Growth through Learning

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Abstract

An agent tries to maximize output by searching for the best technology. Each period the ideal technogical decision has an unknown permanent component and unknown tansitory components. The agent searches for the best policy and learns from experience, ultimately approaching material bliss and infinite output. Surprisingly, the long-run growth rate rises with agents' risk aversion. Information has only level effects, but long-run growth is endogenous. Quantitatively the model matches the growth facts only the periods are decades, not years – the fluctuations are low-frequency events in which case the industrial revolution is a plausible right-tail event. Indeed, the growth distribution has a thick right tail. Then a second agent is added to the economy and a free riding problem arises, but thanks to a scale effect in the effects of knowledge, growth is nevertheless faster.

1 Introduction

Many studies find that policies of various kinds affect growth, be they the policies of a country's own government, or those of its colonizing country governments – e.g., Easterly *et al.* (1994).

We shall model a single agent, with an infinite horizon who must choose production and investment and whose beliefs will evolve as a result of his informational investments and his experience. The model yields a long-run growth rate that depends on the speed at which the agent can learn. That speed depends on the agent's investment in costly signals.

We then model several agents that can share information. Each learns about a parameter that is correlated with those that other agents and therefore benefits from seeing signals of others. A free riding problem then arises and investment is below

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its efficient level. The model has a scale effect, i.e., the total information generated grows with the scale of the economy so that two identical countries can, with the same sacrifice of resources per capita, generate twice the information that one country can generate. This is familiar to growth theorists at least since Arrow (1962). We show that equilibrium investment grows with the number of agents, i.e., that the free-riding effect mitigates but does not overturn the scale effect.

In its current form the model can fit the basic growth facts only at low frequencies – growth surprises occur over decades rather than years. All investment in the model is informational investment that helps agents adapt better to technological shocks. A large growth surprise occurs when the technological signal is highly accurate which allows the agents to take better decisions. That such surprises occur over low frequencies gets support from David (1991) who argued that the 1920s and 1990s were the result of developments that took place decades earlier. One can argue similarly about institutional developments or other policy decisions – they probably take years if not decades to make themselves felt fully.

An unusual result is that the growth rate rises along the transition to long-run growth, opposite to Solow (1956), but in accord with evidence shown, e.g., in Tables 1 and 2 of Romer (1986), and in Jones and Romer (2010). A rising growth rate also implies that if countries begin the process at different times, there will for a while be divergence, and Pritchett (1999) argues that this has been the case historically.

Another unusual result is that risk aversion raises growth – opposite to Romer (1986) and Lucas (1988). The difference in the implied relation arises because investing in information plays a dual role: It raises expected output but it also lowers its variance – these two features go hand in hand for a large subset of the parameters. [1] A related result arises in the Aiyagari-Bewley class of models with limited asset structure, risk aversion raises savings for precautionary reasons. In GE models that have a finite steady state this raises steady state capital and lowers the interest rate. In partial equilibrium, however, such as a small open economy facing a fixed world interest rate, Bouccekine, Fabbri and Pintus (2014) show that if there are constant returns to capita, i.e., an Ak technology, risk aversion raises the savings and long-run growth.

The model is solved exactly in the limit – the long-run growth rate is random and i.i.d., so that Gibrat's Law holds in the limit. When we extend the model to more than one country, Gibrat's law continues to hold at the world level, but it fails at the level of the individual country – in the solved example there is then a negative autocorrelation in a country's growth rate. Moreover, the distribution of growth rates has a thick right tail – the growth factor has a distribution resembling the inverse Gaussian but with a thicker tail – and expected output is infinite. This is true in the transition as well.

[1] The opposite was true in Jovanovic and Nyarko (1996) and Jovanovic (2006) where the adoption of a better of technology was the *cause* of aggregate risk; the bigger the technological leap, the higher the risk.

Closely related is Buera, Monge and Primiceri (2011, henceforth BMP) who assume, as I do, that a country's growth depends on policies that it adopts, and that the ideal policy is not known – it has to be learned through experience. BMP analyze market-oriented policies when their own and their neighbors' past experience can inform policy makers' beliefs over state intervention vs. market orientation strategies. Similarly to what is done here, BMP do not use a production function with the usual inputs and focus instead on the policy decision and its influence on output through a channel not precisely specified. Unlike BMP, however, we shall feature investment in information and endogenous long-run growth.

More generally, other papers on adaptive learning share the same general approach in which a policy maker adjusts his actions to information about an unknown parameter: Prescott (1972), and Easley and Kiefer (1988).

Section 2 outlines the model in which a single agent learns from his own experience and from his own research. Section 3 solves for policies, value and for the distribution of the growth rate in the long-run limiting case which has an Ak form. Section 4 deals with the transitional dynamics, and calibrates the parameters to some facts on long-run growth and investment-shares. Section 5 deals with information sharing and the resulting free riding problem in generating information.[2] Section 6 concludes

2 Model

Consider a single agent, with an infinite horizon who must choose production and investment and whose beliefs evolve as a result of his informational investments and his experience.

Utility function.—Let $c \in R_+$ be the consumption = output of a single consumption good, and let lifetime expected utility be

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \quad \text{where} \quad U(c) = \frac{1}{1-\gamma} c^{1-\gamma}, \tag{1}$$

where c_t denotes the agent's consumption at date t.

Production.—Let $\theta \in R$ be a parameter, let ε_t be i.i.d., and let $x_t \in R$ be a "decision" like the setting of a dial. Output is[3]

$$y_t = (\theta + \varepsilon_t - x_t)^{-\rho}. \tag{2}$$

where $\rho > 0$ is an even integer. This production function is unbounded, and the agent could attain infinite output if he happened to set $x_t = \theta + \varepsilon_t$, but since he

[2] Nelson and Phelps (1968) may have been the first to model information sharing in a multi-country environment. Kremer (1993) and BMP (2011) have shown that information sharing matters. A world equilibrium with information sharing is modeled in Eeckhout and Jovanovic (2002).

[3] This combines the essence of BMP's (2.1) and (2.2).

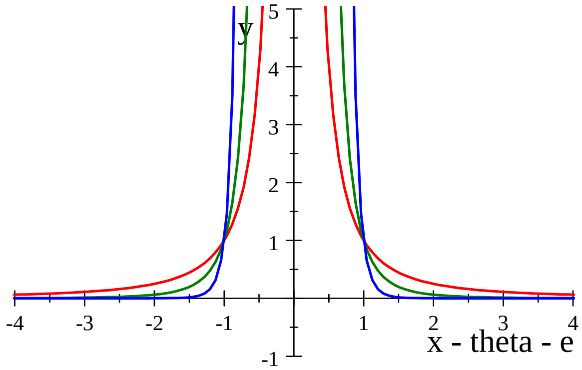

Figure 1: Output as a function of decisions and shocks

knows neither θ nor ε_t, this can happen only by chance and is, in fact, a measure zero event at each t. The larger is ρ, the bigger is the penalty to being wrong about $\theta + \varepsilon$ and the higher the reward to being right, so that ρ is an index of the thickness of the right tail of the distribution of output. The production function leaves out the traditional inputs and focuses instead on policy, which is a real-valued variable, as is the unknown policy target $\theta + \varepsilon_t$.With the possibility of infinite output, we need to ensure that expected utility exists. We therefore assume that $\gamma > 1$; in this case $U \leq 0$. I.e., U is bounded above, and the agent's problem will be well defined.

Investment.—There is no physical capital, and output cannot be stored. Output can, however, be invested in information relevant for next period's production decisions. The income identity is

$$y_t = c_t + n_t, \tag{3}$$

where n_t is the number of independent signals on ε_{t+1}, call these signals, $(\xi_1, ..., \xi_n)$, where I drop the t subscript for the remainder of this paragraph to avoid clutter. Each ξ_i is an independent signal on next period's ε as follows:

$$\xi_i = \varepsilon + \eta_i \tag{4}$$

where $\eta_i \sim N(0, \sigma^2)$ are i.i.d. random variables.

Since the cost per signal is normalized to unity, we may think of $1/\sigma^2$ as the efficiency with which the information sector generates information that raises next-period's productivity of the final-goods sector.[4]

[4]That interpretation would be in the spirit of Greenwood, Hercowicz and Krusell (1997). If we assumed that the cost of each signal is q, then the income identity would read $y_t = c_t + qn_t$, but because of the normality of the signals and of the learning, the model's implications for observables would depend only on the product $q\sigma^2$.

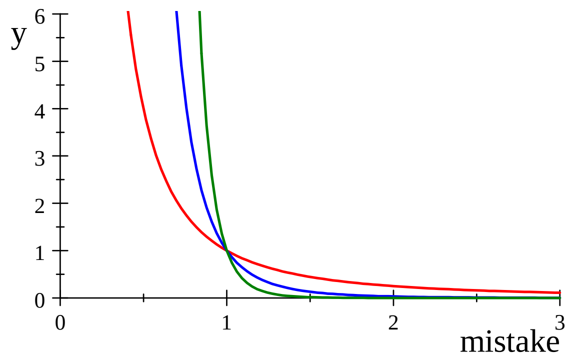

Figure 2: Output and mistakes: $y = w^{-\rho}$ for $\rho = 2$ (red), 5 (blue), 10 (green)

2.1 Learning

Investment in n signals at date t has a short run effect by improving information on ε_{t+1} and thereby raising the quality of next period's decision x_{t+1}. But and a long run effect because θ is fixed and unknown: by having a better measure of ε, the agent will learn about θ more quickly. Learning within the period occurs in two steps. The first is to get a sharper idea of ε given the signals $(\xi_i)_{i=1}^n$ generated at the end of the previous period, and the second is to learn about θ given the output observed at date t distribution of ε generated in step 1.

Step 1.—Form the mean of the sample of n signals:

$$\bar{\xi}_{\mathrm{n}} = \frac{1}{n_t} \sum_{i=1}^{n_t} \xi_i = \varepsilon_{t+1} + \frac{1}{n_t} \sum_{i=1}^{n_t} \eta_i \tag{5}$$

Since each period ε is independent of its past draws, $\bar{\xi}_{\mathrm{n}} \sim N\left(\varepsilon, \sigma^2/n\right)$ is a sufficient statistic for forecasting ε. And since the prior over ε is diffuse, conditional on $\bar{\xi}_{\mathrm{n}}$, ε is distributed normally $N\left(\bar{\xi}_{\mathrm{n}}, \sigma^2/n\right)$. More formally,

$$\lim_{\sigma_\varepsilon \to \infty} \Pr\left(\varepsilon \mid \bar{\xi}_{\mathrm{n}}\right) = \Phi\left(\frac{\varepsilon - \bar{\xi}_{\mathrm{n}}}{\sigma^2/n}\right), \tag{6}$$

where Φ is the standard normal integral.

Step 2.—The agent observes y_t and, additionally, the sign of $\theta + \varepsilon_t$. He then forms the statistic $\zeta = y^{-\rho} + x - \bar{\xi}_{\mathrm{n}} = \theta + \varepsilon - \bar{\xi}_{\mathrm{n}}$ which is distributed $N\left(\theta, \sigma^2/n_t\right)$. The agent then uses ζ to update his beliefs over θ. Let τ denote the beginning-of-period prior variance over θ. The one-step ahead Bayes map says that by the end of the period τ becomes

$$\tau' = \left(\frac{1}{\tau} + \frac{n}{\sigma^2}\right)^{-1} \equiv b\left(\tau, n\right). \tag{7}$$

Therefore a higher investment today, n, raises expected output in the next period because it improves the forecast of ε, but since ε is i.i.d., this effect ends with the next period. However, a higher n raises expected output in all subsequent periods by providing a signal on information about θ.

Now, the following result will help with the Bellman equation.

Lemma 1 *Let $r > 0$ be an even integer. Then*

$$\arg\min_x \int (z - x)^r \, d\Phi\left(\frac{z - \mu}{\sigma_z}\right) = \mu.$$

Proof. The FOC is

$$r \int (z - x)^{r-1} \, d\Phi\left(\frac{z - \mu}{\sigma_z}\right) = 0. \tag{8}$$

The objective is strictly convex in x, so that (8) has exactly one solution for x, which is the globally optimal action. For the normal distribution, all odd moments around the mean are zero and (8) therefore holds at $x = \mu$. Therefore, (12) is the unique solution to (8). ■

2.2 Choice of x

There are no adjustment costs to changing x. Therefore the history of x is irrelevant once beliefs are specified. Since the choice of x does not affect the information about θ or ε, the agent chooses it so as to maximize his current period expected utility conditional on the beginning-of-period priors over θ, which is $N(m, \tau)$, and over ε which is $N\left(\bar{\xi}_n, \sigma^2/n\right)$. Here is where we shall use Lemma 1. All that follows requires that $\gamma \in \{2, 3, ...\}$ be a positive integer and that $\rho \in \{2, 4, ...\}$ be a positive even integer. Under these assumptions $\rho(\gamma - 1)$ is an even, positive integer.

Strictly concave "reduced form" utility.—Since maximization of $E_t(U)$ is dynamically optimal, (12) via Lemma 1 gives the optimal action for a forward looking agent as long as the objective is concave in x. Substitute from (2) into (1) to get

$$U = \frac{1}{1 - \gamma} (\theta - x_t + \varepsilon_t)^{\rho(\gamma - 1)}. \tag{9}$$

Then if

$$\rho(\gamma - 1) > 2, \tag{10}$$

U is strictly concave in x with

$$\frac{\partial U}{\partial x} = \rho(\theta - x + \varepsilon)^{\rho(\gamma-1)-1} \quad \text{and} \quad \frac{\partial^2 U}{\partial x^2} = -\rho^2 \left[(\gamma - 1) - 1\right](\theta - x + \varepsilon)^{\rho(\gamma-1)-2} < 0. \tag{11}$$

The inequality in (11) holds because $\rho(\gamma - 1) - 2$ is an even integer.

Since x maximizes a strictly concave function and, in light of the independence of θ and ε, since $\theta+\varepsilon \sim N\left(m+\bar{\xi}_{\mathrm{n}}, \tau+\sigma^2/n\right)$, we can set $z=\theta+\varepsilon$ and apply Lemma 1 to conclude that

$$x=m+E\left(\varepsilon \mid \bar{\xi}_{\mathrm{n}}\right). \tag{12}$$

Then,

$$\theta+\varepsilon-x \sim N\left(0, \tau+\frac{\sigma^2}{n}\right),$$

so that

$$\zeta=\left(\tau+\frac{\sigma^2}{n}\right)^{-1/2}(\theta+\varepsilon-x) \sim N(0,1). \tag{13}$$

Combining (13) and (2), we can express next-period output as a function of τ, n and ζ alone:

$$y'=\left(\tau+\frac{\sigma^2}{n}\right)^{-\rho/2}\zeta^{-\rho}, \tag{14}$$

where ζ is a standard normal variate.

Bellman equation

Via (12) we have eliminated x from the choice set, and y and τ are beginning-of-period state variables. The only remaining choice variable is n. Let us evaluate the agent's lifetime utility just after the realization of the current output, y, but before he has chosen investment, n.The agent's belief over θ is normal $N(m,\tau)$, but m is not a state because, in light of (12), the agent offsets it by x. Moreover, ε is i.i.d.. The state, then, is the pair (y,τ); the first component tells us his resources that he can consume or invest in forecasting ε, the second the precision with which he can forecast θ. Investment in information, n, is forward looking, of course. Lifetime expected utility, $v(y,\tau)$ satisfies the Bellman equation

$$v(y,\tau)=\max_{n\geq 0}\left\{\frac{(y-n)^{1-\gamma}}{1-\gamma}+\beta\int v\left(y', b(\tau,n)\right)dP\left(y' \mid \tau, n\right)\right\}, \tag{15}$$

with $b(\cdot)$ defined in (7). Using (14), this becomes

$$v(y,\tau)=\max_{n}\left\{\frac{(y-n)^{1-\gamma}}{1-\gamma}+\beta\int v\left(\left(\tau+\frac{\sigma^2}{n}\right)^{-\rho/2}\zeta^{-\rho}, b(\tau,n)\right)\frac{1}{\sqrt{2\pi}}e^{-\zeta^2/2}d\zeta\right\}. \tag{16}$$

The operator on the RHS is a contraction in the sup norm but, since utility is unbounded from below, the space of continuous functions is not complete so we cannot use standard arguments to show that a unique fixed point for v exists. We shall, however, solve for v explicitly for some special cases.

3 Long-run growth with one agent

We wish to impose an Ak property on the model as $\tau \to 0$. Such a property is needed if the model is to have long-run growth in output at a constant rate.

3.1 Information capital and the "Ak" property

An Ak model features two linear processes:

(i) Production of the final good is linear in capital, and

(ii) Additions to the capital stock are linear in output invested.

The present model inherits both (i) and (ii) if an only if $\rho = 2$ and $\tau = 0$. All this is independent of the values of the utility parameters (β, γ) and the value of the information cost σ^2. We do, however, need a diffuse prior over ε, i.e., in the formula for the posterior distribution of ε conditional on the signals we shall assume that $\sigma_\varepsilon^2 \to \infty$.

Let information capital, k, be the precision of the agent's beliefs over the target that x aims to hit, i.e., the inverse of the variance over $\theta + \varepsilon$ pertaining to the next period. In other words, we measure information at the *end* of a period, after beliefs have been updated, and after the signals about next period's ε have come in:

$$k \equiv \frac{1}{\tau + \sigma^2/n}. \tag{17}$$

Property (i).—Substituting from (17) into (14), the latter becomes

$$y' = \zeta^{-\rho} k^{\rho/2}. \tag{18}$$

Linearity of output in k then follows if and only if when $\rho = 2$. This linearity then holds for any τ.

Property (ii).—As a function of investment, k evolves as$\left(\frac{1}{\tau} + \frac{n}{\sigma^2}\right)^{-1} \equiv b(\tau, n)$.

$$k' = \frac{1}{b(\tau, n) + \sigma^2/n} \to \frac{n}{\sigma^2} = \frac{y - c}{\sigma^2} \quad \text{as } \tau \to 0. \tag{19}$$

where we used the income identity (3) updated by a period. In other words, property (ii) holds as $\tau \to 0$, regardless of the values of the parameters. Thus we have proved the following:

Proposition 1 *Let $\rho = 2$. As $\tau \to 0$, the model becomes analogous to an Ak model with full depreciation of physical capital.*

Because ε is serially uncorrelated, information about lagged ε's is irrelevant, which means that past investments in information are of no value in raising future output. This is why the parallel is to the Ak model with full depreciation of capital.[5]

The operator on the RHS of (16) maps continuous functions into other continuous functions, and so as $\tau \to 0$ the solution to 16) converges pointwise to the function $v(y)$ that is the fixed point to the Bellman equation

$$v(y) = \max_n \left\{ \frac{1}{1-\gamma}(y - qn)^{1-\gamma} + \beta \int v\left(n^{\rho/2}\sigma^{-\rho}\zeta^{-\rho}\right) \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2} d\zeta \right\} \tag{20}$$

which, when $\rho = 2$, reduces to

$$v(y) = \max_n \left\{ \frac{1}{1-\gamma}(y - qn)^{1-\gamma} + \beta \int v\left(\frac{n}{\sigma^2\zeta^2}\right) \phi(\zeta)\, d\zeta \right\}$$

where ϕ is the standard normal density. Since preferences are homothetic and since the model has the Ak property, we may guess, following logic similar to that described in Alvarez and Stokey (1998), that the value function and policy function are

$$n = Ay, \quad \text{and} \tag{21}$$

$$v(y) = vy^{1-\gamma}, \tag{22}$$

for some scalars A and v. Feasibility requires that $A < 1$.

Recall that strict concavity of expected utility in x required that condition (10) should hold. When $\rho = 2$, (10) requires that $\gamma > 2$, and we shall maintain this in what follows

Let ω denote the $2(\gamma-1)$'th moment of the standard normal distribution, i.e., $\omega = \int \zeta^{2(\gamma-1)}\phi(\zeta)\, d\zeta$. Then[6]

$$\omega = (2\gamma - 3)!! \tag{23}$$

Proposition 2 *When $\rho = 2$ and $\tau = 0$, (21) and (22) hold, with*

$$A = \left(\omega\beta\sigma^{2(\gamma-1)}\right)^{\frac{1}{\gamma}}, \qquad \textit{and} \tag{24}$$

$$v = \frac{1}{1-\gamma}(1-A)^{-\gamma}. \tag{25}$$

The proof is in the Appendix.

The case $\gamma = 2$.—In this case $\omega = 1$ and (24) and (25) reduce to

$$A = \sqrt{\beta\sigma^2}, \qquad \text{and} \qquad v = -\left(1 - \sqrt{\beta\sigma^2}\right)^{-2}. \tag{26}$$

[5] Less than full depreciation would obtain if $\tau > 0$. Also if the ε's were autocorrelated, but obtaining explicit solutions may prove impossible, at least given the methods used below. An obstacle would be that fact that conditional on past ε's, the prior over ε_{+1} could no longer be diffuse, and optimal investment, n, would no longer be proportional to y.

[6] Here $n!!$ denotes the double factorial, that is, the product of every number from n to 1 that has the same parity as n. In this case $2(\gamma-1)$ is even.

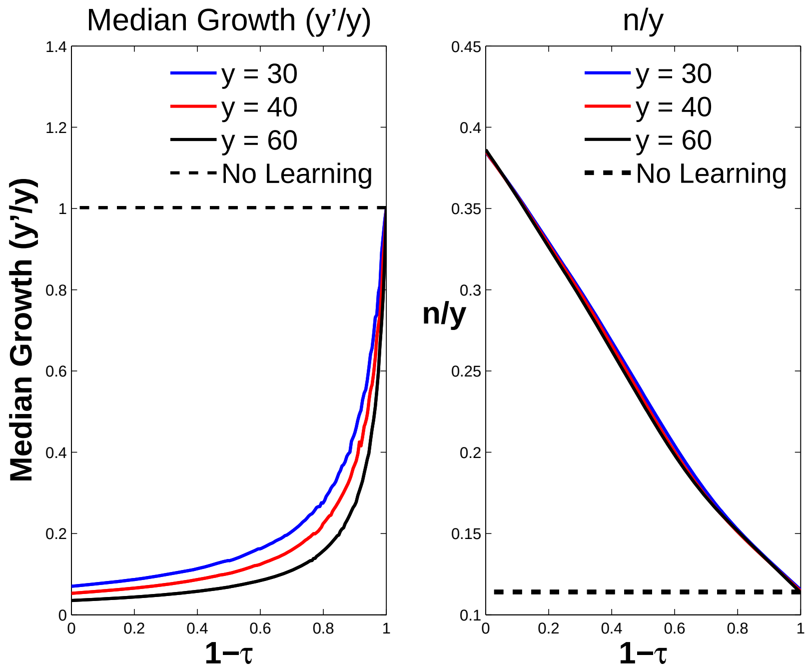

Figure 3: CONVERGENCE AS $\tau \to 0$ OF THE SOLUTIONS TO (16) TO THOSE IN (21) AND (22)

3.2 Convergence of the policy and value functions as $\tau \to 0$.

The first panel of Fig. 3 in which the rate of growth is rising during the transition, and this will be supported by the data in Fig 8 below. For $\tau > 0$ no explicit solution is available to the problem in eq. (16); i.e., we cannot explicitly solve the value and the policy in the transition. In particular, when $\tau > 0$, y does not factor out of the expression for $v(y, \tau)$ in (16). An attempt at a guess and verify for solutions of the form $v(y, \tau) = v(\tau) y^{1-\gamma}$ and $n(y, \tau) = A(\tau) y$ failed because τ has an effect on production that is independent of y. We can, however, check the numerical solution to (16) for convergence to the solutions in (21) and (22). The latter is referred to as the "no learning" case.

A plot of the convergence of the policy and value functions as $\tau \to 0$ is in Fig. 3. Two facts emerge. First, the investment rate is falling on the transition path to its long-run value of A given in eq. (24). In the long run, $\partial b/\partial n = 0$, so that informational investment only helps predict the transitory variable ε_t, which leads to a one-period gain in expected utility. During the transition, however, $\partial b/\partial n < 0$, and a better signal on ε speeds up the inference of θ, which is an additional benefit that spreads into the future.

Second, median growth is negative for a large portion of the transitional dynamics, eventually rising to its long-run value in (28). Plotted on time, as we shall see below,

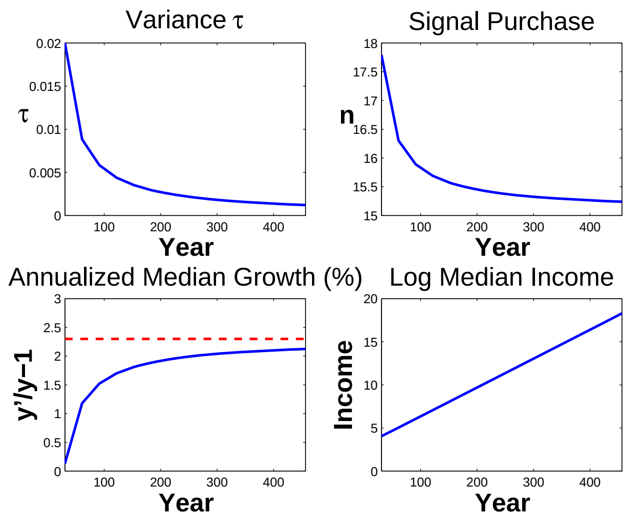

Figure 4: Convergence in real time at the calibrated parameter values

median growth will be positive for a long time as τ gradually approaches zero, i.e., as $1-\tau$ gradually approaches unity on the two horizontal axes in Fig 3. The convergence of τ to zero is arithmetic, and this makes the relevant transitional dynamics more protracted. This is Panel 1 of Fig 4. The plot assumes that $\sigma = 0.53$, and with n around 15 (Panel 2), the effective signal variance, per period is $\sigma^2/n \approx \frac{.53^2}{15} \approx .02$. Panel 3 shows that income growth starts out slightly below zero, and takes 400 years to reach two percent. Panel 4 plots log income which has median growth $\Delta \ln y_t = \ln g_t^{\text{med}}$; initial $y_0 = 4.05$.

3.3 The limiting distribution of g

Denote the growth factor by

$$g = \frac{y'}{y}$$

using (14) evaluated at $\tau = 0$ and $\rho = 2$ and using (21) and then (24) yields $g = \frac{n}{\sigma^2}\zeta^{-2}\frac{1}{y} = \frac{A}{\sigma^2}\zeta^{-2}$, where $A = \left(\omega\beta\sigma^{2(\gamma-1)}\right)^{\frac{1}{\gamma}}$. Substituting for A we have the following result:

Proposition 3 *The limiting growth factor is i.i.d., and equal to*

$$g_t = \omega^{1/\gamma}\left(\frac{\beta}{\sigma^2}\right)^{1/\gamma}\zeta_t^{-2}. \tag{27}$$

where $\zeta_t \sim N(0,1)$ *is i.i.d..*

Therefore the growth factor is increasing in the ratio β/σ^2. The effect of β is standard in Ak models, and the positive effect of more accurate signals, i.e. of a fall in σ^2 is intuitive – a perfect signal would yield infinite output. Since the median of a $\chi^2_{(1)}$ variable is 0.47, we have the corollary

Corollary 1 *The median growth factor is*

$$\textit{median } \frac{y'}{y} = 2.13\omega^{1/\gamma}\left(\frac{\beta}{\sigma^2}\right)^{1/\gamma}. \tag{28}$$

Proof. A $\chi^2_{(1)}$ r.v. has a median of 0.47, and $(0.47)^{-1} = 2.13$. Using (27),

$$\text{median } \frac{y'}{y} = 2.13\frac{A}{\sigma^2}. \tag{29}$$

Substituting for A from (24), and noting that $\left(\sigma^{2(\gamma-1)}\right)^{\frac{1}{\gamma}}/\sigma^2 = \left(\sigma^{2(\gamma-1)}\sigma^{-2\gamma}\right)^{\frac{1}{\gamma}} = \sigma^{-2/\gamma}$ ■

We shall refer to the expression in (28) as g^{med}. Since $\omega = \int \zeta^{2(\gamma-1)}\phi(\zeta)\,d\zeta$ depends only on γ and not on any other parameter, g^{med} is increasing in the ratio β/σ^2, and independent of y.

3.4 The effect of γ on investment and growth

The effect of γ on growth is ambiguous in general. In (27), the term $\omega^{1/\gamma}$ is increasing in γ. Turning to (23) note that since $2\gamma - 3$ is always an odd integer, $(2\gamma - 3)!!$ is a product of all the odd integers between $2\gamma - 3$ and one. Thus in terms of some integer-valued γ, we may summarize the relation in the following table:

γ	2	3	4	5	6	7	10
ω	1	3	15	105	1095	10395	3.4×10^7
$\omega^{1/\gamma}$	1	1.4	1.9	2.5	3.2	3.7	5.7

..

Table 1: THE RELATION BETWEEN γ AND $\omega^{1/\gamma}$

Returning to eq. (27), the term $(\beta/\sigma^2)^{1/\gamma}$ is increasing in γ if $\beta > \sigma^2$. In this case a rise in γ raises growth. This opposite from deterministic growth models of Romer (1990) and Lucas (1988), but in accord with an Ak version Bewley-Aiyagari type of model in Boucekkine, Fabbri and Pintus (2014). On the one hand, investment in information, i.e. a higher n, raises expected output and a higher γ means that the additional output is worth less – the standard effect of γ. On the other hand, a higher

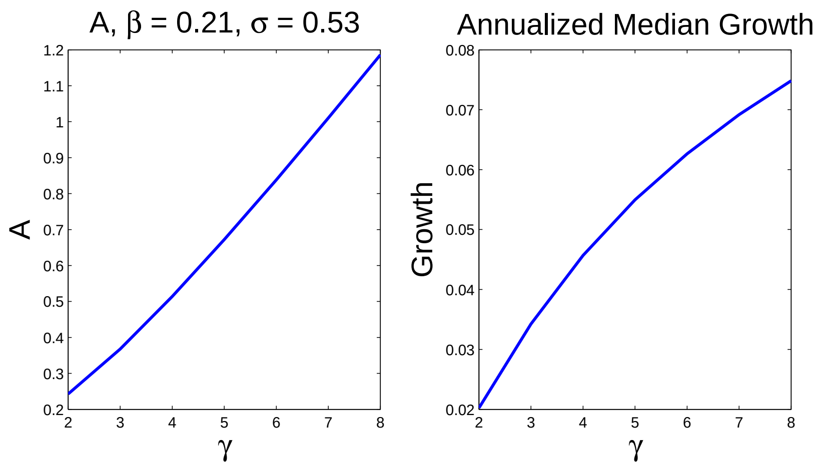

Figure 5: The effect of γ

n also lowers the variance of output, and this effect is especially welcome when the agent is risk averse. In other words, growth is tied to reductions in risk, risk aversion rises with γ, and for a large set of parameters this effect dominates. At the parameter values used in the calibration and listed in eq. (33), investment and growth both rise with γ

As eq. (28) will show, the median of the economy's growth factor is proportional to A, and therefore these plots also apply to growth. In contrast to the deterministic model in which investment is in capital, this model produces growth that *rises* with γ.

The left panel of Fig 5 shows that for $\gamma > 6.9$, the constraint on non-negativity of consumption is violated. That is, the constraint $n \leq y$ fails, and the solutions are not valid beyond that point. Large risk aversion induces too much investment in information. We shall return to this in the calibration.

3.5 The shape of the growth distribution in the long run

Since $\zeta\tilde{}N(0,1)$, ζ^2 has a chi squared distribution with one degree of freedom, and a density

$$\frac{1}{\sqrt{2\pi}}\zeta^{-1/2}e^{-\zeta/2}.$$

Therefore

$$\Psi(g) \equiv \Pr(\tilde{g} \leq g) = \int_{A/\sigma^2 g}^{\infty} \frac{1}{\sqrt{2\pi}}\zeta^{-1/2}e^{-\zeta/2}d\zeta$$

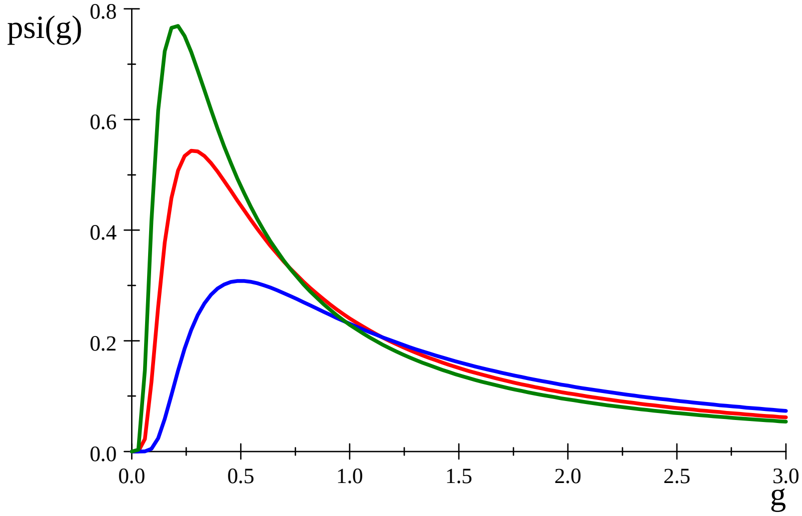

Figure 6: $\psi(g)$ FOR $A/\sigma^2 = 1.5$ (BLUE), 0.85 (RED) AND 0.6 (GREEN)

and therefore the density of g is the derivative w.r.t. g:

$$
\begin{aligned}
\psi(g) &= \frac{A}{\sigma^2} g^{-2} \frac{1}{\sqrt{2\pi}} \left(\frac{A}{\sigma^2} g^{-1} \right)^{-1/2} \exp\left(-\frac{A}{2\sigma^2 g} \right) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{A}{\sigma^2} \right)^{\frac{1}{2}} g^{-3/2} \exp\left(-\frac{A}{2\sigma^2 g} \right)
\end{aligned}
$$

This has exactly the same functional form as the density for the first passage of a Brownian motion through a flat boundary. In that context the variable g would be a waiting time variable.[7]

This distribution has just one parameter, namely A/σ^2. The middle, red line plots ψ at $\frac{A}{\sigma^2} = \frac{0.24}{(.53)^2} = 0.85$. This is at the parameters for the calibrated parameters in eq. (33). When the parameter is lower at 0.6 we have the green line, and when it is higher at 1.5, we get the blue line

3.6 The right tail of the distribution of growth

A fat tail in the growth distribution implies there will occasionally be growth realizations significantly higher than average. This may help explain episodes such as the industrial revolution which was a period of rapid growth and technological improvement. Technically, we ask if asymptotically the tail is Pareto and, if so, of what

[7]Presumably this relates to the normality of the Brownian motion incrfements. On first-passage distributions see, e.g., http://en.wikipedia.org/wiki/First-hitting-time_model

thickness. The right-tail thickness is the value κ for which the following statement is true:

$$\lim_{g\to\infty} g^{\kappa}\left(1-\Psi\left(g\right)\right) = \lim_{g\to\infty} \frac{1-\Psi\left(g\right)}{g^{-\kappa}} \to \text{constant} > 0.$$

By l'Hôpital's rule, this is the same as $\lim_{g\to\infty} \frac{\psi(g)}{\kappa g^{-\kappa-1}}$

$$\lim_{g\to\infty} \frac{\psi\left(g\right)}{\kappa g^{-\kappa-1}} = \lim_{g\to\infty} \frac{\frac{1}{\sqrt{2\pi}}\left(\frac{A}{\sigma^2}\right)^{\frac{1}{2}} g^{-3/2} \exp\left(-\frac{A}{2\sigma^2 g}\right)}{\kappa g^{-\kappa-1}}$$

which implies that

$$\kappa = 1/2.$$

I.e., it has a thick right tail and g therefore has an infinite mean.

3.7 Gibrat's Law without information sharing

As $\tau \to 0$ this becomes a stochastic Ak model. Growth rates are identically and independently distributed over periods. Eventually, then, output obeys Gibrat's law. The distribution of growth rates is identical from period to period.

That holds for a single agent. In a world composed of agents who were informationally isolated this would match experience to the extent that there is little overall tendency for followers to catch up with leaders or fall further behind

In would, however, imply a steadily rising dispersion in the world income distribution and this is counterfactual (Sala-i-Martin 2006). To hold the distribution together we add another agent and information sharing – Kremer (1993) and BMP have stressed its importance. We shall add information sharing in Sec. 5. There we shall find that country growth rates are negatively autocorrelated, even though the world as a whole obeys Gibrat's Law.

4 Transition dynamics

The experiment here is to set the parameters to fit long-run growth facts and then, using those parameters, to simulate the transition and check for a resemblance between model and data. The data show an essentially zero rate of growth 260 years ago and reaching a seemingly long-run growth rate of two percent per year. The world's experience over the past three centuries is summarized, in terms of levels, in Fig. 7. It appears to show that growth rates have risen and, as Pritchett (1999) argues, that there has been much divergence at various points in history.

Log income plots show that growth rates gave risen slightly, as shown by the overall convexity of the eight lines plotted in Fig. 8, with an overall slope of 0.02 per

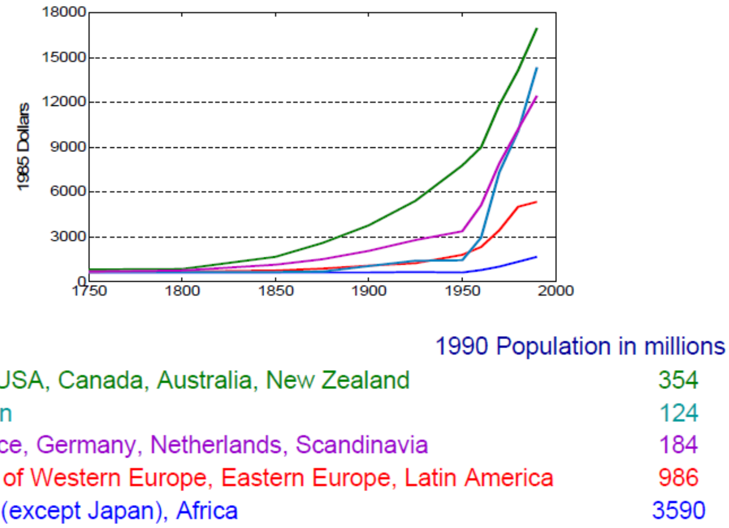

Figure 7: GDP PER CAPITA SINCE 1750, FIVE REGIONS

year.[8] The secular rise in growth rates is also supported by data from Madison. This secular rise in the rate of growth at the world level supports the model's implication in the first panel of Fig. 3 in which the rate of growth is rising during the transition. Rappaport (2006) finds non-monotonic or declining rates of growth post 1945, but one would expect that the war led to destruction of physical capital, but less so of human capital, i.e., a lower production share of the capital that needed rebuilding.

4.1 Estimation

For 166 countries and regions in a sample that has data for 1990-2008, and 149 countries in a the sample starting 1979 -1989,[9] to calculate 30, 27, 20, and 10 year growth factors g_i over the period immediately preceding 2008.[10]

We shall estimate $\psi(g)$ via maximum likelihood the distribution of growth factors g_i treating the observations as independent. That is, we calculate

$$\hat{\theta}_{\text{ML}} \equiv \arg\max_{\theta} \prod_{i=1}^{N} \psi(g_i; \theta),$$

[8]The convergence of the series is, however, driven in part by selection bias because the eight countries in the graph the world's leaders in 2000. De Long made this point in a similar context.

[9]Seventeen countries became independent of the USSR in 1990, so their data are included for 10 year growth factors, whereas the USSR data for 30, 27, 20 year growth rate. The source of the data is the Maddison Project Database, from http://www.ggdc.net/maddison/maddison-project/data.htm)

[10]More precisely, we use the sample from 1979-2008 for 30 years, 1982-2008 for 27, 1989-2008 for 20 years, and 1999 - 2008 for 10 years. This ensures that no country in the sample has missing data.

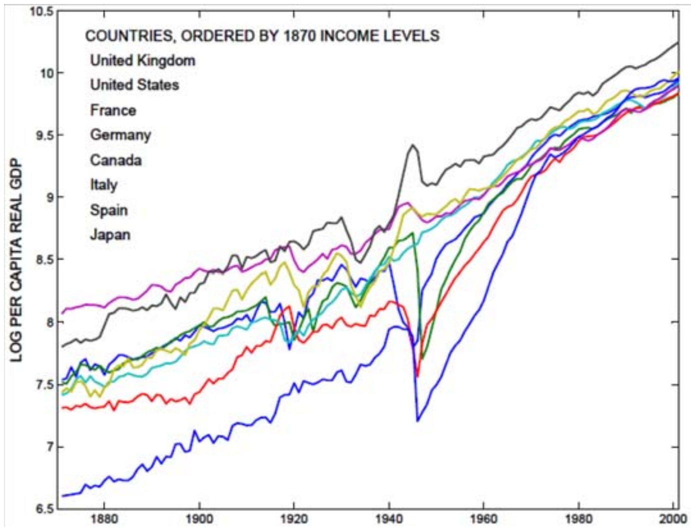

Figure 8: Log income, eight countries

where $\psi(g,\theta) = \frac{1}{\sqrt{2\pi}}\theta^{\frac{1}{2}} g^{-3/2} \exp\left(-\frac{\theta}{2g}\right)$, where $\theta = A/\sigma^2$.

Thus ψ is a one-parameter distribution which, according to (29), has a median $g^{\text{med}} = 2.13\theta$. We treat the countries' growth rates as i.i.d. variables, which supposes that they share the same parameters, but that they do not share information.

Fig. 9 reports the fit for four alternative time units. The scale in the plots is different on the right-hand and left-0hand axes for ψ and for the actual frequencies. For the frequencies the bin size is ten percentage points. The distributions all add up to unity, but not the entire support of ψ is shown on the horizontal axis.[11]

	$\hat{\theta}_{\text{ML}}$			
years	10	20	27	30
g^{med} annualized	1.027	1.016	1.012	1.011
g^{med} not annualized	1.255	1.251	1.210	1.169

Table 1: ML estimate for different time units

Fitted to the 30-year growth-factor distribution, the ML estimate implies a median growth rate of just 1.1 percent. As the period is shortened, median growth rises. This is probably because the shorter-period frequency distributions are less skewed so that median growth is closer to mean growth.

[11] The outlier with a growth factor of around15 over 27 and 30 years in GDP per capita is Equatorial Guinnea.

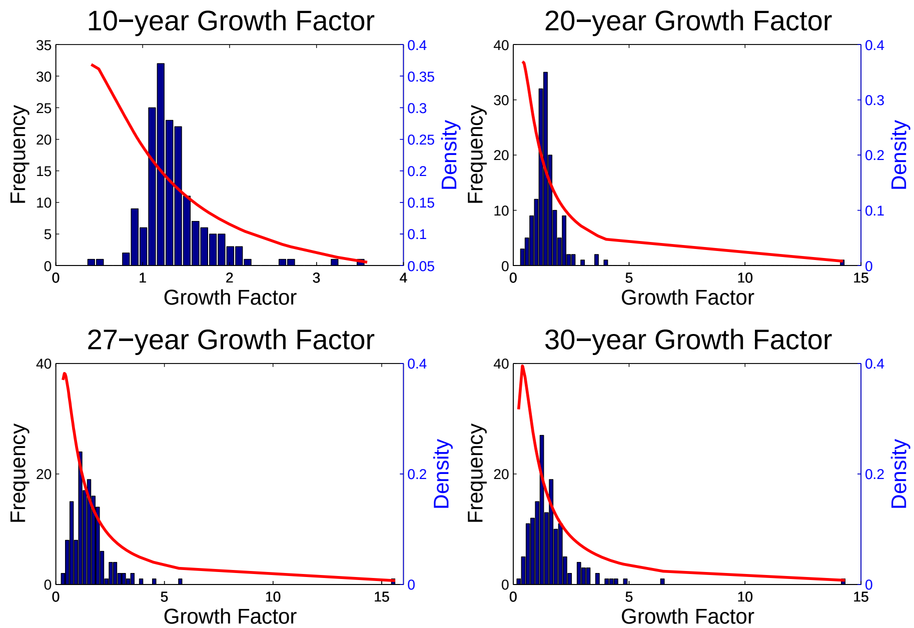

Figure 9: Maximum likelihood fit (not annualized)

4.2 Calibration

We set $\rho = 2$ so that we have constant growth in the limit. We set $\gamma = 2$, i.e., at the smallest value for which the objective is strictly concave w.r.t. x. This allows the model to achieve a level of informational investment well below unity (see the discussion of the left panel of Fig 5).

To fit the pattern shown in Fig. 8 we shall need an extra parameter, namely the frequency at which learning takes place, i.e., the length of a period. Denote this parameter by T. The model is not homogeneous of degree zero in T and σ^{-2}; i.e., implications depend on more than just the ratio T/σ^2 . In other words, having signals with half the precision but twice as often changes the implications. The reasons that are not entirely clear to me at this writing, but presumably this is similar to lengthening the period of production of the capital good – information is the only capital the model has. When there is discounting of the future, the longer is the period of production of capital, the lower is the return to investment and the fraction of income invested goes down.

With $\rho = \gamma = 2$ already specified, this leaves three parameters to be chosen: β, σ^2 and T. They must satisfy the following three restrictions on these parameters:

1. *Relation between T and β.*—Since the growth rate of 2% is annualized, the same must be true of the discount factor which we shall set at 0.95 per year. Then

for a period of length T, discount factor should be

$$\beta = 0.95^T. \tag{30}$$

2. *Annualized growth of* 2%.—Since $\gamma = 2$, eq. (28) has $\omega = \omega^{1/\gamma} = 1$. Then median growth over the period of length T is $2.13\beta^{1/2}/\sigma$. We ask that median income growth compounded over T periods be 1.02^T, which implies that

$$1.02^T = 2.13\frac{\beta^{1/2}}{\sigma} \tag{31}$$

3. *The share of investment in output.*—According to McGrattan and Prescott (2010) and Corrado and Hulten (2009), businesses invest similar amounts in tangible and intangible capital. Taking roughly 10 to 12 percent of GNP for each, the total appears to be 20-24 percent of income. This does not include residential investment. The limiting value is A which is given in (24). With $\gamma = 2$ and the resulting $\omega = 1$,

$$A = \sqrt{\beta\sigma^2} = 0.24 \tag{32}$$

This yields the parameter values

$$\begin{aligned} \beta &= 0.21 \\ \sigma &= .53 \\ T &= 30.7 \end{aligned} \tag{33}$$

Evidently this has to be a low-frequency model if we are to fit both the growth rate and the investment share. David (1991) has argued that absorbing an important new technology – a GPT – takes decades. He argues that the booms of the 1920's and 1990's are traceable to inventions and developments that, in each case, took place several decades earlier. Since ξ_i are technological signals, a particularly productive time is when the $\bar{\xi}$ signal is close to ε itself. This then leads to high growth.

The results are in Fig. 10. Panel 1 starts economies off at the same initial income of $e^{7.5}$ but varies the precision of information on θ. i.e., it varies τ_0. Knowledge of θ implies $\tau_0 = 0$, and such an economy would evolve along the dashed line. The higher is the initial ignorance of θ, and the is lower the growth rate and the distribution of log output diverges. In other words, differences in information cause long-run income-level gaps. But all countries share the same long-run growth rate, and therefore information differences have level effects only. All this is on the assumption that countries do not share the same θ and ε's or that they do not communicate.

Panel 2 is supposed to be the theoretical counterpart of Fig 8. It varies initial income but keeps $\tau > 0$ the same for all three countries plotted. Also plotted is the $\tau_0 = 0$ line for reference – the top dashed line. Since the growth rate depends only on τ and not on y, the three lines are parallel – all 3 countries grow at the same rate, but that rate rises over time. Nevertheless, convergence takes hundreds of years, as seen in Panels 3 and 4 of Fig 4.

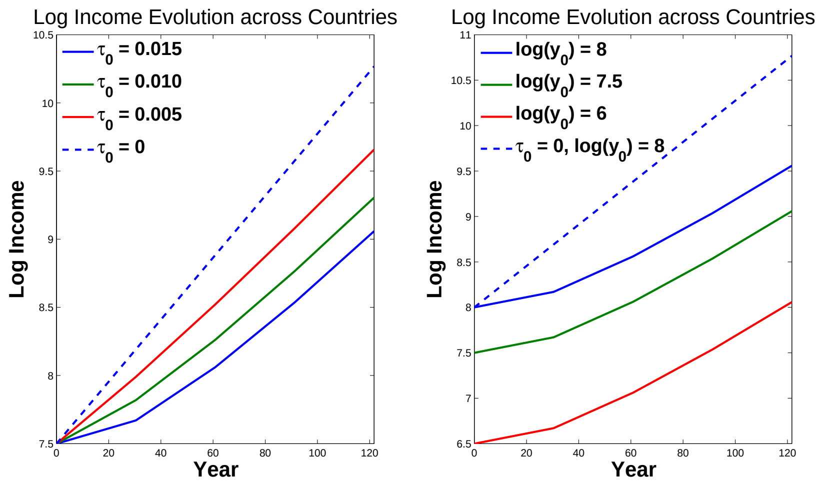

Figure 10: PANEL 1: EQUAL y_0. PANEL 2: EQUAL τ_0

4.2.1 Calibration based on business sector output only

One interpretation of the model is that it really has two sectors: information and all other goods/services. In the starkest case we may think of two identities: $y_i = c_i + n$ and $y_0 = c_0 + x_0$, where i denotes the business sector and 0 denotes non-business services. GDP is $p_i y_i + p_0 y_0$. Utility is defined over both final goods. For an example of this type of accounting, see pp. 101-2 of McGrattan and Prescott (2010).

The BLS (2015) reports that the business sector accounted for about 78 percent of the value of gross domestic product (GDP) in 2000. if n is financed from final goods only, whereas both business and non-business output is consumed, then we should equate A not to 0.24, but to

$$A = \frac{n_t}{y_{i,t} - y_{0,t}} = \frac{n_t}{(0.78)\, GDP_t} = \frac{.24}{.78} = .31$$

With the same two percent target for growth, we now have

$$1.02^T = 2.13 \frac{\beta^{1/2}}{\sigma}, \quad A = \sqrt{\beta \sigma^2} = 0.31, \text{ and } \beta = 0.95^T$$

This yields the parameter values

$$\beta = 0.25, \quad \sigma = 0.62 \quad \text{and} \quad T = 27.1$$

The outcome is depicted in the Appendix Figures 13, 15, and 14. They are quite similar compared to Figs 4, 5, and 10. Although T is smaller, there is little qualitative difference between the model's implications.

5 Information sharing with two agents

Information sharing introduces a free-riding incentive in acquiring signals. Also, the distribution of income becomes a state variable that determines each agent's investment in signals and, therefore, the distribution of growth rates.

Let's assume that an agents' εs have a common component u and an idiosyncratic component ω so that for agent i at date t

$$\varepsilon_{i,t} = u_t + \omega_{i,t}.$$

Both components are i.i.d., mean-zero variates, and normally distributed, where $\sigma_{\mathrm{u}}^2 = \infty$, and where (ω, ω') are independent of each other and have a common variance σ_ω^2. The $\omega_{i,t}$ are nuisance parameters from the point of view of using signals of others. As $\sigma_\omega^2 \to 0$, the signals of different agents become perfect substitutes, whereas when $\sigma_\omega^2 \to \infty$ they are of no value to other agents.

We shall solve the two-agent case which will allow us to discuss inequality in the income distribution and its effects. To ease notation we shall refer to the shocks of the two agents as

$$\varepsilon = u + \omega \quad \text{and} \quad \varepsilon' = u + \omega',$$

and to their incomes and investments as (y, n) and (y', n') respectively. The agents have the same technology for getting signals on ε and ε' respectively, described in (4) and (5).

The information externality.—We shall develop the Bellman equation for the first agent who cares about ε. In terms of the parameter ε, ω, and ω', the Likelihood functions of the two signal averages are $\bar{\xi}_n \sim N\left(\varepsilon, \frac{\sigma^2}{n}\right)$ and $\bar{\xi}'_{n'} \sim N\left(\varepsilon + \omega' - \omega, \frac{\sigma^2}{n'}\right)$. Then conditional on ε alone, the information contained in the other player's signals is distributed

$$\bar{\xi}'_n \sim N\left(\varepsilon, 2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right). \tag{34}$$

The aggregate state is (y, y'). We look for a symmetric Nash equilibrium in Markov strategies

$$n = \phi(y, y') \quad \text{and} \quad n' = \phi(y', y).$$

Let a player's lifetime utility be $v(y, y')$. To write the Bellman equation we need the bivariate distribution of $\left(y_{+1}, y'_{+1}\right)$ as a function of the decisions (n, n'). First define the standard deviations of the univariate distributions for y and y' as

$$s = \left(\frac{n}{\sigma^2} + \frac{1}{2\sigma_\omega^2 + \sigma^2/n'}\right)^{-1/2} \quad \text{and} \quad s' = \left(\frac{n'}{\sigma^2} + \frac{1}{2\sigma_\omega^2 + \sigma^2/n}\right)^{-1/2} \tag{35}$$

Then the following claim is proved in the Appendix:

Lemma 2 *Conditional on (n, n'), we can write*

$$y_{+1} = (s\zeta)^{-2} \quad \text{and} \quad y'_{+1} = (s'\zeta')^{-2} \tag{36}$$

where ζ and ζ' are distributed $N(0,1)$ with a cross-correlation coefficient

$$r = \frac{\sigma^2}{n + 2\frac{\sigma_\omega^2}{\sigma^2} + n'} \left(\frac{n'}{\sigma^2} + \left(2\sigma_\omega^2 + \frac{\sigma^2}{n} \right)^{-1} \right)^{1/2} \left(\frac{n}{\sigma^2} + \left(2\sigma_\omega^2 + \frac{\sigma^2}{n'} \right)^{-1} \right)^{1/2}. \tag{37}$$

Their bivariate normal density is

$$p(\zeta, \zeta') = \frac{1}{2\pi\sqrt{(1-r^2)}} exp \left(-\frac{\zeta^2 - 2r\zeta\zeta' + \zeta'^2}{2(1-r^2)} \right).$$

Corollary 2 *As $\sigma_\omega^2 \to 0$, $s \to s'$ and $r \to 1$,*

This corollary reflects the fact that since the agents share all the information they have and about the same thing, their outputs will be perfectly correlated. Conversely, as $\sigma_\omega^2 \to \infty$, $r \to 0$, which reflects the fact that each agent's information is irrelevant for the other agent.

The Bellman equation.—It reads

$$v(y, y') = \max_n \left\{ \frac{(y-n)^{1-\gamma}}{1-\gamma} + \beta \int v\left((s\zeta)^{-2}, (s'\zeta')^{-2} \right) p(\zeta, \zeta')\, d\zeta d\zeta' \right\} \tag{38}$$

where (s, s') satisfy (35), and where the other player's strategy $n' = \phi(y', y)$ is taken as given

The aggregate law of motion.—Substituting the equilibrium strategies into (36), we get the aggregate law of motion for (y, y') which is a first-order Markov process in $\mathtt{R}_+^2$:

$$\begin{pmatrix} y_{+1} \\ y'_{+1} \end{pmatrix} = \begin{pmatrix} \left(\frac{\phi(y,y')}{\sigma^2} + \frac{1}{2\sigma_\omega^2 + \sigma^2/\phi(y',y)} \right)^{-1/2} \zeta \\ \left(\frac{\phi(y',y)}{\sigma^2} + \frac{1}{2\sigma_\omega^2 + \sigma^2/\phi(y,y')} \right)^{-1/2} \zeta' \end{pmatrix}$$

This may be used to run a regression of a general Nelson-Phelps type, to do with the persistence of leadership using simulated data: Selecting date t leader and labeling his output y_t and that of the follower y'_t, we regress their next-period's differential on today's differential

$$\ln y_{t+1} - \ln y'_{t+1} = \beta_0 + \beta_1 (\ln y_t - \ln y'_t)$$

We expect β_1 to be less than one, increasing in σ_ω^2, and as $\sigma_\omega^2 \to \infty$, we expect that $\beta_1 \to 1$.

5.1 The case $\sigma_\omega^2 = 0$

This is the case when the two agents' signals are perfect substitutes in the sense that each agent finds the other agent's signals as valuable as his own. In this case the countries will converge to each other in one period, and thereafter will forever remain the same. Of course that will not get rid of the free rider problem. Rather, the opposite.

Since the agents will have the same information and the same target, they will take the same decision, and will end up with $y'_{+1} = y_{+1}$. Using Corollary 1, the Bellman equation (38) becomes

$$v(y, y') = \max_n \left\{ \frac{(y-n)^{1-\gamma}}{1-\gamma} + \beta \int v\left((s\zeta)^{-2}, (s\zeta)^{-2}\right) \frac{e^{-\zeta^2/2}}{\sqrt{2\pi}} d\zeta \right\}, \tag{39}$$

subject to

$$s = \left(\frac{n}{\sigma^2} + \frac{\phi(y', y)}{\sigma^2} \right)^{-1/2} \tag{40}$$

because $\sigma_\omega^2 = 0$.

Let $\gamma = 2$. The solution, proved in the Appendix, has the properties summarized in the following claim:

Proposition 4 *When $\gamma = 2$, the solution to (39) has the following properties: Individual investment is*

$$\phi(y, y') = \frac{(1+\lambda)y - \lambda y'}{1+2\lambda} \tag{41}$$

total investment is

$$\phi(y, y') + \phi(y', y) = \frac{1}{1+2\lambda}(y + y') \tag{42}$$

where

$$\lambda = \frac{1}{4}\sqrt{1 + \frac{16}{\beta\sigma^2}} - \frac{3}{4} > 0, \tag{43}$$

and where the inequality in (43) holds if $\beta\sigma^2 < 2$. Finally,

$$v(y, y') = -\frac{1+2\lambda}{2\lambda}\left(\frac{1}{y+y'} + \frac{1}{\lambda(y+y')} \right). \tag{44}$$

A few comments about this proposition:

1. *Free riding.*—The free rider property is evident in (41) where, because $\lambda > 0$, the strategy is declining in the other player's output y'. The strength of the effect rises with $\beta\sigma^2$

2. Eq. (42) states that aggregate investment is a constant fraction, $1/(1+2\lambda)$, of aggregate output given in (42). Now, (36) implies that each country's growth is

proportional to total investment. Therefore Gibrat's law holds at the aggregate level even though it fails at the level of the individual country. That is, growth of each country depends on total output $y + y'$, and not on its division between the players. In other words, each country's growth is a function of total aggregate output, and inequality in is irrelevant. For an individual country, an unusually high growth rate for one country raises total investment less than in proportion, and so growth rates are negatively autocorrelated.

3. The properties described above hold only if $\beta\sigma^2 < 2$, but in fact this is the relevant region of the parameter space. In the calibration in eq. (32) this value was set at 0.24. In any event, when $\gamma = 2$ (which is the case here) the solution to the one-agent problem requires that $\beta\sigma^2 < 1$ – see (26).

4. Since $\frac{\phi}{y} \leq \frac{1+\lambda}{1+2\lambda} < 1$, consumption of both agents is always positive, but the solution ceases to be valid when

$$\frac{y'}{y} > \frac{1+\lambda}{\lambda}$$

because the agent then wants negative investment which is not feasible.

5. Eq. (44) states that v is a function only of the sum $y + y'$. In other words, a mean-preserving spread of the outputs leaves the lifetime utilities of both players unchanged! This surprising conclusion arises because a player's rise in y lowers the other player's n' through the free riding effect by exactly the amount that leaves that player's utility unchanged. As expected, however, v is concave.

5.1.1 The scale effect in equilibrium

Since at least Arrow (1962) growth theory has recognized that technological information is more valuable when used at a larger scale. Here, "scale" means the number of production processes on which information is used to guide decisions, and we expect the point to be valid here too. But could the free rider effect offsets this fact? It is logically possible for equilibrium investment in the two-agent economy to fall short of that in the one-agent economy.

We ask, then, whether two identical countries generate more signals than a single agent would with the same income per head. That is, when $y = y'$, does total investment exceed investment in a single economy with the same income per head? The answer is yes

Proposition 5 *When $\gamma = 2$ and $\sigma_\omega^2 = 0$, a two-agent economy generates more signals and higher income growth per head than a one-agent economy*

Proof. According to (41) and (43),

$$\begin{aligned}
\frac{n+n'}{y} &= \frac{2\phi(y,y)}{y} = 2\left(1 + 2\left(-\frac{3}{4} + \frac{1}{4}\sqrt{1 + \frac{16}{\beta\sigma^2}}\right)\right)^{-1} = \left(\frac{1}{2} + -\frac{3}{4} + \frac{1}{4}\sqrt{1 + \frac{16}{\beta\sigma^2}}\right)^{-1} \\
&= \left(\sqrt{\frac{1}{16} + \frac{1}{\beta\sigma^2}} - \frac{1}{4}\right)^{-1}
\end{aligned}$$

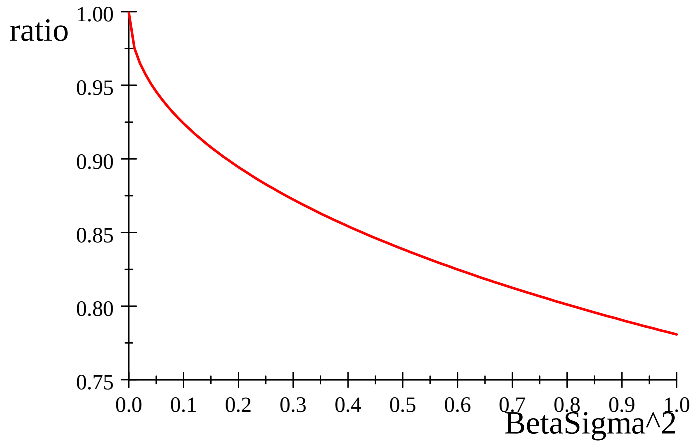

Figure 11: INVESTMENT IN A ONE-AGENT ECONOMY RELATIVE TO TOTAL INVESTMENT IN A TWO-AGENT ECONOMY

whereas in the single agent case (26) says $n/y = \sqrt{\beta\sigma^2}$. The ratio of the one economy divided by the two economies' total investment is

$$\sqrt{\beta\sigma^2}\left(\sqrt{\frac{1}{16}+\frac{1}{\beta\sigma^2}}-\frac{1}{4}\right)=\sqrt{\frac{\beta\sigma^2}{16}+1}-\frac{\sqrt{\beta\sigma^2}}{4}<1$$

whenever $\beta\sigma^2 > 0$ because $\sqrt{1+x} < 1+\sqrt{x}$ for any $x > 0$. ■

Plotting the ratio in Fig. 11 we find it to be less than one and decreasing in $\beta\sigma^2$; the latter must be below 1 for the one-agent solution to be valid.The figure shows that in a one agent economy investment is at least seventy eight percent of what it would be in a two-agent economy.

5.1.2 Numerical simulation

We use the parameter values

$$\sigma = 1,\ \sigma_w^2 = 4, \beta = 0.95,\ \gamma = 2$$

TABLE 2: PARAMETER VALUES FOR FIG. 12

The free-riding effect derived in (41) for the case $\sigma_\omega^2 = 0$ is present even when σ_ω^2 is positive – this is seen in Panel 1 of Fig. 12. Panel 2 shows that the concavity of v in y shown in (44) for the case $\sigma_\omega^2 = 0$ is present even when σ_ω^2 is positive, but the concavity in y' appears not to.

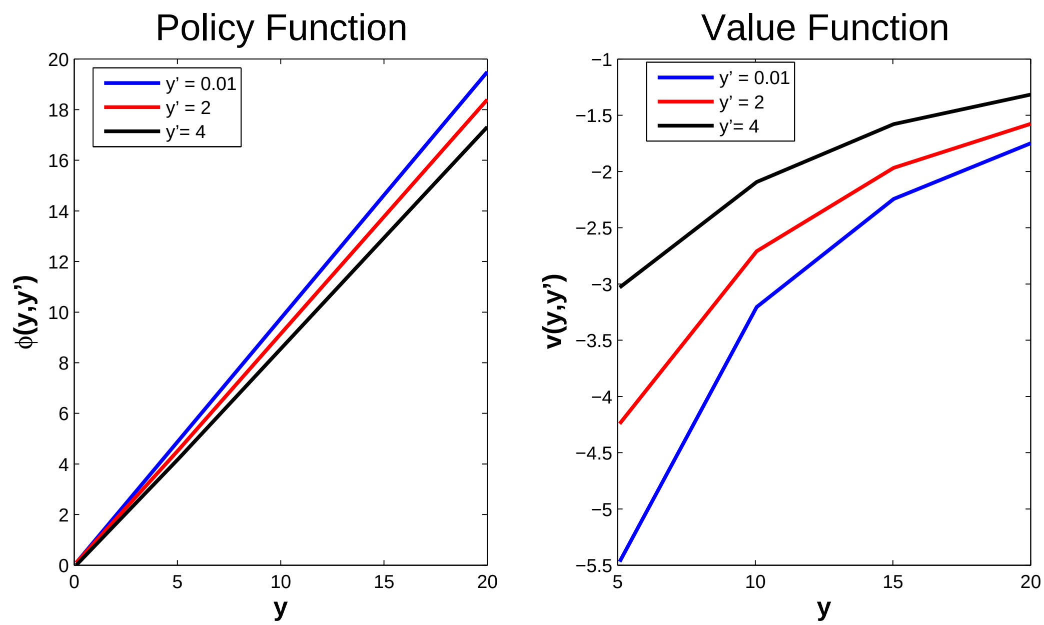

Figure 12: $\phi\left(y, y'\right)$ AND $v\left(y, y'\right)$ WITH TWO AGENTS.

6 Conclusions

The present model focuses on beliefs over possible policies and decisions and their effects on output and growth. As beliefs evolve, output rises. We assume that agents use a model that nests the true model and that they learn from experience. It turns out that information differences have level effects only. All countries share the same long-run growth rate.

The model has led to four conclusions. First, the process of informational investment and the use of the information to raise output makes sense quantitatively only if the periods are decades, not years – the fluctuations are low-frequency events in which case the industrial revolution is a plausible right-tail event. Indeed, the growth distribution has a thick right tail. Second, the long-run growth rate depends on the technology for accumulating signals and the degree of risk aversion – the latter in a way that is counter to the standard model. Risk aversion raises growth because growth is due to better information which also reduces aggregate risk. Third, an informational advantage has a level effect, but no growth effect. Gibrat's Law holds in the limit which is not unlike world experience – there is little overall tendency for followers to catch up with leaders or fall further behind.

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Appendix: Proofs

6.1 Proof of Proposition 2

The Bellman equation reads

$$vy^{1-\gamma} = \frac{1}{1-\gamma}(1-A)^{1-\gamma} y^{1-\gamma} + v\beta \int \left(\frac{1}{\sqrt{n}}\sigma\zeta\right)^{-2(1-\gamma)} \phi(\zeta)\, d\zeta$$

where ϕ is the standard normal density. Then

$$vy^{1-\gamma} = \frac{1}{1-\gamma}(1-A)^{1-\gamma} y^{1-\gamma} + v\omega\beta \left(\frac{1}{\sqrt{Ay}}\sigma\right)^{2(\gamma-1)}$$

where ω is given in (23).

Next, verify that $n = Ay$ will satisfy the FOC.—The Bellman equation reads

$$vy^{1-\gamma} = \frac{1}{1-\gamma}(y-n)^{1-\gamma} + v\omega\beta\sigma^{2(\gamma-1)} n^{1-\gamma}$$

The FOC then reads

$$-(y-n)^{-\gamma} + (1-\gamma)\, v\omega\beta\sigma^{2(\gamma-1)} n^{-\gamma} = 0$$

Substituting $n = Ay$, it reads

$$(1-A)^{-\gamma} y^{-\gamma} = (1-\gamma)\, v\omega\beta\sigma^{2(\gamma-1)} A^{-\gamma} y^{-\gamma}.$$

This leaves us with two equations

$$\begin{aligned} (1-A)^{-\gamma} &= (1-\gamma)\, v\omega\beta\sigma^{2(\gamma-1)} A^{-\gamma} \\ v &= \frac{1}{1-\gamma}(1-A)^{1-\gamma} + v\omega\beta\sigma^{2(\gamma-1)} A^{(1-\gamma)}. \end{aligned}$$

From the second equation

$$v = \frac{1}{1-\gamma}(1-qA)^{1-\gamma}\left(1-\omega\beta\sigma^{2(\gamma-1)} A^{(1-\gamma)}\right)^{-1}. \tag{45}$$

Then we can substitute to get a single equation in the unknown A

$$(1-A)^{-\gamma} = (1-A)^{1-\gamma}\left(1-\omega\beta\sigma^{2(\gamma-1)} A^{(1-\gamma)}\right)^{-1} \omega\beta\sigma^{2(\gamma-1)} A^{-\gamma}$$

i.e.,

$$1 = (1-A)\left(1-\omega\beta\sigma^{2(\gamma-1)} A^{1-\gamma}\right)^{-1} \omega\beta\sigma^{2(\gamma-1)} A^{-\gamma}$$

Evidently, since we can write

$$\frac{1}{1-A} = \frac{\omega\beta\sigma^{-2(1-\gamma)} A^{-\gamma}}{1-\omega\beta\sigma^{-2(1-\gamma)} A^{1-\gamma}}$$

the equation holds if we set

$$1 = \omega\beta\sigma^{2(\gamma-1)} A^{-\gamma},$$

so that A satisfies (24). Substituting into (45), we get

$$\begin{aligned} v &= \frac{1}{1-\gamma}(1-A)^{1-\gamma}\left(1-\omega\beta\sigma^{2(\gamma-1)} A^{1-\gamma}\right)^{-1} \\ &= \frac{1}{1-\gamma}(1-A)^{1-\gamma}(1-A)^{-1}, \end{aligned}$$

i.e., (25).

6.2 Proof of Lemma 1

We first derive the bivariate distribution of $\varepsilon - x$ and $\varepsilon' - x'$. Conditional on $\left(\bar{\xi}_n, \bar{\xi}'_{n'}\right)$, the posterior mean of ε is

$$E\left(\varepsilon \mid \bar{\xi}_n, \bar{\xi}'_{n'}\right) \equiv \frac{\left(\frac{\sigma^2}{n}\right)^{-1}\bar{\xi}_n + \left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)^{-1}\bar{\xi}'_{n'}}{\left(\frac{\sigma^2}{n}\right)^{-1} + \left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)^{-1}} = \frac{\left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)\bar{\xi}_n + \frac{\sigma^2}{n}\bar{\xi}'_{n'}}{\frac{\sigma^2}{n} + 2\sigma_\omega^2 + \frac{\sigma^2}{n'}}$$

Then

$$\begin{aligned}
\varepsilon - x &= \varepsilon - E\left(\varepsilon \mid \bar{\xi}_n, \bar{\xi}'_{n'}\right) \\
&= \varepsilon - \frac{\left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)\left(\varepsilon + \frac{1}{n}\sum_{j=1}^{n}\eta_j\right) + \frac{\sigma^2}{n}\left(\varepsilon' + \frac{1}{n'}\sum_{j=1}^{n'}\eta'_j\right)}{\frac{\sigma^2}{n} + 2\sigma_\omega^2 + \frac{\sigma^2}{n'}} \\
&= \varepsilon - \frac{\left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)\left(\varepsilon + \frac{1}{n}\sum_{j=1}^{n}\eta_j\right) + \frac{\sigma^2}{n}\left(\varepsilon + \omega - \omega' + \frac{1}{n'}\sum_{j=1}^{n'}\eta'_j\right)}{\frac{\sigma^2}{n} + 2\sigma_\omega^2 + \frac{\sigma^2}{n'}} \\
&= -\frac{\left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)\frac{1}{n}\sum_{j=1}^{n}\eta_j + \frac{\sigma^2}{n}\left(\omega - \omega' + \frac{1}{n'}\sum_{j=1}^{n'}\eta'_j\right)}{\frac{\sigma^2}{n} + 2\sigma_\omega^2 + \frac{\sigma^2}{n'}} \\
&= -\frac{\left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)\bar{\eta}_n + \frac{\sigma^2}{n}\left(\omega - \omega' + \bar{\eta}_{n'}\right)}{\frac{\sigma^2}{n} + 2\sigma_\omega^2 + \frac{\sigma^2}{n'}}
\end{aligned}$$

Similarly,

$$\varepsilon' - x' = -\frac{\left(2\sigma_\omega^2 + \frac{\sigma^2}{n}\right)\bar{\eta}_{n'} + \frac{\sigma^2}{n'}\left(\omega' - \omega + \bar{\eta}_n\right)}{\frac{\sigma^2}{n'} + 2\sigma_\omega^2 + \frac{\sigma^2}{n}}$$

The denominators are the same. Since they are zero mean, $\mathrm{Cov}(\varepsilon - x)(\varepsilon' - x') = E(\varepsilon - x)(\varepsilon' - x')$. Since $E(\eta\eta') = E(\eta\omega) = 0$, and letting

$$D = \left(\frac{\sigma^2}{n'} + 2\sigma_\omega^2 + \frac{\sigma^2}{n}\right)^{-2} = \sigma^{-4}\left(\frac{1}{n'} + 2\frac{\sigma_\omega^2}{\sigma^2} + \frac{1}{n}\right)^{-2},$$

we have

$$\begin{aligned}
E(\varepsilon - x)(\varepsilon' - x') &= -D\left(\frac{\sigma^2}{n'}\right)\left(\frac{\sigma^2}{n}\right)2\sigma_\omega^2 + D\left[\left(2\sigma_\omega^2 + \frac{\sigma^2}{n}\right) + \left(2\sigma_\omega^2 + \frac{\sigma^2}{n'}\right)\right]\frac{\sigma^2}{n'}\left(\frac{\sigma^2}{n}\right) \\
&= D\left(\frac{\sigma^2}{n'}\right)\left(\frac{\sigma^2}{n}\right)\left(2\sigma_\omega^2 + \frac{\sigma^2}{n'} + \frac{\sigma^2}{n}\right) = C_{12} \qquad (46)
\end{aligned}$$

Then

$$\begin{aligned}
C_{12} &= \left(\frac{1}{n'} + 2\frac{\sigma_\omega^2}{\sigma^2} + \frac{1}{n}\right)^{-2}\left(\frac{1}{n'}\right)\left(\frac{1}{n}\right)\left(2\sigma_\omega^2 + \frac{\sigma^2}{n'} + \frac{\sigma^2}{n}\right) \\
&= \left(\frac{1}{n'} + 2\frac{\sigma_\omega^2}{\sigma^2} + \frac{1}{n}\right)^{-2}\left(\frac{1}{n'}\right)\left(\frac{\sigma^2}{n}\right)\left(2\frac{\sigma_\omega^2}{\sigma^2} + \frac{1}{n'} + \frac{1}{n}\right) \\
&= \left(\frac{1}{n'} + 2\frac{\sigma_\omega^2}{\sigma^2} + \frac{1}{n}\right)^{-1}\left(\frac{1}{n'}\right)\left(\frac{\sigma^2}{n}\right) = \frac{\sigma^2}{n + 2\frac{\sigma_\omega^2}{\sigma^2} + n'},
\end{aligned}$$

so that

$$r=\frac{\sigma^2}{n+2\frac{\sigma_\omega^2}{\sigma^2}+n'}\left(\left(\frac{\sigma^2}{n'}\right)^{-1}+\left(2\sigma_\omega^2+\frac{\sigma^2}{n}\right)^{-1}\right)^{1/2}\left(\left(\frac{\sigma^2}{n}\right)^{-1}+\left(2\sigma_\omega^2+\frac{\sigma^2}{n'}\right)^{-1}\right)^{1/2},$$

i.e., (37). As a check we let $\sigma_\omega^2\to 0$, we see that $C_{12}\to s^2=s'^2=\left[\left(\frac{\sigma^2}{n'}\right)^{-1}+\left(2\sigma_\omega^2+\frac{\sigma^2}{n}\right)^{-1}\right]^{-1}\to \left[\left(\frac{\sigma^2}{n'}\right)^{-1}+\left(\frac{\sigma^2}{n}\right)^{-1}\right]^{-1}=\frac{1}{(n+n')/\sigma^2}=C_{12}$. I.e., the correlation coefficient of (ζ,ζ') converges to 1, and $y'_{+1}=y_{+1}$ w.p. 1. Now, since $\varepsilon-x=s\zeta$, where s is defined in (35),

$$\begin{aligned}y_{+1} &= \left[\left(\frac{\sigma^2}{n}\right)^{-1}+\left(2\sigma_\omega^2+\frac{\sigma^2}{n'}\right)^{-1}\right]\zeta^{-2}=\left(\left[\left(\frac{\sigma^2}{n}\right)^{-1}+\left(2\sigma_\omega^2+\frac{\sigma^2}{n'}\right)^{-1}\right]^{-1/2}\zeta\right)^{-2}\\ &\equiv (s\zeta)^{-2}\end{aligned}$$

and similarly $y'_{+1}=(s'\zeta')^{-2}$, i.e., (36).

6.3 Proof of Proposition 4

Our task is to show that if v is given in (44) with λ given in (43), then (i) the policy maximizing the RHS of (39) is indeed as given in (41), and (ii) if we substitute from (44) and (41) into the RHS of (39), we obtain the expression in (44) on the LHS of (39).

Since $\zeta\sim N(0,1)\Rightarrow\int\zeta^2\frac{e^{-\zeta^2/2}}{\sqrt{2\pi}}d\zeta=1$, (39) reads

$$\frac{a}{y+y'}+\frac{b}{y+y'}=\max_n\left\{-\frac{1}{y-n}+\beta s^2(a+b)\right\}. \tag{47}$$

where Matching coefficients, we obtain

$$a=\frac{1+2\lambda}{2\lambda}\qquad\text{and}\qquad b=\frac{a}{\lambda}$$

Let

$$B=\beta\sigma^2(a+b)<0. \tag{48}$$

Then (47) reads

$$\frac{a}{y+y'}+\frac{b}{y+y'}=\max_n\left\{-\frac{1}{y-n}+\frac{B}{n+\phi(y',y)}\right\} \tag{49}$$

The FOC is

$$-\frac{1}{(y-n)^2}-\frac{B}{(n+\phi(y',y))^2}=0,$$

i.e.,

$$\frac{1}{(y-n)^2} = \frac{-B}{(n+\phi(y',y))^2}$$

Then

$$y-n = \frac{n+\phi(y',y)}{\sqrt{-B}} = \lambda(n+\phi(y',y))$$

where $-B > 0$, and where

$$\lambda = (-B)^{-1/2} > 0. \tag{50}$$

This gives

$$\phi(y,y') = \frac{1}{1+\lambda}(y - \lambda\phi(y',y))$$

Using symmetry of ϕ,

$$\phi(y',y) = \frac{1}{1+\lambda}(y' - \lambda\phi(y,y'))$$

This gives two equations in two unknowns. Eliminating $\phi(y',y)$ leaves us with

$$\phi(y,y') = \frac{1}{1+\lambda}\left(y - \lambda\frac{1}{1+\lambda}(y' - \lambda\phi(y,y'))\right)$$

i.e.,

$$(1+\lambda)\phi(y,y') = y - \frac{\lambda}{1+\lambda}(y' - \lambda\phi(y,y')),$$

i.e.,

$$\left(1+\lambda - \frac{\lambda^2}{1+\lambda}\right)\phi(y,y') = y - \frac{\lambda}{1+\lambda}y'$$

Now $1+\lambda-\frac{\lambda^2}{1+\lambda} = \frac{1+2\lambda}{1+\lambda}$, and therefore

$$\frac{1+2\lambda}{1+\lambda}\phi(y,y') = y - \frac{\lambda}{1+\lambda}y'$$

i.e.,

$$(1+2\lambda)\phi(y,y') = (1+\lambda)y - \lambda y'$$

i.e., (41) which, since $\lambda > 0$ shows the free-riding effect.[12]

It remains to be seen whether (49) holds, i.e., whether when we substitute the policy ϕ for n on the RHS of (47), the Bellman equation holds for all (y, y').

$$\frac{a}{y+y'} + \frac{b}{y+y'} = -\frac{1}{y-\phi(y,y')} - \frac{\lambda^{-2}}{\phi(y,y')+\phi(y',y)}. \tag{51}$$

[12]Now Gibrat's law no longer holds at the country level because a high realization of y that is not accompanied by an equiproportional rise in y' will cause ϕ/y to rise, and so will (stochastically) the growth rate

The RHS of (51) reads

$$\begin{aligned}
\frac{a}{y+y'}+\frac{b}{y+y'} &= -\frac{1}{y-\frac{(1+\lambda)y-\lambda y'}{1+2\lambda}}-\frac{1+2\lambda}{\lambda^2(y+y')} \\
&= -(1+2\lambda)\left(\frac{1}{(1+2\lambda)y-(1+\lambda)y+\lambda y'}+\frac{1}{\lambda^2(y+y')}\right) \\
&= -(1+2\lambda)\left(\frac{1}{\lambda(y+y')}+\frac{1}{\lambda^2(y+y')}\right)
\end{aligned}$$

But λ itself is defined in terms of (a,b). From (48) and (50) we have

$$B=-\beta\sigma^2\left(\frac{1+2\lambda}{2\lambda}+\frac{1+2\lambda}{2\lambda^2}\right)=\frac{-\beta\sigma^2}{2\lambda^2}(1+2\lambda)(1+\lambda)$$

But from (50), $B=-\lambda^{-2}$. Substituting for B, we have

$$-\lambda^{-2}=-\beta\sigma^2\frac{(1+2\lambda)(1+\lambda)}{2\lambda^2}$$

or

$$2=\beta\sigma^2(1+2\lambda)(1+\lambda)$$

or

$$\beta\sigma^2 2\lambda^2+3\beta\sigma^2\lambda+\left(\beta\sigma^2-2\right)=0$$

or

$$\begin{aligned}
\lambda &= \frac{-3\beta\sigma^2\pm\sqrt{(3\beta\sigma^2)^2-4(2\beta\sigma^2)(\beta\sigma^2-2)}}{2(\beta\sigma^2 2)} \\
&= \frac{-3\beta\sigma^2\pm\sqrt{9(\beta\sigma^2)^2-8\beta\sigma^2(\beta\sigma^2-2)}}{4\beta\sigma^2} \\
&= \frac{-3\beta\sigma^2\pm\sqrt{(\beta\sigma^2)^2+16\beta\sigma^2}}{4\beta\sigma^2} \\
&= \frac{-3\pm\sqrt{1+\frac{16}{\beta\sigma^2}}}{4} \\
&= \frac{1}{4}\sqrt{1+\frac{16}{\beta\sigma^2}}-\frac{3}{4}
\end{aligned} \tag{52}$$

From (50) we know that $\lambda>0$ which means that only the larger root is a valid solution and only when

$$1+\frac{16}{\beta\sigma^2}>9\Longleftrightarrow 2>\beta\sigma^2 \tag{53}$$

which is always met when the individual problem has an interior solution for which (26) implies we need $\beta\sigma^2 < 1$. If the signals are poor, σ^2 is large and the inequality is violated. Investment is

$$\phi\left(y, y'\right) = \frac{\left(1+\lambda\right)y - \lambda y'}{1+2\lambda}$$

i.e., (41).

Numerical Algorithm for Fig. 12

1. Guess function (using basis functions) for $v^{(I)}\left(y, y'\right)$ for each $\left(y, y'\right)$

2. (a) For each guess of $v^{(I)}\left(y, y'\right)$, guess a function $\phi^{(k,I)}\left(y, y'\right)$ such that

$$\phi^{(k+1,I)}\left(y, y'\right) = \arg\max_{n}\left\{\frac{\left(y-n\right)^{1-\gamma}}{1-\gamma} + \beta\int v\left(\left(s\zeta\right)^{-2}, \left(s'\zeta'\right)^{-2}\right)p\left(\zeta, \zeta'\right)d\zeta d\zeta'\right\}$$
$$s.t\ n' = \phi^{(k)}\left(y', y\right)$$

2. (b) the loop stopped if

$$\sup\left\|\phi^{(k+1,I)}\left(y, y'\right) - \phi^{(k,I)}\left(y, y'\right)\right\| < \epsilon$$

denote the stopped policy function as $\phi^{(I)}$

3. Using $\phi^{(I)}\left(y, y'\right)$, solve the value function

$$v^{(I+1)}\left(y, y'\right) = \frac{\left(y - \phi^{(I)}\left(y, y'\right)\right)^{1-\gamma}}{1-\gamma} + \beta\int v^{(I)}\left(\left(s\zeta\right)^{-2}, \left(s'\zeta'\right)^{-2}\right)p\left(\zeta, \zeta'\right)d\zeta d\zeta'$$

4. The algorithm stops if

$$\sup\left\|v^{(I+1)}\left(y, y'\right) - v^{(I)}\left(y, y'\right)\right\| < \epsilon$$

if not, go back to step 1 with the new guess of $v^{(I+1)}\left(y, y'\right)$

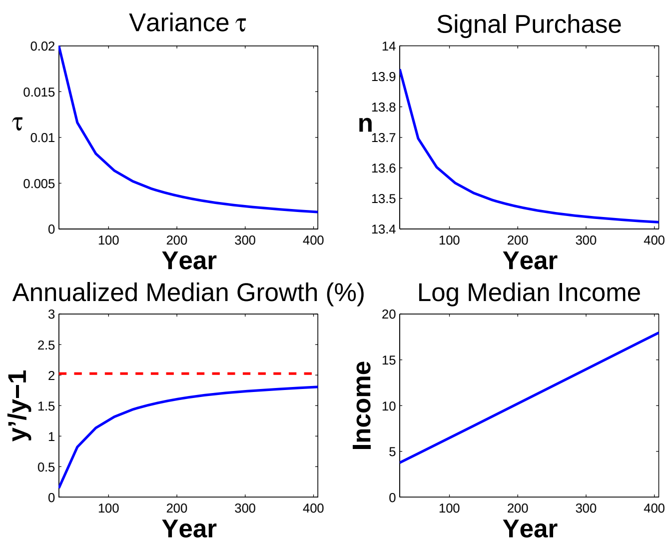

Figure 13: Convergence in real time in the alternative calibration

6.4 Figures for the calibration based on business sector output only

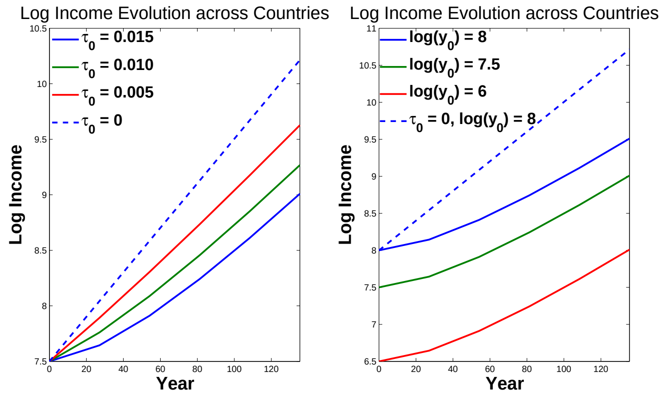

Figure 14: SECOND CALIBRATION: PANEL 1: EQUAL y_0. PANEL 2: EQUAL τ_0

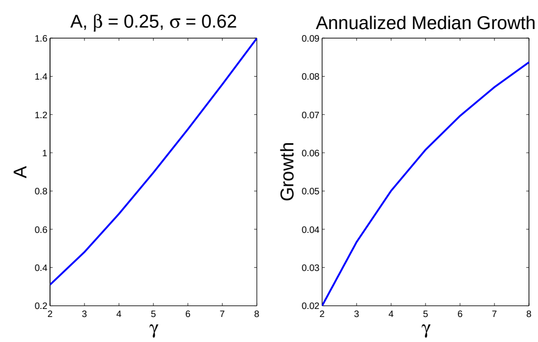

Figure 15: THE EFFECT OF γ IN THE ALTERNATIVE CALIBRATION