Endogenous Ranking and Equilibrium Lorenz Curve Across (ex-ante) Identical Countries: A Generalization

By Kiminori Matsuyama[1]

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Abstract: This paper proposes a symmetry-breaking model of trade with a finite number of identical countries and a continuum of tradeable consumption goods, which differ in their dependence on nontradeable intermediate inputs, "producer services". Productivity of each country is endogenous due to country-specific external economies of scale in its service sector. It is shown that, in any stable equilibrium, the countries sort themselves into specializing in different sets of tradeable goods and that a strict ranking of countries in per capita income, TFP, the service sector share, and the capital-labor ratio emerge endogenously. Furthermore, the distribution of country shares, the Lorenz curve, is unique and analytically solvable in the limit, as the number of countries grows unbounded. Using this limit as an approximation allows us to study what determines the shape of distribution, perform various comparative statics and to evaluate the welfare effects of trade. In doing so, this paper extends the analysis of Matsuyama (2013) for more general and flexible forms of scale economies. It turns out that the technique introduced in Matsuyama (2013) is useful for the equilibrium characterization in this general case as well. Although some results of comparative statics and on welfare inevitably need to be modified, they change in ways that illuminate the underlying mechanism of symmetry-breaking.

Keywords: Endogenous Comparative Advantage, Endogenous Dispersion, External Economies of Scale, Globalization and Inequality, Symmetry-Breaking, Polarization, Power-law, Lorenz-dominant shifts, Log-supermodularity, Log-submodularity

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[1]Email: k-matsuyama@northwestern.edu; Homepage: http://faculty.wcas.northwestern.edu/~kmatsu/. This paper is prepared for International Workshop in Economic Theory: "Where Do We Stand?", held on June 28-29, 2013, at Ca' Foscari, in Venice. I thank Guido Cazzavillan and Department of Economics at University of Venice for the hospitality during my stay and the Workshop participants for their comments. The final version also benefits from the feedback received during the seminars given at Chicago Fed, Institut für Höhere Studien in Vienna, and Northwestern University.

1. Introduction

Rich countries tend to have higher TFPs and higher capital-labor ratios than the poor. Such empirical regularities are generally viewed as a causality running from TFPs and/or capital-labor ratios to per capita income, often under the maintained hypotheses that these countries offer independent observations and that cross-country variation in per capita income would disappear without any variations in some exogenous variables. However, there is a complementary approach, popular in trade and economic geography, that suggests a two-way causality. According to this approach, trade (and factor mobility) among countries/regions, even if they were ex-ante identical, could lead to the instability of the symmetric equilibrium in which they would remain identical. With such *symmetry-breaking*, cross-sectional dispersion and correlation in per capita income, TFPs, and capital-labor ratios, emerge endogenously as only stable patterns.[2] This approach does not try to argue that countries are ex-ante homogenous, nor that exogenous heterogeneities are unimportant. On the contrary, it suggests that even small heterogeneity or shocks could be amplified to create large productivity and income differences, which makes this approach appealing as a possible explanation for "Great Divergence" and "Growth Miracles."

The existing studies of symmetry-breaking, however, demonstrate this insight in a two-country/region setup, which makes it unclear what the message of this approach is when applied to a multi-country/region world. For example, does a symmetry-breaking mechanism split the world into the rich and poor clusters, as the narrative of this literature, such as "core-periphery," or "polarization" might suggest? Or does it keep splitting the world into finer clusters until the distribution becomes more disperse, possibly generating a power-law like distribution, as observed in the size distribution of metropolitan areas? More generally, which features of the economic environment determine the shape of distribution? Not only the existing studies on symmetry-breaking are unable to answer these questions, but also generate little analytical results on comparative statics and welfare. Motivated by these concerns, Matsuyama (2013) has

[2] See Fujita, Krugman, and Venables (1999) and Combes, Mayer, and Thisse (2008) in economic geography and Ethier (1982), Helpman (1986, p.344-346), Krugman and Venables (1995) and Matsuyama (1996) in international trade. The view that trade itself could magnify inequality among nations was discussed informally by Myrdal (1957) and Lewis (1977). See Matsuyama (2011) for more references. *Symmetry-breaking* is a circular mechanism that generates stable asymmetric outcomes in the symmetric environment due to the instability of the symmetric outcome. Although most prominent in economic geography, it has found applications in other areas of economics: see a New Palgrave entry on "symmetry-breaking" by Matsuyama (2008) as well as a related entry on "emergence" by Ioannides (2008).

recently proposed an analytically tractable symmetry-breaking model of trade as a framework in which one could address these issues. The present paper offers a generalization of that analysis.

More specifically, consider a world with a (large but) finite number of (ex-ante) identical countries. In each country, the representative household supplies a single composite of primary factors, such as capital, labor, land, etc., and has Cobb-Douglas preferences over a continuum of tradeable goods, as in Dornbusch, Fischer, and Samuelson (1977). These tradeables are produced by two nontradeable inputs, the (composite) primary factor and the intermediate inputs, "producer services," which are in turn produced by the (composite) primary factor. The two key assumptions are that tradeable sectors differ in their dependence on its service sector and that productivity of the service sector is endogenous due to country-specific external economies of scale in its service sector. This creates a circular mechanism between the patterns of trade and cross-country productivity differences. Having a larger and hence more productive service sector not only makes a country more productive. It also gives a country comparative advantage in tradeable sectors that are more dependent on those services. This in turn means a larger demand for services and, as a result, the country ends up having a larger and more productive service sector.

With (a continuum of) tradable goods vastly outnumbering (a finite number of) countries, this circular mechanism sorts different countries into specializing in different sets of tradeable goods (*endogenous comparative advantage*) and leads to a *strict ranking* of countries in per capita income, TFP, the service sector share in GDP, and (in an extension that allows for variable factor supply) capital-labor ratio in any stable equilibrium. Furthermore, the equilibrium distribution of country shares, the Lorenz curve, is unique and analytically tractable in the limit, as the number of countries grows unbounded. Using this limit as an approximation allows us to study, among other things, what determines the shape of distribution and how various forms of globalization or technical change affect inequality across countries, and to evaluate the welfare effects of trade (e.g., when trade is Pareto-improving, and when it is not, what fraction of countries might lose from trade).

The present paper generalizes the analysis of Matsuyama (2013), which models the service sector as a Dixit-Stiglitz (1977) type monopolistic competitive sector, whose sectoral TFP responds to a change in the variety of differentiated services that become available. Although such a specification has advantage of offering a micro foundation for scale economies and endogenous productivity, it comes at the cost of imposing the restriction that the degree of scale economies, measured by the elasticity of the service sector TFP with respect to a change in its total input, has to be constant; particularly, it has to be independent of the scale of the service sector. The alternative specification in this paper- the service sector is competitive but subject to country-specific external economies of scale-, allows for the degree of scale economies being a function of the scale of the service sector. Even under such more general and flexible forms of scale economies, the technique introduced in Matsuyama (2013) turns out to be useful for characterizing the equilibrium distribution. Of course, some results on comparative statics and welfare effects inevitably need to be modified, but they change in ways that illuminate the underlying mechanisms of symmetry-breaking. This paper reproduces all of the results of Matsuyama (2013) as a special case of the present model. It also resuscitates many discussions that existed in earlier versions of Matsuyama (2013) but had to be removed from its published version due to the space constraint.

Section 2.1 introduces the baseline model, which assumes that all consumption goods are tradeable and all primary factors are in fixed supply. Section 2.2 derives a single-country (or autarky) equilibrium. Section 2.3 derives a stable equilibrium with any finite number of countries, whose associated Lorenz curve is characterized by the second-order difference equation with two terminal conditions. Section 2.4 explains why any equilibrium in which some countries remain identical ex-post is unstable. Section 2.5 shows that, as the number of countries grows unbounded, the Lorenz curve converges to the unique and analytically solvable solution of the second-order differential equation with two terminal conditions. Armed with the explicit formula for the limit Lorenz curve, this subsection examines when the distribution is bimodal or satisfies a power-law. It also shows how various parameters causes a Lorenz-dominant shifts of the distribution, leading to greater or less inequality across countries, by making use of log-super(sub)modularity. Section 2.6 studies the welfare effects of trade. Section 3 discusses two extensions. In section 3.1, a fraction of the consumption goods are assumed to be nontradeable. This extension allows us to study the effects of globalization through trade in goods. In section 3.2, one of the primary factors is allowed to vary in supply either through factor mobility and accumulation. This extension not only generates the correlation between the capital-labor ratio and per capita income and TFP, but also it allows us to study the effects of technical change that increases the relative importance of human capital in production and of globalization through factor mobility.

2. Baseline Model:

2.1 Key Elements of the Model

Following Matsuyama (2013), consider the world economy with J (ex-ante) identical countries, where J is a positive integer, and a continuum of tradeable consumption goods, indexed by $s \in [0,1]$. There may be multiple nontradeable primary factors of production, such as capital (K), labor (L), etc., which can be aggregated to a single composite as $V = F(K, L, \ldots)$. For now, it is assumed that these component factors are in fixed supply and that the representative consumer of each country is endowed with the same quantity of V.

The representative consumer has Cobb-Douglas preferences over a continuum of tradeable consumption goods. This can be expressed by an expenditure function, $E = \exp\left[\int_0^1 \log(P(s))dB(s)\right]U$, where U is utility; $P(s) > 0$ the price of good-s; $B(s) = \int_0^s \beta(u)du$ the expenditure share of goods in $[0,s]$, satisfying $B'(s) = \beta(s) > 0$, $B(0) = 0$, and $B(1) = 1$. By denoting the aggregate income by Y, the budget constraint is then written as

$$(1) \qquad Y = \exp\left[\int_0^1 \log(P(s))dB(s)\right]U .$$

Each tradeable good is produced competitively by combining the two types of nontradeble inputs; the composite of the primary factors and the intermediate inputs, "producer services", using a Cobb-Douglas technology with $\gamma(s) \in [0,1]$ being the share of producer services in sector-s. The unit production cost of good-s can thus be expressed as:

$$(2) \qquad C(s) = \zeta(s)(\omega)^{1-\gamma(s)}(P_N)^{\gamma(s)},$$

where ω is the price of the (composite) primary factor; P_N the price of services; and $\zeta(s)$ is a scale parameter. Services are produced competitively, using the (composite) primary factor as the only input, but its technology is subject to country-specific external economies of scale. More specifically, its unit cost function is given by:

$$(3) \qquad P_N = \frac{\omega}{A(n)},$$

where $A(n)$ is TFP in the service sector, which is treated as a constant by each firm in the service sector but is increasing in its total input in the sector, *n*. For technical reasons, we assume that $A(n)$ is continuously differentiable, so that the elasticity of the sectoral TFP with respect to *n*,

(4) $$\theta(n) \equiv \frac{A'(n)n}{A(n)} > 0,$$

is a well-defined and continuous function of *n*, which shall be referred to as *the degree of scale economies*. Combining (2) and (3) yields,

(5) $$C(s) = \zeta(s)\frac{\omega}{(A(n))^{\gamma(s)}},$$

which shows that the extent to which each tradeable goods sector benefits from a higher TFP in the service sector depends on the share of services in its production, $\gamma(s)$. With no loss of generality, we may order the tradeable goods so that $\gamma(s)$ is increasing in $s \in [0,1]$, so that high-indexed sectors benefit more from scale economies in the service sector. Again, for technical reasons, it is assumed to be continuously differentiable.

Before proceeding, note that we may set,

(6) $B(s) = s$ for all $s \in [0,1]$,

so that $\beta(s) = 1$ for all $s \in [0,1]$ by choosing the tradeable goods indices, without further loss of generality.[3] In words, we measure the size of (a set of) sectors by the expenditure share of the goods produced in these sectors. With this indexing, the size of sectors whose γ is less than or equal to $\gamma(s)$ is equal to *s*. (In other words, $\gamma(s)$ is the inverse of the cumulative distribution of γ across sectors.) Furthermore, a country's share in the world income is equal to the measure of the tradeables for which the country ends up having comparative advantage in equilibrium.

2.2 Single-Country (or Autarky) Equilibrium ($J = 1$)

First, let us look at the equilibrium allocation of a one-country world ($J = 1$). This can also be viewed as the equilibrium allocation of a country in autarky in a world with multiple countries, which would serve as the benchmark for evaluating the welfare effects of trade.

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[3] To see this, starting from any indexing of the goods $s' \in [0,1]$ satisfying i) $\tilde{\gamma}(s') \in [0,1]$ is increasing in $s' \in [0,1]$, ii) $\tilde{\beta}(s') > 0$ for $s' \in [0,1]$, and iii) $\int_0^1 \tilde{\beta}(s')ds' = 1$, re-index the goods by $s = \tilde{B}(s') \equiv \int_0^{s'} \tilde{\beta}(u)du$. Then, $\gamma(s) \equiv \tilde{\gamma}(\tilde{B}^{-1}(s))$ is increasing in $s \in [0,1]$ and $ds = \tilde{\beta}(s')ds'$, so that $\beta(s) = 1$ for $s \in [0,1]$.

Because of Cobb-Douglas preferences, the economy must produce all the consumption goods in the absence of trade, which means that their prices must be equal to their costs; that is,

(7) $$P(s) = C(s) = \frac{\zeta(s)\omega}{(A(n))^{\gamma(s)}} \qquad \text{for all } s \in [0,1]$$

Since the representative consumer spends $\beta(s)Y = Y$ on good-s, and sector-s spends $100\gamma(s)$% of its revenue on services, the equilibrium size of the services sector, measured in its total input payment, is

(8) $$\omega^A n^A = \int_0^1 \gamma(s)\beta(s)Yds = \Gamma^A Y, \quad \text{where} \qquad \Gamma^A \equiv \int_0^1 \gamma(s)ds.$$

(Superscript A stands for autarky, here.) Since the aggregate income is accrued entirely to the primary factors in equilibrium,[4]

(9) $$Y^A = \omega^A V = \omega^A F(K,L,\ldots),$$

the equilibrium value of n in autarky is given as follows:

(10) $$n^A = \Gamma^A V$$

Eq. (9) shows that, with all the primary factors, capital (K), labor (L), etc. being aggregated into the composite, $V = F(K,L,\ldots)$, the equilibrium price of the composite factor, ω^A, is nothing but the aggregate TFP, as measured in GDP accounting. Eq. (10) shows that the equilibrium share of the service sector, n^A/V, is equal to the share of services in the total expenditure, which is equal to Γ^A in autarky.

2.3 Stable Multi-Country Equilibrium ($J \geq 2$)

Let us now turn to the case with $J \geq 2$. Since these countries are ex-ante identical, they share the same values for all the exogenous parameters. However, the stability of equilibrium requires that no two countries share the same value of n, as explained later. This allows us to rank the countries such that $\{n_j\}_{j=1}^J$ is (strictly) increasing in j. (The subscript here indicates the rank, not the identity, of a country in a particular equilibrium.) Then, from (5), the relative cost between the j-th and the $(j+1)$-th countries,

[4] Note that the service sector firms in this model earn zero profit because they are competitive with the linear technology, even though the sector as a whole is subject to economies of scale. The service sector firms in the model of Matsuyama (2013) also earn zero profit because of the free entry condition, even though they are monopolistically competitive.

$$(11)\qquad \frac{C_j(s)}{C_{j+1}(s)} = \left(\frac{A(n_j)}{A(n_{j+1})}\right)^{-\gamma(s)} \left(\frac{\omega_j}{\omega_{j+1}}\right)$$

is increasing in s for any j = 1, 2, ..., $J-1$, for any combination of the factor prices $\{\omega_j\}_{j=1}^J$, as illustrated by upward-sloping curves in Figure 1. In words, a country with a larger service sector has comparative advantage in higher-indexed sectors, which are more dependent on services. Furthermore, $\{\omega_j\}_{j=1}^J$ must adjust in equilibrium so that each country becomes the lowest cost producer (and hence the exporter) of a positive measure of the tradeable goods.[5] This implies that a sequence, $\{S_j\}_{j=0}^J$, defined by $S_0 = 0$, $S_J = 1$, and

$$\frac{C_j(S_j)}{C_{j+1}(S_j)} = \left(\frac{A(n_j)}{A(n_{j+1})}\right)^{-\gamma(S_j)} \left(\frac{\omega_j}{\omega_{j+1}}\right) = 1 \qquad (j = 1, 2, ..., J-1),$$

is increasing in j.[6] As showed in Figure 1, the tradeable goods, [0,1], are partitioned into J subintervals of positive measure, (S_{j-1}, S_j), such that the j-th country becomes the lowest cost producer (and hence its sole producer and exporter) of $s \in (S_{j-1}, S_j)$.[7] Note also that the definition of $\{S_j\}_{j=1}^{J-1}$ can be rewritten to obtain:

$$(12)\qquad \left(\frac{\omega_{j+1}}{\omega_j}\right) = \left(\frac{A(n_{j+1})}{A(n_j)}\right)^{\gamma(S_j)} > 1 \qquad (j = 1, 2, ..., J-1)$$

so that $\{\omega_j\}_{j=1}^J$ is also increasing in j; that is, a country with a large service sector is more productive and richer because it has a more productive service sector.

Since the j-th country specializes in (S_{j-1}, S_j), $100(S_j - S_{j-1})$% of the world income, Y^W, is spent on its tradeable sectors, so that its aggregate income is

$$(13)\qquad Y_j = \omega_j V = (S_j - S_{j-1})Y^W \qquad (j = 1, 2, ..., J)$$

Furthermore, since its sector-s in (S_{j-1}, S_j) spends $100\gamma(s)$% of its revenue on its services, the total input payment of its service sector is

[5] Otherwise, the factor price would be zero for a positive fraction (at least $1/J$) of the world population.

[6] To see why, $S_j \geq S_{j+1}$ would imply $C_j(s) > \min\{C_{j-1}(s), C_{j+1}(s)\}$ for all $s \in [0,1]$, hence that the j-th country is not the lowest cost producers of any tradeable good, a contradiction.

[7] In addition, S_0 is produced and exported by the 1st country and S_J by the J-th country. The borderline sector, S_j (j = 1, 2,…, $J-1$), could be produced and exported by either j-th or (j+1)-th country or both. This type of indeterminacy is inconsequential and ignored in the following discussion.

(14) $n_j\omega_j = \left[\int_{S_{j-1}}^{S_j}\gamma(s)ds\right]Y^W = (S_j - S_{j-1})\Gamma_j Y^W$ (j = 1, 2, ...,J)

where

(15) $\Gamma_j \equiv \Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}}\int_{S_{j-1}}^{S_j}\gamma(s)ds$. ($j$ = 1, 2, ...,J)

is the average of γ over (S_{j-1}, S_j). Note that $\{\Gamma_j\}_{j=1}^J$ is increasing in j, since γ is increasing in s.

From (13) and (14),

(16) $n_j = \Gamma_j V$; (j = 1, 2, ...,J)

Because $\{\Gamma_j\}_{j=1}^J$ is increasing in j, eq.(16) shows that $\{n_j\}_{j=1}^J$ is also increasing in j, as has been assumed. Eq.(13) shows that ω_j represents TFP of the j-th poorest country, and $s_j \equiv S_j - S_{j-1}$, the size of the tradeable sectors in which this country has comparative advantage, is also equal to its share in the world income. Thus, $S_j = \sum_{k=1}^{j} s_k$ is the cumulative share of the j poorest countries in world income. By combining (12), (13), (15), and (16), we obtain:

Proposition 1: Let S_j be the cumulative share of the j poorest countries in world income. Then, $\{S_j\}_{j=0}^J$ solves the second-order difference equation with two terminal conditions:

(17) $\frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{A(V\Gamma(S_j, S_{j+1}))}{A(V\Gamma(S_{j-1}, S_j))}\right)^{\gamma(S_j)} > 1$ with $S_0 = 0$ & $S_J = 1$,

where $\Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}}\int_{S_{j-1}}^{S_j}\gamma(s)ds$.

Figure 2 illustrates a solution to eq. (17) by means of the Lorenz curve, $\Phi^J : [0,1] \to [0,1]$, defined by the piece-wise linear function, with $\Phi^J(j/J) = S_j$. From the Lorenz curve, we can recover $\{s_j\}_{j=0}^J$, the distribution of the country shares and *vice versa*.[8] A few points deserve emphasis. First, because $\Gamma_j \equiv \Gamma(S_{j-1}, S_j)$ is increasing in j, $s_{j+1}/s_j \equiv (S_{j+1} - S_j)/(S_j - S_{j-1}) >$ 1. Hence, the Lorenz curve is kinked at j/J for each j = 1, 2, ..., J−1. In other words, the

[8] This merely states that there is a one-to-one correspondence between the distribution of income shares and the Lorenz curve. With J ex-ante identical countries, there are J! (factorial) equilibria for each Lorenz curve.

ranking of the countries is strict.[9] Second, since both income and TFP are proportional to $s_j \equiv S_j - S_{j-1}$, the Lorenz curve here also represents the Lorenz curve for income and TFP. Third, we could also obtain the ranking of countries in other variables of interest that are functions of $\{s_j\}_{j=0}^{J}$. For example, the *j*-th country's share in world trade, is equal to $\{s_j - (s_j)^2\}/\sum_{k=1}^{J}\{s_k - (s_k)^2\}$, which is increasing in j. The *j*-th country's trade dependence, defined by the volume of trade divided by its GDP, is equal to $1 - s_j$, which is decreasing in *j*.

Note also that Proposition 1 of Matsuyama (2013) can be obtained as a special case of the above proposition, with $A(n) \propto (n)^{\theta}$ (i.e., $\theta(n) = \theta > 0$), which simplifies the second-order difference equation, eq. (17), to:

$$\text{(18)} \qquad \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)}\right)^{\theta\gamma(S_j)} > 1.$$

This shows that the constant degree of scale economies was crucial for the equilibrium distribution in Matsuyama (2013) to be *scale-free*, i.e., independent of *V*.

2.4 Symmetry-Breaking: Instability of Equilibria without Strict Ranking of Countries

In characterizing the above equilibrium, we have started by imposing the condition that no two countries share the same value of *n*, and hence the countries could be ranked strictly so that $\{n_j\}_{j=1}^{J}$ is increasing in *j*, and then verified later that this condition holds in equilibrium. Indeed, there are also equilibria, in which some countries share the same value of *n*, and without strict ranking, $\{n_j\}_{j=1}^{J}$ is merely nondecreasing in *j*. For example, consider the case of $J = 2$ and suppose $n_1 = n_2$ in equilibrium. Then, from (5), $C_1(s)/C_2(s) = \omega_1/\omega_2$, which is independent of *s*. Thus, the condition under which each country produces a positive measure of goods is satisfied only if $C_1(s)/C_2(s) = \omega_1/\omega_2 = 1$. This means that, in this equilibrium, the consumers everywhere is indifferent as to which country they purchase tradeable goods from. In other words, the patterns of trade are indeterminate. *If* exactly 50% of the world income is spent on each country's tradeable goods sectors, and *if* this spending is distributed across the two

[9] This is in sharp contrast to Matsuyama (1996), which shows a non-degenerate distribution of income across countries, but with a clustering of countries that share the same level of income. The crucial difference is that the countries outnumber the tradeable goods in the model of Matsuyama (1996), while it is assumed more realistically, here and in Matsuyama (2013), that the tradable goods outnumber the countries.

countries in such a way that the services sector of each country ends up receiving exactly $\Gamma^A/2$ fraction of the world spending, then they would operate exactly at the same scale, $n_1 = n_2 = n^A$. Thus, the two countries remain identical ex-post. However, it is easy to see that this symmetric equilibrium, which replicates the autarky equilibrium in each country, is "sitting on the knife-edge", in that the required spending patterns described above must be met exactly in spite of the consumers' indifference. Furthermore, this symmetric equilibrium is unstable with respect to the usual Marshallian quantity adjustment dynamics.[10] To see this, imagine that small random perturbations make n_1 slightly lower than n_2. This would leads to a change in the spending patterns that reduces demand for the service sector in country 1 relative to that in country 2, creating a force towards a further decline in n_1 and a further increase in n_2, pushing the world further away from the symmetric allocation. This logic carries over to the case of $J > 2$ with $n_j = n_{j+1}$ for some j, because that would imply $C_j(s)/C_{j+1}(s) = \omega_j/\omega_{j+1} = 1$ so that, for a positive measure of goods, the consumers would be indifferent between buying from the j-th or (j+1)-th country, which would generate the same knife-edge property. For this reason, we restrict ourselves only to the equilibrium with a strict ranking of countries.[11]

[10] Alternatively, one could introduce dynamics through learning-by-doing, as done in Matsuyama (2002), by assuming that TFP of the service sector depends on its (discounted) cumulative total input. In the model of Matsuyama (2013), where the service sector is monopolistically competitive, one could also introduce dynamics through firms entering (exiting) in response to positive (negative) profit.

[11] The logic behind the instability of equilibrium without strict ranking of countries is similar to that of the mixed strategy equilibrium in games of strategic complementarity, particularly the game of the battle of the sexes. The assumption of a finite number of countries is crucial, but the assumption of zero trade cost in tradeable goods is not. To understand the latter, consider the case of the constant degree of scale economies $\theta(n) = \theta$ and $J = 2$. Imagine that these two countries are ex-ante identical, but *repeatedly* hit by small random shocks such that the realized parameter values cause the ratio of n^A of the two countries to fluctuate over a small support around one, $[e^{-\varepsilon}, e^{\varepsilon}]$ with $\varepsilon > 0$. (For example, the relative size of the two countries, V_1/V_2, might fluctuate around one, due to small shocks to exogenous components of TFP.) Now, assume an iceberg trade cost, such that one unit of the good shipped shrinks to $e^{-\delta} < 1$ when it arrives, where $\delta > 0$. Then, one could show that, even with a small $\varepsilon > 0$, symmetry breaking occurs eventually if $\delta < \varepsilon\theta(\gamma(1)-\gamma(0))/2$. (The logic here should be familiar to those who are exposed to the notion of stochastic stability of dynamical systems with random perturbations, where the long run stability of equilibrium depends on the size of its basin of attraction.) This extension also suggests that the world undergo a symmetry-breaking bifurcation, when the trade cost declines from $\delta > \varepsilon\theta(\gamma(1)-\gamma(0))/2$ to $\delta < \varepsilon\theta(\gamma(1)-\gamma(0))/2$. Of course, this means that, with a small positive cost $\delta > 0$, infinitesimal perturbations $\varepsilon \to 0$ cannot break symmetry. However, this is a mere technicality with no substantive issue at stake. Symmetry-breaking captures the idea that the symmetric outcome is more vulnerable to small shocks than the asymmetric outcomes, so that the asymmetric outcomes are likely to be observed. What matters is that, the smaller the trade cost, smaller shocks are enough to break symmetry.

Incidentally, there are a couple of ways to extend the model that could ensure that equilibrium without strict ranking is unstable even to infinitesimal shocks in spite of a small trade cost. For example, the present model assumes for simplicity, like any standard Ricardian model, that the goods produced by different countries are perfect substitutes within each sector. This means that introducing a small iceberg trade cost causes a discontinuous shift in the demand across countries, which is why small (but not infinitesimally) shocks are needed for breaking symmetry. Instead, consider the Armington specification that the goods produced by different countries within each sector are

The symmetry-breaking mechanism that renders equilibrium without strict ranking unstable and leads to the emergence of strict ranking across ex-ante identical countries is a *two-way causality* between the patterns of trade and cross-country productivity differences. A country with a larger services sector not only has higher TFP, but also comparative advantage in tradeable sectors that depend more on services. Having comparative advantage in such sectors means a larger demand for such services, which leads to an even larger services sector and hence higher TFP. Since tradeable goods differ in their dependence of services, some countries end up becoming less productive and poorer than others. Although many similar symmetry-breaking mechanisms exist in the trade and geography literature, they are usually demonstrated in models with two countries/regions. One advantage of the present model is that, with a continuum of goods, the logic extends to any finite number of countries/regions.

2.5 Equilibrium Lorenz Curve: Limit Case ($J \to \infty$)

Even though eq. (17) fully characterizes the equilibrium distribution of country shares, it is not analytically solvable. Of course, one could try to solve it numerically. However, numerical methods are not useful for answering the question of the uniqueness of the solution or for determining how the solution depends on the parameters of the model. Instead, in spirit similar to the central limit theorem, let us approximate the equilibrium Lorenz curve with a large but a finite number of countries by $\lim_{J\to\infty} \Phi^J = \Phi$.[12] It turns out that, as $J \to \infty$, eq.(17) converges to the second-order differential equation with two terminal conditions, whose solution is unique and can be solved analytically.[13] This allows us to study not only what determines the shape of the Lorenz curve, but also conduct various comparative statics, and to evaluate the welfare effects of trade.

Here's how to obtain the limit Lorenz curve, $\lim_{J\to\infty} \Phi^J = \Phi$. The basic strategy is to take Taylor expansions on both sides of eq. (17).[14] First, by setting $x = j/J$ and $\Delta x = 1/J$,

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highly but not perfectly substitutable. This would make the property of the model continuous at zero trade cost, so that even infinitesimal perturbations would be enough for symmetry-breaking even in spite of a small trade cost.

[12] Note that this is different from assuming a continuum of countries, as eq.(17) is derived under the assumption that there are a finite number of countries.

[13] From this, the Lorenz curve is unique for a sufficiently large J. I conjecture that the Lorenz curve is unique for any finite J, but the proof has been elusive, and it remains an open question.

[14] Initially, I used a different approximation strategy to obtain the limit Lorenz curve in Proposition 2 of Matsuyama (2013). I am grateful to Hiroshi Matano for showing me this (more efficient) method, without which I might not have been able to achieve the two extensions in Matsuyama (2013) and generalizations studied in this paper.

$$S_{j+1} - S_j = \Phi(x+\Delta x) - \Phi(x) = \Phi'(x)\Delta x + \Phi''(x)\tfrac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

$$S_j - S_{j-1} = \Phi(x) - \Phi(x-\Delta x) = \Phi'(x)\Delta x - \Phi''(x)\tfrac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

from which the LHS of eq. (17) can be written as:

$$\text{LHS} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o(|\Delta x|).$$

Likewise,

$$\Gamma(S_j, S_{j+1}) = \frac{\int_{\Phi(x)}^{\Phi(x+\Delta x)} \gamma(s)ds}{\Phi(x+\Delta x) - \Phi(x)} = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$\Gamma(S_j, S_{j-1}) = \frac{\int_{\Phi(x-\Delta x)}^{\Phi(x)} \gamma(s)ds}{\Phi(x) - \Phi(x-\Delta x)} = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

so that

$$A\left(V\Gamma(S_j, S_{j+1})\right) = A\left(V\gamma(\Phi(x))\right) + \frac{V}{2}A'\left(V\gamma(\Phi(x))\right)\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$A\left(V\Gamma(S_{j-1}, S_j)\right) = A\left(V\gamma(\Phi(x))\right) - \frac{V}{2}A'\left(V\gamma(\Phi(x))\right)\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|),$$

from which the RHS of eq.(17) can be written as:

$$\text{RHS} = \left(\frac{A(V\Gamma(S_j, S_{j+1}))}{A(V\Gamma(S_{j-1}, S_j))}\right)^{\gamma(S_j)} = \left(1 + \frac{VA'(V\gamma(\Phi(x)))}{A(V\gamma(\Phi(x)))}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)\right)^{\gamma(\Phi(x))}$$
$$= 1 + \theta(V\gamma(\Phi(x)))\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

where we recall $\theta(n) \equiv \dfrac{A'(n)n}{A(n)} > 0$ is the degree of scale economies. By equating the LHS and RHS of eq.(17),

$$1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o(|\Delta x|) = 1 + \theta(V\gamma(\Phi(x)))\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|).$$

By letting $\Delta x = 1/J \to 0$, we obtain the second-order differential equation:

(19) $$\frac{\Phi''(x)}{\Phi'(x)} = \theta(V\gamma(\Phi(x)))\gamma'(\Phi(x))\Phi'(x)$$

By integrating the above once,

$$\log\left(\Phi'(x)\right) - \frac{\Theta(V\gamma(\Phi(x)))}{V} = c_0 \iff \exp\left(\frac{-\Theta(V\gamma(\Phi(x)))}{V}\right)\Phi'(x) = e^{c_0}$$

where c_0 is an integral constant and

$$\Theta(n) \equiv \int_0^n \theta(u)du$$

is an increasing and continuously differentiable function. By integrating the above once again,

$$\int_0^{\Phi(x)} \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)ds = c_1 + e^{c_0}x,$$

where c_1 is another integral constant.

From the two terminal conditions, $\Phi(0) = 0$ and $\Phi(1) = 1$, the two integral constants, c_0 and c_1, are pinned down as:

$$e^{c_0} = \int_0^1 \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)ds; \qquad c_1 = 0.$$

From this, we obtain to

$$\int_0^{\Phi(x)} \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)ds = \left[\int_0^1 \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)ds\right]x.$$

This can be further rewritten as follows:

Proposition 2: The limit equilibrium Lorenz curve, $\lim_{J\to\infty} \Phi^J = \Phi$, is given by:

(20) $$x = H\left(\Phi(x)\right) \equiv \int_0^{\Phi(x)} h(s)ds, \text{ where } h(s) \equiv \frac{\hat{h}(s)}{\int_0^1 \hat{h}(u)du} \text{ with } \hat{h}(s) \equiv \exp\left(\frac{-\Theta(V\gamma(s))}{V}\right)$$

Figure 3 illustrates eq. (20). As shown in the left panel, $h(s)$ is positive and decreasing in $s \in [0, 1]$. Thus, its integral, $x = H(s)$, is increasing and concave. Furthermore, $h(s)$ is normalized in such a way that $H(0) = 0$ and $H(1) = 1$, as shown in the right panel. Hence, its inverse function, the Lorenz curve, $s = \Phi(x) = H^{-1}(x)$ is increasing, convex, with $\Phi(0) = 0$ and $\Phi(1) = 1$. It is also worth noting that the Lorenz curve may be viewed as the one-to-one mapping between a set of countries (on the *x*-axis) and a set of the goods they produce (on the *s*-axis).

Proposition 2 of Matsuyama (2013) can be obtained as a special case of the above proposition, where $A(n) \propto (n)^{\theta}$, which implies $\theta(n) = \theta > 0$ or $\Theta(n) = \theta n$, and hence

$$\hat{h}(s) \equiv \exp(-\theta\gamma(s)).$$

Again, this shows that the constant degree of scale economies was crucial for the equilibrium distribution in Matsuyama (2013) to be *scale-free*, i.e., independent of *V*.

From $s = \Phi(x) = H^{-1}(x)$, one could obtain GDP of the country at 100*x*% (with World GDP normalized to one), $y = \Phi'(x)$, its cumulative distribution function (cdf), $x = \Psi(y) \equiv (\Phi')^{-1}(y)$, and its probability density function (pdf), $\psi(y) = \Psi'(y)$. Table shows one such calculation for $\theta(n) = \theta > 0$ (i.e., $A(n) \propto (n)^{\theta}$), and an algebraically tractable one-parameter (λ) family of γ functions, which turns out to generate power-law (e.g., truncated Pareto).[15] Note that the power in the pdf is $\lambda/\theta - 2$.

Table: Power-Law Examples

	Example 1: $\gamma(s) = s$	Example 2: $\gamma(s) = \log\left[1+(e^{\theta}-1)s\right]^{\frac{1}{\theta}}$	Example 3: $\gamma(s) = \log\left[1+(e^{\lambda}-1)s\right]^{\frac{1}{\lambda}}$ $(\lambda \neq 0; \neq \theta)$
Inverse Lorenz Curve: $x = H(s)$	$\dfrac{1-e^{-\theta s}}{1-e^{-\theta}}$	$\log\left[1+(e^{\theta}-1)s\right]^{\frac{1}{\theta}}$	$\dfrac{\left[1+(e^{\lambda}-1)s\right]^{1-\frac{\theta}{\lambda}}-1}{e^{\lambda-\theta}-1}$
Lorenz Curve: $s = \Phi(x)$	$\log\left[1-(1-e^{-\theta})x\right]^{-\frac{1}{\theta}}$	$\dfrac{e^{\theta x}-1}{e^{\theta}-1}$	$\dfrac{\left[1+(e^{\lambda-\theta}-1)x\right]^{\frac{\lambda}{\lambda-\theta}}-1}{e^{\lambda}-1}$
Cdf: $x = \Psi(y) = (\Phi')^{-1}(y)$	$\dfrac{1}{1-e^{-\theta}} - \dfrac{1}{\theta y}$	$\dfrac{1}{\theta}\log\left(\dfrac{e^{\theta}-1}{\theta}y\right)$	$\dfrac{\left(\dfrac{y}{y_{Min}}\right)^{\frac{\lambda}{\theta}-1}-1}{e^{\lambda-\theta}-1} = 1 - \dfrac{1-\left(\dfrac{y}{y_{Max}}\right)^{\frac{\lambda}{\theta}-1}}{1-e^{\theta-\lambda}}$
Pdf: $\psi(y) = \Psi'(y)$	$\dfrac{1}{\theta y^2}$	$\dfrac{1}{\theta y}$	$\left[\dfrac{(\lambda/\theta)-1}{(y_{Max})^{(\lambda/\theta)-1}-(y_{Min})^{(\lambda/\theta)-1}}\right](y)^{\frac{\lambda}{\theta}-2}$
Support: $[y_{Min}, y_{Max}]$	$\dfrac{1-e^{-\theta}}{\theta} \leq y \leq \dfrac{e^{\theta}-1}{\theta}$	$\dfrac{\theta}{e^{\theta}-1} \leq y \leq \dfrac{\theta e^{\theta}}{e^{\theta}-1}$	$\left(\dfrac{\lambda}{e^{\lambda}-1}\right)\left(\dfrac{e^{\lambda-\theta}-1}{\lambda-\theta}\right) \leq y \leq \left(\dfrac{\lambda}{e^{\lambda}-1}\right)\left(\dfrac{e^{\lambda-\theta}-1}{\lambda-\theta}\right)e^{\theta}$

[15] Example 1 and Example 2 may be viewed as the limit cases of Example 3, as $\lambda \to 0$ and $\lambda \to \theta$, respectively. In addition to being algebraically tractable, the power-law examples have some empirical appeal when "countries" are interpreted as "cities" or "metropolitan areas": see, e.g., Gabaix and Ioannides (2004). I am grateful to Fabrizio Perri for suggesting to me to construct power-law examples.

Intuitively, with a smaller λ, the use of local services is more concentrated at higher-indexed sectors, so that only a smaller fraction of countries specializes in such "desirable" tradeables. Hence, a smaller λ makes the pdf decline more sharply at the upper end.

Eq.(20) can also be used to investigate when a symmetry-breaking mechanism of this kind leads to, say, a bimodal distribution, as the narrative in much of this literature, ("core-periphery" or "polarization") seems to suggest. When the increasing and continuously differentiable function $\Theta(V\gamma(\bullet))$ can be approximated by a two-step function, the corresponding pdf becomes bimodal, as illustrated in Figure 4.[16] This could occur in two different ways. First, $\gamma(\bullet)$ may be approximated by a two-step function.[17] Thus, the world becomes polarized into the rich core and the poor periphery, when the tradeable goods can be classified into two categories in such a way that they are roughly homogeneous within each category. Second, $\Theta(\bullet)$ may be approximated by a two-step function, or equivalently, $\theta(\bullet)$ has a sharp single peak, that is to say, when the scale economies in the service sector are subject to the single threshold externalities.[18] Generally, a symmetry-breaking mechanism of this kind leads to *N* "clusters" of countries if $\Theta(V\gamma(\bullet))$ can be approximated by an *N*-step function.

Another advantage of Eq.(20) is that one could easily see the effect of changing *V*. To see this, note that the numerator of $h(s)$, $\hat{h}(s) \equiv \exp\left(-\Theta(V\gamma(s))/V\right)$, satisfies

$$\operatorname{sgn}\left\{\frac{\partial^2 \log(\hat{h}(s))}{\partial V \partial s}\right\} = -\operatorname{sgn}\left\{\theta'(V\gamma(s))\gamma(s)\gamma'(s)\right\} = -\operatorname{sgn}\left\{\theta'(V\gamma(s))\right\}.$$

In words, $\hat{h}(s)$ is *log-submodular* in *V* and *s,* for $\theta'(\bullet) > 0$ and *log-supermodular* in *V* and *s,* if $\theta'(\bullet) < 0$.[19] For example, the arrows in Figure 3 illustrates the effect of a higher *V* for $\theta'(\bullet) > 0$.

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[16] Formally, consider a sequence of (increasing and continuously differentiable) $\Theta(V\gamma(\bullet))$ functions that converges point-wise to a two-step function, $\Theta(V\gamma(s)) = \Theta_L$ for $s \in [0, \bar{s})$ and $\Theta(V\gamma(s)) = \Theta_H > \Theta_L$ for $s \in (\bar{s}, 1]$. Then, the sequence of the corresponding cdf's converges to the cdf, $\Psi(y) = 0$ for $y < 1-(1-\bar{s})(1-e^{-\theta\Delta})$; $\Psi(y) = [1+(1/\bar{s}-1)e^{-\theta\Delta}]^{-1}$ for $1-(1-\bar{s})(1-e^{-\theta\Delta}) < y < 1+\bar{s}(e^{\theta\Delta}-1)$, and $\Psi(y) =$ 1 for $y > 1+\bar{s}(e^{\theta\Delta}-1)$, where $\Delta \equiv \Theta_H - \Theta_L > 0$.

[17] Note that this is different from assuming that γ is a two-step function, which is equivalent to assume that there are only two tradeable goods. Then, the equilibrium distribution would not be unique; see Matsuyama (1996). To obtain the uniqueness, it is essential that γ is increasing, which means that the set of the tradeable goods is a continuum, and hence outnumbers the set of the countries for a large but finite number, *J*.

[18] Obviously, this second possibility is absent in Matsuyama (2013), with the constant degree of scale economies.

[19] See Topkis (1998) for mathematics of super-(and sub-)modularity and Costinot (2009) for a recent application to international trade.

In this case, $\hat{h}(s)$ is log-submodular, which means that a higher V shifts the graph of $\hat{h}(s)$ down everywhere but proportionately more at a higher s. Since $h(s)$ is a rescaled version of $\hat{h}(s)$ to keep the area under the graph unchanged, the graph of $h(s)$ is rotated "clockwise" by a higher V, as shown in the left panel. This "single-crossing" in $h(s)$ implies that a higher V makes the Lorenz curve more "curved" and move further away from the diagonal line everywhere (i.e., without any "Lorenz-crossing"), as shown in the right panel. In short, a higher V causes a Lorenz-dominant shift of the Lorenz curve away from the diagonal. Thus, any Lorenz-consistent inequality measure, such as the generalized Kuznets Ratio, the Gini index, the coefficients of variations, etc. all agree that a higher V leads to greater inequality, when the degree of scale economies is an increasing function. Likewise, the arrows in Figure 3 also illustrates the effect of a lower V for the log-supermodular case, $\theta'(n) < 0$. In other words, a higher V in this case causes a Lorenz-dominant shift of the Lorenz curve in the direction opposite of the arrows, that is, closer to the diagonal, hence less inequality.[20]

2.6 Welfare Effects of Trade

The mere fact that trade creates ranking of countries, making some countries poorer than others, does not necessarily imply that trade make them poorer. It could simply mean that they gain less from trade than others. To see whether trade would make some countries poorer, we need to compare the utility levels under trade and under autarky. From eq.(1), the welfare under autarky is

$$\log\left(U^A\right) = \log\left(\omega^A V\right) - \int_0^1 \log\left(P^A(s)\right) ds.$$

Likewise, the welfare of the country that ends up being the j-th poorest can be written as

$$\log\left(U_j\right) = \log\left(\omega_j V\right) - \int_0^1 \log\left(P(s)\right) ds,$$

where the tradeable goods prices satisfy

$$\frac{P(s)}{P^A(s)} = \left(\frac{\omega_k}{\omega^A}\right)\left(\frac{A(n_k)}{A(n^A)}\right)^{-\gamma(s)} = \left(\frac{\omega_k}{\omega^A}\right)\left(\frac{A(V\Gamma_k)}{A(V\Gamma^A)}\right)^{-\gamma(s)}$$

for $s \in (S_{k-1}, S_k)$ for $k = 1, 2, \ldots, J$.

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[20] Likewise, any upward shift in $\Theta(\bullet)$ would lead to greater inequality. The effect of θ, discussed in Matsuyama (2013) may be viewed as a special case of this observation with $\Theta(n) \equiv \theta n$.

Combining these equations yields

$$\log\left(\frac{U_j}{U^A}\right)=\log\left(\frac{\omega_j}{\omega^A}\right)-\int_0^1\log\left(\frac{P(s)}{P^A(s)}\right)ds$$

$$=\log\left(\frac{\omega_j}{\omega^A}\right)-\sum_{k=1}^{J}\left[\int_{S_{k-1}}^{S_k}\log\left(\frac{\omega_k}{\omega^A}\right)ds-\int_{S_{k-1}}^{S_k}\gamma(s)\log\left(\frac{A(V\Gamma_k)}{A(V\Gamma^A)}\right)ds\right],$$

which can be further rewritten as follows:

Proposition 3 (*J*-country case): The country that ends up being the *j*-th poorest under trade gains from trade if and only if:

$$\log\left(\frac{U_j}{U^A}\right)=\sum_{k=1}^{J}\log\left(\frac{\omega_j}{\omega_k}\right)(S_k-S_{k-1})+\sum_{k=1}^{J}\Gamma_k\log\left(\frac{A(V\Gamma_k)}{A(V\Gamma^A)}\right)(S_k-S_{k-1})>0 \tag{21}$$

Eq. (21) in Proposition 3 offers a decomposition of the welfare effects of trade. The first term is the country's TFP relative to the world average, and it is increasing in j, negative at $j = 1$ and positive at $j = J$. The second term captures the usual gains from trade (i.e., after controlling for the income and TFP differences across countries) and it is always positive. However, aside from rather obvious statements like "a country gains from trade if the second term dominates the first term," or "a country gains from trade if its income (and TFP) ends up being higher than the world average," Proposition 3 offers little insight without an explicit solution for eq. (17).

As $J\to\infty$, the task of evaluating the overall welfare effect becomes greatly simplified. By setting $x^* = j/J$ and $x = k/J$ in eq. (21) and noting that $\omega_j/\omega_k\to\Phi'(x^*)/\Phi'(x)$ and $S_k-S_{k-1}\to\Phi'(x)dx$ as $J\to\infty$, eq.(21) converges to:

$$\log\left(\frac{U(x^*)}{U^A}\right)=\int_0^1\log\left(\frac{\Phi'(x^*)}{\Phi'(x)}\right)\Phi'(x)dx+\int_0^1\gamma(\Phi(x))\log\left(\frac{A(V\gamma(\Phi(x)))}{A(V\Gamma_A)}\right)\Phi'(x)dx$$

Since eq. (19) implies $\log\left(\Phi'(x)\right)-\frac{\Theta(V\gamma(\Phi(x)))}{V}=c_0$, this can be rewritten as:

$$\log\left(\frac{U(x^*)}{U^A}\right)=\int_0^1\left(\frac{\Theta(V\gamma(\Phi(x^*))}{V}-\frac{\Theta(V\gamma(\Phi(x))}{V}\right)d\Phi+\int_0^1\gamma(\Phi(x))\log\left(\frac{A(V\gamma(\Phi(x)))}{A(V\Gamma_A)}\right)d\Phi,$$

$$=\frac{\Theta(V\gamma(s^*))}{V}-\int_0^1\left(\frac{\Theta(V\gamma(s))}{V}\right)ds+\int_0^1\gamma(s)\log\left(\frac{A(V\gamma(s))}{A(V\Gamma_A)}\right)ds.$$

To summarize:

Proposition 4 (Limit case; $J \rightarrow \infty$): The country that ends up being at 100x^* percentile under trade gains from trade if and only if:

$$\log\left(\frac{U(x^*)}{U^A}\right) = \frac{\Theta(V\gamma(s^*))}{V} - \int_0^1 \left(\frac{\Theta(V\gamma(s))}{V}\right) ds + \int_0^1 \gamma(s) \log\left(\frac{A(V\gamma(s))}{A(V\Gamma_A)}\right) ds > 0, \tag{22}$$

where $s^* = \Phi(x^*)$ or $x^* = \Phi^{-1}(s^*)$.

Proposition 4 offers a decomposition of the welfare effects of trade, similar to Proposition 3. In fact, Proposition 4 allows us to say a lot more about the overall welfare effects of trade, as shown in the following two Corollaries.

Corollary 1: All countries gain from trade if and only if

$$\int_0^1 \left(\frac{\Theta(V\gamma(s))}{V}\right) ds - \frac{\Theta(V\gamma(0))}{V} < \int_0^1 \gamma(s) \log\left(\frac{A(V\gamma(s))}{A(V\Gamma_A)}\right) ds .$$

To see what is involved in this condition, consider the case $A(n) \propto (n)^{\theta}$, or $\theta(n) = \theta$, as in Matsuyama (2013). Then, this can be rewritten as:

$$1 - \frac{\gamma(0)}{\Gamma^A} \leq \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A}\right) \log\left(\frac{\gamma(s)}{\Gamma^A}\right) ds \tag{23}$$

= Diversity (**Theil index/entropy**) of γ.

Note that the sufficient and necessary condition under which all countries gain from trade, (23), depends solely on $\gamma(\bullet)$. In particular, it is independent of θ, which only plays a role of magnifying the gains and losses from trade. LHS of eq.(23) shows how much the share of the service sector, and hence productivity, declines in the country that ends up being the poorest. RHS of eq. (23) is the Theil index (or entropy) of γ, which measures its dispersion. This corollary thus states that trade is Pareto-improving (i.e., even the country that ends up being the poorest would benefit from trade) when the tradeable goods are sufficiently diverse, as measured by the Theil index of γ, and hence the gains from specialization (by making countries ex-post heterogeneous) is sufficiently large.

Corollary 2: Suppose that the condition of Corollary 1 fails. Then, for $s_c > 0$, defined by

$$\frac{\Theta(V\gamma(s_c))}{V} \equiv \int_0^1 \left(\frac{\Theta(V\gamma(s))}{V}\right) ds - \int_0^1 \gamma(s) \log\left(\frac{A(V\gamma(s))}{A(V\Gamma_A)}\right) ds .$$

a): All countries producing and exporting goods $s \in [0, s_c)$ lose from trade, while all countries producing and exporting goods $s \in (s_c, 1]$ gain from trade.

b): Consider a shift parameter, σ > 0, such that $A(n;\sigma) = [A(n;1)]^{\sigma}$.[21] Then, s_c is independent of σ, and the fraction of the countries that lose, $x_c = H(s_c;\sigma)$, is increasing in σ with $\lim_{\sigma\to 0} x_c = s_c$ and $\lim_{\sigma\to\infty} x_c = 1$.

Corollary 2a) follows immediately from eq.(22) and the definition of s_c. The independence of s_c from σ follows from that $A(n;\sigma) = [A(n;1)]^{\sigma}$ implies $\Theta(n;\sigma) = \sigma\Theta(n;1)$, and hence the definition of s_c can be rewritten as

$$\frac{\Theta(V\gamma(s_c);\sigma)}{V} - \int_0^1 \left(\frac{\Theta(V\gamma(s);\sigma)}{V}\right) ds + \int_0^1 \gamma(s)\log\left(\frac{A(V\gamma(s);\sigma)}{A(V\Gamma_A;\sigma)}\right) ds =$$

$$\sigma\left[\frac{\Theta(V\gamma(s_c);1)}{V} - \int_0^1 \left(\frac{\Theta(V\gamma(s);1)}{V}\right) ds + \int_0^1 \gamma(s)\log\left(\frac{A(V\gamma(s);1)}{A(V\Gamma_A;1)}\right) ds\right] = 0 .$$

Corollary 2 is illustrated in the right panel of Figure 3. Corollary 2a) states that all countries that end up specializing in [0, s_c) lose from trade and they account for x_c fraction of the world.[22] Corollary 2b) states that, a higher σ shifts the Lorenz curve but s_c is unaffected so that x_c goes up vertically. As varying σ from 0 to ∞, x_c increases from s_c to 1. So, perhaps surprisingly, when $\gamma(\bullet)$ and $A(\bullet;1)$ satisfy the condition that *some* countries lose from trade, *virtually all* countries could lose from trade, as σ goes to ∞.

3. Two Extensions

The above model can be generalized in many directions. This section offers two extensions. The first allows a fraction of the consumption goods within each sector to be nontradeable. By reducing the fraction, this extension enables us to examine how inequality

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[21] Obviously, for the case discussed in Matsuyama (2013), $A(n) \propto (n)^{\theta}$, hence θ works as such a shift parameter.

[22] At one of the seminars where Matsuyama (2013) was presented, someone commented that Corollary 2a) implies a complete unraveling of trade if the countries choose whether to trade. He argued that, after $x_c > 0$ fraction of the countries decide to go back to autarky, the x_c fraction of the remaining countries would also go back to autarky, and this process would continue unless no countries would be left to trade. This comment is, however, false, because Corollary 2a) is a limit result, which offers a good approximation for a sufficiently large *J.* As more countries leave and *J* becomes smaller, this approximation becomes less and less accurate. Indeed, with a small *J,* even the country that ends up being the poorest might produce a sufficiently wide set of tradeables, which could ensure that even the poorest gains from trade, in which case no country has an incentive to stop trading. Indeed, in her insightful comments, Emily Blanchard suggested to me that, under the condition that ensures $x_c > 0$, the model of Matsuyama (2013)—hence this model as well—, may offer a model of the equilibrium size of the trading bloc. It seems, however, that characterizing the equilibrium size would be feasible only numerically.

across countries is affected by globalization through trade in goods.[23] The second allows variable supply in one of the components in the composite of primary factors, either through factor accumulation or factor mobility. This extension not only generates the correlation between the capital-labor ratio and per capita income and TFP. By changing the share of the variable primary factor in the composite, this extension also allows us to examine how inequality across countries is affected by technological change that increases importance of human capital or by globalization through trade in factors.

3.1 Nontradeable Consumption Goods: Globalization through Trade in Goods

In the model of section 2, all consumption goods are assumed to be tradeable. Assume now that each sector-s produces many varieties, a fraction τ of which is tradeable and a fraction $1-\tau$ is nontradeable, and that they are aggregated by Cobb-Douglas preferences.[24] The expenditure function is now obtained by replacing $\log(P(s))$ with $\tau\log(P_T(s))+(1-\tau)\log(P_N(s))$ for each $s \in [0,1]$, where $P_T(s) = Min\{C_j(s))\}$ is the price of each tradeable good in sector-s, common across all countries, $P_N(s) = C_j(s)$ is the price of each nontradeable good in sector-s, which is equal to the unit of cost of production in each country.

Instead of going through the entire derivation of the equilibrium, only the key steps will be highlighted below. Again, let $\{n_j\}_{j=1}^{J}$ be a monotone increasing sequence. As before,

$$(12)\qquad \frac{\omega_{j+1}}{\omega_j} = \left(\frac{A(n_{j+1})}{A(n_j)}\right)^{\gamma(S_j)} > 1. \qquad (j = 1, 2, ..., J-1)$$

$$(13)\qquad Y_j = \omega_j V = (S_j - S_{j-1})Y^W. \qquad (j = 1, 2, ..., J)$$

However, the equilibrium size of the service sector is now given by, instead of (16):

$$(24)\qquad n_j = \left(\tau\Gamma_j + (1-\tau)\Gamma^A\right)V.$$

Combining these equations yields

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[23]Emi Nakamura suggested another interpretation of this thought experiment. By increasing τ one could also see how inequality across regions change, as one looks at more and more disaggregated levels of geographical units.

[24]This specification assumes that the share of nontradeable producer services in sector-s is $\gamma(s)$ for both nontradeable and tradeable consumption goods. This assumption is made because, when examining the effect of globalization by changing τ, we do not want the distribution of γ across all tradeable consumption goods to change. However, for some other purposes, it would be useful to consider the case where the distribution of γ among nontradeable consumption goods differ systematically from those among tradeable consumption goods. For example, Matsuyama (1996) allows for such possibility to generate a positive correlation between per capita income and the nontradeable consumption goods prices across countries, similar to the Balassa-Samuelson effect.

Proposition 5 (*J*-country case): Let S_j be the cumulative share of the *J* poorest countries. Then, $\{S_j\}_{j=0}^{J}$ solves the 2rd order difference equation with two terminal conditions:

$$\frac{S_{j+1}-S_j}{S_j - S_{j-1}} = \left(\frac{A\left(V[\tau\Gamma(S_j,S_{j+1})+(1-\tau)\Gamma^A]\right)}{A\left(V[\tau\Gamma(S_{j-1},S_j)+(1-\tau)\Gamma^A]\right)}\right)^{\gamma(S_j)} > 1 \text{ with } S_0 = 0 \;\&\; S_J = 1,$$

where $\Gamma(S_{j-1},S_j) \equiv \frac{1}{S_j - S_{j-1}}\int_{S_{j-1}}^{S_j}\gamma(s)ds$.

This equilibrium converges to a collection of *J* identical single-country (autarky) equilibria as $\tau \to 0$ and to the *J*-country trade equilibrium shown in Proposition 1, as $\tau \to 1$.

Now, let us calculate the limit Lorenz curve by following the steps similar to those used in section 2.5. As before, by setting $x = j/J$ and $\Delta x = 1/J$,

$$\text{LHS} = \frac{S_{j+1}-S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o(|\Delta x|).$$

$$\Gamma(S_j,S_{j+1}) = \frac{\int_{\Phi(x)}^{\Phi(x+\Delta x)}\gamma(s)ds}{\Phi(x+\Delta x)-\Phi(x)} = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$\Gamma(S_{j-1},S_j) = \frac{\int_{\Phi(x-\Delta x)}^{\Phi(x)}\gamma(s)ds}{\Phi(x)-\Phi(x-\Delta x)} = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

so that

$$\begin{aligned}
&A\left(V[\tau\Gamma_j + (1-\tau)\Gamma^A]\right) \\
&\quad = A\left(V[\tau\gamma(\Phi(x)) + (1-\tau)\Gamma^A]\right) + \frac{V\tau}{2}A'\left(V[\tau\gamma(\Phi(x)) + (1-\tau)\Gamma^A]\right)\frac{d(\gamma(\Phi(x)))}{dx}\Delta x + o(|\Delta x|) \\
&A\left(V[\tau\Gamma_{j-1} + (1-\tau)\Gamma^A]\right) \\
&\quad = A\left(V[\tau\gamma(\Phi(x)) + (1-\tau)\Gamma^A]\right) - \frac{V\tau}{2}A'\left(V[\tau\gamma(\Phi(x)) + (1-\tau)\Gamma^A]\right)\frac{d(\gamma(\Phi(x)))}{dx}\Delta x + o(|\Delta x|)
\end{aligned}$$

from which

$$\begin{aligned}
\text{RHS} &= 1 + \frac{V\tau\gamma(\Phi(x))A'\left(V[\tau\gamma(\Phi(x)) + (1-\tau)\Gamma^A]\right)}{A\left(V[\tau\gamma(\Phi(x)) + (1-\tau)\Gamma^A]\right)}\frac{d(\gamma(\Phi(x)))}{dx}\Delta x + o(|\Delta x|) \\
&= 1 + \frac{\theta\left(V[\tau\gamma(\Phi(x)) + (1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau\gamma(\Phi(x))}\frac{d(\gamma(\Phi(x)))}{dx}\Delta x + o(|\Delta x|)
\end{aligned}$$

Combining these yields

$$1+\frac{\Phi''(x)}{\Phi'(x)}\Delta x+o(|\Delta x|) = 1+\frac{\theta\left(V[\tau\gamma(\Phi(x))+(1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau\gamma(\Phi(x))}\frac{d(\gamma(\Phi(x)))}{dx}\Delta x+o(|\Delta x|)$$

Hence, as $J\to\infty$, $\Delta x=1/J\to 0$,

$$\frac{\Phi''(x)}{\Phi'(x)} = \frac{\theta\left(V[\tau\gamma(\Phi(x))+(1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau\gamma(\Phi(x))}\frac{d(\gamma(\Phi(x)))}{dx}$$

Integrating once,

$$\log(\Phi'(x))-\int_0^{\gamma(\Phi(x))}\frac{\theta\left(V[\tau v+(1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau v}dv=c_0 \Leftrightarrow \exp\left[-\int_0^{\gamma(\Phi(x))}\frac{\theta\left(V[\tau v+(1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau v}dv\right]\Phi'(x)=e^{c_0}.$$

Integrating once more,

$$\int_0^{\Phi(x)}\exp\left[-\int_0^{\gamma(s)}\frac{\theta\left(V[\tau+(1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau v}dv\right]ds=e^{c_0}x+c_1$$

From $\Phi(0)=0$ & $\Phi(1)=1$, the two integral constants are pinned down to obtain:

Proposition 6 (Limit Case; $J\to\infty$): The limit equilibrium Lorenz curve, Φ, is given by:

$$x=H(\Phi(x);\tau)\equiv\int_0^{\Phi(x)}h(s;\tau)ds,$$

where $h(s;\tau)\equiv\dfrac{\hat{h}(s;\tau)}{\int_0^1\hat{h}(u;\tau)du}$ with $\hat{h}(s;\tau)\equiv\exp\left[-\int_0^{\gamma(s)}\dfrac{\theta\left(V[\tau v+(1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau v}dv\right]$.

Proposition 5 of Matsuyama (2013) can be obtained as a special case of the above.

Again, Figure 3 illustrates the above proposition. For each τ, $h(s;\tau)$ is positive, and decreasing in s, and it is normalized so that its integral from 0 to 1 is equal to 1. Thus, $H(s;\tau)$ is increasing and concave in s, with $H(0;\tau)=0$ and $H(1;\tau)=1$. Hence, $\Phi(x;\tau)=H^{-1}(x;\tau)$ is increasing and convex in *x*, with $\Phi(0;\tau)=0$ and $\Phi(1;\tau)=1$. Note also $\lim_{\tau\to 0}\hat{h}(s;\tau)=1$. Thus, as $\tau\to 0$, each country converges to the same single-country (autarky) equilibrium and hence the Lorenz curve converges to the diagonal line, and inequality disappears. Likewise, $\lim_{\tau\to 1}\hat{h}(s;\tau)=$ $\exp[-\Theta(V\gamma(s))/V]$. Thus, as $\tau\to 1$, the Lorenz curve converges to the one shown in Proposition 2.

As before, let us look at the log-super(sub)modularity of $\hat{h}(s)$ to see the effects of parameter changes. From

$$\frac{\partial \log(\hat{h}(s;\tau))}{\partial s} = -\frac{\theta\left(V[\tau\gamma(s)+(1-\tau)\Gamma^A]\right)}{1+(1-\tau)\Gamma^A/\tau\gamma(s)}\gamma'(s),$$

we have

$$\mathrm{sgn}\left\{\frac{\partial^2 \log(\hat{h}(s;\tau))}{\partial V \partial s}\right\} = -\mathrm{sgn}\left\{\theta'\left(V[\tau\gamma(s)+(1-\tau)\Gamma^A]\right)\right\}.$$

Again, for $\theta'(\bullet) > 0$, $\hat{h}(s)$ is *log-submodular* in *V* and *s*, so that a higher *V* causes a Lorenz-dominant shift away from the diagonal line, implying a greater inequality across countries, while for $\theta'(\bullet) < 0$, $\hat{h}(s)$ is *log-supermodular* in *V* and *s*, so that a higher *V* causes a Lorenz-dominant shift toward the diagonal line, implying a less inequality acourss countries. For $\theta(n) = \theta$, the Lorenz curve is independent of *V* but it also implies $\partial^2 \log(\hat{h}(s;\tau))/\partial\theta\partial s < 0$, so that a higher θ causes a Lorenz-dominant shift away from the diagonal, i.e., a greater inequality. Thus, the effects of *V* obtained in section 2 for $\tau = 1$ extends for $\tau < 1$.

For the effect of τ, the above Proposition suggests that changing from $\tau = 0$ to $\tau > 0$ clearly causes a Lorenz-dominant shift away from the diagonal, hence a greater inequality. However, this does not necessarily mean that any small increase in τ would cause a Lorenz-dominant shift. Matsuyama (2013) already shows that $\partial^2 \log(\hat{h}(s;\tau))/\partial\tau\partial s < 0$ for $\theta(n) = \theta$. Another sufficient condition for $\partial^2 \log(\hat{h}(s;\tau))/\partial\tau\partial s < 0$ is $\theta'(n) > 0$ for $n < V\Gamma^A$ and $\theta'(n) < 0$ for $n > V\Gamma^A$. When these conditions are met, a higher τ causes a Lorenz-dominant shift away from the diagonal, suggesting that globalization through trade in goods leads to greater inequality across countries.

3.2 Variable Factor Supply: Effects of Factor Mobility and/or Factor Accumulation

Returning to the case where $\tau = 1$, this subsection instead allows the available amount of the composite primary factors, *V*, to vary across countries by endogenizing the supply of one of the component factors, *K*, as follows:

(25) $V_j = F(K_j, L)$ with $\omega_j F_K(K_j, L) = \rho$.

where $F_K(K_j, L)$ is the first derivative of F with respect to K, satisfying $F_{KK} < 0$. In words, the supply of K in the j-th country responds to its TFP, ω_j, such that its factor price is equalized across countries at a common value, ρ. This can be justified in two different ways.

A. Factor Mobility: Imagine that L represents (a composite of) factors that are immobile across borders and K represents (a composite of) factors that are freely mobile across borders, which seek higher return until its return is equalized in equilibrium.[25] According to this interpretation, ρ is an equilibrium rate of return determined endogenously, although it is not necessary to solve for it when deriving the Lorenz curve.[26]

B. Factor Accumulation: Reinterpret the structure of the economy as follows. Time is continuous. All the tradeable goods, $s \in [0,1]$, are intermediate inputs that goes into the production of a single final good, Y_t, with the Cobb-Douglas function, $Y_t = \exp\left[\int_0^1 \log(X_t(s))ds\right]$ so that its unit cost is $\exp\left[\int_0^1 \log(P_t(s))ds\right]$. The representative agent in each country consumes and invests the final good to accumulate K_t, so as to maximize $\int_0^\infty u(C_t)e^{-\rho t}dt$ s.t. $Y_t = C_t + \dot{K}_t$, where ρ is the subjective discount rate common across countries. Then, the steady state rate of return on K is equalized at ρ.[27] According to this interpretation, K may include not only physical capital but also human capital, and the Lorenz curve derived below represents steady state inequality across countries.

Again, only the key steps will be shown. Let $\{n_j\}_{j=1}^{J}$ be monotone increasing. From (12) and (25),

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[25]Which factors should be considered as mobile or immobile depends on the context. If "countries" are interpreted as smaller geographical units such as "metropolitan areas," K may include not only capital but also labor, with L representing the immobile "land." Although labor is commonly treated as an immobile factor in the trade literature, we will later consider the possibility of trade in factors, in which case certain types of labor should be included among mobile factors.

[26]Also, $Y_j = V_j = \omega_j F(K_j, L)$ should be now interpreted as GDP of the economy, not GNP, and K_j is the amount of K used in the j-th country, not the amount of K owned by the representative agent in the j-th country. This also means that the LHS of the budget constraint in the j-th country, eq.(1), should be its GNP, not its GDP (Y_j). However, calculating the distributions of GDP (Y_j), TFP (ω_j), and K_j/L does not require to use the budget constraint for each country, given that all consumption goods are tradeable ($\tau = 1$). The analysis would be more involved if $\tau < 1$.

[27]The intertemporal resource constraint assumes not only that K is immobile but also that international lending and borrowing is not possible. Of course, these restrictions are not binding in steady state, because the rate of return is equalized across countries at ρ

(26) $$\frac{F_K(K_j,L)}{F_K(K_{j+1},L)} = \frac{\omega_{j+1}}{\omega_j} = \left(\frac{A(n_{j+1})}{A(n_j)}\right)^{\gamma(S_j)} > 1$$

which implies that $\{\omega_j\}_{j=1}^J$, $\{K_j\}_{j=1}^J$, $\{V_j\}_{j=1}^J$, and $\{Y_j\}_{j=1}^J$ are all monotone increasing in *j*, hence they are (perfectly) positively correlated. Since V_j is now endogenous, eq.(13) and eq.(15) need be modified as:

(27) $$n_j = \Gamma_j V_j = \Gamma_j F(K_j, L)$$

(28) $$Y_j = \omega_j V_j = \omega_j F(K_j, L) = (S_j - S_{j-1})Y^W.$$

These equations can so be summarized as:

$$\frac{Y_{j+1}}{Y_j} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \frac{\omega_{j+1}V_{j+1}}{\omega_j V_j} = \frac{V_{j+1}}{V_j}\left(\frac{A(\Gamma_{j+1}V_{j+1})}{A(\Gamma_j V_j)}\right)^{\gamma(S_j)} > 1.$$

To see what is involved, suppose $V = F(K, L) = ZK^\alpha$ with $0 < \alpha < 1/(1+\theta(\bullet))$. Then,

Proposition 7 (*J*-country case): Let S_j be the cumulative share of the *J* poorest countries in income. Then, $\{S_j\}_{j=0}^J$ solves:

$$\frac{K_{j+1}}{K_j} = \frac{Y_{j+1}}{Y_j} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\omega_{j+1}}{\omega_j}\right)^{\frac{1}{1-\alpha}} = \left(\frac{A\left(Z(K_{j+1})^\alpha \Gamma(S_j, S_{j+1})\right)}{A\left(Z(K_j)^\alpha \Gamma(S_{j-1}, S_j)\right)}\right)^{\frac{\gamma(S_j)}{1-\alpha}} > 1 \text{ with } S_0 = 0 \ \& \ S_J = 1,$$

where $\Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s)ds$.

Two remarks are in order. First, $\{S_j\}_{j=0}^J$ here represents the cumulative shares in *Y/L* and in *K/L*, not in TFP. However, the distribution of TFP can be obtained from the distribution of *Y/L* (or *K/L*) and *vice versa*, using a monotone transformation, $\omega_{j+1}/\omega_j = (Y_{j+1}/Y_j)^{1-\alpha}$. Second, the above condition does not fully characterize the equilibrium distribution. Generally, we need another condition to pin down the *level* of *Y* (or *K/L*), because the symmetry-breaking mechanism is not scale-free under a variable degree of scale economies, and the supply of primary factors is now variable. For $A(n) \propto (n)^\theta$, this condition can be rewritten as the second order difference equation in $\{S_j\}_{j=0}^J$ as follows, which fully characterizes the equilibrium Lorenz curve:

Corollary 3 (the *J*-country case): Let $A(n) \propto (n)^{\theta}$. Then, $\{S_j\}_{j=0}^{J}$ solves:

$$\frac{S_{j+1}-S_j}{S_j-S_{j-1}} = \left(\frac{\Gamma(S_j,S_{j+1})}{\Gamma(S_{j-1},S_j)}\right)^{\frac{\theta\gamma(S_j)}{1-\alpha-\alpha\theta\gamma(S_j)}} > 1 \text{ with } S_0 = 0 \ \& \ S_J = 1,$$

where $\Gamma(S_{j-1},S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s)ds$.

Again, let us calculate the limit Lorenz curve. As before, by setting $x = j/J$ and $\Delta x = 1/J$,

$$\text{LHS} = \frac{K_{j+1}}{K_j} = \frac{S_{j+1}-S_j}{S_j-S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o(|\Delta x|)$$

$$\Gamma(S_j,S_{j+1}) = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$\Gamma(S_{j-1},S_j) = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|).$$

Furthermore, we have

$$(K(x+\Delta x))^{\alpha} = (K(x))^{\alpha}\left(1 + \alpha\frac{\Phi''(x)}{\Phi'(x)}\Delta x\right) + o(|\Delta x|),$$

from which

$$A\big(Z(K_{j+1})^{\alpha}\Gamma(S_j,S_{j+1})\big)$$

$$= A\big(Z(K(x))^{\alpha}\gamma(\Phi(x))\big)\left(1 + \theta\big(Z(K(x))^{\alpha}\gamma(\Phi(x))\big)\left(\alpha\frac{\Phi''(x)}{\Phi'(x)} + \frac{\gamma'(\Phi(x))\Phi'(x)}{2\gamma(\Phi(x))}\right)\Delta x\right) + o(|\Delta x|)$$

$$A\big(Z(K_j)^{\alpha}\Gamma(S_{j-1},S_j)\big)$$

$$= A\big(Z(K(x))^{\alpha}\gamma(\Phi(x))\big)\left(1 - \theta\big(Z(K(x))^{\alpha}\gamma(\Phi(x))\big)\left(\frac{\gamma'(\Phi(x))\Phi'(x)}{2\gamma(\Phi(x))}\right)\Delta x\right) + o(|\Delta x|)$$

from which

$$\text{RHS} = \left(\frac{A\big(Z(K_{j+1})^{\alpha}\Gamma(S_j,S_{j+1})\big)}{A\big(Z(K_j)^{\alpha}\Gamma(S_{j-1},S_j)\big)}\right)^{\gamma(S_j)/(1-\alpha)}$$

$$= 1+\frac{\gamma(\Phi(x))}{1-\alpha}\theta\Big(Z\big(K(x)\big)^{\alpha}\gamma(\Phi(x))\Big)\left(\alpha\frac{\Phi''(x)}{\Phi'(x)}+\frac{\gamma'(\Phi(x))\Phi'(x)}{\gamma(\Phi(x))}\right)\Delta x+o(|\Delta x|)$$

Combining these and let $J\to\infty$, $\Delta x = 1/J \to 0$ yields

$$\frac{\Phi''(x)}{\Phi'(x)}=\frac{\theta\Big(Z\big(\overline{K}\Phi'(x)\big)^{\alpha}\gamma(\Phi(x))\Big)}{1-\alpha-\alpha\gamma(\Phi(x))\theta\Big(Z\big(\overline{K}\Phi'(x)\big)^{\alpha}\gamma(\Phi(x))\Big)}\frac{d\gamma(\Phi(x))}{dx}$$

where use has been made of

$$\frac{K'(x)}{K(x)}=\frac{\Phi''(x)}{\Phi'(x)} \text{ or } K(x)=\overline{K}\Phi'(x),$$

where $\overline{K}$ is the average of *K*. By setting $V = Z(\overline{K})^{\alpha}$,

Proposition 8 (Limit Case, $J \to \infty$): The limit equilibrium Lorenz curve in income, Φ, solves:

$$\frac{\Phi''(x)}{\Phi'(x)}=\frac{\theta\Big(V\big(\Phi'(x)\big)^{\alpha}\gamma(\Phi(x))\Big)}{1-\alpha-\alpha\gamma(\Phi(x))\theta\Big(V\big(\Phi'(x)\big)^{\alpha}\gamma(\Phi(x))\Big)}\gamma'(\Phi(x))\Phi'(x)$$

with $\Phi(0)=0$ & $\Phi(1)=1$.

Again, some remarks are in order. First, Φ here represents the Lorenz curve in *Y/L* and in *K/L*, not in TFP. The Lorenz curve in TFP can be obtained from Φ, using a monotone transformation, $\Phi^{\omega}(x)=\int_0^x\big(\Phi'(u)\big)^{1-\alpha}du\Big/\Big[\int_0^1\big(\Phi'(u)\big)^{1-\alpha}du\Big]$. Second, the above differential equation generally depends on $V = Z(\overline{K})^{\alpha}$. Under Interpretation A (Factor Mobility), $\overline{K}$ may be interpreted as the initial endowment of the mobile factor in each country and hence be treated as exogenous, so that the above equation fully characterizes the equilibrium Lorenz curve. However, under Interpretation B (Factor Accumulation), $\overline{K}$ is endogenous so that we need another condition to pin it down. Again, this is because the symmetry-breaking mechanism is not scale-free under a variable degree of scale economies. Third, for $\alpha > 0$, the above differential equation does not have a closed form solution for a general $A(n)$ or $\theta(n)$, because $\Phi'(x)$ appears inside $\theta(\bullet)$.[28] For $A(n)\propto(n)^{\theta}$, $\theta(n)=\theta n$, which enables us to solve it explicitly as follows:

[28] For $\alpha = 0$, it is equivalent to eq. (19).

Corollary 4 (Limit Case, *J* → ∞): Let $A(n) \propto (n)^{\theta}$. Then, the limit equilibrium Lorenz curve, Φ, solves:

$$\frac{\Phi''(x)}{\Phi'(x)} = \frac{\theta}{1-\alpha-\alpha\theta\gamma(\Phi(x))}\frac{d\gamma(\Phi(x))}{dx}$$

with $\Phi(0)=0$ & $\Phi(1)=1$, whose unique solution is:

$$x = H\left(\Phi(x)\right) \equiv \int_0^{\Phi(x)} h(s)ds, \text{ where } h(s) \equiv \frac{\hat{h}(s)}{\int_0^1 \hat{h}(u)du} \quad \text{with } \hat{h}(s) \equiv \left(1-\frac{\alpha\theta}{1-\alpha}\gamma(s)\right)^{1/\alpha}$$

This reproduces Proposition 6 of Matsuyama (2013). Again, Figure 3 illustrates this. For each $\alpha < 1-1/\sigma = 1/(1+\theta)$, $h(s;\alpha)$ is positive, and decreasing in s, and it is normalized so that its integral from 0 to 1 is equal to 1. Thus, $H(s;\alpha)$ is increasing and concave in s, with $H(0;\alpha)=0$ and $H(1;\alpha)=1$. Hence, $\Phi(x;\alpha) = H^{-1}(x;\alpha)$ is increasing and convex in *x*, with $\Phi(0;\alpha)=0$ and $\Phi(1;\alpha)=1$. It is also easy to check $\lim_{\alpha\to 0}\hat{h}(s;\alpha) \equiv \exp\left(-\theta\gamma(s)\right)$.

Again, the effect of α can be seen by noting that

$$\hat{h}(s;\alpha) \equiv \left(1-\frac{\alpha\theta}{1-\alpha}\gamma(s)\right)^{1/\alpha},$$

is *log-submodular* in α and s (and in θ and s). Thus, a higher α (and a higher θ) causes a Lorenz-dominant shift away from the diagonal, as shown in the arrows in the right panel of Figure 3. This result thus suggests that skill-biased technological change that increases the share of human capital and reduces the share of raw labor in production, or globalization through trade in some factors, both of which can be interpreted as an increase in α, could lead to greater inequality across countries.

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Figure 1: Comparative Advantage and Patterns of Trade in the *J*-country World

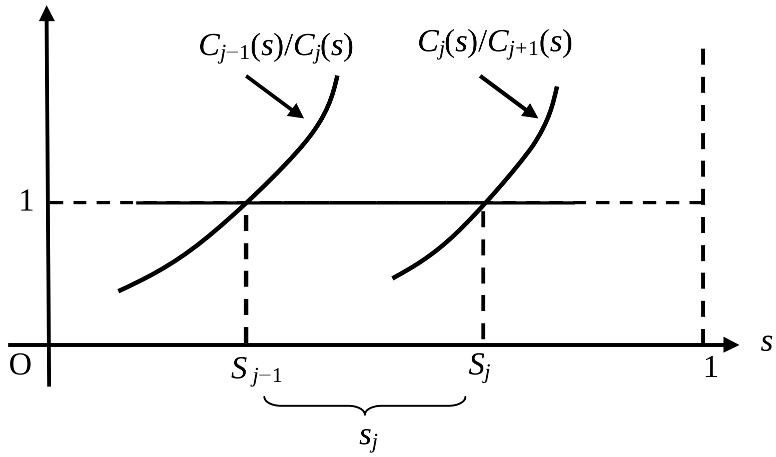

Figure 2: Equilibrium Lorenz curve, Φ^J: A Graphic Illustration for $J = 4$

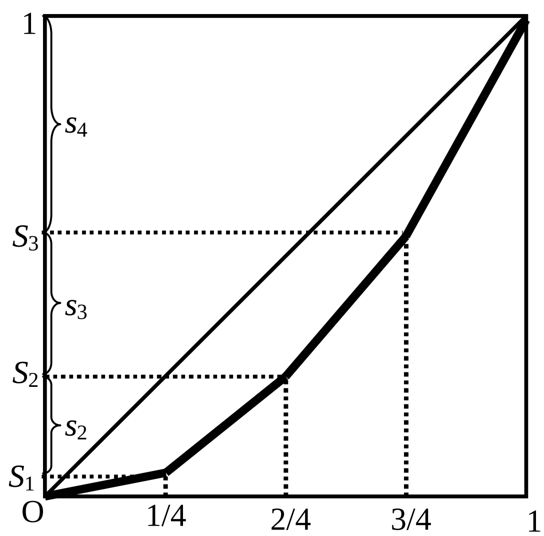

Figure 3: Limit Equilibrium Lorenz Curve, Φ(*x*), and its Lorenz-dominant Shift

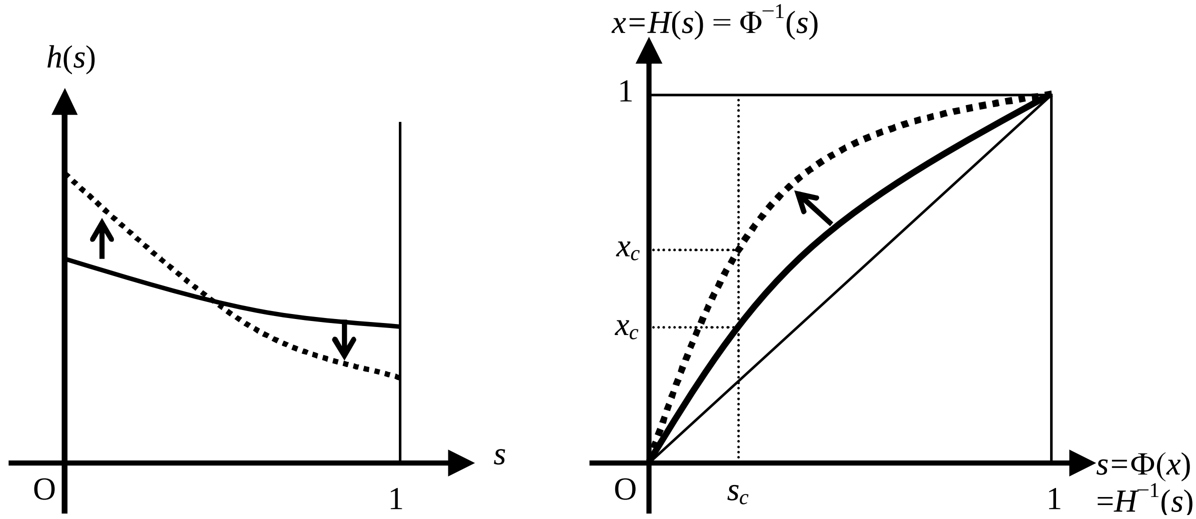

Figure 4: Polarization or Bimodal Distribution