Welfare and Tax Policies in a Simple Neoclassical Growth Model with Non Unitary Discounting*

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November 26, 2014

Abstract

When an individual uses different discount rates for different sources of utilities, we call it non unitary discounting. We show that a decision making of the individual becomes time inconsistent. We examine a simple neoclassical growth model with endogenous labor supply in which an individual discounts the utility of consumption and utility of leisure differently. We derive competitive equilibria in which individuals behave in a time consistent way. We investigate welfare performances of the economy by comparing the allocation of competitive equilibria and that by a central planner. The planner cannot commit its initial decisions like the individual. Thus, the planner must solve the allocation problem in a time consistent way. The welfare performance of the allocation by the central planner dominates that of the competitive equilibria from an initial point of view; however, the opposite result obtains from a future point of view. We finally examine whether a government can reconcile this welfare conflict by using tax policies.

JEL classification: E21; H21; O41; Z00

*We would like to thank Takao Asano, Shinsuke Ikeda, Yoshiyasu Ono, Hirokazu Ishise, Ken-ichi Hashimoto, Katsuhiko Hori, Keizo Mizuno, and seminar participants at Okayama, Osaka, Kobe, Tezukayama, and Kwansei Gakuin Universities for many helpful comments. Koichi Futagami gratefully acknowledge financial support from G-COE (Osaka University) 'Human Behavior and Socioeconomic Dynamics.'

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1 Introduction

> A father: Could you stop playing football and mow the yard tomorrow? After completing the job, I will give you \$20.
>
> His son: Really? I will. Then, I can buy a new game software!
>
> **Tomorrow has come.**
>
> The father: Why are you going out to play football? Mow the yard! You promised yesterday, didn't you?
>
> His son: Sorry daddy. I think \$20 is not enough for the job.

Why did the son break his promise? Is this because he is a liar? We can give this an another reason. He is not a liar, however, he discounts reward of \$20, R (or purchase of the software) and benefits of playing football, F, differently. Suppose that, on the first day he applied a lower discount factor β_l for the benefits of playing football and a higher discount factor β_c for the purchase of the software, that is, $\beta_l < \beta_c$. Then, $\beta_l F < \beta_c R$ and $F > R$ are possible simultaneously. If so, the boy can accept his father's job offer on the first day and can simultaneously break it on the next day because the cost outweighs the reward on the day.

This behavior is related to the *domain effect*; that is, the discount rates (or factors) may differ for different domains.[1] In the above example, different discount rates are applied to reward (utility) and effort (disutility). An experimental study by Soman (1989), indeed reported that the behavior like the above boy was often observed among his trial subjects.

When an individual uses different discount rates for different sources of utilities, we call it *non-unitary discounting*. Non-unitary discounting caused by the domain effect clearly provokes time inconsistent decision making of individuals.[2] The time inconsistent decision making arises because an individual cannot commit to his or her future decisions. Therefore, we must treat an individual in different point of time as different individuals who do not follow decisions made by a current individual. We call the decision maker at time t self t. Then we consider choices of each self to be the outcome of an intrapersonal game. We are interested in welfare performances of the economy in which individuals with non-unitary discounting inhabit.

We examine a simple neoclassical growth model with endogenous labor supply in which a representative individual discounts the utility of consumption and

[1] The sign effect is also considered to be important. The sign effect refers to the finding that gains are discounted at a higher rate than losses. Loewenstein (1987), for example, found that on average, 30 undergraduate students discounted obtaining four dollars at higher rates than losing four dollars.

[2] Soman (2004) and Zauberman and Lynch (2005) also showed that people used different rates to discount time and money. Ubfal (2011), by using surveys in rural Uganda, shows that a discount rate for entertainment is lower than a discount rate for money.

utility of leisure differently.[3] The individual plays an intrapersonal game against his or her future selves. We use the recursive methods to analyze the model and derive Markov perfect solution of the market equilibrium This solution becomes time consistent. Specific functional forms for the instantaneous utility function and the production function is used to obtain a closed form solutions.

In order to conduct welfare analyses by using the above framework, we consider a central planner that can command consumption and labor supply decisions of the individual. The planner knows resource constraint of the economy. However, the planner faces the same time inconsistent problem as the individual. Future planners may not follow decisions made by the initial planner. Thus, the planner must solve Markov (time consistent) solution in a similar way to the individual. We compare the welfare level of the market equilibrium with that of the planner's solution. We find that the planner's solution gives a higher welfare level than the market equilibrium for the current self; however, the planner's solution can give a lower welfare level than the market equilibrium for the future selves. This implies that the allocation by the central planner cannot dominate that by the market. This result is in sharp contrast with that of Krusell et al (2002). They examine a simple neoclassical growth model with quasi-geometric discounting. As is well known, such an individual makes time inconsistent decisions like the individual with non-unitary discounting. They show that the market can do its job better than a central planner for all selves; that is, the market equilibrium attains a higher welfare level than the allocation of the central planner. However, in the model with non-unitary discounting, the central planner can do its job better than the market only for selves in early periods. Thus, the central planner must solve a conflict between selves in early periods and ones in the future periods. Therefore, we next examine whether the central planner can solve this problem by using tax policies.

To consider this problem, we derive a time consistent tax policy. We suppose that a government can impose taxes on wage income, interest income, and investment and that there is no government expenditure. We can show that the time consistent tax policy replicates the planning allocation. Therefore, the time consistent tax policy improves the welfare level of the selves in early periods, however, worsens the welfare level of the selves in future periods. Consequently, if the government has to use the time consistent tax policy, then such policy cannot solve the above trade off.

There are only a few theoretical researches related to non-unitary discounting. Futagami and Hori (2010) derive an optimal tax policy by using a continuous time version of a dynamic general equilibrium model without capital accumulation. However, the government in the model is supposed to be commit

[3] Samuleson (1937) proposed the discounted utility model in which an individual maximizes $\int_t^s u_v e^{-\rho(v-t)} dv$ where u_v is the instantaneous utility at time v and ρ is the subjective discount rate. Even if the individual obtain his or her utility from more than two sources like consumption and leisure, a single discount rate has been utilized in all studies since Samuelson (1937). However, as Frederick et al. (2002) criticizes, analyses based on a single discount rate lose their foundation if people apply different discount rates to different sources of utilities.

to her current decisions. In contrast to their analysis, Hori and Futagami (2013) examine the relationships among income, patience and consumption growth, and investigate the interactions among development, patience and saving behaviors.[4]

The rest of the paper is structured as follows. Section 2 sets up the simple neoclassical growth model and explains the mechanism of the time inconsistency of the present model. Section 3 solves the problem of the central planner. Section 4 derives the competitive equilibrium allocation and examines its characters. Section 5 examines welfare properties. Section 6 derives the time consistent tax policy. Section 7 concludes.

2 A Simple Neoclassical Growth Model with Non-Unitary Discounting

2.1 Model

There exists a representative individual in our economy. Thus, the population size is normalized to be one. The individual derives his or her utility from consumption and leisure. We use specific functional forms for the instantaneous utility functions. We assume $\ln c$ and $\ln(1-l)$ for the utility functions for individual consumption, c and individual leisure, $1-l$, respectively. Here, $l \in [0,1]$ stands for labor supply. The individual applies a discount factor $\beta_c \in (0,1)$ to the utility from consumption and $\beta_l \in (0,1)$ to the utility from leisure. Thus, the individual tries to maximize the following:

$$\sum_{t=0}^{\infty} \left[\beta_c^t \ln(c_t) + \beta_l^t \zeta \ln(1-l_t)\right], \tag{1}$$

where $\zeta > 0$ stands for a weight on the utility from leisure. We assume that the individual discounts each utility in the geometric way to focus on the non-unitary discounting. The budget constraint of the individual becomes

$$k_{t+1} = r_t k_t + w_t l_t - c_t, \tag{2}$$

where k stands for the individual holding of capital stock. r and w represent the rental price of capital and the wage rate, respectively. We assume that capital fully depreciates after production in each period.

The production function is supposed to take a Cobb-Douglas form, $Y = AK^{\alpha}L^{1-\alpha}$ where $0 < \alpha < 1$, Y, K, and L are an aggregate output, aggregate capital stock and an aggregate labor supply. Therefore, the resource constraint becomes

$$K_{t+1} = AK_t^{\alpha}L_t^{1-\alpha} - C_t. \tag{3}$$

[4] The model proposed in Banerjee and Mullainathan (2010) is also similar to our model. Their model provides many highly important implications. In their model, there are two types of consumption goods, and the agents do not care about future consumption of one of the two goods but do care about future consumption of the other. In our framework, this is equivalent to the case where either β_c or β_l is equal to zero. Moreover, they do not examine welfare effects of taxes.

2.2 Time Inconsistent Decision Making

We first describe the decision making of the individual who does not take care of the possibility of time inconsistency. To solve the problem, the individual maximizes (1) subject to (2) by constructing the Lagrangian function as follows:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta_c^t \ln(c_t) + \beta_l^t \zeta \ln(1 - l_t)\right] + \sum_{t=0}^{\infty} \lambda_t \left[r_t k_t + w_t l_t - c_t - k_{t+1}\right]. \qquad (4)$$

The solution of this problem is derived in Appendix A and becomes

$$k_{t+1} = \alpha\beta_c A k_t^{\alpha} \left(\frac{1-\alpha}{(1-\alpha) + \zeta(1-\alpha\beta_c)}\right)^{1-\alpha}, \qquad (5)$$

$$l_t = \frac{(1-\alpha)}{(1-\alpha) + \zeta(1-\beta_c\alpha)\left(\frac{\beta_l}{\beta_c}\right)^t}. \qquad (6)$$

We can see that when $\beta_l > (<)\beta_c$, labor supply decreases (increases) as time goes by and approaches to zero (one). When $\beta_l > (<)\beta_c$ the individual puts more (less) weight on the utility from the future leisure than that from the future consumption, he or she will supply less (more) labor as time goes by.

Does the individual follow this decisions after the first period ($t \geq 1$)? The answer is NO. If the individual solves the problem at time t once again, he or she does not supply l_t but supply

$$l_0 = \frac{(1-\alpha)}{(1-\alpha) + \zeta(1-\beta_c\alpha)}.$$

For, he or she constructs the same Lagrangian function (4) at time t once again. Let us, for example, consider the case of $\beta_l < \beta_c$. Such an individual puts less weight on the future leisure. Consequently, he or she initially plans to supply more labor in the future periods. However, when the future period comes, he or she wants to enjoy more leisure like the boy in the introduction and does not work a lot.

3 The Problem of the Central Planner

We first solve the problem of the central planner. The planner can recognize the resource constraint (3). The planner knows the aggregate variables must coincide with their corresponding variables of each individual, that is, $k = K$ and $l = L$. The current planner perceives that planners in the future may not follow decisions made by the current planner. Thus, the current planner must play an intrapersonal game with the future planners.

Assume that the current planner thinks that saving decision and labor supply by any future planners is respectively given by: $k' = g(k)$ and $l(k)$.

We now describe the problem of the planner. Note that the objective, (1) can be expressed as follows:

$$
\begin{aligned}
& \sum_{t=0}^{\infty} \left[\beta_c^t \ln(c_t) + \beta_l^t \zeta \ln(1-l_t)\right] \\
= \quad & \ln(c_1) + \beta_c \left[\ln(c_2) + \beta_c \ln(c_3) + \beta_c^2 \ln(c_4) + \cdots\right] \\
& + \zeta \ln(1-l_1) + \beta_l \zeta \left[\ln(1-l_2) + \beta_l \ln(1-l_3) + \beta_l^2 \ln(1-l_4) + \cdots\right].
\end{aligned}
$$

Thus, the problem of the current planner is given by

$$V_0(k) = \max_{k',l} \left[\ln(Ak^\alpha l^{1-\alpha} - k') + \zeta \ln(1-l) + \beta_c V_c(k') + \beta_l \zeta V_l(k')\right],$$

where

$$V_c(k) = \ln(Ak^\alpha l^{1-\alpha} - g(k)) + \beta_c V_c(g(k)), \tag{7}$$

$$V_l(k) = \ln(1-l) + \beta_l V_l(g(k)). \tag{8}$$

We denote a solution of this current planner's problem as $\widetilde{g}(k)$. We will obtain a solution of an intrapersonal game among planners when the stationary condition holds, that is, $\widetilde{g}(k) = g(k)$ for all k. Moreover, we will see that l takes a constant value over time in its derivation. We present its derivation in Appendix B. The solutions of labor supply and saving decision are given by:

$$
\begin{aligned}
k' = g(k) &= s^* A k^\alpha l^{*1-\alpha}, \\
s^* &= \beta_c \alpha, \\
l^* &= \frac{1-\alpha}{1-\alpha+\zeta(1-\alpha\beta_c)}.
\end{aligned}
$$

Note that this solution does not depend on the discount factor β_l. Moreover, the solution in Section 3 shows that this solution of the central planner coincides with that of the following problem:[5]

$$\sum_{t=1}^{\infty} \beta_c^t \left[\ln(c_t) + \zeta \ln(1-l_t)\right],$$

$$k' = Ak^\alpha l^{1-\alpha} - c.$$

4 Recursive Competitive Equilibrium

We next consider the competitive equilibrium. Individuals take factor prices, r and w as given under this setting. Individuals behave considering these prices depend on the aggregate capital, K and the aggregate labor L. Thus, we denote

[5] We can confirm this result by setting $\beta_c = \beta_l$ in equation (6). This result obtains because of the assumption of the specific functional forms, that is, the log utility function and the Cobb-Douglas production function.

them as $r(K,L)$ and $w(K,L)$. We denote capital stock hold by each individual as k. Each individual makes his decisions taking account of the law of motion of aggregate capital, $K' = G(K,L)$ and the decision rules of their future selves; that is, saving decision, $k' = g(k,K,L)$ and labor supply decision, $l(k,K,L)$.

The current self's problem can be stated as follows:

$$V_0(k,K,L) = \max_{k',l} \left[\ln(rk + wl - k') + \zeta \ln(1-l) + \beta_c \cdot V_c(k',K',L') + \beta_l \cdot \zeta V_l(k',K',L')\right]. \tag{9}$$

We denote the solution of this problem as $\widetilde{g}(k,K,L)$ and $\widetilde{l}(k,K,L)$. The value functions $V_c(k,K,L)$ and $V_l(k,K,L)$ satisfies the following relationships:

$$V_c(k,K,L) = \ln c + \beta_c V_c(g(k,K,L),\ K',L'), \tag{10}$$

$$V_l(k,K,L) = \ln(1-l) + \beta_l V_l(g(k,K,L),\ K',L'). \tag{11}$$

The competitive equilibrium is defined as

1. $g(k,K,L)$ and $l(k,K,L)$ are also the solution of the maximization problem (9) given $V_c(k,K,L)$ and $V_l(k,K,L)$.

2. $V_c(k,K,L)$ and $V_l(k,K,L)$ are the solutions of the functional equations (10) and (11), respectively.

3. The law of motion of individuals' capital coincides with that of the aggregate capital; $g(K,K,L) = G(K,L)$.

4. The labor supply of individuals coincides with the aggregate labor supply; $l(K,K,L) = L$.

5. The factor prices are given by $r = \alpha A \left(\frac{K}{L}\right)^{\alpha-1}$ and $w = (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha}$.

The solution of the market equilibrium is given by:

$$K' = g(K,K,L^E) = s^E A K^{\alpha} (L^E)^{1-\alpha},$$

$$s^E = \frac{\beta_c(1-\beta_l) + \zeta\beta_l(1-\beta_c)}{1-\beta_l+\zeta(1-\beta_c)}\alpha,$$

$$L^E = \frac{(1-\alpha)\left[1-\beta_l+\zeta(1-\beta_c)\right]}{(1+\zeta)\left[(1-\alpha)\{1-\beta_l+\zeta(1-\beta_c)\}+\alpha\zeta(1-\beta_l)(1-\beta_c)\right]}.$$

When $\beta_c = \beta_l (\equiv \beta)$, $s^E = \beta\alpha$ and $L^E = l^*$. We can summarize the preceding arguments as the following lemma.

Lemma 1 *s^E and L^E are increasing functions of β_i $(i = c, l)$. c^E is a decreasing function of β_i $(i = c, l)$; that is,*

$$\frac{\partial s^E}{\partial \beta_c} > 0,\ \frac{\partial s^E}{\partial \beta_l} > 0,\ \frac{\partial L^E}{\partial \beta_c} > 0,\ \frac{\partial L^E}{\partial \beta_l} > 0.$$

The solution of the competitive equilibrium coincides with that of the problem of the central planner when $\beta_c = \beta_l (\equiv \beta)$.

Proof. See Appendix C. ■

5 Welfare

We conduct welfare comparison between the solution of the central planner and that of the competitive equilibrium. Note that due to the time inconsistency, the optimal solution of each individual may not coincide. Because both the solutions are constant over time, we calculate the value function of the current self under the following policy functions:

$$c = Ak^\alpha l^{1-\alpha} - k' = (1-s)Ak^\alpha l^{1-\alpha},$$

$$k' = sAk^\alpha l^{1-\alpha},$$

Note that labor supply l is here supposed to be constant over time. By using this value function, we evaluate it at the respective solutions.

We can calculate the value function $V_0(k)$ of self 0 as follows:[6]

$$\begin{aligned} V_0(k) &= \sum_{t=0}^{\infty}\left[\beta_c^t \ln c_t + \beta_l^t \zeta \ln(1-l_t)\right] \\ &= \frac{1}{1-\beta_c}\ln(1-s) + \frac{1}{(1-\beta_c)(1-\alpha\beta_c)}\ln A + \frac{\alpha\beta_c}{(1-\beta_c)(1-\alpha\beta_c)}\ln s \\ &\quad + \frac{\alpha}{1-\alpha\beta_c}\ln k + \frac{1-\alpha}{(1-\beta_c)(1-\alpha\beta_c)}\ln l + \frac{\zeta}{1-\beta_l}\ln(1-l). \end{aligned}$$

Differentiating this with respect to s, we obtain

$$\frac{\partial V_0(k)}{\partial s} = \frac{1}{1-\beta_c}\frac{-1}{1-s} + \frac{\alpha\beta_c}{(1-\beta_c)(1-\alpha\beta_c)}\frac{1}{s}.$$

Differentiating this with respect to l, we obtain

$$\frac{\partial V_0(k)}{\partial l} = \frac{1-\alpha}{(1-\beta_c)(1-\alpha\beta_c)}\frac{1}{l} + \frac{\zeta}{1-\beta_l}\frac{-1}{1-l}.$$

Therefore, the value function attains its maximum when $s^{OP} = \alpha\beta_c$ and $l^{OP} = \frac{(1-\alpha)(1-\beta_l)}{(1-\alpha)(1-\beta_l)+\zeta(1-\beta_c)(1-\alpha\beta_c)}$. It is easy to show that $\frac{\partial l^{OP}}{\partial \beta_l} = \frac{\partial l^{OP}}{\partial \beta_l} < 0$. When $\beta_c = \beta_l (\equiv \beta)$, $l^{OP} = l^*$. Therefore, from Lemma 1 we can state:

Proposition 2 *When $\beta_l \gtreqless \beta_c$, $L^E \gtreqless l^* \gtreqless l^{OP}$. The saving rate of the central planner is the same as s^{OP}. Moreover, $\beta_l \gtreqless \beta_c$, $s^E \gtreqless s^* = \beta_c$.*

Figure 1 depicts this proposition. Because the saving rate of the central planner and that of the optimal level of the current self are same and does not depend on β_l, the horizontal line depicts this rate. The upward sloping curve depicts the saving rate of the market equilibrium. These lines cross each other when $\beta_c = \beta_l$.

[6] See Appendix D for its derivation.

This shows that the central planner can do its job better for the current self than the market. This is because inefficiency arises when individuals discounts future utilities differently. When $\beta_c > (<) \beta_l$, future selves supply less (more) labor than the current self (self 0) prefers. Please remember that the current self wants to increase (decrease) his or her labor supply gradually as time goes by; however, due to the time inconsistency of decision making in the model with non-unitary discounting, the future selves supply labor less (more). Thus, when $\beta_c > (<) \beta_l$, the current self (self 0) saves less (more) and supply less (more) labor to make the future selves supply more (less) labor

This result is contrary to that of Krusell et al (2002). They showed that the market can do its job better for the current self than the central planner. It is not always the case that the market can do its job better than the central planner when individuals have preferences of non-unitary discounting.

Does the above situation apply to the other selves? To consider this question, we next examine the value function of the selves in the later periods. Specifically, we focus on the value function of the selves in the far future; that is, this implies examining the welfare level of the steady state. When the saving rate and labor supply are constant over time, the steady state level of capital stock is given by

$$k^* = sAk^{*\alpha} l^{1-\alpha} \rightarrow k^* = (sA)^{\frac{1}{1-\alpha}} l.$$

Therefore, the steady state level of consumption becomes

$$c = (1-s)A\left[(sA)^{\frac{1}{1-\alpha}} l\right]^{\alpha} l^{1-\alpha} = (1-s)s^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} l.$$

Thus the welfare level at the steady state becomes

$$\begin{aligned} V_\infty(k) &= \sum_{t=1}^{\infty}\left[\beta_c^{t-1}\ln c + \beta_l^{t-1}\zeta\ln(1-l)\right] \\ &= \frac{1}{1-\beta_c}\ln(1-s)s^{\frac{\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} l + \frac{\zeta}{1-\beta_l}\ln(1-l). \end{aligned}$$

Maximizing conditions with respect to s and l are

$$\frac{-1}{1-s} + \frac{\alpha}{1-\alpha}\frac{1}{s} = 0,$$

and

$$\frac{1}{1-\beta_c}\frac{1}{l} + \frac{\zeta}{1-\beta_l}\frac{-1}{1-l} = 0.$$

Thus, its maximizers are

$$s^\infty = \alpha,$$

$$l^\infty = \frac{1-\beta_l}{1-\beta_l+\zeta(1-\beta_c)} = \frac{x_c+1}{x_c+1+\zeta(x_l+1)}. \tag{12}$$

We have here used the following transformation of variables. Note that the rates of time preference are defined as $\rho_i = \frac{1}{\beta_i} - 1$ $(i = c, l)$. By defining $x_i \equiv \frac{1}{\rho_i}$

$(i = c, l)$, we can obtain the above expression. This saving rate coincides with that at the golden rule under the Cobb-Douglas production function and is always larger than s^E and s^*.

Subtracting the labor supply in the market allocation (see (61) in Appendix C) from this result, we obtain

$$l^\infty - L^E = \frac{(1-\alpha)\left[x_c + 1 + \zeta(x_l + 1)\right]\zeta(x_c - x_l) + (x_c + 1)(1+\zeta)\alpha\zeta}{[x_c + 1 + \zeta(x_l + 1)](1+\zeta)\left[(1-\alpha)\left\{x_c + 1 + \zeta(x_l + 1)\right\} + \alpha\zeta\right]}.$$

Therefore, when $x_c \geq x_l$ (that is, $\beta_c \geq \beta_l$), $l^\infty > L^E$. l^∞ obviously decreases with x_l (β_l) and approaches to zero when x_l (β_l) goes to infinity (one). On the other hand, L^E is an increasing function of x_l (β_l) and is equal to l^* when $x_c = x_l$ ($\beta_c < \beta_l$). This implies that there exits $\widehat{x}$ ($\widehat{\beta}$) ($> x_c$ (β_c)) at which $l^\infty = L^E$. Moreover, when $\beta_l = \widehat{\beta}(> \beta_c)$, $s^\infty > s^E > s^*$. Figure 2 depicts this situation. Figure 2 indicates that the market can do its job better than the central planner for the far future selves when $\beta_l = \widehat{\beta}(> \beta_c)$.

We can explain this result as follows. The selves in the far future hope the selves in early periods to save more (the most preferred saving rate coincides with that at the golden rule). When $\beta_c < \beta_l$, the current self saves more to make the future selves supply less labor as stated above. This can improve the welfare level of the selves in the far future under the market allocation. However, when β_l takes a sufficiently large value, the current self supplies too much labor. This can worsen the welfare level of the selves in the far future under the market allocation.

6 Tax Policies

In this section, we investigate tax policies of a government. The government can impose taxes on wage income, interest income, and investment. The government cannot commit to its future ones' decisions. We suppose that the government at time t represents the interest of self t. Thus, the government faces the same problem as the individual. Before solving a time consistent tax policy, we seek a tax policy that sets the tax rates constant over time. We call this policy a time constant tax policy in the following analyses.

6.1 Time Constant Tax Policy

We assume that the government imposes taxes on wage income, interest income, and investment. The respective tax rates (τ_r, τ_w, τ_i) are assumed to be constant over time. The budget constraint of the individual becomes

$$c = (1-\tau_r)rk + (1-\tau_w)wl - (1+\tau_i)k'.$$

We further assume that there is no government expenditure and its budget must be balanced in each period. Consequently, the following relationship holds:

$$\tau_r rK + \tau_w wL + \tau_i K' = 0.$$

By substituting the factor prices into this, we obtain the following relationship.

$$\alpha\tau_r + (1-\alpha)\tau_w + \tau_i s = 0. \tag{13}$$

Therefore, one of the tax rates is determined by the other two rates. We choose the rates of wage tax and of the investment tax as independent rates. We denote them as $\boldsymbol{\tau} = (\tau_w, \tau_i)$.

The current self's problem can be stated as follows:

$$V_0(k,K,L) = \max_{k',l} \left[\begin{array}{c} \ln((1-\tau_r)rk + (1-\tau_w)wl - (1+\tau_i)k') + \zeta \ln(1-l) \\ +\beta_c \cdot V_c(k',K',L') + \beta_l \cdot \zeta V_l(k',K',L') \end{array} \right]. \tag{14}$$

The solution of this problem is denoted by $\widetilde{g}(k,K,L;\boldsymbol{\tau})$ and $\widetilde{l}(k,K,L;\boldsymbol{\tau})$. The value functions are defined by the following functional equations.

$$V_c(k,K,L) = \ln c + \beta_c V_c(g(k,K,L;\boldsymbol{\tau}),\ K',L'), \tag{15}$$

$$V_l(k,K,L) = \ln(1-l) + \beta_l V_l(g(k,K,L;\boldsymbol{\tau}),\ K',L'), \tag{16}$$

where $g(k,K,L;\boldsymbol{\tau})$ stands for the perceived saving decision made by the future selves.

The derivation of the solution proceeds in a similar way to that in section 3. The solution of this problem is given by:

$$k' = \widetilde{g}(k,K,L;\boldsymbol{\tau}) = AK^{\alpha}L^{1-\alpha} \left[\frac{\frac{\Phi}{\zeta}\frac{1-\tau_r}{1+\tau_i}\alpha\frac{k}{K} + \frac{1-\tau_w}{1+\tau_i}(1-\alpha)\frac{\Phi}{\zeta L} - \left(1+\frac{1}{\zeta}\right)\varphi s}{1+\frac{1+\Phi}{\zeta}} \right], \tag{17}$$

$$\widetilde{l}(k,K,L;\boldsymbol{\tau}) = \frac{-(1-\tau_r)\alpha\frac{L}{K}k + (1-\tau_w)(1-\alpha)\frac{1+\Phi}{\zeta} - (1+\tau_i)\varphi \cdot sL}{(1-\tau_w)(1-\alpha)\left(1+\frac{1+\Phi}{\zeta}\right)}, \tag{18}$$

where $\Phi \equiv \beta_c d_c + \beta_l \zeta d_l$. The value functions become

$$V_c(k,K) = a_c + b_c \ln K + d_c \ln(k+\varphi K),\ \ V_l(k,K) = a_l + b_l \ln K + d_l \ln(k+\varphi K), \tag{19}$$

where

$$b_c = \frac{\alpha - 1}{(1-\beta_c\alpha)(1-\beta_c)}, \tag{20}$$

$$d_c = \frac{1}{1-\beta_c}, \tag{21}$$

$$\varphi = \frac{(1-\tau_w)(1-\alpha)}{[(1-\tau_r)\alpha - (1+\tau_i)s]L}, \tag{22}$$

$$b_l = \frac{-1}{1-\beta_l}, \tag{23}$$

$$d_l = \frac{1}{1-\beta_l}. \tag{24}$$

We can also obtain the equations for the saving rate and the labor supply as follows:

$$s = \frac{\frac{\Phi}{\zeta}\frac{1-\tau_r}{1+\tau_i}\alpha\frac{k}{K} + \frac{1-\tau_w}{1+\tau_i}(1-\alpha)\frac{\Phi}{\zeta L} - \left(1+\frac{1}{\zeta}\right)\varphi\cdot sL}{1+\frac{1+\Phi}{\zeta}}, \tag{25}$$

$$l = \frac{-(1-\tau_r)\alpha\frac{L}{K}k + (1-\tau_w)(1-\alpha)\frac{1+\Phi}{\zeta} - (1+\tau_i)\varphi\cdot sL}{(1-\tau_w)(1-\alpha)\left(1+\frac{1+\Phi}{\zeta}\right)}. \tag{26}$$

By using (22) and rearranging these, we can finally obtain the solutions as follows:

$$s^{ET} = \frac{\frac{\Phi}{\zeta}\left[1+\tau_w\frac{1-\alpha}{\alpha}\right]\alpha}{\frac{\Phi}{\zeta}+\left(1+\frac{1}{\zeta}\right)(1+\tau_i)}, \tag{27}$$

and

$$L^{ET} = \frac{(1-\tau_w)(1-\alpha)\frac{1+\zeta+\Phi}{\zeta(1+\zeta)}}{1+\alpha s\tau_i+\frac{(1-\tau_w)(1-\alpha)(1+\Phi)}{\zeta}}. \tag{28}$$

We first consider effects of tax rates on the saving rate and the labor supply. From (27), it is obvious that an increase in the tax rate on investment reduces the saving rate and the tax rate on wage income raises the saving rate. Second, we examine effects the tax rates on the labor supply. An increase in the tax rate on investment raises the saving rate and thus reduces the labor supply. An increase in the tax rate on wage income decreases the labor supply.

6.2 Time Consistent Tax Policy

The government cannot commit to its future decisions in a similar sense to the individual. We make a conjecture that tax rates chosen by the future governments are constant over time. We can verify this conjecture in the following analysis.

We first consider one-period deviation by the current government from the equilibrium path following Krusell et al (2002). We denote tax rates in the future as τ_j $(j = r, w, i)$. $\widetilde{\boldsymbol{\tau}} = (\widetilde{\tau}_w, \widetilde{\tau}_i)$ represents one period deviation of the tax rates. $\widetilde{G}(K, \widetilde{L}, \widetilde{\boldsymbol{\tau}})$ stands for the law of motion for the one-period deviation. $\widetilde{g}(k, K, \widetilde{L}; \widetilde{\boldsymbol{\tau}})$ are $\widetilde{L}(k, K, \widetilde{L}; \widetilde{\boldsymbol{\tau}})$ the individuals's responses against the one-period deviation by the current government.

$\widetilde{g}(k, K, \widetilde{L}; \widetilde{\boldsymbol{\tau}})$ and $\widetilde{L}(k, K, \widetilde{L}; \widetilde{\boldsymbol{\tau}})$ solutions of the following problem:

$$V_0(k,K,L) = \max_{k',l}\left[\begin{array}{c}\ln((1-\widetilde{\tau}_r)rk + (1-\widetilde{\tau}_w)wl - (1+\widetilde{\tau}_i)k') + \zeta\ln(1-l) \\ +\beta_c\cdot V_c(k', \widetilde{G}(K,\widetilde{L},\widetilde{\boldsymbol{\tau}}), L') + \beta_l\cdot\zeta V_l(k', \widetilde{G}(K,\widetilde{L},\widetilde{\boldsymbol{\tau}}), L')\end{array}\right]. \tag{29}$$

The value functions satisfy the following:

$$V_c(k,K,L) = \ln((1-\tau_r)rk+(1-\tau_w)wl-(1+\tau_i)k')+\beta_c V_c(g(k,K,L;\boldsymbol{\tau}),\ G(K,L),L'),$$

$$V_l(k,K,L)=\ln(1-l)+\beta_l V_l(g(k,K,L;\boldsymbol{\tau}),\ G(K,L),L').$$

These value functions are already obtained in the previous subsection. Thus, the parameters of the above value functions are also derived in the previous subsection.

We use the following guess for the motion of aggregate capital.

$$\widetilde{G}(K,\widetilde{L},\widetilde{\boldsymbol{\tau}})=\widetilde{s}AK^{\alpha}\widetilde{L}^{1-\alpha}.$$

The solution of (29) is similar to (17) except that the tax rates are given by $\widetilde{\tau}_r,\widetilde{\tau}_w$, and $\widetilde{\tau}_i$, and is given by

$$k'=AK^{\alpha}\widetilde{L}^{1-\alpha}\left[\frac{\frac{\Phi}{\zeta}\frac{1-\widetilde{\tau}_r}{1+\widetilde{\tau}_i}\alpha\frac{k}{K}+\frac{1-\widetilde{\tau}_w}{1+\widetilde{\tau}_i}(1-\alpha)\frac{\Phi}{\zeta\widetilde{L}}-\left(1+\frac{1}{\zeta}\right)\varphi\widetilde{s}}{1+\frac{1+\Phi}{\zeta}}\right], \tag{30}$$

where the parameters of the value functions are already obtained in the previous subsection. Consistency requires

$$\widetilde{s}=\frac{\frac{\Phi}{\zeta}\frac{1-\widetilde{\tau}_r}{1+\widetilde{\tau}_i}\alpha\frac{k}{K}+\frac{1-\widetilde{\tau}_w}{1+\widetilde{\tau}_i}(1-\alpha)\frac{\Phi}{\zeta\widetilde{L}}-\left(1+\frac{1}{\zeta}\right)\varphi\widetilde{s}}{1+\frac{1+\Phi}{\zeta}}. \tag{31}$$

The labor supply for the one-period deviation is also given by

$$\widetilde{L}=\frac{-(1-\widetilde{\tau}_r)\alpha\widetilde{L}+(1-\widetilde{\tau}_w)(1-\alpha)\frac{1+\Phi}{\zeta}-(1+\widetilde{\tau}_i)\varphi\cdot s\widetilde{L}}{(1-\widetilde{\tau}_w)(1-\alpha)\left(1+\frac{1+\Phi}{\zeta}\right)}. \tag{32}$$

For a later use, we rewrite this as follows:

$$1+\widetilde{\tau}_i\widetilde{s}+(1+\widetilde{\tau}_i)\varphi\widetilde{s}+(1-\widetilde{\tau}_w)(1-\alpha)\frac{1+\Phi}{\zeta}=(1-\widetilde{\tau}_w)(1-\alpha)\frac{1+\Phi}{\zeta\widetilde{L}},$$

where we have used consistency conditions, $k=K$ and $l=\widetilde{L}$. Because φ depends on $\boldsymbol{\tau}=(\tau_w,\tau_i)$ and s that are chosen by the future governments and the future selves respectively, we can denote the solutions of (31) and (32) as

$$\widetilde{s}(\widetilde{\tau}_w,\widetilde{\tau}_i;\tau_w,\tau_i,s)\text{ and }\widetilde{L}(\widetilde{\tau}_w,\widetilde{\tau}_i;\tau_w,\tau_i,s).$$

Note here that (31) and (32) are the same as (25) and (26) when $\widetilde{\boldsymbol{\tau}}=\boldsymbol{\tau}$ and $\widetilde{s}=s$. Therefore, it is satisfied that $\widetilde{s}(\tau_w,\tau_i;\tau_w,\tau_i,s^{ET})=s^{ET}$ and $\widetilde{L}(\tau_w,\tau_i;\tau_w,\tau_i,s^{ET})=L^{ET}$.

We next consider the problem of the current government. The government maximizes the following:

$$\begin{aligned}&V_0(K,K,L)\\ =\ &\max\left[\begin{array}{c}\ln\left(AK^{\alpha}\widetilde{L}(\tau_w,\tau_i;\tau_w,\tau_i,s^{ET})^{1-\alpha}-\widetilde{s}(\widetilde{\tau}_w,\widetilde{\tau}_i;\tau_w,\tau_i,s^{ET})AK^{\alpha}\widetilde{L}(\tau_w,\tau_i;\tau_w,\tau_i,s^{ET})^{1-\alpha}\right)\\ +\zeta\ln(1-\widetilde{L}(\tau_w,\tau_i;\tau_w,\tau_i,s^{ET}))\\ +\beta_c\left(a_c+b_c\ln K'+d_c\ln(1+\varphi)K')\right)+\beta_l\zeta\left(a_l+b_l\ln K'+d_l\ln(1+\varphi)K')\right)\end{array}\right].\end{aligned}$$

Because $K' = \widetilde{G}(K, \widetilde{L}, \widetilde{\boldsymbol{\tau}}) = \widetilde{s}AK^{\alpha}\widetilde{L}^{1-\alpha}$, we can express this as follows:

$$
\begin{aligned}
V_0(K, K, L) &= \ln(1-\widetilde{s}) + \zeta \ln(1-\widetilde{L}) + (1-\alpha)\ln \widetilde{L} \\
&\quad + \beta_c(b_c + d_c)\{\ln \widetilde{s} + (1-\alpha)\ln \widetilde{L}\} + \beta_l(b_l + d_l)\{\ln \widetilde{s} + (1-\alpha)\ln \widetilde{L}\} \\
&\quad + \text{other terms} \\
&= \ln(1-\widetilde{s}) + \zeta \ln(1-\widetilde{L}) + (1-\alpha)\ln \widetilde{L} + \beta_c(b_c + d_c)\{\ln \widetilde{s} + (1-\alpha)\ln \widetilde{L}\} \\
&\quad + \text{other terms},
\end{aligned}
$$

where the arguments of $\widetilde{s}$ and $\widetilde{L}$ are abbreviated to save space. The second equality comes from the fact, $b_l + d_l = 0$ (see (54) and (55) in Appendix C). By choosing $\widetilde{\tau}_w$ and $\widetilde{\tau}_i$, the current government tries to maximize this. This problem is equivalent to choosing $\widetilde{s}$ and $\widetilde{L}$. The maximizing conditions are given by

$$
\frac{-1}{1-\widetilde{s}} + \beta_c(b_c + d_c)\frac{1}{\widetilde{s}} = 0,
$$

$$
-\frac{\zeta}{1-\widetilde{L}} + \left[1 + \beta_c(b_c + d_c)\right](1-\alpha)\frac{1}{\widetilde{L}} = 0.
$$

By substituting (51) and (52), that is, $b_c = \frac{\alpha-1}{(1-\beta_c\alpha)(1-\beta_c)}$ and $d_c = \frac{1}{1-\beta_c}$ into this, we obtain

$$
\widetilde{s} = \beta_c\alpha \text{ and } \widetilde{L} = \frac{1-\alpha}{1-\alpha+\zeta(1-\beta_c\alpha)}.
$$

Note that these coincide with the solutions of the central planner.

From these arguments, the government must choose the tax rates that enforce these values. Moreover, because time consistency requires $\widetilde{\boldsymbol{\tau}} = \boldsymbol{\tau}$, $\widetilde{s}(\tau_w, \tau_i; \tau_w, \tau_i, s^{ET}) = s^{ET}$ and $\widetilde{L}(\tau_w, \tau_i; \tau_w, \tau_i, s^{ET}) = L^{ET}$. Therefore, from (27) and (28), the time consistent tax rates must satisfy the following:

$$
s^{ET} = \frac{\frac{\Phi}{\zeta}\left[1 + \tau_w \frac{1-\alpha}{\alpha}\right]\alpha}{\frac{\Phi}{\zeta} + \left(1 + \frac{1}{\zeta}\right)(1+\tau_i)} = \beta_c\alpha,
$$

$$
L^{ET} = \frac{(1-\tau_w)(1-\alpha)\frac{1+\zeta+\Phi}{\zeta(1+\zeta)}}{1 + \alpha\beta_c\tau_i + \frac{(1-\tau_w)(1-\alpha)(1+\Phi)}{\zeta}} = \frac{1-\alpha}{1-\alpha+\zeta(1-\beta_c\alpha)}.
$$

These define linear equations with respect to τ_w and τ_i. Noting that the definition of Φ, we can find a unique solution for these equations as follows:

$$
\tau_w = 0,\ 1+\tau_i = \frac{\Phi}{1+\zeta}\frac{1-\beta_c}{\beta_c} = \frac{\frac{\beta_c}{1-\beta_c} + \zeta\frac{\beta_l}{1-\beta_l}}{1+\zeta}\frac{1-\beta_c}{\beta_c} = \frac{1 + \zeta\frac{\beta_l}{1-\beta_l}\frac{1-\beta_c}{\beta_c}}{1+\zeta}.
$$

By the definition of the rate of time preference, $\rho_i = \frac{1}{\beta_i} - 1$ $(i = c, l)$, we can obtain the following relationship:

$$
1+\tau_i = \frac{1+\zeta\frac{\rho_c}{\rho_l}}{1+\zeta}.
$$

$$\tau_i \ \substack{\geq \\ <} \ 0 \ \leftrightarrow \ \rho_c \ \substack{\geq \\ <} \ \rho_l \ \left(\Leftrightarrow \beta_c \ \substack{\leq \\ >} \ \beta_l \right).$$

From the budget constraint of the government, $\alpha\tau_r + (1-\alpha)\tau_w + \tau_i s\alpha = 0$, we can also obtain the following relationship:

$$\tau_r \ \substack{< \\ \geq} \ 0 \ \leftrightarrow \ \rho_c \ \substack{\geq \\ <} \ \rho_l \ \left(\Leftrightarrow \beta_c \ \substack{\leq \\ >} \ \beta_l \right).$$

These results show that when the individual discounts future consumption by a higher rate than future leisure, the government must impose tax on investment and pay subsidy on interest income and vice versa in order to attain time consistency. The government should not impose tax on wage income or pay subsidy on wage income.

As stated above, the time consistent tax policy replicates the planning allocation. This implies that the time consistent tax policy improves the welfare level of the selves in early periods, however, can worsen the welfare levels of the selves in future periods.

7 Concluding Remarks

We have considered the economy in which individuals with non-unitary discounting inhabit and investigated the welfare performances of the market compared to the allocation by the central planner. The results show that the allocation by the central planner overcomes that by the market from a view point of selves in early periods; however, the opposite obtains from a view point of selves in the far future periods. This conflict cannot be resolved by tax policies of the government if she cannot commit to her future decisions.

8 Appendix A: Time inconsistent decision making

The first order conditions of the problem are given by

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta_c^t \frac{1}{c_t} - \lambda_t = 0, \ t = 0, 1, 2, \cdots, \tag{33}$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = \beta_l^t \zeta \frac{-1}{1 - l_t} + \lambda_t w_t = 0 \ t = 0, 1, 2, \cdots, \tag{34}$$

$$\frac{\partial \mathcal{L}}{\partial k_t} = \lambda_t r_t - \lambda_{t-1} = 0, \ t = 1, 2, \cdots. \tag{35}$$

Because the factor prices are given by $r = \alpha A \left(\frac{K}{L}\right)^{\alpha-1}$ and $w = (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha}$, (33), (34),and (35) can be rearranged as follows:

$$\beta_c c_{t-1} \alpha A \left(\frac{K_t}{L_t}\right)^{\alpha-1} = c_t, \tag{36}$$

$$\zeta^{-1}\left(\frac{\beta_c}{\beta_l}\right)^t (1-\alpha) A \left(\frac{K_t}{L_t}\right)^{\alpha} = \frac{c_t}{1-l_t}. \tag{37}$$

Note that, in equilibrium, $K = k$ and $L = l$ hold.

In the following, we use a guess-and-verify method. We first make a guess for the following saving decision of the individual:

$$k_{t+1} = s_I A k_t^{\alpha} l_t^{1-\alpha}.$$

Substituting this guess into (36) results in

$$\beta_c (1 - s_I) A k_{t-1}^{\alpha} l_{t-1}^{1-\alpha} \alpha A \left(\frac{k_t}{l_t}\right)^{\alpha-1} = (1 - s_I) A k_t^{\alpha} l_t^{1-\alpha},$$

which implies that

$$k_{t+1} = \beta_c \alpha A k_t^{\alpha} l_t^{1-\alpha}.$$

Substituting $c_t = (1- \beta_c \alpha) A k_t^{\alpha} l_t^{1-\alpha}$ into (37), we can obtain

$$l_t = \frac{(1-\alpha)}{(1-\alpha) + \zeta(1-\beta_c\alpha)\left(\frac{\beta_l}{\beta_c}\right)^t}.$$

9 Appendix B: The solution of the central planner

The first order conditions of the problem become

$$-\frac{1}{Ak^{\alpha}l^{1-\alpha} - k'} + \beta_c V_c'(k') + \beta_l \zeta V_l'(k') = 0, \tag{38}$$

$$\frac{(1-\alpha)Ak^{\alpha}l^{-\alpha}}{Ak^{\alpha}l^{1-\alpha}-k'}-\zeta\frac{1}{1-l}=0. \tag{39}$$

We use the following guesses for the value functions $V_c(k)$ and $V_l(k)$.

$$V_c(k)=a_c+b_c\ln k,\;\; V_l(k)=a_l+b_l\ln k$$

From (38), we obtain

$$k'=\widetilde{g}(k)=\frac{\beta_c b_c+\beta_l\zeta b_l}{1+(\beta_c b_c+\beta_l\zeta b_l)}Ak^{\alpha}l^{1-\alpha}. \tag{40}$$

Therefore, consumption becomes

$$c=\frac{1}{1+(\beta_c b_c+\beta_l b_l)}Ak^{\alpha}l^{1-\alpha}.$$

From (39) and (40), we obtain

$$l^*=\frac{(1-\alpha)(1+\beta_c b_c+\beta_l\zeta b_l)}{\zeta+(1-\alpha)[1+(\beta_c b_c+\beta_l\zeta b_l)]}. \tag{41}$$

From (7) and (8), the consistency condition $\widetilde{g}(k)=g(k)$ requires

$$\begin{aligned}V_c \;\;&\equiv\;\; a_c+b_c\ln k=\ln\left(\frac{1}{1+(\beta_c b_c+\beta_l b_l)}Ak^{\alpha}l^{1-\alpha}\right)\\ &\quad+\beta_c\left[a_c+b_c\ln\left(\frac{\beta_c b_c+\beta_l\zeta b_l}{1+(\beta_c b_c+\beta_l b_l)}Ak^{\alpha}l^{1-\alpha}\right)\right],\end{aligned}$$

$$V_l\equiv a_l+b_l\ln k=\ln(1-l^*)+\beta_l\zeta\left[a_l+b_l\ln\left(\frac{\beta_c b_c+\beta_l\zeta b_l}{1+(\beta_c b_c+\beta_l b_l)}Ak^{\alpha}l^{1-\alpha}\right)\right].$$

Because the coefficients of both the side must be equal, we can show that $b_l=0$ and

$$b_c=\frac{\alpha}{1-\beta_c\alpha}.$$

By substituting these into (41), we finally obtain

$$l^*=\frac{1-\alpha}{(1-\alpha)+\zeta(1-\alpha\beta_c)}.$$

Substituting these results into (40), we obtain

$$k'=\alpha\beta_c Ak^{\alpha}\left(\frac{1-\alpha}{(1-\alpha)+\zeta(1-\alpha\beta_c)}\right)^{1-\alpha}.$$

10 Appendix C

We first solve the problem of the current self. The first order conditions are

$$\frac{-1}{rk+wl-k'} + \beta_c V_{ck}(k',K',L') + \beta_l \zeta V_{lk}(k',K',L') = 0, \tag{42}$$

$$\frac{w}{rk+wl-k'} - \zeta \frac{1}{1-l} = 0. \tag{43}$$

From (43), we obtain

$$k' = rk + w\left[\left(1+\frac{1}{\zeta}\right)l - \frac{1}{\zeta}\right].$$

where $V_{ck}(k,K,L) \equiv \frac{\partial V_c(k,K,L)}{\partial k}$ and $V_{lk}(k,K,L) \equiv \frac{\partial V_l(k,K,L)}{\partial k}$. We next use the following guesses for the value functions and the transition equation of the aggregate state, respectively:[7]

$$V_c(k,K,L) = a_c + b_c \ln K + d_c \ln(k+\varphi(L)\cdot K),\ \ V_l(k,K,L) = a_l + b_l \ln K + d_l \ln(k+\varphi(L)\cdot K), \tag{44}$$

$$K' = sAK^{\alpha}L^{1-\alpha}. \tag{45}$$

We further guess that the aggregate labor supply is constant over time. Thus, φ also becomes constant over time. Consequently, we can rewrite (42) as follows:

$$\frac{1}{rk+wl-k'} = \frac{\beta_c d_c + \beta_l \zeta d_l}{k'+\varphi K'} \rightarrow k' + \varphi K' = (\beta_c d_c + \beta_l \zeta d_l)(rk+wl-k').$$

By using (43) and (45), we can rewrite this as follows:

$$rk + w\left[\left(1+\frac{1}{\zeta}\right)l - \frac{1}{\zeta}\right] + \varphi \cdot sAK^{\alpha}L^{1-\alpha} = (\beta_c d_c + \beta_l \zeta d_l)\frac{w}{\zeta}(1-l).$$

Because the self can perceive the factor prices, substituting the factor prices into this leads to

$$\begin{aligned} & \alpha A\left(\frac{K}{L}\right)^{\alpha-1} k + (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha}\left[\left(1+\frac{1}{\zeta}\right)l - \frac{1}{\zeta}\right] + \varphi \cdot sA\left(\frac{K}{L}\right)^{\alpha} L \\ = \ & (\beta_c d_c + \beta_l \zeta d_l)\frac{1}{\zeta}(1-\alpha)A\left(\frac{K}{L}\right)^{\alpha}(1-l). \end{aligned}$$

This results in

$$\alpha \frac{L}{K}k - (1-\alpha)\frac{1+\Phi}{\zeta} + \varphi \cdot sL = -(1-\alpha)\left(1+\frac{1+\Phi}{\zeta}\right)l,$$

where $\Phi \equiv \beta_c d_c + \beta_l \zeta d_l$. We can solve this for l as follows:

$$\widetilde{l}(k,K,L) = \frac{-\alpha \frac{L}{K}k + (1-\alpha)\frac{1+\Phi}{\zeta} - \varphi \cdot sL}{(1-\alpha)\left(1+\frac{1+\Phi}{\zeta}\right)}. \tag{46}$$

[7] We owe these specifications to Krusell et. al (2002).

Therefore, we obtain

$$1-\widetilde{l}(k,K,L)=\frac{\alpha\frac{L}{K}k+(1-\alpha)+\varphi\cdot sL}{(1-\alpha)\left(1+\frac{1+\Phi}{\zeta}\right)}. \tag{47}$$

By using this and (43), we obtain the followings:

$$c=\frac{w}{\zeta}(1-l)=AK^{\alpha}L^{1-\alpha}\frac{\alpha\frac{k}{K}+\frac{1-\alpha}{L}+\varphi s}{(1+\zeta+\Phi)}, \tag{48}$$

$$k'=\widetilde{g}(k,K,L)=AK^{\alpha}L^{1-\alpha}\left[\frac{\frac{\Phi}{\zeta}\alpha\frac{k}{K}+(1-\alpha)\frac{\Phi}{\zeta L}-\left(1+\frac{1}{\zeta}\right)\varphi s}{1+\frac{1+\Phi}{\zeta}}\right], \tag{49}$$

$$k'+\varphi K'=AK^{\alpha}L^{1-\alpha}\frac{\Phi}{\zeta}\left[\frac{\alpha\frac{k}{K}+\frac{1-\alpha}{L}+\varphi s}{1+\frac{1+\Phi}{\zeta}}\right]. \tag{50}$$

From the guesses and the functional equation, (10), we obtain the following relationship:

$$\begin{aligned}
& a_c+b_c\ln K+d_c\ln(k+\varphi K)\\
= \;& \ln AK^{\alpha}L^{1-\alpha}\frac{\alpha\frac{k}{K}+\frac{1-\alpha}{L}+\varphi s}{(1+\zeta+\Phi)}\\
& +\beta_c\left[a_c+b_c\ln sAK^{\alpha}L^{1-\alpha}+d_c\ln\left(AK^{\alpha}L^{1-\alpha}\frac{\Phi}{\zeta}\left[\frac{\alpha\frac{k}{K}+\frac{1-\alpha}{L}+\varphi s}{1+\frac{1+\Phi}{\zeta}}\right]\right)\right].
\end{aligned}$$

Note that we have used the consistency condition, $\widetilde{g}(k,K,L)=g(k,K,L)$. By rearranging this yields

$$\begin{aligned}
& a_c+b_c\ln K+d_c\ln(k+\varphi K)\\
= \;& \ln AK^{\alpha-1}L^{1-\alpha}\frac{\alpha\left[k+\frac{1}{\alpha}\left(\frac{1-\alpha}{L}+\varphi s\right)K\right]}{(1+\zeta+\Phi)}\\
& +\beta_c\left[a_c+b_c\ln sAK^{\alpha}L^{1-\alpha}+d_c\ln\left(AK^{\alpha-1}L^{1-\alpha}\frac{\Phi}{\zeta}\alpha\left[\frac{k+\frac{1}{\alpha}\left(\frac{1-\alpha}{L}+\varphi s\right)K}{1+\frac{1+\Phi}{\zeta}}\right]\right)\right].
\end{aligned}$$

Comparison of the coefficients of both the side leads to

$$K\colon\; b_c=(\alpha-1)+\beta_c[b_c\alpha+d_c(\alpha-1)].$$

$$k+\varphi K\colon\; d_c=1+\beta_c d_c.$$

$$\varphi\colon\; \varphi=\frac{1}{\alpha}\left(\frac{1-\alpha}{L}+\varphi s\right).$$

Therefore, we obtain the followings:

$$b_c=\frac{\alpha-1}{(1-\beta_c\alpha)(1-\beta_c)}, \tag{51}$$

$$d_c = \frac{1}{1-\beta_c}, \tag{52}$$

$$\varphi = \frac{1-\alpha}{(\alpha - s)L}. \tag{53}$$

Substituting (47) and 50) into (11) leads to

$$\begin{aligned}
& a_l + b_l \ln K + d_l \ln(k + \varphi K) \\
= \quad & \ln\left(\frac{\left(k + \frac{1}{\alpha}\left[\frac{1-\alpha}{L} + \varphi s\right] K\right)\alpha\frac{L}{K}}{(1-\alpha)\left(1 + \frac{1+\Phi}{\zeta}\right)}\right) \\
& + \beta_l \left[a_l + b_l \ln sAK^{\alpha}L^{1-\alpha} + d_l \ln\left(AK^{\alpha-1}L^{1-\alpha}\frac{\Phi}{\zeta}\alpha\left[\frac{k + \left(\frac{1-\alpha}{L} + \varphi s\right)K}{1 + \frac{1+\Phi}{\zeta}}\right]\right)\right].
\end{aligned}$$

Comparison of the coefficients of both the side leads to

$$K\colon\ b_l = -1 + \beta_l[b_l\alpha + d_l(\alpha - 1)].$$

$$k + \varphi K\colon\ d_l = 1 + \beta_l d_l.$$

$$\varphi\colon\ \varphi = \frac{1}{\alpha}\left(\frac{1-\alpha}{L} + \varphi s\right).$$

Therefore, we obtain the followings:

$$b_l = -\frac{1}{1-\beta_l}, \tag{54}$$

$$d_l = \frac{1}{1-\beta_l}, \tag{55}$$

$$\varphi = \frac{1-\alpha}{(\alpha - s)L}.$$

Substituting (53) into (46), (48), and (49), we obtain

$$l = \frac{-\alpha\frac{L}{K}k + (1-\alpha)\frac{1+\Phi}{\zeta} - \frac{(1-\alpha)s}{\alpha - s}}{(1-\alpha)\left(1 + \frac{1+\Phi}{\zeta}\right)},$$

$$1 - l = \frac{\alpha\frac{L}{K}k + (1-\alpha) + \frac{(1-\alpha)s}{\alpha - s}}{(1-\alpha)\left(1 + \frac{1+\Phi}{\zeta}\right)},$$

$$c = AK^{\alpha}L^{1-\alpha}\frac{\alpha\frac{k}{K} + \frac{1-\alpha}{L} + \frac{(1-\alpha)s}{(\alpha-s)L}}{(1 + \zeta + \Phi)},$$

and

$$k' = AK^{\alpha}L^{1-\alpha}\left[\frac{\frac{\Phi}{\zeta}\alpha\frac{k}{K} + (1-\alpha)\frac{\Phi}{\zeta L} - \left(1 + \frac{1}{\zeta}\right)\frac{(1-\alpha)s}{(\alpha-s)L}}{1 + \frac{1+\Phi}{\zeta}}\right].$$

Consistency requires $k = K$ and $g(K, K, L) = G(K, L)$. Therefore, the following relationship must hold:

$$s = \frac{\alpha \frac{\Phi}{\zeta} + (1-\alpha)\frac{\Phi}{\zeta L} - \left(1 + \frac{1}{\zeta}\right)\frac{(1-\alpha)s}{(\alpha - s)L}}{1 + \frac{1+\Phi}{\zeta}}. \tag{56}$$

We solve the saving rate in the competitive equilibrium as follows. Rearranging (56) results in

$$\begin{aligned}
s\left(1 + \frac{1+\Phi}{\zeta}\right) &= \alpha\frac{\Phi}{\zeta} + \left[\frac{\Phi}{\zeta} - \left(1 + \frac{1}{\zeta}\right)\frac{s}{(\alpha - s)}\right]\frac{(1-\alpha)}{L} \\
s\left(1 + \frac{1}{\zeta}\right) - (\alpha - s)\frac{\Phi}{\zeta} &= \left[\frac{\Phi}{\zeta} - \left(1 + \frac{1}{\zeta}\right)\frac{s}{(\alpha - s)}\right]\frac{(1-\alpha)}{L} \\
(\alpha - s)\left[\frac{s}{\alpha - s}\left(1 + \frac{1}{\zeta}\right) - \frac{\Phi}{\zeta}\right] &= \left[\frac{\Phi}{\zeta} - \left(1 + \frac{1}{\zeta}\right)\frac{s}{(\alpha - s)}\right]\frac{(1-\alpha)}{L}
\end{aligned}$$

$$\left[\frac{\Phi}{\zeta} - \left(1 + \frac{1}{\zeta}\right)\frac{s}{(\alpha - s)}\right]\left[\frac{(1-\alpha)}{L} + (\alpha - s)\right] = 0.$$

Because $s < \alpha$ holds as confirmed later, we obtain the following saving rate as the solution.[8]

$$s = \frac{\frac{\Phi}{\zeta}\alpha}{1 + \frac{1}{\zeta}(1+\Phi)} = \frac{\Phi\alpha}{\zeta + 1 + \Phi}. \tag{57}$$

Substituting (52) and (55) into $\Phi \equiv \beta_c d_c + \beta_l \zeta d_l$, we obtain

$$s^E = \frac{(\beta_c d_c + \beta_l \zeta d_l)\alpha}{\zeta + 1 + \beta_c d_c + \beta_l \zeta d_l} = \frac{\beta_c(1-\beta_l) + \zeta\beta_l(1-\beta_c)}{(1-\beta_l) + \zeta(1-\beta_c)}\alpha. \tag{58}$$

It is easily confirmed that $s < \alpha$.

We next calculate the labor supply l. From the consistency, $k = K$ and $l = L$, (46) results in

$$L = \frac{(1-\alpha)(1+\zeta+\Phi)}{(1+\zeta)\left[\zeta + (1-\alpha)(1+\Phi)\right]}.$$

Using the definition of $\Phi \equiv \beta_c d_c + \beta_l \zeta d_l$ and substituting (52) and (55) into this, we obtain

$$L^E = \frac{(1-\alpha)\left[1 - \beta_l + \zeta(1-\beta_c)\right]}{(1+\zeta)\left[(1-\alpha)\{1 - \beta_l + \zeta(1-\beta_c)\} + \alpha\zeta(1-\beta_c)(1-\beta_c)\right]}, \tag{59}$$

and

$$1 - L^E = \frac{\zeta\left\langle(1-\alpha)\left[1 - \beta_l + \zeta(1-\beta_c)\right] + \alpha(1+\zeta)(1-\beta_c)(1-\beta_c)\right\rangle}{(1+\zeta)\left[(1-\alpha)\{1 - \beta_l + \zeta(1-\beta_c)\} + \alpha\zeta(1-\beta_c)(1-\beta_c)\right]}.$$

[8] See appendix for its derivation.

We next derive the consumption. From (48), we can calculate the amount of consumption as follows:

$$c^E = \frac{w}{\zeta}(1-l)$$

$$= (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} \frac{(1-\alpha)\left[1-\beta_l+\zeta(1-\beta_c)\right]+\alpha(1+\zeta)(1-\beta_c)(1-\beta_c)}{(1+\zeta)\left[(1-\alpha)\{1-\beta_l+\zeta(1-\beta_c)\}+\alpha\zeta(1-\beta_c)(1-\beta_c)\right]}. \tag{60}$$

We next examine characters of the above solution. We can define the rate of time preference as $\rho_i = \frac{1}{\beta_i} - 1$ $(i = c, l)$. By using these rates, we can rewrite (58) as follows:

$$s^E = \frac{\frac{1}{\rho_c} + \zeta\frac{1}{\rho_l}}{\frac{1+\rho_c}{\rho_c} + \zeta\frac{1+\rho_l}{\rho_l}}\alpha.$$

By defining $x_i \equiv \frac{1}{\rho_i}$ $(i = c, l)$, we can rewrite this as follows:

$$s^E = \frac{x_c + \zeta x_l}{x_c + 1 + \zeta(x_l + 1)}\alpha.$$

In a similar way, we can express L^E and c^E as follows:

$$\begin{aligned} L^E &= \frac{(1-\alpha)\left[\frac{1+\rho_c}{\rho_c} + \zeta\frac{1+\rho_l}{\rho_l}\right]}{(1+\zeta)\left[(1-\alpha)\left\{\frac{1+\rho_c}{\rho_c} + \zeta\frac{1+\rho_l}{\rho_l}\right\} + \alpha\zeta\right]} \\ &= \frac{(1-\alpha)\left[x_c + 1 + \zeta(x_l + 1)\right]}{(1+\zeta)\left[\zeta + (1-\alpha)(1 + x_c + \zeta x_l)\right]}. \end{aligned} \tag{61}$$

$$\begin{aligned} c^E &= (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} \frac{(1-\alpha)\left[\frac{1+\rho_c}{\rho_c} + \zeta\frac{1+\rho_l}{\rho_l})\right] + \alpha(1+\zeta)}{(1+\zeta)\left[(1-\alpha)\{\frac{1+\rho_c}{\rho_c} + \zeta\frac{1+\rho_l}{\rho_l}\} + \alpha\zeta\right]} \\ &= (1-\alpha)A\left(\frac{K}{L}\right)^{\alpha} \frac{(1-\alpha)\left[x_c + 1 + \zeta(x_l + 1))\right] + \alpha(1+\zeta)}{(1+\zeta)\left[(1-\alpha)\left\{x_c + 1 + \zeta(x_l + 1)\right\} + \alpha\zeta\right]}. \end{aligned}$$

We next examine how the discount factors affect the solutions. Note that $\frac{dx_i}{d\beta_i} > 0$ $(i = c, l)$. By differentiating s^E and L^E with respect to β_c and β_l, we obtain

$$\frac{\partial s^E}{\partial \beta_c} = \frac{(1+\zeta)\alpha}{[x_c + 1 + \zeta(x_l + 1)]^2}\frac{dx_c}{d\beta_c} > 0,$$

$$\frac{\partial s^E}{\partial \beta_l} = \frac{\zeta(1+\zeta)\alpha}{[x_c + 1 + \zeta(x_l + 1)]^2}\frac{dx_l}{d\beta_l} > 0,$$

$$\frac{\partial L^E}{\partial \beta_c} = \frac{1-\alpha}{1+\zeta}\frac{\alpha\zeta}{\left[(1-\alpha)\left\{x_l + 1 + \zeta(x_l + 1)\right\} + \alpha\zeta\right]^2}\frac{dx_c}{d\beta_c} > 0,$$

$$\frac{\partial L^E}{\partial \beta_l} = \frac{1-\alpha}{1+\zeta}\frac{\alpha\zeta^2}{\left[(1-\alpha)\left\{x_l + 1 + \zeta(x_l + 1)\right\} + \alpha\zeta\right]^2}\frac{dx_l}{d\beta_l} > 0.$$

Thus, increases in β_i $(i = c, l)$ reduces the wage rate and this clearly reduces c^E.

11 Appendix D

We calculate the welfare level of the current self when the saving rate and labor supply are constant over time as follows.

$$
\begin{aligned}
V_0(k) &= \sum_{t=1}^{\infty} \left[\beta_c^{t-1} \ln c_t + \beta_l^{t-1} \zeta \ln(1-l_t)\right] \\
&= \ln\left[(1-s)Ak^{\alpha}l^{1-\alpha}\right] + \beta_c \ln\left[(1-s)Ak'^{\alpha}l^{1-\alpha}\right] + \beta_c^2 \ln\left[(1-s)Ak''^{\alpha}l^{1-\alpha}\right] + \cdots \\
&\quad + \zeta \ln(1-l) + \beta_l \zeta \ln(1-l) + \beta_l \zeta \ln(1-l) + \beta_l^2 \zeta \ln(1-l) + \cdots \\
&= \ln\left[(1-s)Ak^{\alpha}l^{1-\alpha}\right] + \beta_c \ln\left[(1-s)A\left(sAk^{\alpha}l^{1-\alpha}\right)^{\alpha} l^{1-\alpha}\right] \\
&\quad + \beta_c^2 \ln\left[(1-s)A\left(s(A\left(sAk^{\alpha}l^{1-\alpha}\right)^{\alpha} l^{1-\alpha}\right)^{\alpha} l^{1-\alpha}\right] + \frac{\zeta}{1-\beta_l}\ln(1-l) \\
&= \sum_{t=1}^{\infty} \beta_c^{t-1} \ln(1-s) + \sum_{t=1}^{\infty} \beta_c^{t-1} \left(\sum_{j=1}^{t} \alpha^{j-1}\right) \ln A + \sum_{t=1}^{\infty} \beta_c^{t} \left(\sum_{j=1}^{t} \alpha^{j}\right) \ln s \\
&\quad + \alpha \sum_{t=1}^{\infty} (\alpha\beta_c)^{t-1} \ln k + (1-\alpha) \sum_{t=1}^{\infty} \beta_c^{t-1} \left(\sum_{j=1}^{t} \alpha^{j-1}\right) \ln l + \frac{\zeta}{1-\beta_l}\ln(1-l) \\
&= \frac{1}{1-\beta_c}\ln(1-s) + \frac{1}{(1-\beta_c)(1-\alpha\beta_c)}\ln A + \frac{\alpha\beta_c}{(1-\beta_c)(1-\alpha\beta_c)}\ln s + \frac{\alpha}{1-\alpha\beta_c}\ln k \\
&\quad + \frac{1-\alpha}{(1-\beta_c)(1-\alpha\beta_c)}\ln l + \frac{\zeta}{1-\beta_l}\ln(1-l).
\end{aligned}
$$

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