to the lower bound for the first time in the Post WWII history among major advanced economies.[11]

Figure 4: Output Gap, Inflation, Policy Rate and Long-Run Inflation Expectations†




†Vertical lines mark the year when ZLB started binding. Horizontal lines represent target values for the respective variables.

We set the time discount rate to 0.99875 so that the contribution of the discount rate to the long-run equilibrium real rate is 50 basis points. We set the target inflation in the

[10]The long-run inflation expectations measured by PCE inflation is available only from 2007. The average differential between CPI and PCE inflation rates over the past two decades is about 50 basis points.

[11]Some of the earliest research on the ELB were initiated within the Federal Reserve System in this period. See, for example, Clouse, Henderson, Orphanides, Small, and Tinsley (2003), Reifschneider and Williams (2000), and Wolman (1998).

interest-rate feedback rule to 2 percent as this is the FOMC's official target rate of inflation. We set the trend growth rate of productivity to 1.25 percent so that the policy rate is 3.75 percent at the economy's deterministic steady state, slightyl higher than the median estimate of the long-run level of the federal funds rate among FOMC participatns, according to the most recent Summary of Economic Projections.

Table 3: **Parameter Values for the Empirical Model**

| Parameter | Description | Parameter Value |
|---|---|---|
| $\beta$ | Discount rate | 0.99875 |
| $a$ | Trend growth rate of productivity | $\frac{1.25}{400}$ |
| $\chi_c$ | Inverse intertemporal elasticity of substitution for $C_t$ | 1.0 |
| $\zeta$ | Degree of consumption habits | 0.5 |
| $\chi_n$ | Inverse labor supply elasticity | 0.5 |
| $\theta_p$ | Elasticity of substitution among intermediate goods | 11 |
| $\theta_w$ | Elasticity of substitution among intermediate labor | 4 |
| $\varphi_p$ | Price adjustment cost | 400 |
| $\varphi_w$ | Wage adjustment cost | 300 |
| Interest-rate feedback rule | | |
| $400(\bar{\Pi}-1)$ | (Annualized) target rate of inflation | 2.0 |
| $\rho_R$ | Interest-rate smoothing parameter in the Taylor rule | 0.85 |
| $\phi_\pi$ | Coefficient on inflation in the Taylor rule | 2.5 |
| $\phi_y$ | Coefficient on the output gap in the Taylor rule | 0.25 |
| $400(R_{ELB}-1)$ | (Annualized) effective lower bound | 0.13 |
| Shocks | | |
| $\rho_d$ | AR(1) coefficient for the discount factor shock | 0.85 |
| $\sigma_{\epsilon,\delta}$ | The standard deviation of shocks to the discount factor | $\frac{0.67}{100}$ |
| | *The implied prob. that the policy rate is at the lower bound | 10.0% |

In the household utility function, the degree of consumption habits, the inverse Frisch labor elasticity, and the inverse intertemporal elasticity of substitution are set to 0.5, 0.5 and 1, respectively. These are all within the range of standard values found the literature.

Following Erceg and Lindé (2014), parameters governing the steady-state markups for intermediate goods and the intermediate labor inputs are set to 11 and 4. We set the parameters governing the price adjustment costs for prices and wages to 400 and 300. In a hypothetical log-linear environment, these values would correspond to 85 percent probabilities that prices and wages cannot adjust each quarter in the Calvo version of the model. While these values are relatively high, they are broadly in line with the values ued in Erceg and Lindé (2014). High degrees of stickiness in prices and wages help the model to match the moderate decline in inflation while the federal funds rate was constrained at the ELB.

The coefficients on inflation and the output gap in the interest-rate feedback rule are set to 2.5 and 0.25. The coefficient on the output gap, 0.25, is standard. The coefficient on inflation is a bit higher compared to the values commonly used in the literature. A higher coefficient serves two purposes. First, it reduces the volatility of inflation, relative to the volatility of the output gap. Second, a higher value makes the existence of the equilibrium more likely.[12] Erceg and Lindé (2014) argue that a high inflation coefficient of this magnitude is consistent with a IV-type regression estimate of this coefficient based on a recent sample.

The interest-rate smoothing parameter for the policy rule is set to 0.85. This value is on the higher end of the standard values used in the literature. A higher degree of interest-rate smoothing serves two purposes. First, it increases the expected duration of the lower bound episodes, improving the model's implication in this dimension. Second, the presence of the lagged shadow policy rate makes monetary policy accommodative in the sense that the policy rate is kept at the lower bound for long, which tempers the declines in inflation and the output gap while the lower-bound is an constraint. Finally, the lower bound on the policy rate is set to 0.13 percent, the average of the annualized federal funds rate during the recent lower bound episode (from 2009:Q1 to 2015:Q2).

The persistence of the discount rate shock is set to 0.85. This is a bit higher than the common value of 0.8 used in the existing studies of the models with occasionally binding lower bound constraints.[13] However, a higher persistence increases the expected duration of being at the lower-bound, as the higher interes-rate smoothing parameter in the policy rule does. The standard deviation of the discount rate shock is chosen so that the unconditional probability of being at the interest-rate lower bound is 10 percent.

Table 4 shows the key statistics of the output gap, inflation and the policy rate in the model and in the data. The measure of the output gap is based on the CBO estimate of potential output. As for the measure of inflation, we use core PCE Price Index inflation as this is the measure U.S. policymakers focus on.

Table 4: **Key Moments**

| Moment | Variable | Model | Data† (1995Q3–2015Q2) |
|---|---|---|---|
| St.Dev.(·) | Output gap | 2.9 | 2.6 |
| | Inflation | 0.35 | 0.52 |
| | Policy rate | 2.15 | 2.34 |
| E(X\|ELB) | Output gap | −3.7 | −5.2 |
| | Inflation | 1.19 | 1.48 |
| | Policy rate | 0.13 | 0.13 |
| ELB | Frequency | 10.0% | 36% |
| | Expected/Actual Duration | 8.4 quarters | 26 quarters |

†Inflation rate is computed as the annualized quarterly percentage change (log difference) in the personal consumption expenditure core price index. The annualized federal funds rate is used as the measure for the nominal interest rate.

[12] Richter and Throckmorton (2015) have shown that the model with occasionally binding ELB constraints may not have minimum-state-variable solutions when this coefficient is low even if the Taylor-principle is satisfied.

[13] See, for example, Adam and Billi (2007), Nakov (2008), and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015).

The standard deviation of the output gap in the model is 2.9, which is in line with the sample standard deviation from the data. The ELB conditional mean of the output gap in the model is [-3.7] percent, which is above the estimate from the data. The standard deviation of inflation in the model is [0.35] percent, which is lower than what's observed in the data, while the ELB conditional mean of inflation in the model is [1.19] percent, which is somewhat lower than what's observed in the data. It should be noted that there is a tension in matching the standard deviation and the conditional average at the ELB for the output gap and inflation. Since the ELB conditional means of the output gap and inflation are averages of the left-tail of the unconditional distributions of teh output gap and inflation, an increase in the standard deviation of these variables would necessarily imply decreases in the ZLB conditional means. In our case, if we increase the standard deviation of inflation in our model to get closer to what's observed in the data, the ELB conditional mean of the output gap would increase further away from its empirical counterpart and the ELB conditional mean of inflation would fall further below its empirical counterpart. Our choice reflects a compromise of staying close to the data in these two features.

As previously explained, the standard deviation of the discount rate shock was chosen so that the probability of being at the ELB is 10 percent. With this ELB frequency, the standard deviation of the federal funds rate in the model is about [2.15] percent, slightly lower than in the data. The model-implied expected duration of the ELB episode is about 2 years. While this is substantially higher than what is found in other existing models with occasionally binding ELB constraints, it is substantially lower than the empirical counterpart in the most recent ELB episode in the U.S. Thus, in our model, the recent lower-bound episode can be interpreted as surprisingly long. Consistent with this interpretation, the data on liftoff expectations shows that market participants have underpredicted how long the policy rate will be kept at the ELB throughout the recent ELB episode, as shown in the Appendix A.

### 4.3 Results

Table 5: **The Risky Steady State in the Empirical Model**

| | Inflation | Output gap | Policy rate |
|---|---|---|---|
| Deterministic steady state | 2 | 0 | 3.75 |
| Risky steady state | 1.77 | 0.17 | 3.33 |
| (Wedge) | (−0.23) | (0.17) | (−0.42) |
| Risky steady state w/o the ELB | 1.91 | 0.03 | 3.54 |
| (Wedge) | (−0.09) | (0.03) | (−0.21) |
| $E[\cdot \mid R_t > R_{ELB}]$ | 1.80 | 0.57 | 3.74 |

Table 5 shows the risky and deterministic steady-state values of inflation, the output gap, and the policy rate of our empirical model. For this model, the risky steady state is computed by simulating the model for a long period while setting the realization of the exogenous disturbances to zero. All (stationarized) endogenous variables eventually converge in that simulation, and that point of convergence is the risky state of the economy. By construction, the deterministic steady state of inflation is given by the target rate of inflation and the output gap is zero at the deterministic steady state. As explained earlier, parameter values ($\beta$, $\chi_c$ and $a$) are chosen so that the deterministic steady state of the policy rate is 3.75 percent.

Consistent with our earlier analyses based on a stylized model, inflation and the policy rate are lower, and the output gap is higher, at the risky steady state than at the deterministic steady state. Inflation falls [19] basis points below the target rate of inflation at the risky steady state. This is large given the small standard deviation of inflation. The policy rate at the risky steady state falls [39] basis points below its deterministic counterpart. While this is a small number relative to its standard deviation, it is nevertheless significant in light of recent discussions among economists and policymakers regarding the long-run equilibrium policy rate.[14] Finally, the output wedge between the deterministic and risky steady states is small, with the output gap standing at [0.18] percentage point at the risky steady state.

As explained in the previous section, the wedge between the deterministic and risky steady states is not only driven by the lower bound constraint on policy rates, but is also affected by other nonlinear features of the model. To decompose these two effects, the fourth line of Table 5 shows the risky steady state of the model without the lower bound constraint. While the output gap is [0.03] percent, inflation and the policy rate are [1.92] and [3.55] percent, respectively. Thus, the most of the wedge in the model with the lower bound is attributed to the nonlinearity associated with the lower bound constraint, as opposed to other nonlinear features of the model.

# 5 Long-Run Interest Rates

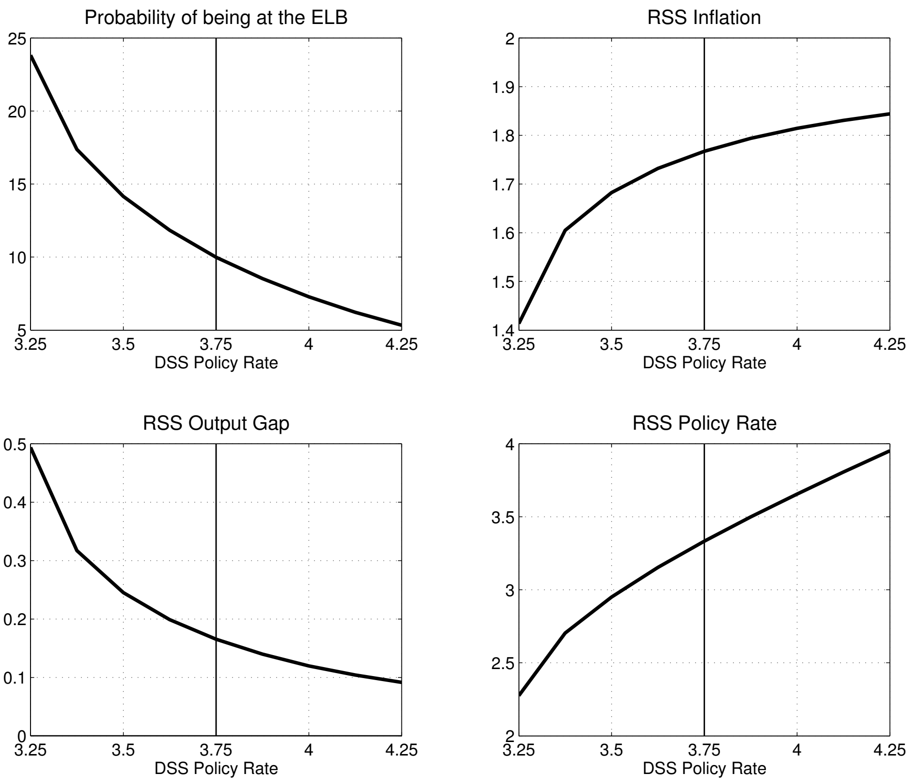
There are substantial uncertainties surrounding the level of the long-run real equilibrium interest rate. Many economists have recently argued that various structural factors—including a lower trend growth rate of productivity, demographic trends, and global factors—make the long-run equilibrium interest rate lower in the future than the average level prevailed before the Great Recession.[15] A lower long-run equilibrium interest rates means that the probability of being at the ELB is higher, which should increase the magnitude of undershooting of the inflation target at the risky steady state.

Figure 5 shows how sensitive the risky steady state is to alternative assumptions about the long-run equilibrium interest rate. In this exercise, we vary the long-run deterministic steady-state policy rate by varying the trend growth rate. As shown in the top-left panel, the probability of the policy rate being at the ELB increases as the DSS policy rate declines. With the DSS policy rate at [3] percent, the probability of being at the ELB is [20] percent. A higher probability of being at the lower bound increases the wedge between the deterministic

[14] See, for example, Hamilton, Harris, Hatzius, and West (2015).

[15] See, for example, the Council of Economic Advisers (2015) and IMF (2014).

Figure 5: Long-Run Interest Rates and the Risky Steady State†




†DSS stands for "deterministic steady state." Vertical lines mark the DSS policy rate in the baseline calibration.

and risky steady states. With the DSS policy rate at [3] percent, the RSS inflation, output, and the policy rates are about [1.6] percent, [0.3] percent, and [2.5] percent.[16]

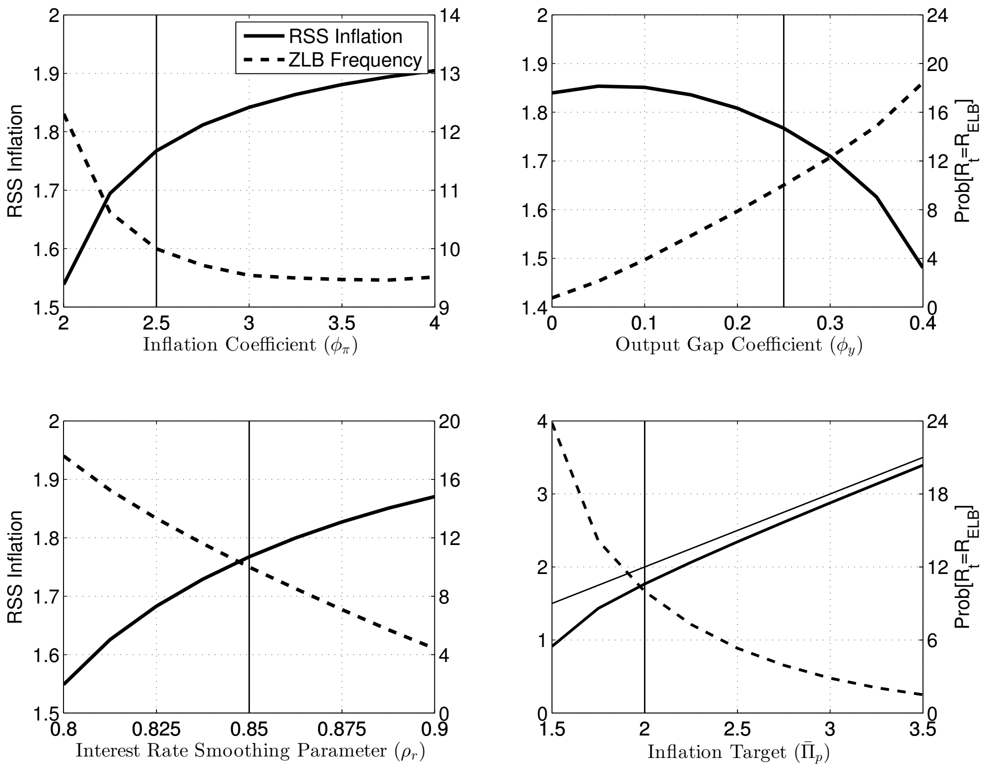
# 6 Policy Implications

We have shown that the deflationary bias is nontrivial at the risky steady state in our empirical model. In our model where the prices and wages are indexed to the target rate of inflation, such undershooting of the inflation target is undesirable. A natural question is what the central bank can do to mitigate the deflationary bias.

Figure 6 show how the probability of being at the ELB and the risky steady-state inflation depends on parameters governing the interest-rate feedback rule. According to the top-left panel, a higher coefficient on inflation reduces the probability of being at the ELB, increasng the risky steady state closer to the inflation target. On the other hand, a higher coefficient on the output gap increases the probability of being at the ELB and reduces the RSS inflation. These results are consistent with the analytical results from a two-state shock model in Nakata and Schmidt (2014).

[16]Note that an increase in the output gap does not necessarily mean an increase in the level of output because output measures are stationarized by the trend growth rate.

Figure 6: Alternative Policy Parameters†




†In each panel, the vertical line marks the baseline parameter value.

The bottom-left panel shows that a higher interes-rate smoothing parameter reduces the probability of being at the ELB, thus reducing the deflationary bias at the risky steady state. This makes sense as a higher inertia in the policy rule limits the response of the policy rate to fluctuations in the demand shock, lwoering the standard deviation of the policy rate. Finally, and perhaps not surprisingly, a higher inflation target reduces the probability of being at the ELB and reduces the wedge between the DSS and RSS inflation rates, as shown in the bottom-right panel. With the inflation target at 1.5 percent, the deflationary bias at the risky steady state is more than 50 basis points. With the inflation target at 2.5 percent, the deflationary bias at the risky steady state is about 10 basis points. This final exercise demonstrates the importance of taking into account the lower bound risk in the cost-benefit analysis of raising the inflation target.[17]

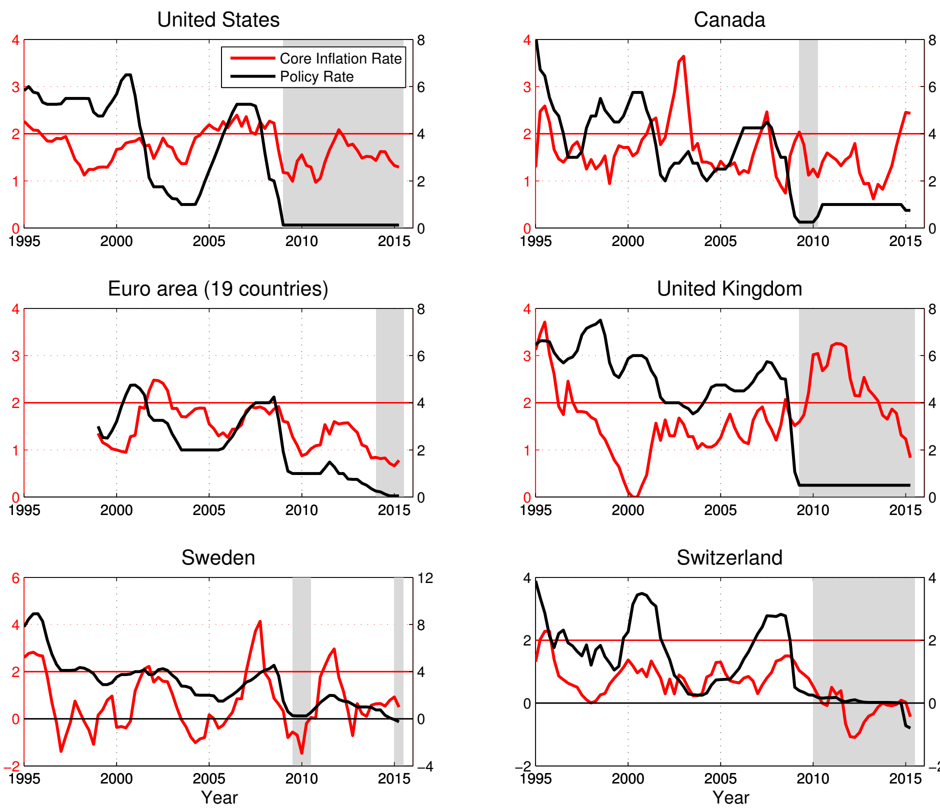
# 7 Discussion

In our empirical model, the deflationary bias is sufficiently large so that the conditional average inflation away from the ELB is also below the long-run inflation target, as shown in

[17]The computation of the optimal inflation target is often conducted under the assumption of perfect-foresight. See, for example, Williams (2009) and Coibion, Gorodnichenko, and Wieland (2012).

the table 5. In this section, we show that this feature of our empirical model is consistent with the fact that some central banks on average undershot their inflation target even before the policy rate became constrained by the ELB.

Figure 7: Policy Rates and Inflation in Six Economies[†]




[†]Shaded regions mark the ELB era. Horizontal lines represent inflation target rate. For Sweden, horizontal line at zero policy rate is added to help readers better observe its policy rate evolution.

Figure 7 shows the evolution of inflation and policy rate past two decades in six select economies (U.S., Canada, Euro Area, U.K., Sweden, and Switzerland). We exclude Japan because the Japanese economy is better described by the deflationary equilibrium that fluctuates around deflationary steady state.[18] In all of the six economies, the policy rate became constrained at the ELB for the first time in the post WWII era, the long-run inflation expectations are low and stable, and the central bank has an explicit inflation target of 2 percent.

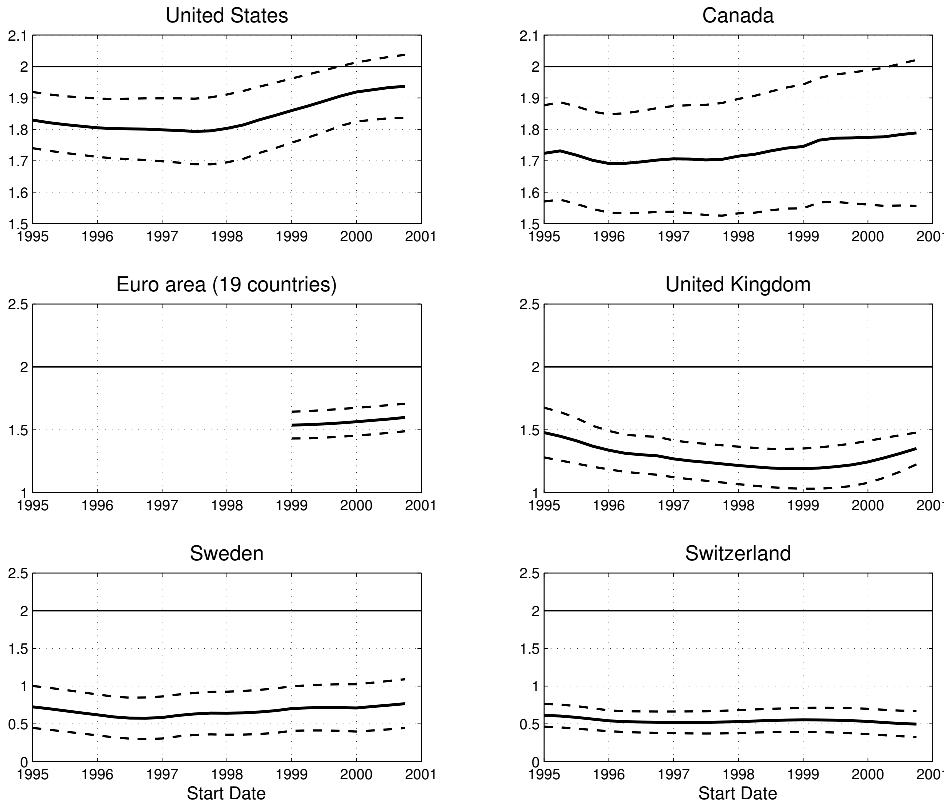
Figure 8 shows the average inflation in these six economies over the period when the policy rate was above the ELB. Since the sample period is short, the figure shows how the conditional average depends on the starting date of the sample as well as the confidence band for plus/minus two standard errors.

According to the figure, inflation averaged below the inflation target while the policy rate was away from ELB the all six countries.[19] In the Euro Area, the United Kingdom, Sweden,

[18]Aruoba, Cuba-Borda, and Schorfheide (2014) provides an empirical support for this view.

[19]The United Kingdom is an interesting case because inflation rate were on average above the inflation

Figure 8: Average Inflation Rates Away From the Lower Bound
with Alternative Starting Dates†




†Horizontal lines mark the inflation target rate. Dashed lines represent the 95% confidence interval (2 standard errors) of the actual average inflation.

and Switzerland, the conditional average inflation whent the policy rate is away from the ELB are substantially below the target rate of 2 percent, regardless of the starting date of the sample. In the U.S. and Canada, inflation averaged about 20 and 30 basis point below the 2 percent target while the ELB was not binding. The conditional averages in these two countries are closer to 2 percent and the confidence band inclues 2 percent when the starting date of the sample is around 2000. Thus, we cannot completely rule out the possibility that the observed undershooting of the inflation target is statistically insignificant. However, the figures for the U.S. and Canada clearly demonstrate the overall tendency for the central banks to undershoot the long-run target by non-trivial amount in these countries.

We concude our discussion by noting that this feature of the data—systematic undershooting of the target inflation even before the policy rate is above the ELB—cannot be explained by the deflationary steady state of the sticky-price economy (Benhabib, Schmitt-Grohe, and Uribe (2001) and Armenter (2014)). In the deflationary steady state, inflation is below the target, but the policy rate is at the ELB. To our knowledge, we are the first to point out that models with an occasionally binding ELB constraint is consistent with this

target while the policy rate is at the ELB.

feature of the data.

# 8 Conclusion

In this paper, we have examined the implications of the effective lower bound constraint on interest rates for the steady state of the economy. Using an empirically rich model calibrated to capture key features of the U.S. economy over the past two decades, we show that inflation falls below the target rate of inflation by about 25 basis points at the risky steady state. The inflation wedge between the deterministic and risky steady states can exceed 50 basis points under an alternative plausible assumptions about the long-run growth rate of the economy and monetary policy parameters. Our analysis provides a potential explanation for why some central banks undershot the inflation target in some countries even before the policy rates became constrained by the ELB constraint.

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# For Online Publication: Technical Appendix

## A Expected Time Until the Liftoff

In this section, I present the survey-based measures of the expected time until the liftoff to support the claim that the market participants consistently underestimated the duration of the lower bound episode since the federal funds rate hit the lower bound in late 2008. The surveys I examine are (i) the Blue Chip Surveys, (ii) the Survey of Professional Forecasters, and (iii) the Primary Dealers Survey.

The evidence from all three surveys is consistent with the claim that the market participants have consistently underestimated the duration of the lower bound episode. In particular, for the first two years of the lower bound episode, the market participants expected that the federal funds rate to stay at the ELB only for additional few quarters.[20]

### A.1 Blue Chip Surveys

The Blue Chip Surveys consists of two monthly surveys, the Blue Chip Economic Indicators Survey and the Blue Chip Financial Forecasts Survey. These two surveys ask their participants (about 50 financial institutions for each survey) their forecast paths of various macroeconomic variables, including the 3-month Treasury Bill rate in the Economic Indicators Survey and the federal funds rate in the Financial Forecasts Survey. The near-term forecast horizon is up until the end of next calendar year and the frequency of the projection is quarterly. Thus, the forecast path of the Treasury rate or the federal funds rate can tell us the expected time until the liftoff when the participants expect the first liftoff to occur within two years.

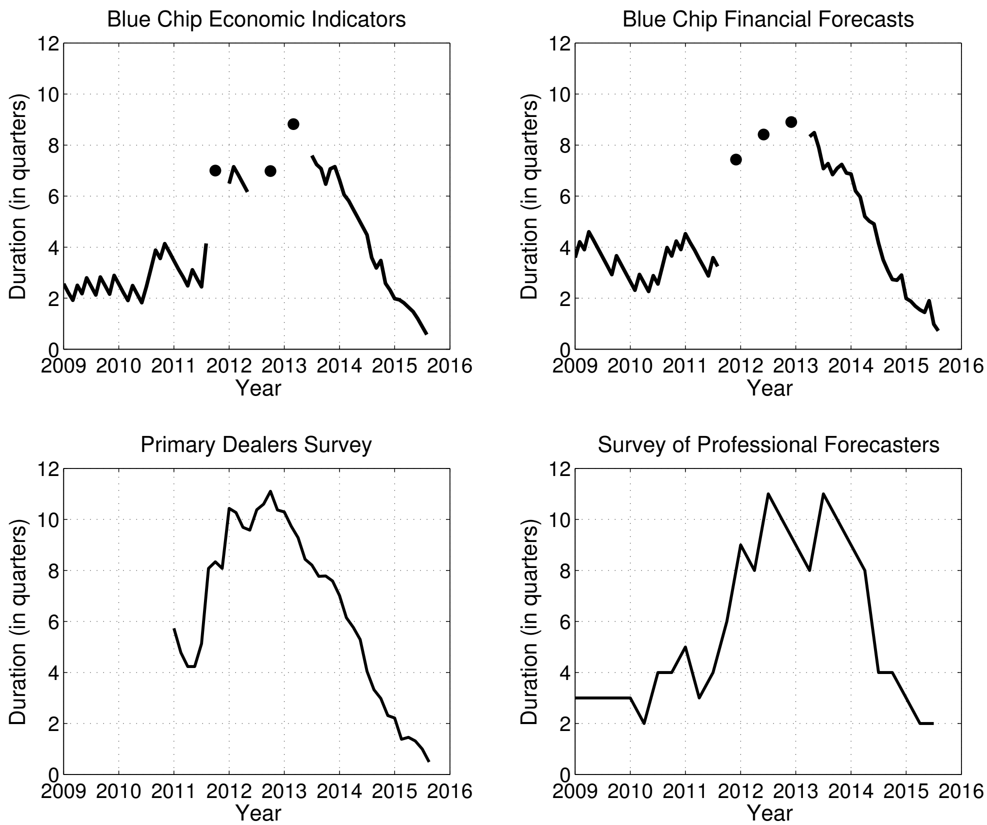
Twice a year, the surveys ask longer-run projections of certain variables in the special question section (March and October for the Economic Indicators and June and December for the Financial Forecasts). The longer-run forecasts are in annual frequency for next 5 to 6 years. Towards the end of the lower bound episode, the Surveys also asked the participants to provide the expected liftoff date in the special questions section.

For each survey, I combine these various pieces of information in the following way to construct a series for the expected period until the liftoff. First, I use the average probability distribution over the timing of the liftoff to compute the expected time until the liftoff whenever that information is available. Second, if the probability distribution is not available, then I use the information from the near-term forecasts. The time of liftoff is defined to be the first quarter when the median federal funds rate forecast exceeds 37.5 basis points. Finally, when the policy rate is projected to stay at the ELB until the end of the near-term

[20] While not shown, the expected duration of the lower bound episode based on the expected policy path implied by the federal funds rate futures is also consistent with this claim.

forecast horizon, I use the information from the long-run projections if the Survey has that information and leave the series blank when the Long-Range section is not available.

Figure 9: Expected Time Until Liftoff




Top two panels in figure 9 show the evolutions of the expected period until the liftoff based on the Blue Chip Economic Indicators Survey and the Blue Chip Financial Forecasts Survey. According to both panels, the market participants expected the lower bound episode to be transitory in the early stage of the lower bound episode. The market's expectation shifted in the second half of 2011, with the expectated duration of staying at the ELB exceeding 2 years. Since late 2012 or early 2013, the market participants started to gradually reduce its expectation for the additional duration of the lower bound episode.

## A.2 Primary Dealers Survey

The Primary Dealers Survey (the PD Survey in the remainder of the text), conducted by the Federal Reserve Bank of New York, asks primary dealers about their policy expectations eight times a year. The survey asks its participants their probability distribution over the liftoff timing (quarter or FOMC meeting). I compute the expected time until the liftoff using the average probability distribution over the liftoff timing. The results of the PD Survey are publicly available since January 2011.

The bottom-left panel of figure 9 shows the evolution of the expected period until the liftoff based on the PD Survey. Consistent with the measures based on the Blue Chip, the expected duration of the additional period of the lower bound episode increase markedly in the second half of 2011. The expected duration hovers around 10 quarters during 2012, and has declined steadily since then.

### A.3 Survey of Professional Forecasters

The Survey of Professional Forecasters (the SPF in the remainder of the text) is a quarterly survey of about 40 individuals in academia, financial industries, and policy institutions, administered by the Federal Reserve Bank of Philadelphia. Like the Blue Chip Surveys, the SPF asks its participants their projections of various macroeconomic variables, including 3-month Treasury rate. For the near-term projection that extends to the end of the next calendar year, the forecasts are available in quarterly frequency. For the longer horizon, the forecast is available in annual frequency.

The bottom-right panel of figure 9 shows the evolution of the expected period until the liftoff based on the SPF. Consistent with the Blue Chip Surveys and the Primary Dealers Survey, the SPF shows that the market anticipated the lower bound episode to last for only about one additional year until the second half of 2011. The expected duration averages about 9 quarters in 2012 and 2013. The expected duration started declining in the second half of 2013 and has come down to 2 quarters in February 2015.

## B Sensitivity Analyses

### B.1 Price Indexation

In this subsection, we examine the implications of price indexation for the risky steady state. Figures 10 and 11 show how the key features of the model vary with the degree of price indexation. Note that, as we vary the degree of price indexation, we adjust the standard deviation of the discount rate shock so that the probability of being at the ELB stays unchanged at 10 percent.

As shown in Figure 10, the volatility of inflation increases, and the volatility of the output gap decreases, with the degree of price indexation. The volatility of the policy rate as well as the expected duration of the ZLB episode are not affected by the degree of price indexation. According to Figure 11, inflation, output, and the policy rate at the risky steady state increases, decreases, and increases with the degree of price indexation, respectively. However, the magnitudes are small. Thus, price-setters do not need to be fully forward-looking for the ZLB risk to reduce the risky steady state inflation by non-trivial amount.

Figure 10: Moments with
Alternative Degrees of Price Indexation

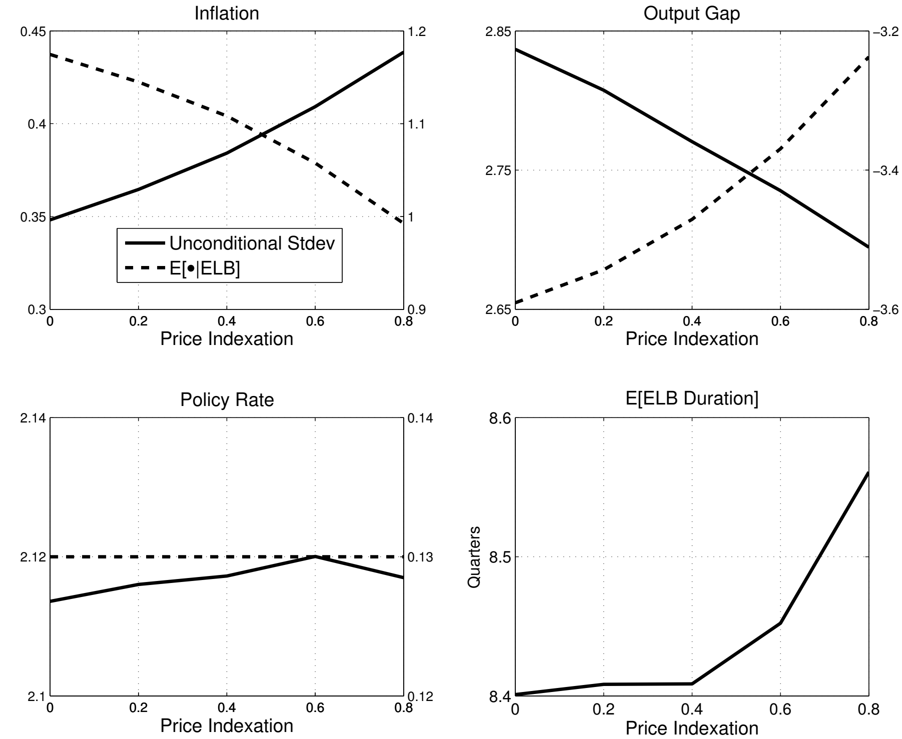
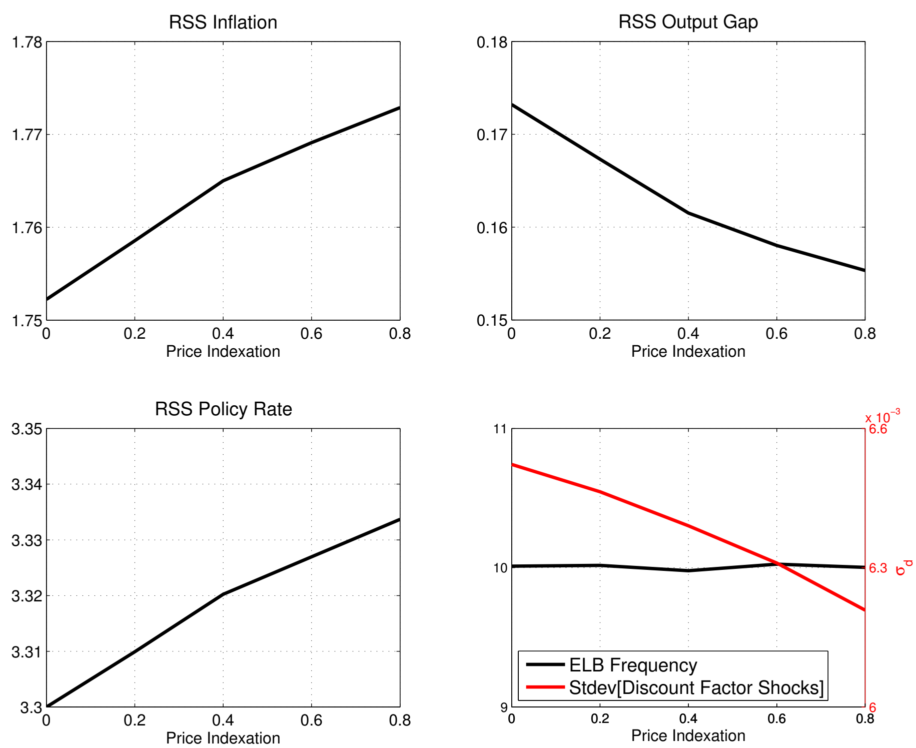



Figure 11: Risky Steady State with
Alternative Degrees of Price Indexation



## B.2 Wage Indexation

In this subsection, we examine the implications of wage indexation for the risky steady state. Figures 12 and 13 show how the key features of the model vary with the degree of wage indexation. Note that, as we vary the degree of wage indexation, we adjust the standard deviation of the discount rate shock so that the probability of being at the ELB stays unchanged at 10 percent.

As shown in Figure 12, the volatilities of inflation, the output gap, and the policy rate as well as the expected duration of the ELB episode decrease with the degree of wage indexation. According to Figure 11, inflation, output, and the policy rate at the risky steady state decreases, increases, and decreases with the degree of wage indexation, respectively. The effects of wage indexation on the risky steady state is qunatitatively large. For example, inflation is about 75 basis points and the policy rate is about 2.25 percent at the risky steady state when the degree of wage indexation is 0.8. Thus, even when wage-setters are not fully forward-looking, the wedge between the risky and deterministic steady states can be large.

Figure 12: Moments with
Alternative Degrees of Wage Indexation

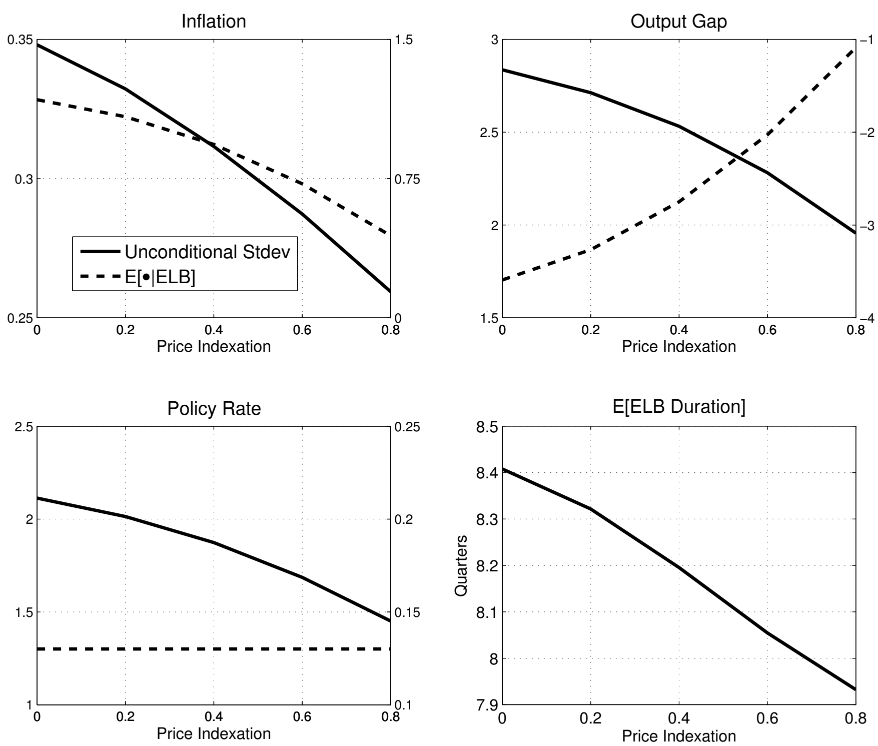
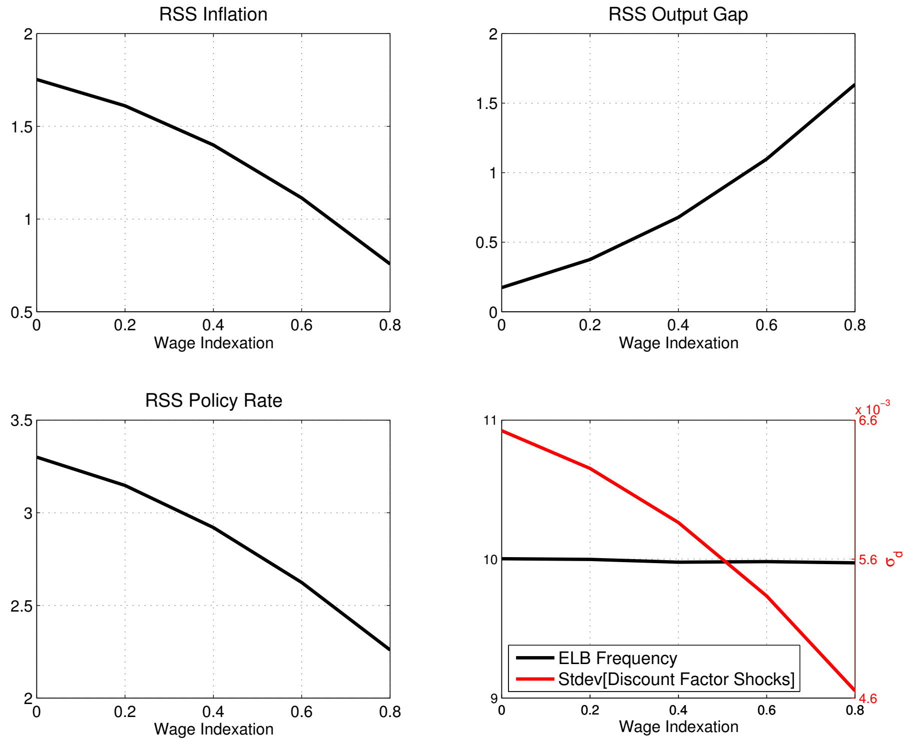


Figure 13: Risky Steady State with
Alternative Degrees of Wage Indexation




## B.3 Model with Monetary Policy Shocks

In this subsection, I introduce monetary policy shocks into the baseline empirical model. The standard deviation of the monetary policy shock is set to the standard deviation of the residuals in the interest-rate feedback rule computed using the U.S. data before the federal funds rate hit the ELB ($\sigma_r = \frac{0.11}{100}$). The standard deviation of the discount rate shock is adjusted so that the probability of being at the ZLB remain 10 percent.

As shown in Table 6, The moments and risky steady states of this model are similar to those of the baseline empirical model. This reflects the fact that the magnitude of the monetary policy shock is very small, accounting for only less than 5 percent of the total variation in the policy rate.

Table 6: Model with Monetary Policy Shocks

(a) Moments

| Moment | Variable | Model (original) | Model (w/ MP shocks) | Data (1995Q3–2015Q2) |
|---|---|---|---|---|
| St.Dev.(·) | Output gap | 2.9 | 2.8 | 2.6 |
| | Inflation | 0.35 | 0.34 | 0.52 |
| | Policy rate | 2.15 | 2.15 | 2.34 |
| E(X\|ELB) | Output gap | −3.7 | −3.3 | −5.2 |
| | Inflation | 1.19 | 1.25 | 1.48 |
| | Policy rate | 0.13 | 0.13 | 0.13 |
| ELB | Frequency | 10.0% | 10.0% | 36% |
| | Expected/Actual Duration | 8.4 quarters | 6.3 quarters | 26 quarters |
| • | $100\sigma_{\epsilon,\delta}$ | 0.67 | 0.65 | • |

(b) Risky Steady State†

| | Inflation | Output Gap | Policy Rate |
|---|---|---|---|
| DSS | 2 | 0 | 3.75 |
| RSS (w/ $\sigma_r = \frac{0.11}{100}$ ) | 1.77 | 0.15 | 3.33 |
| (Wedge) | (−0.23) | (0.15) | (−0.42) |
| RSS (w/o $\sigma_r$ ) | 1.77 | 0.17 | 3.33 |
| (Wedge) | (−0.23) | (0.17) | (−0.42) |

†DSS stands for 'Deterministic Steady State' while RSS stands for 'Risky Steady State.'

## B.4 Model with TFP Shocks

In this subsection, I introduce TFP shocks into the baseline empirical model in such a way that the standard deviations of output, inflation, and the policy rate are roughly unchanged from the baseline empirical model. To do so, I adjust a few parameters ($\varphi_p = 400$ and $\phi_\pi = 3$) and reduce the size of the standard deviation of the TFP shock so that the volatility of output explained by the TFP shock accounts for 10 percent, as opposed to 25 percent, of the total output volatility. This model with this calibration are used in the non-technical summary of this paper.

As shown in Table 7, the moments and risky steady states of this model is broadly similar to those of the baseline empirical model.

Table 7: Model with TFP Shocks

(a) Moments

| Moment | Variable | Model (original) | Model (w/ TFP shocks) | Data (1995Q3–2015Q2) |
|---|---|---|---|---|
| St.Dev.(·) | Output gap | 2.9 | 2.7 | 2.6 |
| | Inflation | 0.35 | 0.34 | 0.52 |
| | Policy rate | 2.15 | 2.06 | 2.34 |
| E(X\|ELB) | Output gap | −3.7 | −3.5 | −5.2 |
| | Inflation | 1.19 | 1.32 | 1.48 |
| | Policy rate | 0.13 | 0.13 | 0.13 |
| ELB | Frequency | 10.0% | 10.0% | 36% |
| | Expected/Actual Duration | 8.4 quarters | 7.8 quarters | 26 quarters |
| • | $100\sigma_{\epsilon,\delta}$ | 0.67 | 0.59 | • |

(b) Risky Steady State†

| | Inflation | Output Gap | Policy Rate |
|---|---|---|---|
| DSS | 2 | 0 | 3.75 |
| RSS (new) | 1.73 | 0.22 | 3.17 |
| (Wedge) | (−0.27) | (0.22) | (−0.58) |
| RSS (original) | 1.77 | 0.17 | 3.33 |
| (Wedge) | (−0.23) | (0.17) | (−0.42) |

†DSS stands for 'Deterministic Steady State' while RSS stands for 'Risky Steady State.'

# C Details of the Stylized Model

This section describes a stylized DSGE model with a representative household, a final good producer, a continuum of intermediate goods producers with unit measure, and government policies.

## C.1 Household

The representative household chooses its consumption level, amount of labor, and bond holdings so as to maximize the expected discounted sum of utility in future periods. As is common in the literature, the household enjoys consumption and dislikes labor. Assuming that period utility is separable, the household problem can be defined by

$$\max_{C_t,N_t,B_t} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[ \prod_{s=0}^{t-1} \delta_s \Big] \Big[ \frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1+\chi_n}}{1+\chi_n} \Big] \tag{23}$$

subject to the budget constraint

$$P_tC_t + R_t^{-1}B_t \leq W_tN_t + B_{t-1} + P_t\Phi_t \tag{24}$$

or equivalently

$$C_t + \frac{B_t}{R_tP_t} \leq w_tN_t + \frac{B_{t-1}}{P_t} + \Phi_t \tag{25}$$

where $C_t$ is consumption, $N_t$ is the labor supply, $P_t$ is the price of the consumption good, $W_t$ ($w_t$) is the nominal (real) wage, $\Phi_t$ is the profit share (dividends) of the household from the interdemidiate goods producers, $B_t$ is a one-period risk free bond that pays one unit of money at period t+1, and $R_t^{-1}$ is the price of the bond.

The discount rate at time t is given by $\beta\delta_t$ where $\delta_t$ is the discount factor shock altering the weight of future utility at time t+1 relative to the period utility at time t. This shock follows an AR(1) process:

$$\delta_t - 1 = \rho(\delta_{t-1} - 1) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon) \tag{26}$$

This increase in $\delta_t$ is a preference imposed by the household to increase the relative valuation of future utility flows, resulting in decreased consumption today (when considered in the absence of changes in the nominal interest rate).

## C.2 Firms

There is a final good producer and a continuum of intermediate goods producers indexed by $i \in [0, 1]$. The final good producer purchases the intermediate goods $Y_{i,t}$ at the intermediate price $P_{i,t}$ and aggregates them using CES technology to produce and sell the final good $Y_t$ to the household and government at price $P_t$. Its problem is then summarized as

$$\max_{Y_{i,t}, i\in[0,1]} P_tY_t - \int_0^1 P_{i,t}Y_{i,t}di \tag{27}$$

subject to the CES production function

$$Y_t = \Big[\int_0^1 Y_{i,t}^{\frac{\theta-1}{\theta}} di\Big]^{\frac{\theta}{\theta-1}}. \tag{28}$$

Intermediate goods producers use labor to produce the imperfectly substitutable intermediate goods according to a linear production function ($Y_{i,t} = N_{i,t}$) and then sell the product to the final good producer. Each firm maximizes its expected discounted sum of future profits[21]

[21] Each period, as it is written below, is in *nominal* terms. However, we want each period's profits in *real* terms so the profits in each period must be divided by that period's price level $P_t$ which we take care of further along in the document.

by setting the price of its own good. We can assume that each firm receives a production subsidy $\tau$ so that the economy is fully efficient in the steady state.[22] In our baseline, however, we set $\tau = 0$. Price changes are subject to quadratic adjustment costs.

$$\max_{P_{i,t}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[\prod_{s=0}^{t-1} \delta_s\Big] \lambda_t \Big[P_{i,t}Y_{i,t} - (1-\tau)W_t N_{i,t} - P_t \frac{\varphi}{2}\big[\frac{P_{i,t}}{P_{i,t-1}\bar{\Pi}} - 1\big]^2 Y_t\Big] \tag{29}$$

such that

$$Y_{i,t} = \left[\frac{P_{i,t}}{P_t}\right]^{-\theta} Y_t.^{23} \tag{30}$$

$\lambda_t$ is the Lagrange multiplier on the household's budget constraint at time t and $\beta^{t-1}\Big[\prod_{s=0}^{t-1}\delta_s\Big]\lambda_t$ is the marginal value of an additional profit to the household. The positive time zero price is the same across firms (i.e. $P_{i,0} = P_0 > 0$).

## C.3 Government policies

It is assumed that the monetary authority determines nominal interest rates according to a Taylor rule

$$R_t = \max\left[1, \quad \frac{\bar{\Pi}}{\beta}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}\right] \tag{31}$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ and $\bar{Y}$ is the steady state level of output. This equation will be modified in order to do an extensive sensitivity analysis of policy inertia and other rule specifications.

## C.4 Market clearing conditions

The market clearing conditions for the final good, labor and government bond are given by

$$Y_t = C_t + \int_0^1 \frac{\varphi}{2}\Big[\frac{P_{i,t}}{P_{i,t-1}\bar{\Pi}} - 1\Big]^2 Y_t di \tag{32}$$

$$N_t = \int_0^1 N_{i,t} di \tag{33}$$

and

$$B_t = 0. \tag{34}$$

[22] $(\theta - 1) = (1-\tau)\theta$ which implies zero profits in the zero inflation steady state. In a welfare analysis, this would extract any inflation bias from the second-order approximated objective welfare function. $\tau$ therefore represents the size of a steady state distortion (see Chapter 5 Appendix, Galí (2008)).

[23] This expression is derived from the profit maximizing input demand schedule when solving for the final good producer's problem above. Plugging this expression back into the CES production function implies that the final good producer will set the price of the final good $P_t = \Big[\int_0^1 P_{i,t}^{1-\theta} di\Big]^{\frac{1}{1-\theta}}$.

### C.5 Recursive equilibrium

Given $P_0$ and a two-state Markov shock process establishing $\delta_t$ and $\gamma_t$, an equilibrium consists of allocations $\{C_t, N_t, N_{i,t}, Y_t, Y_{i,t}, G_t\}_{t=1}^{\infty}$, prices $\{W_t, P_t, P_{i,t}\}_{t=1}^{\infty}$, and a policy instrument $\{R_t\}_{t=1}^{\infty}$ such that (i) given the determined prices and policies, allocations solve the problem of the household, (ii) $P_{i,t}$ solves the problem of firm $i$, (iii) $R_t$ follows a specified rule, and (iv) all markets clear.

Combining all of the results from (i)-(v), a symmetric equilibrium can be characterized recursively by $\{C_t, N_t, Y_t, w_t, \Pi_t, R_t\}_{t=1}^{\infty}$ satisfying the following equilibrium conditions:

$$C_t^{-\chi_c} = \beta\delta_t R_t \mathrm{E_t} C_{t+1}^{-\chi_c}\Pi_{t+1}^{-1} \tag{35}$$

$$w_t = N_t^{\chi_n} C_t^{\chi_c} \tag{36}$$

$$\frac{Y_t}{C_t^{\chi_c}}\left[\varphi\left(\frac{\Pi_t}{\bar{\Pi}} - 1\right)\frac{\Pi_t}{\bar{\Pi}} - (1-\theta) - \theta(1-\tau)w_t\right] = \beta\delta_t \mathrm{E_t}\frac{Y_{t+1}}{C_{t+1}^{\chi_c}}\varphi\left(\frac{\Pi_{t+1}}{\bar{\Pi}} - 1\right)\frac{\Pi_{t+1}}{\bar{\Pi}} \tag{37}$$

$$Y_t = C_t + \frac{\varphi}{2}\left[\frac{\Pi_t}{\bar{\Pi}} - 1\right]^2 Y_t \tag{38}$$

$$Y_t = N_t \tag{39}$$

$$R_t = \max\left[1, \quad \frac{\bar{\Pi}}{\beta}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}\right] \tag{40}$$

Equation 35 is the consumption Euler equation, Equation 36 is the intratemporal optimality condition of the household, Equation 37 is the optimal condition of the intermediate good producing firms (forward-looking Phillips Curve) relating today's inflation to real marginal cost today and expected inflation tomorrow, Equation 38 is the aggregate resource constraint capturing the resource cost of price adjustment, and Equation 39 is the aggregate production function. Equation 40 is the interest-rate feedback rule.

# D Details of the Empirical Model

This section describes an extension of the stylized model with a representative household, a final good producer, a continuum of intermediate goods producers with unit measure, and the government.

## D.1 Household markets

### D.1.1 Labor packer

The labor packer buys labor $N_{h,t}$ from households at their monopolistic wage $W_{h,t}$ and resells the packaged labor $N_t$ to intermediate goods producers at $W_t$. The problem can be written as

$$\max_{N_{h,t},h\in[0,1]} W_tN_t - \int_0^1 W_{h,t}N_{h,t}df \tag{41}$$

subject to the following CES technology

$$N_t = \left[\int_0^1 N_{h,t}^{\frac{\theta^w-1}{\theta^w}} dh\right]^{\frac{\theta^w}{\theta^w-1}}. \tag{42}$$

The first order condition implies a labor demand schedule

$$N_{h,t} = \left[\frac{W_{h,t}}{W_t}\right]^{-\theta^w} N_t.^{24} \tag{43}$$

$\theta^w$ is the wage markup parameter.

### D.1.2 Household

The representative household chooses its consumption level, amount of labor, and bond holdings so as to maximize the expected discounted sum of utility in future periods. As is common in the literature, the household enjoys consumption and dislikes labor. Assuming that period utility is separable, the household problem can be defined by

$$\max_{C_{h,t},w_{h,t},B_{h,t}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1}\left[\prod_{s=0}^{t-1}\delta_s\right]\left[\frac{(C_{h,t}-\zeta C_{t-1}^a)^{1-\chi_c}}{1-\chi_c} - A_t^{1-\chi_c}\frac{N_{h,t}^{1+\chi_n}}{1+\chi_n}\right] \tag{44}$$

subject to the budget constraint

$$P_tC_{h,t}+R_t^{-1}B_{h,t} \le W_{h,t}N_{h,t}-W_t\frac{\varphi_w}{2}\left[\frac{W_{h,t}}{aW_{h,t-1}\left(\bar{\Pi}^w\right)^{1-\iota_w}\left(\Pi_{t-1}^w\right)^{\iota_w}}-1\right]^2 N_t+B_{h,t-1}+P_t\Phi_t-P_tT_t \tag{45}$$

or equivalently

$$C_{h,t}+\frac{B_{h,t}}{R_tP_t} \le w_{h,t}N_{h,t}-w_t\frac{\varphi_w}{2}\left[\frac{w_{h,t}}{aw_{h,t-1}}\frac{\Pi_t^p}{\left(\bar{\Pi}^w\right)^{1-\iota_w}\left(\Pi_{t-1}^w\right)^{\iota_w}}-1\right]^2 N_t+\frac{B_{h,t-1}}{P_t}+\Phi_t-T_t \tag{46}$$

and subject to the labor demand schedule

$$N_{h,t} = \left[\frac{W_{h,t}}{W_t}\right]^{-\theta^w} N_t. \tag{47}$$

–or equivalently

$$N_{h,t} = \left[\frac{w_{h,t}}{w_t}\right]^{-\theta^w} N_t. \tag{48}$$

[24] This implies that the labor packer will set the wage of the packaged labor to $W_t = \left[\int_0^1 W_{h,t}^{1-\theta^w} dh\right]^{\frac{1}{1-\theta^w}}$.

where $C_{h,t}$ is the household's consumption, $N_{h,t}$ is the labor supplied by the household, $P_t$ is the price of the consumption good, $W_{h,t}$ ($w_{h,t}$) is the nominal (real) wage set by the household, $W_t$ ($w_t$) is the market nominal (real) wage, $\Phi_t$ is the profit share (dividends) of the household from the intermediate goods producers, $B_{h,t}$ is a one-period risk free bond that pays one unit of money at period t+1, $T_t$ are lump-sum taxes or transfers, and $R_t^{-1}$ is the price of the bond. $C_{t-1}^a$ represents the aggregate consumption level from the previous period that the household takes as given. The parameter $0 \leq \zeta < 1$ measures how important these external habits are to the household. Because we are including wage indexation, measured by the parameter $\iota_w$, we assume the household takes as given the previous period wage inflation, $\Pi_{t-1}^w$, where $\Pi_t^w = \frac{W_t}{aW_{t-1}} = \frac{w_t P_t}{aw_{t-1}P_{t-1}} = \frac{w_t}{aw_{t-1}}\Pi_t^p$.

The discount rate at time t is given by $\beta\delta_t$ where $\delta_t$ is the discount factor shock altering the weight of future utility at time t+1 relative to the period utility at time t. $\delta_t$ is assumed to follow an AR(1) process

$$(\delta_t - 1) = \rho_\delta(\delta_{t-1} - 1) + \epsilon_t^\delta \quad \forall t \geq 2 \tag{49}$$

and $\delta_1$ is given. The innovation $\epsilon_t^\delta$ is normally distributed with mean zero and standard deviation $\sigma_\delta$. It may therefore be interpreted that an increase in $\delta_t$ is a preference imposed by the household to increase the relative valuation of the future utility flows, resulting in decreased consumption today (when considered in the absence of changes in the nominal interest rate).

$A_t$ is a non-stationary total factor productivity shock that also augments labor in the utility function in order to accommodate the necessary stationarization of the model later on. See the next section for more details on this process.

## D.2 Producers

### D.2.1 Final good producer

The final good producer purchases the intermediate goods $Y_{f,t}$ at the intermediate price $P_{f,t}$ and aggregates them using CES technology to produce and sell the final good $Y_t$ to the household and government at price $P_t$. Its problem is then summarized as

$$\max_{Y_{f,t}, f\in[0,1]} P_t Y_t - \int_0^1 P_{f,t} Y_{f,t} di \tag{50}$$

subject to the CES production function

$$Y_t = \left[\int_0^1 Y_{f,t}^{\frac{\theta^p - 1}{\theta^p}} di\right]^{\frac{\theta^p}{\theta^p - 1}}. \tag{51}$$

$\theta^p$ is the price markup parameter.

#### D.2.2 Intermediate goods producers

There is a continuum of intermediate goods producers indexed by $f \in [0,1]$. Intermediate goods producers use labor to produce the imperfectly substitutable intermediate goods according to a linear production function ($Y_{f,t} = A_t N_{f,t}$) and then sell the product to the final good producer. Each firm maximizes its expected discounted sum of future profits[25] by setting the price of its own good. Any price changes are subject to quadratic adjustment costs. $\varphi_p$ will represent an obstruction of price adjustment, the firm indexes for prices—measured by $\iota_p$—and takes as given previous period inflation $\Pi^p_{t-1}$, and $\bar{\Pi}^p$ represents the monetary authority's inflation target.

$$\max_{P_{f,t}} \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[\prod_{s=0}^{t-1} \delta_s\right] \lambda_t \left[P_{f,t}Y_{f,t} - W_t N_{f,t} - P_t \frac{\varphi_p}{2} [\frac{P_{f,t}}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi^p_{t-1}\right)^{\iota_p} P_{f,t-1}} - 1]^2 Y_t\right] \tag{52}$$

such that

$$Y_{f,t} = \left[\frac{P_{f,t}}{P_t}\right]^{-\theta^p} Y_t.^{26} \tag{53}$$

$\lambda_t$ is the Lagrange multiplier on the household's budget constraint at time t and $\beta^{t-1}\left[\prod_{s=0}^{t-1} \delta_s\right]\lambda_t$ is the marginal value of an additional profit to the household. The positive time zero price is the same across firms (i.e. $P_{i,0} = P_0 > 0$).

$A_t$ represents total factor productivity which follows a random walk with drift:

$$\ln(A_t) = \ln(a) + \ln(A_{t-1}) + a_t. \tag{54}$$

$a$ is the unconditional rate of growth of productivity. $a_t$ is a productivity shock following an AR(1) process:

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \epsilon^A_t. \tag{55}$$

where $\epsilon^A_t$ is normally distributed with mean zero and standard deviation $\sigma_A$. This growth factor will imply that some of the variables will acquire a unit root, meaning the model will have to be stationarized. Monetary policy will also have to accommodate this growth factor as well.

[25]NOTE: Each period, as it is written below, is in *nominal* terms. However, we want each period's profits in *real* terms so the profits in each period must be divided by that period's price level $P_t$ which we take care of further along in the document.

[26]This expression is derived from the profit maximizing input demand schedule when solving for the final good producer's problem above. Plugging this expression back into the CES production function implies that the final good producer will set the price of the final good $P_t = \left[\int_0^1 P_{f,t}^{1-\theta^p} di\right]^{\frac{1}{1-\theta^p}}$.

### D.3 Government policies

It is assumed that the monetary authority determines nominal interest rates according to a truncated notional inertial Taylor rule augmented by a speed limit component.

$$R_t = \max\left[1, R_t^*\right] \tag{56}$$

where

$$\frac{R_t^*}{\bar{R}} = \left(\frac{R_{t-1}^*}{\bar{R}}\right)^{\rho_R} \left(\frac{\Pi_t^p}{\bar{\Pi}^p}\right)^{(1-\rho_r)\phi_\pi} \left(\frac{Y_t}{A_t \bar{Y}}\right)^{(1-\rho_r)\phi_y} \exp(\epsilon_t^R) \tag{57}$$

where $\Pi_t^p = \frac{P_t}{P_{t-1}}$ is the inflation rate between periods $t-1$ and $t$, $\bar{R} = \frac{\bar{\Pi}^p a^{\chi_c}}{\beta}$ (see the section on stationarization to see why), and $\epsilon_t^R$ represents white noise monetary policy shocks with mean zero and standard deviation $\sigma_R$.

### D.4 Market clearing conditions

The market clearing conditions for the final good, labor and government bond are given by

$$\begin{aligned} Y_t = C_t + \int_0^1 \frac{\varphi_p}{2} \left[ \frac{P_{f,t}}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p} P_{f,t-1}} - 1 \right]^2 Y_t df + ... \\ ... + \int_0^1 w_t \frac{\varphi_w}{2} \left[ \frac{w_{h,t}}{a w_{h,t-1}} \frac{\Pi_t^p}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_{t-1}^w\right)^{\iota_w}} - 1 \right]^2 N_t dh \end{aligned} \tag{58}$$

$$N_t = \int_0^1 N_{f,t} di \tag{59}$$

$$C_t^a = C_t = \int_0^1 C_{h,t} dh \tag{60}$$

and

$$B_t = \int_0^1 B_{h,t} dh = 0. \tag{61}$$

### D.5 An equilibrium

Given $P_0$ and stochastic processes for $\delta_t$, an equilibrium consists of allocations $\{C_t, N_t, N_{f,t}, Y_t, Y_{f,t}, G_t\}_{t=1}^\infty$, prices $\{W_t, P_t, P_{f,t}\}_{t=1}^\infty$, and a policy instrument $\{R_t\}_{t=1}^\infty$ such that

(i) allocations solve the problem of the household given prices and policies

$$\partial C_{h,t} : (C_{h,t} - \zeta C_{t-1}^a)^{-\chi_c} - \lambda_t = 0 \tag{62}$$

$$\partial w_{h,t} : \theta^w A_t^{1-\chi_c} \frac{N_t^{1+\chi_n}}{w_t} \left( \frac{w_{h,t}}{w_t} \right)^{-\theta^w(1+\chi_n)-1}$$
$$+(1-\theta^w)\lambda_t \left( \frac{w_{h,t}}{w_t} \right)^{-\theta^w} N_t$$
$$-\lambda_t w_t \varphi_w \left( \frac{w_{h,t}}{a w_{h,t-1}} \frac{\Pi_t^p}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_{t-1}^w\right)^{\iota_w}} - 1 \right) N_t \frac{\Pi_t^p}{a w_{h,t-1} \left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_{t-1}^w\right)^{\iota_w}} \tag{63}$$
$$+\beta\delta_t \mathrm{E}_t \lambda_{t+1} w_{t+1} \varphi_w \left( \frac{w_{h,t+1}}{a w_{h,t}} \frac{\Pi_{t+1}^p}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_t^w\right)^{\iota_w}} - 1 \right) N_{t+1} \frac{w_{h,t+1}}{a w_{h,t}^2} \frac{\Pi_{t+1}^p}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_t^w\right)^{\iota_w}} = 0$$

$$\partial B_{h,t} : -\frac{\lambda_t}{R_t P_t} + \beta\delta_t \mathrm{E_t} \frac{\lambda_{t+1}}{P_{t+1}} = 0 \tag{64}$$

(ii) $P_{f,t}$ solves the problem of firm $i$

By making the appropriate substitution (the intermediate goods producer's constraints in place of $Y_{f,t}$ and subsequently in for $N_{f,t}$) and by dividing each period's profits by that period's price level $P_t$ so as to put profits in real terms (and thus make profits across periods comparable) we get the following:

$$\partial P_{f,t} : \lambda_t \frac{Y_t}{P_t} \Big[ \frac{P_t}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p} P_{f,t-1}} \varphi_p \left( \frac{P_{f,t}}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p} P_{f,t-1}} - 1 \right) - (1-\theta^p) \left( \frac{P_{f,t}}{P_t} \right)^{-\theta^p}$$
$$-\theta^p \frac{w_t}{A_t} \left( \frac{P_t}{P_{f,t}} \right)^{1+\theta^p} \Big] = \beta\delta_t \mathrm{E_t} \frac{\lambda_{t+1} Y_{t+1}}{P_{t+1}} \left[ P_{t+1} \varphi_p \left( \frac{P_{f,t+1}}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_t^p\right)^{\iota_p} P_{f,t}} - 1 \right) \frac{P_{f,t+1}}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_t^p\right)^{\iota_p} P_{f,t}^2} \right] \tag{65}$$

(iii) $P_{f,t} = P_{j,t} \quad \forall i \neq j$

$$\frac{Y_t}{\lambda_t^{-1}} \left[ \varphi_p \left( \frac{\Pi_t^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p}} - 1 \right) \frac{\Pi_t^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p}} - (1-\theta^p) - \theta^p \frac{w_t}{A_t} \right] = \ldots$$
$$\ldots = \beta\delta_t \mathrm{E_t} \frac{Y_{t+1}}{\lambda_{t+1}^{-1}} \varphi_p \left( \frac{\Pi_{t+1}^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_t^p\right)^{\iota_p}} - 1 \right) \frac{\Pi_{t+1}^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_t^p\right)^{\iota_p}} \tag{66}$$

(iv) $R_t$ follows a specified rule

and

(v) all markets clear.

Combining all of the results derived from the conditions and exercises in (i)-(v), a symmetric equilibrium can be characterized recursively by $\{C_t, N_t, Y_t, w_t, \Pi_t^p, R_t\}_{t=1}^{\infty}$ satisfying the following equilibrium conditions:

$$\lambda_t = \beta\delta_t R_t \mathrm{E_t} \lambda_{t+1} \left(\Pi^p_{t+1}\right)^{-1} \tag{67}$$

$$\lambda_t = (C_t - \zeta C_{t-1})^{-\chi_c} \tag{68}$$

$$\begin{aligned}\frac{N_t}{\lambda_t^{-1}}\left[\varphi_w\left(\frac{\Pi^w_t}{\left(\bar{\Pi}^w\right)^{1-\iota_w}\left(\Pi^w_{t-1}\right)^{\iota_w}}-1\right)\frac{\Pi^w_t}{\left(\bar{\Pi}^w\right)^{1-\iota_w}\left(\Pi^w_{t-1}\right)^{\iota_w}}-(1-\theta^w)-\theta^w\frac{A_t^{1-\chi_c}N_t^{\chi_n}}{\lambda_t w_t}\right] = ...\\ ... = \beta\delta_t \mathrm{E_t}\frac{N_{t+1}}{\lambda_{t+1}^{-1}}\varphi_w\left(\frac{\Pi^w_{t+1}}{\left(\bar{\Pi}^w\right)^{1-\iota_w}\left(\Pi^w_t\right)^{\iota_w}}-1\right)\frac{\Pi^w_{t+1}}{\left(\bar{\Pi}^w\right)^{1-\iota_w}\left(\Pi^w_t\right)^{\iota_w}}\frac{w_{t+1}}{w_t}\end{aligned} \tag{69}$$

$$\Pi^w_t = \frac{w_t}{a w_{t-1}}\Pi^p_t \tag{70}$$

$$\begin{aligned}\frac{Y_t}{\lambda_t^{-1}}\left[\varphi_p\left(\frac{\Pi^p_t}{\left(\bar{\Pi}^p\right)^{1-\iota_p}\left(\Pi^p_{t-1}\right)^{\iota_p}}-1\right)\frac{\Pi^p_t}{\left(\bar{\Pi}^p\right)^{1-\iota_p}\left(\Pi^p_{t-1}\right)^{\iota_p}}-(1-\theta^p)-\theta^p\frac{w_t}{A_t}\right] = ...\\ ... = \beta\delta_t \mathrm{E_t}\frac{Y_{t+1}}{\lambda_{t+1}^{-1}}\varphi_p\left(\frac{\Pi^p_{t+1}}{\left(\bar{\Pi}^p\right)^{1-\iota_p}\left(\Pi^p_t\right)^{\iota_p}}-1\right)\frac{\Pi^p_{t+1}}{\left(\bar{\Pi}^p\right)^{1-\iota_p}\left(\Pi^p_t\right)^{\iota_p}}\end{aligned} \tag{71}$$

$$Y_t = C_t + \frac{\varphi_p}{2}\left[\frac{\Pi^p_t}{\left(\bar{\Pi}^p\right)^{1-\iota_p}\left(\Pi^p_{t-1}\right)^{\iota_p}}-1\right]^2 Y_t + \frac{\varphi_w}{2}\left[\frac{\Pi^w_t}{\left(\bar{\Pi}^w\right)^{1-\iota_w}\left(\Pi^w_{t-1}\right)^{\iota_w}}-1\right]^2 w_t N_t \tag{72}$$

$$Y_t = A_t N_t \tag{73}$$

$$R_t = \max\left[1, R^*_t\right] \tag{74}$$

where

$$\frac{R^*_t}{\bar{R}} = \left(\frac{R^*_{t-1}}{\bar{R}}\right)^{\rho_R}\left(\frac{\Pi^p_t}{\bar{\Pi}^p}\right)^{(1-\rho_r)\phi_\pi}\left(\frac{Y_t}{A_t\bar{Y}}\right)^{(1-\rho_r)\phi_y}\exp(\epsilon^R_t) \tag{75}$$

and given the following processes ($\forall t \geq 2$):

$$(\delta_t - 1) = \rho_\delta(\delta_{t-1} - 1) + \epsilon^\delta_t \tag{76}$$

and

$$\ln(A_t) = \ln(a) + \ln(A_{t-1}) + a_t. \tag{77}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \epsilon^A_t. \tag{78}$$

### D.6 A stationary equilibrium

Let $\tilde{Y}_t = \frac{Y_t}{A_t}$, $\tilde{C}_t = \frac{C_t}{A_t}$, $\tilde{w}_t = \frac{w_t}{A_t}$, and $\tilde{\lambda}_t = \frac{\lambda_t}{A_t^{-\chi_c}}$ be the stationary representations of output, consumption, real wage, and marginal utility of consumption respectively. The stationary symmetric equilibrium can now be characterized by the following system of equations.

$$\tilde{\lambda}_t = \frac{\beta}{a^{\chi_c}}\delta_t R_t \mathrm{E_t}\tilde{\lambda}_{t+1}\left(\Pi^p_{t+1}\right)^{-1}\exp(-\chi_c\epsilon^A_{t+1}) \tag{79}$$

$$\tilde{\lambda}_t = (\tilde{C}_t - \tilde{\zeta}\tilde{C}_{t-1}\exp(-\epsilon_t^A))^{-\chi_c}, \quad \tilde{\zeta} = \frac{\zeta}{a} \tag{80}$$

$$\begin{aligned}
&\frac{N_t \tilde{w}_t}{\tilde{\lambda}_t^{-1}} \left[ \varphi_w \left( \frac{\Pi_t^w}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_{t-1}^w\right)^{\iota_w}} - 1 \right) \frac{\Pi_t^w}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_{t-1}^w\right)^{\iota_w}} - (1-\theta^w) - \theta^w \frac{N_t^{\chi_n}}{\tilde{\lambda}_t \tilde{w}_t} \right] = ... \\
&... = \frac{\beta \varphi_w}{a^{\chi_c - 1}} \delta_t \mathrm{E}_t \frac{N_{t+1}\tilde{w}_{t+1}}{\lambda_{t+1}^{-1}} \left( \frac{\Pi_{t+1}^w}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_t^w\right)^{\iota_w}} - 1 \right) \frac{\Pi_{t+1}^w}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_t^w\right)^{\iota_w}} \exp\left( (1-\chi_c)\, \epsilon_{t+1}^A \right)
\end{aligned} \tag{81}$$

$$\Pi_t^w = \frac{\tilde{w}_t}{\tilde{w}_{t-1}} \Pi_t^p \exp\left(\epsilon_t^A\right) \tag{82}$$

$$\begin{aligned}
&\frac{\tilde{Y}_t}{\tilde{\lambda}_t^{-1}} \left[ \varphi_p \left( \frac{\Pi_t^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p}} - 1 \right) \frac{\Pi_t^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p}} - (1-\theta^p) - \theta^p \tilde{w}_t \right] = ... \\
&... = \frac{\beta \varphi_p}{a^{\chi_c - 1}} \delta_t \mathrm{E}_t \frac{\tilde{Y}_{t+1}}{\tilde{\lambda}_{t+1}^{-1}} \left( \frac{\Pi_{t+1}^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_t^p\right)^{\iota_p}} - 1 \right) \frac{\Pi_{t+1}^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_t^p\right)^{\iota_p}} \exp\left( (1-\chi_c)\, \epsilon_{t+1}^A \right)
\end{aligned} \tag{83}$$

$$\tilde{Y}_t = \tilde{C}_t + \frac{\varphi_p}{2} \left[ \frac{\Pi_t^p}{\left(\bar{\Pi}^p\right)^{1-\iota_p} \left(\Pi_{t-1}^p\right)^{\iota_p}} - 1 \right]^2 \tilde{Y}_t + \frac{\varphi_w}{2} \left[ \frac{\Pi_t^w}{\left(\bar{\Pi}^w\right)^{1-\iota_w} \left(\Pi_{t-1}^w\right)^{\iota_w}} - 1 \right]^2 \tilde{w}_t N_t \tag{84}$$

$$\tilde{Y}_t = N_t \tag{85}$$

and

$$R_t = \max\left[1, R_t^*\right] \tag{86}$$

where

$$\frac{R_t^*}{\bar{R}} = \left( \frac{R_{t-1}^*}{\bar{R}} \right)^{\rho_R} \left( \frac{\Pi_t^p}{\bar{\Pi}^p} \right)^{(1-\rho_r)\phi_\pi} \left( \frac{\tilde{Y}_t}{\bar{Y}} \right)^{(1-\rho_r)\phi_y} \exp\left(\epsilon_t^R\right) \tag{87}$$

and given the following processes ($\forall t \geq 2$):

$$(\delta_t - 1) = \rho_\delta(\delta_{t-1} - 1) + \epsilon_t^\delta \tag{88}$$

and

## D.7 Stationary deterministic steady-state values

For each variable, $X_t$, we denote its corresponding *stationary* deterministic steady-state value as $\bar{X}$. The following is a list of analytical expressions for the stationary steady states for each of the variables of the model.

$$\bar{\Pi}^p = \bar{\Pi}^p, \quad \text{(this parameter is set exogenously by the monetary authority)}$$

$$\bar{\Pi}^w = \bar{\Pi}^p$$

$$\bar{R} = \frac{a^{\chi_c}\bar{\Pi}^p}{\beta}$$

$$\bar{w} = \frac{\theta_p - 1}{\theta_p}$$

$$\bar{C} = \left( \frac{\bar{w}\left(\theta_w - 1\right)}{\theta_w \left(1 - \tilde{\zeta}\right)^{\chi_c}} \right)^{\frac{1}{\chi_c + \chi_n}}$$

$$\bar{\lambda} = \left[ \left(1 - \tilde{\zeta}\right) \bar{C} \right]^{-\chi_c}$$

$$\bar{N} = \bar{Y} = \bar{C}$$

# E Solution Method

We describe our solution method using the stylized model analyzed in the main text. The extension of the method to the empirical model is straightforward.

The problem is to find a set of policy functions, $\{C(\cdot),\, N(\cdot),\, Y(\cdot),\, w(\cdot),\, \Pi(\cdot),\, R(\cdot)\}$, that solves the following system of *functional* equations.

$$C(\delta_t)^{-\chi_c} = \beta \delta_t R(\delta_t) E_t C(\delta_{t+1})^{-\chi_c} \Pi(\delta_{t+1})^{-1} \tag{89}$$

$$w(\delta_t) = N(\delta_t)^{\chi_n} C(\delta_t)^{\chi_c} \tag{90}$$

$$\frac{N(\delta_t)}{C(\delta_t)^{\chi_c}} \Big[ \varphi \Big( \frac{\Pi(\delta_t)}{\bar{\Pi}} - 1 \Big) \frac{\Pi(\delta_t)}{\bar{\Pi}} - (1-\theta) - \theta w(\delta_t) \Big] ...$$

$$... = \beta \delta_t E_t \frac{N(\delta_{t+1})}{C(\delta_{t+1})^{\chi_c}} \varphi \Big( \frac{\Pi(\delta_{t+1})}{\bar{\Pi}} - 1 \Big) \frac{\Pi(\delta_{t+1})}{\bar{\Pi}} \tag{91}$$

$$Y(\delta_t) = C(\delta_t) + \frac{\varphi}{2} \Big[ \frac{\Pi(\delta_t)}{\bar{\Pi}} - 1 \Big]^2 Y(\delta_t) \tag{92}$$

$$Y(\delta_t) = N(\delta_t) \tag{93}$$

$$R(\delta_t) = \max \left[ 1, \frac{\bar{\Pi}}{\beta} \left[ \frac{\Pi(\delta_t)}{\bar{\Pi}} \right]^{\phi_\pi} \left[ \frac{Y(\delta_t)}{\bar{Y}} \right]^{\phi_y} \right] \tag{94}$$

Substituting out $w(\cdot)$ and $N(\cdot)$ using equations (90) and (93), this system can be reduced to a system of four functional equations for $C(\cdot)$, $Y(\cdot)$, $\Pi(\cdot)$, and $R(\cdot)$.

$$C(\delta_t)^{-\chi_c} = \beta \delta_t R(\delta_t) E_t C(\delta_{t+1})^{-\chi_c} \Pi(\delta_{t+1})^{-1} \tag{95}$$

$$\frac{Y(\delta_t)}{C(\delta_t)^{\chi_c}} \Big[ \varphi \Big( \frac{\Pi(\delta_t)}{\bar{\Pi}} - 1 \Big) \frac{\Pi(\delta_t)}{\bar{\Pi}} - (1-\theta) - \theta Y(\delta_t)^{\chi_n} C(\delta_t)^{\chi_c} \Big] ...$$

$$... = \beta \delta_t E_t \frac{Y(\delta_{t+1})}{C(\delta_{t+1})^{\chi_c}} \varphi \Big( \frac{\Pi(\delta_{t+1})}{\bar{\Pi}} - 1 \Big) \frac{\Pi(\delta_{t+1})}{\bar{\Pi}} \tag{96}$$

$$Y(\delta_t) = C(\delta_t) + \frac{\varphi}{2} \Big[ \frac{\Pi(\delta_t)}{\bar{\Pi}} - 1 \Big]^2 Y(\delta_t) \tag{97}$$

$$R(\delta_t) = \max \left[ 1, \frac{\bar{\Pi}}{\beta} \left[ \frac{\Pi(\delta_t)}{\bar{\Pi}} \right]^{\phi_\pi} \left[ \frac{Y(\delta_t)}{\bar{Y}} \right]^{\phi_y} \right] \tag{98}$$

Following the idea of Christiano and Fisher (2000) and Gust, López-Salido, and Smith (2012), I decompose these policy functions into two parts using an indicator function: One in which the policy rate is allowed to be less than zero, and the other in which the policy rate is assumed to be zero. That is, for any variable Z,

$$Z(\cdot) = \mathbb{1}_{\{R(\cdot)\geq 1\}} Z_{unc}(\cdot) + (1 - \mathbb{1}_{\{R(\cdot)\geq 1\}}) Z_{zlb}(\cdot). \tag{99}$$

The problem then becomes finding a set of a *pair* of policy functions, $\{[C_{unc}(\cdot),\ C_{zlb}(\cdot)],\ [Y_{unc}(\cdot),\ Y_{zlb}(\cdot)],\ [\Pi_{unc}(\cdot),\ \Pi_{zlb}(\cdot)],\ [R_{unc}(\cdot),\ R_{zlb}(\cdot)]\}$ that solves the system of functional equations above. This method can achieve a given level of accuracy with a considerable less number of grid points relative to the standard approach.[27]

The time-iteration method starts by specifying a guess for the values policy functions take on a finite number of grid points. The values of the policy function that are not on any of the grid points are interporate or extrapolated linearly. Let $X(\cdot)$ be a vector of policy functions that solves the functional equations above and let $X^{(0)}$ be the initial guess of such policy functions.[28] At the $s$-th iteration and at each point of the state space, we solve the system of nonlinear equations given by equations (95)-(98) to find today's consumption, output, inflation, and the policy rate, given that $X^{(s-1)}(\cdot)$ is in place for the next period. In solving the system of nonlinear equations, I use Gaussian quadrature to evaluate the expectation terms in the consumption Euler equation and the Phillips curve, and the value of future variables not on the grid points are evaluated with linear interpolation. The system is solved numerically by using a nonlinear equation solver, `dneqnf`, provided by the IMSL Fortran Numerical Library. For all models, I use 10 grid points for the Gaussian quadrature. If the updated policy functions are sufficiently close to the guessed policy functions, then the algorithm ends. Otherwise, using the updated policy functions just obtained as the guess for the next period's policy functions, I iterate on this process until the difference between the guessed and updated policy functions is sufficiently small ($\left\| vec(X^s(\delta) - X^{s-1}(\delta)) \right\|_\infty < 1\text{E-}11$ is used as the convergence criteria). I used equally spaced 1001 grid points on the interval between $[1 - 4\sigma_\delta, 1 + 4\sigma_\delta]$. The solution method can be extended to models with multiple exogenous shocks and endogenous state variables in a straightforward way.

[27] A systematic analysis of the benefits of using the Christiano-Fisher approach is available upon request.

[28] For all models and all variables, I use flat functions at the deterministic steady-state values as the initial guess.