Handling Non-Invertibility: Theories and Applications*

Bill Dupor and Jing Han

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Abstract

Existing research provides no systematic, limited information procedure for handling non-invertibility, despite the well-known inference problem it causes as well as its presence in many types of dynamic systems. Non-invertibility means that structural shocks cannot be recovered from a history of observed variables. It arises from a form of delayed responses due to, among other things, time-to-plan, sticky information or news shocks. Structural VARs rule out non-invertibility by assumption. Inference about structural responses can, in turn, be incorrect. We develop a four-step procedure to partially, and sometimes fully, identify structural responses whether or not non-invertibility is present. Our method combines structural VAR restrictions, e.g. recursive identification, with "agnostic" identification, e.g. sign restrictions and bounds on forecast error contributions. In two model-generated examples, our procedure recovers the structural responses where structural VARs cannot. Also, we apply our procedure to real world data. We show that non-invertibility is unlikely in Fisher's (2006) study of technology shocks in the U.S.

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"Do you mean now?" – *Baseball player and manager Yogi Berra, when asked for the time.*

1 Introduction

Suppose a police officer on foot patrol happens upon a dead man with a knife in his back. An autopsy firmly establishes that the time of death was 5:00 AM earlier that day. Detectives would like to know when he was stabbed. With no witnesses, the stabbing could have occurred at 4:59 AM with the victim dying very quickly. Or, the stabbing could have occurred the evening before with the victim could have died very slowly. There are other possibilities, and thus, the time of the crime is not well identified.

A time series analyst often faces a similar problem. Suppose the analyst observes a series of outcomes (e.g. real GDP), each of which is indexed by a known time. Suppose the analyst does not observe the sequence of impulses (e.g. preference shocks) or their associated times. A current change in an observable might be due to immediate response to a contemporaneous impulse. Or, the current change might be a delayed response to an impulse that occurred long ago. To the analyst, this is known as the "non-invertibility identification problem." It is distinct from the "simultaneous equation problem" that arises with multiple unobserved shocks.[1]

The police detective and the time series analyst have different standard operating procedures for dealing with this identification problem. The police detective would look for other evidences to inform when the shock (i.e. the stabbing) occurred, such as the stiffness of the dead body. Faced with the same crime, on the other hand, the time series analyst typically would usually assume that stabbing occurred at 4:59 because this is the response with the shortest delay from impulse to observable. In technical language, the analyst has dealt with the non-invertibility problem by assuming the invertible representation, i.e. the one with minimal delay, is the correct one. In non-technical terms, the analyst has done shabby police work.

In this paper, we develop a procedure for handling the identification problem with-

[1] In most problems, a researcher must deal with both the simultaneous equations problem and non-invertibility problem. Dealing with both is a part of our paper.

out assuming that responses to structural shocks occur with minimal delay. Rather, we follow the police detective's method. We ask whether other evidence, including the co-movement of the observable with other observables or the sign of impulse responses, are consistent or inconsistent with restrictions implied by economic theory. We wish to use as few clues given by economic theory as possible.

This paper addresses non-invertibility in a limited information framework. We treat non-invertibility in a similar manner to the one that researchers already use in VARs to deal with the simultaneous equations identification problem. That is, compute all of the stochastic processes consistent with the data and then apply identifying restrictions from economic theory to exclude some (and potentially all but one) of these processes.

Our procedure has four steps.

Step One: *Estimate a reduced-form VARMA(1,1) on the observables.*

We begin by assuming the time series has a state-space representation. Many dynamic economic models is consistent with this form. A large set of processes can be written as VARMA(1,1) by stacking the state space. To be concrete, let Y_t represent a vector of k observable, stationary variables. In some very general conditions, observable variables have a VARMA(1,1) representation.

Step Two: *Calculate all covariance equivalent representations.*

With k observable variables, there are at most 2^k state-space forms that have the identical covariance functions, modulus the simulatenous equations problem. One of these state-space forms will be invertible, i.e. have minimal delay. However, there is no rationale for simply choosing this one over a non-invertible representation without futher identification restrictions.

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction on each representation.*

This step mimics that of the SVAR approach. A shock of interest might be to technology or monetary policy. Short-run restrictions (e.g. output does not respond to current monetary policy changes) and long-run restrictions (e.g. only technological change affects long-run labor productivity) are examples of SVAR-type restrictions. This step is necessary because non-invertibility neither mitigates nor intensifies the simultaneous equations problem.

Step Four: *Impose agnostic restrictions on each representation, delivered from step three, to further rule out structural respones.*

Uhlig uses the phrase "agnostic restrictions" to describe identifying assumptions of the kind implemented in Faust (1998), Scholl and Uhlig (2005) and Uhlig (2005).[2] For example, a positive innovation to the structural shock might be required to: (i) have a non-negative long-run effect on a particular observable; (ii) imply a positive response to an observable at the two-year horizon; (iii) explain the variation in one variable within a certain range.

After step four, the researcher is left with one or multiple structural impulse responses to the shock of interest. When only one response remains, the impulse response is fully identified. When multiple remain, the impulse response is partially identified. In either case, the invertible form may or may not belong to the set. If the invertible form is consistent with the restrictions from step four, then it will be a valid structural response. Importantly, our procedure does not a priori choose this response.

The problem of non-invertibility has received great attention in economics and time series analysis. In an introductory chapter of his textbook, Hamilton (1994, pg. 64) discusses the issue and presents practical reasons for preferring the invertible representation.[3] Sargent (1987) presents another textbook discussion. FRSW (2006) explain that non-invertibility is induced by missing variables.

Economists have pointed out that non-invertibility arises in many environments. Model features that can induce non-invertibility in the structural responses include: permanent income economies (Hansen and Sargent 1991 and FRSW 2006); learning-by-doing (Lippi and Reichlin 1993); anticipated fiscal policy shocks (Leeper, Walker and Yang 2009); anticipated technology shocks (Blanchard et. al. 2009). Non-invertibility can also arise from sticky information, time-to-plan and Townsend-type economies with "forecasting

[2]Examples of other papers using agnostic identification include: Cardoso-Mendonca, Medrano and Sachsida (2008) and Owyang (2002).

[3]We discuss these reasons and how our addresses them in section two.

the forecasts of others."

Most of the researches listed above emphasize the difficulties non-invertibility brings to empirical studies, which share the same spirit as the story we show in the beginning of this paper. Non-invertibility does not only mis-specify the timing of a certain structural shock (as in Hansen and Sargent (1991))but also entangle identifications of different shocks (as in Leeper et al (2009)). Sims (2009) is an exception. Using data simulated from a calibrated DSGE model, he finds that the presence of non-invertibility introduces very little bias in the estimates delivered by a simple SVAR analysis.

Alessi et all (2008) present a comprehensive review and history of developments related on non-invertibility in structural estimation. Despite these extensive discussions of the problem and its practical relevance, there are few solutions. To our knowledge, our four step procedure is the first systematic, limited information method for dealing with non-invertibility.

In existing research, three methods for handling non-invertibility have been offered. Each differs from ours in separate and important ways. These methods are: (i) using observed shocks rather than idnetified shocks; (ii) using full information estimation of a correctly specified DSGE model rather than our limited information approach; (iii) standard SVAR estimation augmented with something akin to our step three.

First, numerous researchers use data where shocks are directly observable. If the shock and its arrival time are known, the identification problem disappears. Case studies applied to particular changes in tax policy are well-suited for this approach. However, in most cases, shocks are not directly observed.

Second, FRSW's method draws upon their discussion of the danger in using SVARS. SVARs always choose the invertible representation of a time series. When the actual structural response is non-invertible, the SVAR leads to incorrect inference. Rather than an SVAR, they recommend correctly specifying a full dynamic, stochastic general equilibrium (DSGE) model and using a full information technique. Our limited information procedure is less likely to suffer from misspecification than using a fully specified model.

FRSW also provide a condition to use, case-by-case, to determine whether an SVAR would generate incorrect inferences. To check this condition, one uses the estimates or calibration of the DSGE model relevant for the particular time series. However, with a correctly specified DSGE model in hand, one should use all of the information in the DSGE model rather than the limited information SVAR on efficiency grounds.

Third, Lippi and Reichlin (1994) suggest a limited information approach. It is the closest anticedent of our work. They compute the structural impulse response using a VAR and a standard rotation restriction. The estimated structural response is by construction invertible, as discussed in FSRW. Recognizing that non-invertible solutions are also consistent with the observed data, they then do a visual inspection of roots from the estimated VAR in search of an MA structure. Based on the inspection, they plot both non-invertible and invertible structural responses implied by their VAR. This is similar to our step three. As they explain, their method is only suitable for a two variable system. On the other hand, our procedure works for a system with more variables because we estimate the MA component directly (i.e. our step one). Also, our procedure allows us to exclude some of the potential structural responses (i.e. our step four) in a systematic manner.

More recently, Mertens and Ravn (2010) brings DSGE models, SVAR analysis and the method proposed by Lippi and Reichlin (1994) together in an inventive way, to address non-invertibility. They specify and calibrate a DSGE model with news shocks, and then use it to determin the placement of the non-invertibility in the system's moving-average structure, along with the magnitude of the roots associated with the non-invertibility. In their exercise, they calibrate the values of the roots associated with the non-invertibility, while our procedure calculate these roots based on the data. Moreover, their procedure can only analyze a single shock with non-invertibility, while our procedure is suitable for cases with multiple non-invertible shocks.

The next section contains scalar and bivariate examples the features of non-invertibility that our method will exploit. Section 3 presents the four-step procedure along with its theoretical justification. Section 4 applies the procedure to two sets of model-generated data and section 5 applies the procedure to two real world applications. Section 6 concludes.

2 Introductory Examples: When Non-invertibility Emerges

Non-invertibility arises in many situations. In this section, we will use two simple examples to illustrate: (i) how non-invertibility emerges from those models; (ii) how non-invertibility affect the dynamic of the model and economists' inference.

First illustration: a scalar observable that is iid

Suppose an economist knows a scalar variable y_t to be Gaussian iid with expectation zero and positive variance v_0. He also knows that there is single unobserved shock, which drives the observed variable via a linear relationship. This is probably the simplest structural estimation problem imaginable.

$$E\left(y_t y_{t-j}\right) = \begin{cases} v_0 & \text{if } j = 0 \\ v_1 & \text{if } j = 1 \\ 0 & \text{if} j > 1 \end{cases} \tag{1}$$

The economist asks, how might the unobserved shock influence y_t? We interpret the economist's question as equivalent to: what are all moving average representations that are consistent with y_t? In particular, let us restrict attention to MA(1) processes. A general expression for an MA(1) is:

$$y_t = \theta_{j,0} w_t^j + \theta_{j,1} w_{t-1}^j \tag{2}$$

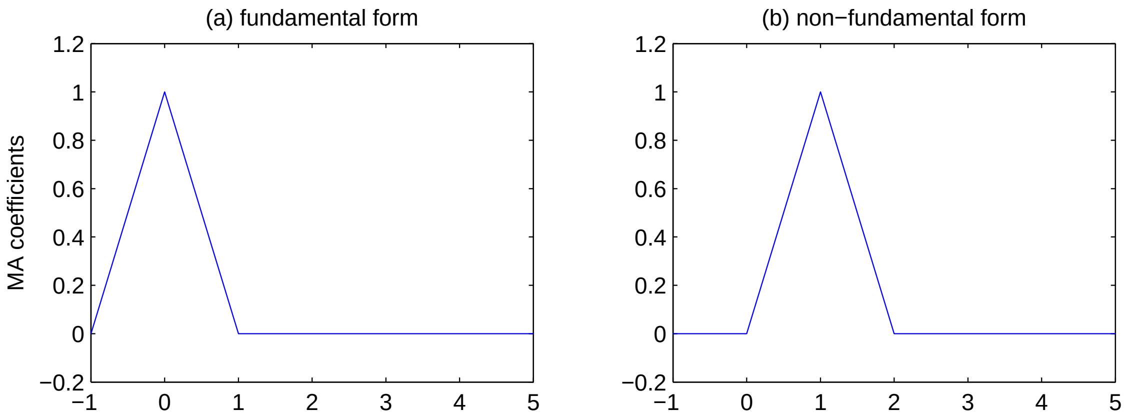
Here, j indexes a particular representation. Each particular j corresponds to a different process $\left\{w_t^j\right\}$ as well as a pair $\left(\theta_{j,0}, \theta_{j,1}\right)$.[4]

What restrictions do the moments given by (1) put on $\left(\theta_{j,0}, \theta_{j,1}, \left\{w_t^j\right\}\right)$? We can find all such restrictions by matching moments from (1) with those implied by (2). These imply two independent restrictions:

$$\left(\theta_{j,0}\right)^2 + \left(\theta_{j,1}\right)^2 = v_0 \tag{3}$$

[4] According to the definition of a moving average process, $\left\{w_t^j\right\}$ is a mean zero, white noise process for all j. As a normalization and without loss of generality, assume w_t^j has unit variance for all j. Hamilton and Sargent contain textbook treatments of non-invertibility. Each assumes $\theta_{j,0} = 1$ as a normalization and allow the variance of w_t^j to be free.

Figure 1: Two covariance-equivalent structural forms; scalar observable is iid



Notes:

$$\theta_{j,0}\theta_{j,1} = v_1 \tag{4}$$

We know that $v_1 = 0$. As such, $\theta_{j,0} = 0$ and/or $\theta_{j,1}$ must equal zero. If $\theta_{j,0} = 0$, then $\theta_{j,1} = \sqrt{v_0}$ by equation (3).[5] Similarly, if $\theta_{j,1} = 0$, then $\theta_{j,0} = \sqrt{v_0}$. Note that both coefficients cannot be zero because $v_0 > 0$.

Figure 1 plots out two covariance-equivalent sets of impulse response functions. The lack of identifyability is straightforward. If the economist sees y_t increase, the increase could be due to an instantaneous response to a shock this period (as in panel (a)) or the increase could be due to one period lagged response to a shock in the previous period (as in panel (b)). Because y_t is observed to be iid, the economist does know that the impulse response is zero at all but one horizon.

It is worth noting that the only reason that there are only two potential responses rather than three or more is because we restricted attention to structural forms that are MA(1). Without this restriction, a third covariance equivalent structural form would be a zero response in every period except period two, when there would be a unity response. For this form, an increase in y_t would correspond to a shock that arrived two periods ago with a lagged effect of two periods. By this same logic, an increase in y_t could be due to a

[5]Here we maintain our sign restriction that $\theta_{j,0} \geq 0$.

shock that happened r periods ago that had its effect with a lag of r periods.

Non-invertibility in Multivariate Enviroment

Our second example is a simple example with two variables. Suppose an economist observes y_{1t} and y_{2t}, output and money growth respectively. Each variable has expectation zero and unit variance. The covariance with each other and at every lead and lag equals zero.

What are the set of MA(1) processes, each indexed by j, that are consistent with the observed covariance structure? In matrix form,

$$y_t = \Gamma_0^j \omega_t^j + \Gamma_1^j \omega_{t-1}^j$$

where Γ_0^j, Γ_1^j are square matrices of dimension two and ω_t^j is 2 by 1.

One obvious structure is that y_{1t} and y_{2t} are each driven by distinct and uncorrelated white noise processes. That is, $\Gamma_0^j = I$ and $\Gamma_0^j = 0$ for $j = 1$. To be concrete, let us give an economic interpretation to these shocks. The first shock ω_{1t}^1 might be called a technology shock and the second shock ω_{2t}^1 might be called a monetary policy shock. We plot the impulse responses for this representation in panels (a) and (b) of figure 2.

With these interpretations, the economist would conclude that monetary policy is neutral and also that monetary policy does not respond to changes in output.

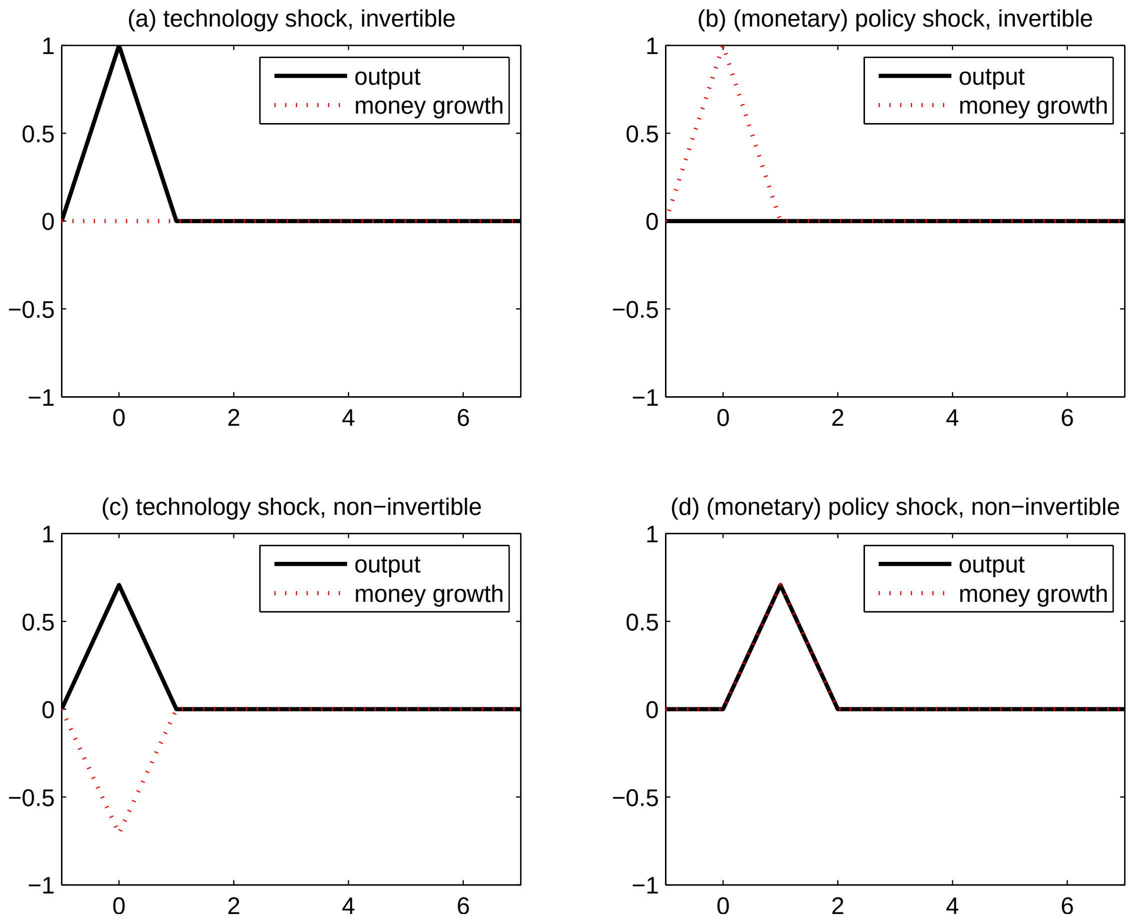
However, there are other MA(1) processes that satisfy the covariance restrictions. Another example appears is

$$\begin{aligned} y_{1t} &= \frac{\sqrt{2}}{2}\left(\omega_{1t}^2 + \omega_{2t-1}^2\right) \\ y_{1t} &= -\frac{\sqrt{2}}{2}\left(\omega_{1t}^2 - \omega_{2t-1}^2\right) \end{aligned}$$

panels (c) and (d) of figure 2.

Examining figure 2(d), the money growth and output impulse responses are zero on impact and positive at horizon one in response to a monetary shock. First, note that, because the output response happens with a one period delay, panel (d) is consistent

Figure 2: Two covariance-equivalent structural forms; bivariate observable with zero covariance between variance and zero covariance at all leads and lags



Notes:

with the typical VAR restriction that output is predetermined relative to a policy shock. Second, panel (d) implies that money growth and output are *perfectly, positively correlated* with respect to the policy shock. At the same time, output and money growth must be uncorrelated. Therefore, money growth and output must be *negatively correlated* with respect to the technology shock in order to offset the positive correlation above. This is clear from panel (c). Third, each of the four impulse responses (in panels (c) and (d)) is non-zero either on impact or at horizon one. This guarantees that there is no serial correlation in the observed money growth and output.

It is important to note that there are more than two impulse responses that generate the same observed population moments for money growth and output. We plot only two sets for the sake of pedagogy. The exact number depends upon how many other restrictions are imposed on the system. In the next section, we show how imposing the standard restrictions from existing VAR research that assumes invertibility leads to $\frac{k(k-1)}{2}$ restrictions where k is the number of observable variables.

Panel (d) is the most straightforward non-fundamental form to interpret. In this case, every impulse response is either zero everywhere or else it is zero at every horizon except at horizon one. The policy shock affects only output, and with a one period delay. The technology shock affects only money growth, and with a one period delay. Because the two shocks are uncorrelated, observed output and money growth are uncorrelated. When there is non-invertibility shown as in panel (d) and the above system, traditional method can only give us panel (a) or (c). It not only just miss the timing of the shocks as both in (a) and (c), it also possibly miss the true effect of shocks,i.d, attributing all output growth to technology shocks as in (a).[6] Comparison of panels (a) and (d) are consistent with observation (i): non-invertibility pushes the strongest impulses to later horizons.

Based on the examples above, we can infer some basis properties of the non-invertible models:

(i) Non-invertible forms likely push strongest impulse to later horizons.

From the very simple example, it is obvious that the magnitude of impulse responses

[6]Note that an instantaneous response of money growth to a technology shock and of output to a policy shock is ruled out, by our upper diagonal assumption on all D_j.

in later periods is larger than those on impact. We name this property as "delayed response". In more genernal cases, it is still the case that non-invertible models have delayed response more often than their invertible counterpart.We will use the following derivation to illustrate why non-invertible models imply such a pattern.

Without loss of generality, we can focus on a VMA(1) model. Any MA(q) model can be re-modelled as a VMA(1) model. Furthermore, it is staightforward to generalize the discussion here to a VARMA(p,q) model or VMA(∞) model.

The model is given by

$$Y_t = Me_t + Ne_{t-1} \tag{5}$$

where M is assumed to be a full rank matrix, e_t is a *i.i.d* shock following a standard normal distribution. Without loss of generality, we normalize the responses on impact as the numeraire. It is straightforward to show that the responses after one period is given by NM^{-1} [7]. The normalized impulse responses at the longer horizon are represented by row vectors of the matrix NM^{-1}. In other words, a weighted average of eigenvalues of NM^{-1} [8]. Since we can always normalize the eigenvector, so the magnitudes of eigen values are

[7] If the model is a VMA(q) model defined as:

$$y_t = N_0 e_t + N_1 e_{t-1} + ... + N_q e_{t-q}. \tag{6}$$

We can always define $Y_t = [y_t' e_t' \ ... \ e_{t-q+2}]'$ and $E_t = [e_t' e_{t-1}' e_{t-2}' \ ... \ e_{t-q+1}]'$, and the model is re-written as

$$Y_t = ME_t + NE_{t-1} \tag{7}$$

. The matrices, M and N are given by

$$M = \begin{bmatrix} N_0 & N_1 & N_2 & ... & N_{q-1} \\ I_k & 0 & 0 & ... & 0 \\ 0 & I_k & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & I_k & 0 \\ \end{bmatrix}$$

$$N = \begin{bmatrix} N_q & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 0 \\ \end{bmatrix}. \tag{8}$$

In this case, only the first k rows in NM^{-1} represent the normalized impulse responses of y_t.

[8] Through some simple but tedious algebra, we can show that $\{NM^{-1}\}_{i,j} = \sum_K^{k=1} a_{i,k} a^{k,j} \lambda_k\}$, where K is the dimension of Y_t, $a_{i,k}$ and $a^{i,k}$ are the kth entry on the ith row of the eigenvector matrix and the inverse

more important factors determining the magnitude of those impulse responses. Compared to the invertible case, there is at least one eigvenvalue is higher in absolute value in each non-invertible case, since this eigenvalue is obtained by flipping the corresponding eigenvalue (inside the unit circle) in the invertible case. Therefore, it is more likely that impulse responses in later periods are higher than responses on impact in non-invertible cases.

Furthermore, this pattern is consistent with the implication from models featuring ”sticky information” or ”news shocks”. In models with sticky information, most agents can only respond to events or shocks several quarters before, thus, the contribution from earlier shocks is bigger at the aggregate level. If the model is featured by ”news shocks”, earlier information is more relevant for current economic situation, so agents act on earlier information rather than more recent information. In the next bullet point, we will elaborate how non-invertibility is implied by those economic models

(ii) Non-invertible forms arise naturally from economic models with ”sticky information” or ”news shocks”.

Non-invertible models correspond to cases where the zeros for the MA polynomials are inside the unit circle. A general VARMA(p,q) model is given by $M(L)Y_t = N(L)\epsilon_t$, where $M(L)$ is the AR polynominal, with an order of p, and $N(L)$ is the MA polynomial with an order of q. Non-invertibility implies that there is at least one z satisfying $N(z) =$ is inside the unit circle. It implies the contribution of some ”old” shocks are higher than their ”recent” conterpart.

This characteristics is shared by economic models featuring ”sticky information” or ”news shocks”. In models with sticky information, most agents can only act on the old information while only a smal fraction of agents can act on the new information. As a consequence, aggregated data respond to ”old” shocks rather the most recent ones. In models with news shocks, agents put more weight on ”old” information than the ”new” information, because the information structure implies the current information only matters for future economic condition, which should be discounted when making decisions.

of eigenvector matrix of NM^{-1}, and λ_k is the kth eigenvalue.

The old information, on the contrary, is more relevant to the current economic condition, so it is optimal to respond to old rather than new information.

(*iii) Non-invertible forms have a "hidden state variable" interpretation.*

A tradtional interpretation of non-invertibility is the story of "missing variables". If all shocks and state variables are observable, there won't be non-invertbility anymore, since the model is just VAR(1).

(*iv) Non-invertible forms likely bear a relationship to zero restrictions in standard structural VARs.*

In the extreme case discussed above, if agents can only respond to shocks in previous periods, we can recover the underlying economic model by imposing restrictions on the Γ_0 matrix, i.e, $\Gamma_0 = 0$. This methodology is not at odds with existing research. When identifying monetary policy shocks, economists assume that every endogenous variable other than the policy variable is unable to respond to current monetary policy shock. In our example, it is equivalent to set $\Gamma_0(1,2) = 0$ and $\Gamma_0(2,2) \neq 0$. This identification scheme is widely used in empirical macroeconomic studies known as "short-run restrictions". Nevertheless, this type of structural models are never categorized as "non-invertible" models. The insight we can get from this approach is to begin with an agnostic setup, i.e., a reduced form model and use economic theory to identify the underlying structural models.

These characteristics are either found in empirical research on real world data or consistent with implications of state-of-the-art business cycle models. In the following section, we develop a systematic approach to study non-invertible models and use three real world application to illustate how this algorithm is applied.

3 Theory and A Four-Step Procedure

A generic covariance-stationary stochastic process is given by:

$$
\begin{aligned}
s_{t+1} &= Qs_t + Ue_{t+1} \\
r_{t+1} &= Ws_t + Ze_{t+1}
\end{aligned}
\tag{9}
$$

where e_{t+1} is k by 1 and $N(0, I)$. We refer to (Q, U, W, Z) as a *state-space form (*with associated shock process e_t) for the stochastic process $\{s_t, r_t\}$. Here, Q, U, W, Z are real-valued. Only r_t is observed by the economist.

In addition, we make the following additional assumptions on the state-space form.

Assumption 1 *The left inverse of W, which we denote* $\bar{W}$*, exists.*

Assumption 2 *All eigenvalues of Q and* $WQ\bar{W}$ *are inside the unit circle*

Assumption 3 *The matrix Z is invertible*

Assumption one requires that there are least as many observables as states. To identify the underlying system, economists need to have enough information,i.e, enough observable variables. This assumption is not as restrictive as it may seem. If the economy is actually driven by a few common factors, e.g. the dynamic factors as those identified by Stock and Watson (2002) or used by Bernanke, Boivin and Giannoni (2006), most multivariate time seris models have more observables than states. Assumption two ensures the observables are stationary. In our exercise, we rule out cases with non-stationary variables. However, it is straightforward to covert non-stataionary variables to stationary ones by detrending them. Our procedure then is ready to go. Assumption three requires there are at least as many observables as structure shocks of concern. This assumption is for technical purposes and not restrictive, since we can add include measurement errors as structural shocks. Fernandez-Villaverde et al (2006) also make this assumption.

In lieu of additional information, the time series analyst knows or can estimate the covariance generating function of the observables. Let this covariance structure be denoted $C_i = E\left(r_t r'_{t-i}\right)$ for all i.

To understand the theory that follows as we as our procedure, it is useful to compute these covariances as functions of the underlying structural form:

$$\begin{aligned} C_0 &= WQ\bar{W}C_0\,(WQ\bar{W})' + ZZ' + WUU'W' \\ &\quad - WQ\bar{W}C_0\,(WQ\bar{W})' \end{aligned}$$

$$C_1 = WQ\bar{W}C_0 + WUZ' - WQ\bar{W}ZZ'$$

$$C_i = (WQ\bar{W})^{i-1}\,C_1 \;\text{ for all } i > 1$$

In the theorem that follows, we find the number of matrix triples $\{A_j, B_j, D_j\}$ corrresponding to covariance equivalent forms and also show how to conveniently compute each of them.

Moving from the structural form to an observationally equivalent one changes the amount of delay in the system, as we saw in the scalar and bivariate examples in section 2. Intuitively, this can be seen in the state space system by examing the MA representation of the original structural system. This MA representation is:

$$r_{t+1} = Ze_{t+1} + W\sum_{i=0}^{\infty} Q^i U e_{t-i}$$

Because the original and observational equivalent state-space forms differ in terms of U and Z, the corresponding impulse responses will differ in magnitude of a shock's instantaneous effect, i.e. e_{t+1}, versus its lagged effect, $e_t, e_{t-1,\ldots}$. Moreover, as seen in the bivariate example of section 2, changing the delay in the response of one variable to a shock has implications for all of the other impulse responses because of the known covariance structure of the observables. The theorem below formalize the relation between the structural form and its covariance-equivalent cousins. Furthermore, it lays out the theoretical foundation for the practical procedure we use to tackle non-invertibilities.

Theorem: If r_t is a length k stochastic process with the structural state-space form (9) and assumptions 1 through 3 are satisfied, then there exists at most 2^k infinite-order covariance equivalent moving average representations for $\{r_t\}$, indexed by j, where the innovations process ε_t^j satisfies $E\left(\varepsilon_t^j \varepsilon_t^{j\prime}\right) = I_k$. Representation j is given by

$$r_{t+1} = (I - AL)^{-1}\left[D_j + \tilde{C}_1 (D_j')^{-1}\right]\varepsilon_{t+1}^j, \tag{10}$$

The coefficient matrices, α and $\tilde{C}_i,\ i = 0, 1$ are:

$$\begin{cases} A &= C_2 {C_1}^{-1} \\ \tilde{C}_1 &= C_1 - AC_0 \\ \tilde{C}_0 &= C_0 - AC_0A' - A\tilde{C}_1' - \tilde{C}_1 A' \end{cases} \tag{11}$$

where C_i is the ith order autocovariance of the observable vector. The matrix, D_j, satisfies:

(i)

$$(D_jD_j')(\tilde{C}_1')^{-1}(D_jD_j') - \tilde{C}_0(\tilde{C}_1')^{-1}(D_jD_j') + \tilde{C}_1 = 0, \tag{12}$$

(ii) $D_j = D_j^c K$, where D_j^c is the lower triangular matrix generated by the Cholesky decomposition of D_jD_j'. The orthonormal matrix, K, is given by $(Z^c)^{-1}Z$, where Z^c is the lower triangular matrix derived from the Cholesky decomposition of ZZ'.

(iii) one of the D_js is invertible and the corresponding MA form matches the Wold representation for r_t.

Proof**:** *First, we prove equation (10) to equation (12) are necessary conditions for a valid representation of the structural form. That is, the MA representation of the structural form satisifies these conditions. We accompolish this component of the proof in a two-part manner*

Part One**:** *The structrual form has a MA representation in the same format as (10).*

Let $\bar{W}$ be the left inverse W, the MA representation of the transition equation of the state-space form is given by:

$$s_{t+1} = (I - QL)^{-1}Ue_{t+1} = \sum_{i=0}^{\infty} Q^i Ue_{t+1-i}.$$

Substituting s_t with its MA representation in the obserable equation from the state-space form, we have:

$$r_{t+1} = W\sum_{i=0}^{\infty} Q^i Ue_{t-i} + Ze_{t+1}, \tag{13}$$

Premultiplying both side by $\bar{W}L$ and rearranging items, we have:

$$\sum_{i=0}^{\infty} Q^i U e_{t-1-i} = \bar{W}(r_t - Ze_t). \tag{14}$$

Hence, equation (13) can be rewritten as:

$$\begin{aligned} r_{t+1} &= W[Ue_t + Q\bar{W}(r_t - Ze_t)] + Ze_{t+1} \\ &= WQ\bar{W}r_t + Ze_{t+1} + (WU - WQ\bar{W}Z)e_t, \end{aligned} \tag{15}$$

The MA representation of model (15) is given by:

$$r_{t+1} = [I - WQ\bar{W}L]^{-1}[Z + W(U - Q\bar{W}Z)L]e_t, \tag{16}$$

In the next step, show that $WQ\bar{W} = A$ and $W(U - Q\bar{W}Z) = \tilde{C}_1(Z')^{-1}$.

Part Two: *We show that the MA representation, equation (16), satisifies (11) and (12). Define C_i to be the ith order autocovariance matrix of r_t. The autocovariance-generating function of a general VARMA(p,q) model $y_t = M(L)y_t + N(L)w_t$, where w_t $N(0,I)$, is given by $G_y(z) = (I - M(z))^{-1}N(z)N(z^{-1})'(I - M'^{-1}))^{-1}$. Therefore, we have:*

$$\begin{aligned} C_0 &= E\{r_t r_t'\} \\ &= WQ\bar{W}C_0(WQ\bar{W})' + ZZ' + WUU'W' \\ &\quad -WQ\bar{W}ZZ'(WQ\bar{W})' \end{aligned} \tag{17}$$

$$\begin{aligned} C_1 &= E\{r_t r_{t-1}\} \\ &= WQ\bar{W}C_0 + WUZ' - WQ\bar{W}ZZ' \end{aligned} \tag{18}$$

$$\begin{aligned} C_i &= E\{r_t r_{t-i}{}'\} \\ &= (WQ\bar{W})^{i-1}C_1,\ \forall i \geq 2 \end{aligned} \tag{19}$$

We further simplify notation by defining $A = WQ\bar{W}$, $B = WU - \alpha Z$ and $D = Z$. Consequently, we have:

$$A = WQ\bar{W} = C_2 {C_1}^{-1}. \tag{20}$$

Based on the definition, $\tilde{C}_0$ and $\tilde{C}_1$ satisfy:

$$\begin{aligned} \tilde{C}_1 &= C_1 - AC_0 \\ &= BD', \end{aligned} \tag{21}$$

$$\begin{aligned} \tilde{C}_0 &= C_0 - AC_0A' - A\tilde{C}_1' - \tilde{C}_1A' \\ &= DD' + BB'. \end{aligned} \tag{22}$$

Therefore, we have:

$$B = W(U - Q\bar{W}Z) = \tilde{C}_1 {Z'}^{-1}. \tag{23}$$

We further substitute B in the equation with $\tilde{C}_0$ to generate the following equation:

$$\tilde{C}_0 = ZZ' + \tilde{C}_1 (ZZ')^{-1} \tilde{C}_1'. \tag{24}$$

Premultiplying both sides with $\tilde{C}_1^{-1}(ZZ')$, we get :

$$(ZZ')(\tilde{C}_1')^{-1}(ZZ') - \tilde{C}_0(\tilde{C}_1')^{-1}(ZZ') + \tilde{C}_1 = 0, \tag{25}$$

Thus, ZZ' satisfies condition (12). Furthermore, as ZZ' is a symmetric positive semi-definite matrix, its Cholesky decomposition generates a lower triangular matrix Z^c such that $Z^c Z^{c\prime} = ZZ'$. Based on Uhlig (2005), there is always an orthonormal matrix, $K = (Z^c)^{-1} Z$.

Next, we show that equation (10) through (12) are also sufficient for a valid covariance equivalent representation : every model satisfying (10)-(12) is a valid representation of the structural form.

It is obvious that the proposed representations have the same first-order unconditional moments as the structural form. Hence, if the second order moments of the proposed models are also the same as those implified by the structural form, we can say the proposed forms are "valid representations" of the structural form. Moreover, if the disturbance is Gaussian, all the implications *of the dynamics of the structural model are captured by the first two moments.*

Based on the construction, the general form of each candidate is:

$$\hat{r}_{t+1} = A\hat{r}_t + Z_j\varepsilon^j_{t+1} + \tilde{C}_1(Z'_j)^{-1}\varepsilon^j_t \tag{26}$$

where A, Z_j and $\tilde{C}_1$ are determined by constructed based on equation (11) and equation (12), and ε^j_t is $N(0, I)$. *Therefore ,the autocovariance of the process $\hat{r}_t$ is:*

$$\begin{aligned}
\hat{C}_0 &= E\{\hat{r}_{t+1}(\hat{r}_{t+1})'\} && (27)\\
&= A\hat{C}_0A' + AZ_j(Z_j)^{-1}\tilde{C}_1 + \left(AZ_j(Z'_j)^{-1}\tilde{C}'_1\right)' + Z_jZ'_j + \tilde{C}_1(Z_jZ_j)^{-1}\tilde{C}'_1 \\
\hat{C}_1 &= E\{\hat{r}_t(\hat{r}_{t-1})'\} && (28)\\
&= A\hat{C}_0 + \tilde{C}_1(Z'_j)^{-1}Z_j \\
\hat{C}_i &= E\{\hat{r}_t(\hat{r}_{t-i})'\} && (29)\\
&= (A)^{i-1}\hat{C}_1,\ i \geq 2
\end{aligned}$$

Since $Z_jZ'_j$ is a solution to equation (25), one can get:

$$\tilde{C}_0 = (Z_jZ'_j) + \tilde{C}_1(Z_jZ'_j)^{-1}\tilde{C}'_1 \tag{30}$$

Therefore, the equation with regard to $\tilde{C}_0$ becomes:

$$\hat{C}_0 = A\hat{C}_0A' + A\tilde{C}_1 + \tilde{C}_1A' + \tilde{C}_0 \tag{31}$$

Hence, the solution of $\hat{C}_0$ is given by

$$vec(\hat{C}_0) = [I - (A \otimes A)]^{-1}vec(A\tilde{C}_1 + \tilde{C}_1A' + \tilde{C}_0) \tag{32}$$

where $vec(\bullet)$ is the vectorization operation turning an m by *n matrix into an mn* by *1 vector. Based on the definition of $\tilde{C}_0$ and $\tilde{C}_1$, we know that*

$$vec(C_0) = [I - (A \otimes A)]^{-1}vec(A\tilde{C}_1 + \tilde{C}_1A' + \tilde{C}_0) \tag{33}$$

Therefore, we reach the conclusion:

$$\hat{C}_0 = C_0. \tag{34}$$

Given the equivalence between C_0 and $\hat{C}_0$, it is easy to see that

$$\hat{C}_1 = A\hat{C}_0 + \tilde{C}_1 = AC_0 + \tilde{C}_1 = C_1 \tag{35}$$

and

$$\hat{C}_i = A^{i-1}\hat{C}_1 = A^{i-1}C_1 = C_i, \ \forall i \geq 2. \tag{36}$$

Hence, we can reach the conclusion that if a model satisfies condition (10) to (12), it shares the same first and second moments with the structural form. Therefore, such a model is a valid representation of the structural form

As for the number of valid Z_js, there are $\begin{pmatrix} 2k \\ k \end{pmatrix}$ solutions to equation (c). The format of $Z_j Z_j$ requires it to be symmetric and positive definite, thus the valid solution is less than $\begin{pmatrix} 2k \\ k \end{pmatrix}$. With an alternative approach, we can show there are 2^k valid representations in total. Furthermore, we show that among all the valid covariance-equivalent representations, there is one presentation which is invertible. The detail of this alternative approach is included in appendix (A)

Q.E.D

This theorem formalizes the relation between models with the same population moments in observables: covariance equivalent invertible and non-invertible forms. It is the source of identification problem with VARs in the presence of non-invertibility. Equation (12) provides a way to find all covariance equivalent representations. Hence, it allows us to dramatically reduce the dimension of the identification problem.

The theorem shows: (a) even if the structural form is non-invertible, economists can still find all ”covariance-equivalent” representations, (b) when there is non-invertibility implied by the structural form, unrestricted full information method does not necessarily identify the right model, since there are multiple peaks of the likelihood function. Each corresponds to a ”covariance-equivalent” form. Those ”covariance-equivalent” forms share the same unconditional moments with the structural form up to the second order. The conditional moments, and especially impulse responses, are quite different. Based on the theorem, we develop our four-step procedure. In the section 4 and 5, we use model-generated data and real-world data to demonstrate the procedure.

Our method will proceed according as follow:

Step One: *Estimate a reduced-form VARMA(1,1) model on the observables*
With Assumptions 1, 2 and 3, the structural model has a unique invertible VARMA(1,1) representation. This VARMA(1,1) model for this innovation form can be consistently estimated with traditional methods.

Step Two: *Calculate all covariance equivalent representations.*
With the same assumptions used in step one, the true model could have multiple non-invertible VARMA(1,1) representations and one invertible representation. All of these representations share the same population moments with the invertible VARMA(1,1) estimated in step one. Each of these model corresponds to a solution of a quadratic matrix equation, whose solution algorithm is offered by Potter (1964).

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction on each representation.*
When the dimension of the observable variables is k, there are at most 2^k solutions for fully specified rotation matrices. There is at least one solution, which is the innovation representation.

Step Four: *Impose agnostic restrictions on each representation, delivered from step three, to rule out futher structural representations.*
Usually there are multiple solutions after step three. More restrictions other than those on the pattern on the rotation matrix help reduce the set of valid models. If there is only one

solution left, the structural modle is fully identified, otherwise, the model is only partially identified.

4 Two Model-Based Implementations of Our Procedure

In this section, we use two model-generated example to illustrate how to use our procedure to identify the true model when traditional methods cannot. The first example is adopted from the permanent income example used by FRSW (2006). In thise case, our procedure identifies the true model, while traditional VAR model cannot do the job. The second example is from the model with news shock in Leeper, Walker and Yang (2009). In general, we achieve a partial idenfication in this example and a full idenfication is achieved only with a very strong restricion. However, we are successful to rule out the (wrong) invertible model in both applications.

4.1 Savings and permanent income in FRSW (2009)

FRSW show how applying structural VAR analysis to data from a permanent income model generates an incorrect conclusion about the consumption response to an income shock. We show how our procedure leads to the correct conclusion.

The economic model has two equations.

$$\begin{aligned} c_{t+1} &= \beta c_t + \sigma_w (1 - R^{-1}) w_{t+1}, \quad (37) \\ z_{t+1} &= y_{t+1} - c_{t+1} = -c_t + \sigma_w R^{-1} w_t, \quad (38) \end{aligned}$$

Equation (37) is the intertemporal Euler equation and equation (38) defines saving. In the model, c_t is the unobserved state, while $z_t = y_t - c_t$ is saving, the only observable in the model. This process invertible, since $Q - UZ^{-1}W = \beta + R - 1 > 1$ as in FRSW, when β is close enough to one. The ARMA(1,1) representation of the observable is given by:

$$z_{t+1} = \beta z_t + \sigma_w R^{-1} w_{t+1} - \sigma_w [1 - R^{-1} + \beta R^{-1}] w_t, \quad (39)$$

which is non-invertible. The innovations representation is:

$$\hat{c}_{t+1} = \beta \hat{c}_t + \sigma_w (\frac{\beta - \beta^2 + 1}{R} - \beta)\epsilon_{t+1} \quad (40)$$

$$z_{t+1} = -\hat{c}_t + \sigma_w (\frac{\beta - 1 + R}{R})\epsilon_{t+1}. \quad (41)$$

Straightforwardly, the ARMA(1,1) model corresponding to the innovation representation is:

$$z_{t+1} = \beta z_t + \sigma_w (\frac{\beta - 1 + R}{R})\epsilon_{t+1} - \frac{\sigma_w}{R}\epsilon_t. \quad (42)$$

The innovation representation is invertible, since $Q - \hat{U}\hat{Z}^{-1}W' = \frac{1}{R+\beta-1} \in (0,1)$. However, since the implied state variable is not the true state variable, i.e, $\hat{c}_t = E\{c_t|z^t\} \neq c_t$, so FRSW warn that inference based on the (estimated) innovation representation is not reliable.

Suppose the economist knows the population moments for savings, z_t. The economist is uninformed regarding consumtion and income. In sample, one could run a vector-autoregression, use spectral techniques or apply the state-space approach to approximate these moments. Our procedure uses the state-space approach.

Step One: *Estimate a reduced-form ARMA(1,1) on the observables.*

Step Two: *Calculate all covariance equivalent representations.*

With only one observable variable, there are only two covariance equivalent MA representations.

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction on each representation.*

We define a positive savings shock a disturbance that increases savings in the period of the shock. Different researchers may have different interpretations as to what exogenous factors drive savings changes, such as shocks to permanent income, transitory income or preferences. Since we have a scalar observable and a scalar shock, there is no simultaneity problem. As such, an SVAR-type restriction is unnecessary here.

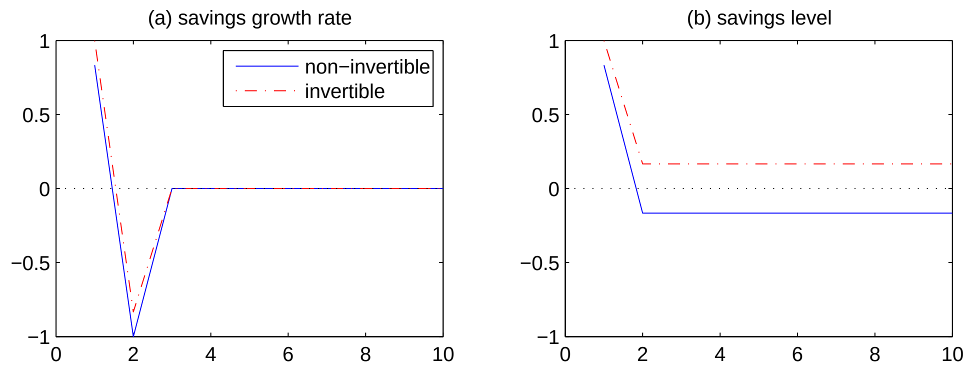
Step Four: *Impose an agnostic restriction on each representation, delivered from step three.*

Before imposing step four, we plot the two impulse responses that come out of step three. These appear in 3 in both the growth rate and level. The solid and dashed lines are, respectively, the invertible and non-invertible responses. Both of these impulse response functions give the same population moments as those from (**??**). The non-invertible response is the true response and the invertible representation is spurious. As FRSW explain, a structural VAR always selects the invertible representation; therefore, in this case it would lead to the incorrect conclusion.

Rather than a priori select the invertible form, we impose an agnostic restriction based on economic theory. We will impose the standard idea that people save now in order to consume more later. Formally, we require that: *if savings is non-zero in at least one period, then it must switch signs at least once*.

Examining figure 3(b), only the invertible response satisfies the agnostic restriction. After step four, we have a single structural impulse response, plotted in figure 4, which is the true repsonse from the economic model. It is exactly the structural model's impulse response.

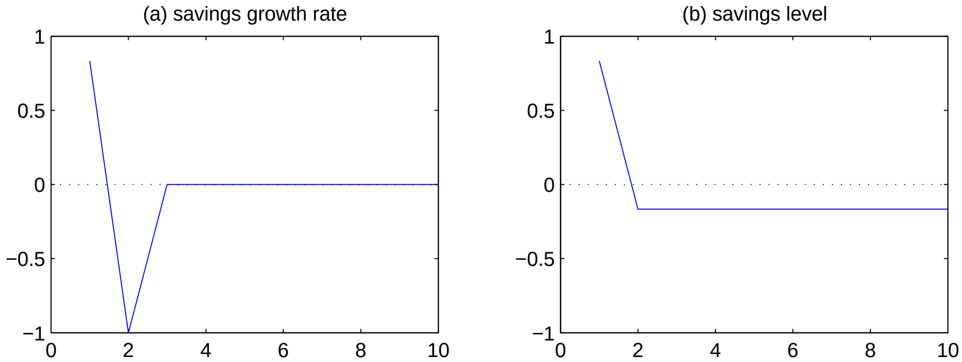
Figure 3: Covariance-equivalent impulse responses to a positive savings shock



Notes: From the permanent income model with $r = 0.2$. Impulse responses to a one unit shock from step three and before application of step four.

In a wide class of models, an individual increases current savings in order to finance greater future consumption. The use of agnostic restrictions is, in our view, very powerful exactly because it implies transparency regarding the source of identification.

Figure 4: Structural impulse response to a positive savings shock that satisfies the step four identification restriction



Notes: From the permanent income model with $r = 0.2$. Impulse responses to a one unit shock after application of step four.

4.2 An anticipated fiscal shock in Leeper, Walker and Yang (2009)

The second model-generated example has anticipated tax shocks as the source of non-invertibility. It is based on Leeper, Walker and Yang (2009, LWY, hereafter). This example has an anticipated fiscal shock: changes in the tax rate are announced two quarters before their implementation.

Consider a neoclassical model with fixed labor supply and full capital depreciation. The capital stock k_t is the single endogenous state variable. In equilibrium, it satisfies

$$(1 - \alpha L)(1 - \theta L^{-1})k_t = -\frac{\tau}{1 - \tau} E_t\{\tau_{t+1}\} + a_t - \theta E_t\{a_{t+1}\}$$

where every variable is the log deviation from its steady-state value. The variables τ_t and a_t are the tax rate and technology level.

LWY further assume there is a random componet to the tax rate, which is announced two periods before the tax implementation. This news is denoted by $\epsilon_{\tau,t}$. The equilibrium law of motion for capital, consumption c_t and output y_t are:

$$\begin{aligned} k_{t+1} &= \alpha k_t + a_{t+1} - \frac{\tau}{1-\tau}(1-\theta)[\theta\epsilon_{\tau,t+1} + \epsilon_{\tau,t}], & (43)\\ c_{t+1} &= \alpha k_t + a_{t+1} + \frac{\tau}{1-\tau}\theta[\theta\epsilon_{\tau,t+1} + \epsilon_{\tau,t}], & (44)\\ y_{t+1} &= \alpha k_t + a_{t+1}. & (45) \end{aligned}$$

LWY show that non-invertibility affects not only the identification of fiscal shocks, but also the identification of the other shock (the technology shock). They assume that the tax rate has both the above anticipated random component as well as a contemporaneous response to technology. The tax rate is: $\tau_t = \psi a_t + \epsilon_{\tau,t-2}$.

LWY demonstrate the non-invertibility problem using a structural VAR where τ_t and k_t observed. In this case, the shocks identified by the structural VAR are not the true shocks, but rather combinations of the technology and tax/news shocks.

Our four-step procedure can identify, at least partially, the structural shocks in the model. It is applied step-by-step below. We requires having enough observable variables, hence, we augment the observable space with consumption, c_t and the shocks with u_t, a measurement error on consumption. The addition of consumption does not remove the non-invertibility.

The state-space representation is:

$$\overbrace{\begin{bmatrix} k_{t+1} \\ \epsilon_{\tau,t+1} \\ \epsilon_{\tau,t} \end{bmatrix}}^{s_{t+1}} = \overbrace{\begin{bmatrix} \alpha & -\frac{\tau(1-\theta)}{1-\tau} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} k_t \\ \epsilon_{\tau,t} \\ \epsilon_{\tau,t-1} \end{bmatrix}}^{s_t} + \overbrace{\begin{bmatrix} 1 & -\frac{\tau\theta(1-\theta)}{1-\tau} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} a_{t+1} \\ \epsilon_{\tau,t+1} \\ u_{t+1} \end{bmatrix}}^{e_{t+1}} \quad (46)$$

$$\overbrace{\begin{bmatrix} \tau_{t+1} \\ k_{t+1} \\ c_{t+1} \end{bmatrix}}^{r_{t+1}} = \overbrace{\begin{bmatrix} 0 & 0 & 1 \\ \alpha & -\frac{\tau(1-\theta)}{1-\tau} & 0 \\ \alpha & \frac{\tau\theta}{1-\tau} & 0 \end{bmatrix}}^{W} \overbrace{\begin{bmatrix} k_t \\ \epsilon_{\tau,t} \\ \epsilon_{\tau,t-1} \end{bmatrix}}^{s_t} + \overbrace{\begin{bmatrix} \psi & 0 & 0 \\ 1 & -\frac{\tau\theta(1-\theta)}{1-\tau} & 0 \\ 1 & \frac{\tau\theta^2}{1-\tau} & 1 \end{bmatrix}}^{Z} \overbrace{\begin{bmatrix} a_{t+1} \\ \epsilon_{\tau,t+1} \\ u_{t+1} \end{bmatrix}}^{e_{t+1}}$$

Our analysis requires setting values for the parameters. We follow LWY for most parameters.[9] In additionl, we normalize the size of fiscal shocks to be 1, and the size of technology shock is set to be $\sigma_a = 0.1$, The standard deviation of the measurement error is 0.05.[10]

By checking the "poor man's invertibility condition" from FRSW, we see that the system is non-invertible. This is because the matrix $Q - UZ^{-1}W$ has eigenvalues outside the unit circle for our parameterization. The three eigenvalues of $Q - UZ^{-1}W$ are .33, -8.98 and -0.45; therefore, there is one dimension of non-invertibility.

The structural VAR approach ignores the embedded non-invertibility. On the other hand, our procedure takes all possible non-invertibilities into consideration.

Step one: *Estimate a reduced-form VARMA(1,1) on the observables.* Denote the VARMA(1,1) representation of the structural model as $r_{t+1} = \overbrace{WQ\bar{W}}^{A} r_t + \overbrace{Z}^{D} e_{t+1} + \overbrace{(WU - WQ\bar{W}Z)}^{B} e_t$ with the following matrices:

$$A = \begin{bmatrix} 0 & \frac{(\tau-1)}{\tau} & \frac{(1-\tau)}{\tau} \\ 0 & \alpha & 0 \\ 0 & \alpha & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \psi\sigma_a & 0 & 0 \\ \sigma_a & \frac{\tau\theta(\theta-1)}{1-\tau} & 0 \\ \sigma_a & \frac{\tau\theta^2}{1-\theta} & \sigma_u \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \theta & \frac{(1-\tau)}{\tau}\sigma_u \\ 0 & \frac{\tau(1-\theta)}{\tau-1} & 0 \\ 0 & \frac{\tau\theta}{1-\theta} & 0 \end{bmatrix}$$

The traditional structural VAR approach can only give the innovation representation, $r_{t+1} = Ar_t + \hat{D}\hat{e}_{t+1} + \hat{B}\hat{e}_t$, of the true model. The AR coefficient matrix, A is consistently identified, but $\hat{D}$ and $\hat{B}$ are biased. In our numerical example, the true VARMA(1,1) representation is:

$$A = \begin{bmatrix} 0 & -3 & 3 \\ 0 & .36 & 0 \\ 0 & .36 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} .12 & 0 & 0 \\ .12 & .065 & 0 \\ .12 & -.024 & .05 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -.27 & -.15 \\ 0 & .24 & 0 \\ 0 & .89 & 0 \end{bmatrix}.$$

[9] We choose $\alpha = .36$, $\beta = .99$, $\tau = .25$.

[10] The size of technology shock is set up to allow the contribution of technology shocks and tax shocks on the variance of consumption is equalized in the long run. This parameterization is purely for analytical simplicity, and it does not affect the result qualitatively

The estimated innovation representation, on the other hand, is given by [11]:

$$A = \begin{bmatrix} 0 & -3 & 3 \\ 0 & .36 & 0 \\ 0 & .36 & 0 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} .29 & 0 & 0 \\ .21 & .14 & 0 \\ -.01 & -.01 & .15 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 & -.12 & -.08 \\ 0 & .13 & .03 \\ 0 & -.04 & 0.01 \end{bmatrix}.$$

The true VARMA(1,1) representation has eigenvalues outside the unit circle, while the innovation representation has no eigenvalues outside the unit circle.[12]

Step two: *Calculate all covariance equivalent representations*

This step finds all the representations with the same autocovariance structure, i.e., the covariance equivalent representations. Each covariance equivalent representation has an associated triple $\{A_j, D_j, B_j\}$. It is easy to verify that $A_j = A$ and every pair of $\{D_j, B_j\}$ satisfies the following equations:

$$\begin{aligned} D_j D_j' + B_j B_j' &= \qquad (47) \\ & \begin{bmatrix} \psi^2\sigma_a^2 + \theta^2 + (\frac{\sigma_u}{\kappa})^2 & \psi\sigma_a^2 + \kappa\theta(1-\theta) & \psi\sigma_a^2 - \kappa\theta^2 \\ \psi\sigma_a^2 + \kappa\theta(1-\theta) & \sigma_a^2 + \kappa^2(1+\theta^2)(1-\theta)^2 & \sigma_a^2 - \kappa^2\theta(1-\theta)(1+\theta^2) \\ \psi{\sigma_a}^2 - \kappa\theta^2 & \sigma_a^2 - \kappa^2\theta(1-\theta)(1+\theta^2) & \sigma_a^2 + \kappa^2\theta^2(1+\theta^2) + \sigma_u^2 \end{bmatrix} \\ B_j D_j' &= \begin{bmatrix} 0 & \kappa\theta^2(1-\theta) & -\kappa\theta^3 - \frac{\sigma_u^2}{\kappa} \\ 0 & \kappa^2\theta(1-\theta)^2 & -\kappa^2\theta^2(1-\theta) \\ 0 & -\kappa^2\theta^2(1-\theta) & \kappa^2\theta^3 \end{bmatrix}, \end{aligned}$$

where $\kappa = \tau/(1-\tau)$. The equation system (48) can be equivalently converted into a quadratic matrix equation in $D_j D_j'$. The solution of this quadratic matrix equation is given in Potter (1964). Since $D_j D_j'$ is a 3×3 matrix for each j, there are at most $2^3 = 8$ different solutions to the quadratic matrix. Under this current parameterization, $D_j D_j'$ has a pair of complex eigenvalues. As such, there are only four real-valued structural responses.

Step three: *Define the structural shock of interest and impose an SVAR-type restriction on each representation.*

[11] Here we only show the result after imposing a short run restricion.

[12] The true model has two eigevalues outside the unit circle, which are complex conjegutes of each other.

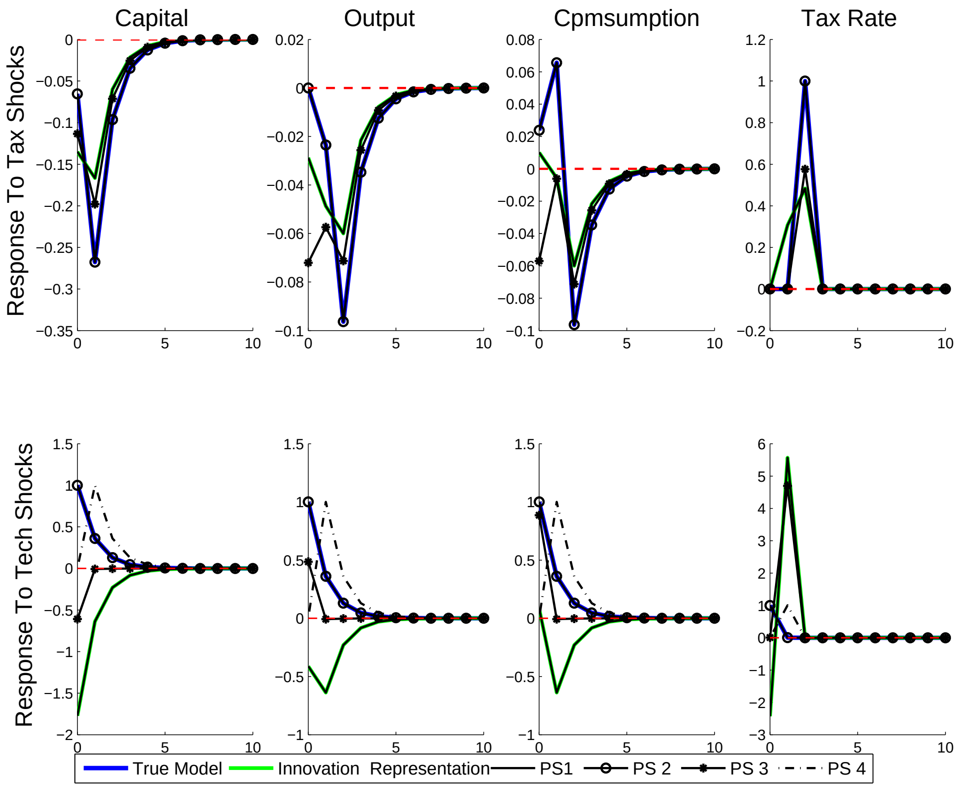
A positive technology shock is defined as a shock which increases consumption and does not reduce the tax rate. Consumption increases because of positive effect of technology shocks on production capacity. Obvioiusly, a positive tax shock increases the tax rate as well but the way it affect capital and consumption is not clear. One possible way to separate the positive tax shock from the positive technology shock is by assuming that an anticipated tax rate change cannot changes the current tax rate. Since we know that measurement error only affects the measurement of consumption, it should not affect the tax rate or capital on impact. Based on the definitions, we can impose a short-run restriction to identify the shocks: a valid D matrix should be lower triangular.

Figure (5) shows impulse responses to a positive tax shock (upper panel) and those to a positive technology shock (lower panel) in all the four possible cases after imposing the short run restriciton. One of them overlaps with the VAR-based inference, which is the (invertible) innovation representation of the model. In response to a positive tax shocks, capital and output falls in all four cases and tax rate increases in all of them. The only difference is the magnitude of responses. When studying the responses to a positive technology shock, capital falls in two cases but rises in other three. Ouput falls in the innovation representation but rises in all the other three cases. The fall in output seems to contradict traditional wisdom, however, there are evidences in existing research to show technology shocks are contrationary. At this stage, we cannot rule out any the four cases for the time being without further justification.

Step Four: *Impose agnostic restrictions on each representation, delivered from step three, to further rule out structural responses.*

In this exercise, we use short-term forecast error variance decomposition to distinguish models. In order to identify the true impulse responses, we employ multiple criteria based on reasonable economic intuition. Firstly, measurement errors should not be important factors to explain volatilities in any of the variables, especially in the longer term. Therefore, we setup a quantitative threshold of 30% for the average contribution of measurement errors on all observable variables. (*criterion one*) Secondly, technology shocks should not be the dominant factor to explain the volatilities in the tax rate, espe-

Figure 5: Response To Tax and Technology Shocks (after step three)



Notes: upper panel responses to a positive tax shock; lower panels responses to a positive technology shock. *PS i*: the *i*th solution based on the Potter equation.

Table 1: Identification Based on Short-Term Variance Decomposition

	Model One	**Model Two**	**Model Three**	**Model Four**
The average contributions on different horizons of identified measurement errors on variables				
tax rate	0	34.82	0	14.78
capital	0	39.32	0	0.51
consumption	7.84	39.45	7.84	70.51
The average contributions of technology on tax rate at different horizons				
	1.42	35.05	1.42	53.24
The contribution of technology shocks on capital and consumption when $h = 1$				
capital	0	37.55	79.11	71.01
consumption	0	48.01	83.23	0.09

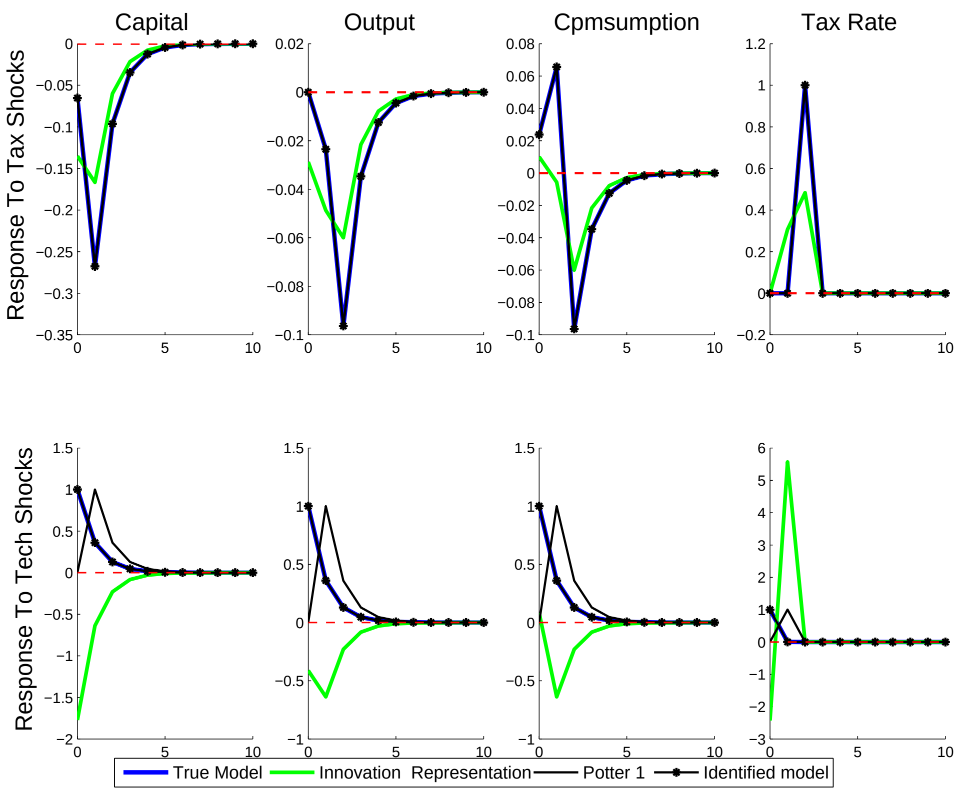
ically in longer time horizons. Quantitatively, we set up the threshold value to be 50% when the the time horizon is longer than two quarters (*criterion two*). The result of this variance decomposition exercise is shown in table (1)

Based on criterion one, case 2 and case 4 are ruled out, since these two models attribute too many variations to measurement errors. In this model, case 4 is corresponding to the innovation representation, in other words, the model identified with traditional SVAR methods. This specification can be ruled out based on our second criterion as well, since technology shocks should not be the main driving force for tax rates. The economic intuition behind the variance decomposition exercise is that mis-identified models do not identify structrural shock correctly, instead, the shocks identified in these models are linear combinations of structural shocks. Leeper et al (2009) makes a similar point from a different perspective. They view this as a failure in idenfication with traditional SVAR methods. Our procedure goes one step further: some mis-idenfication will give wildly implusible variance decomposition. Therefore, we can rule out such mis-identified models.

However, we still cannot achieve full identification in this model. As shown in table 1, we cannot choose between case one and case three based on the first two criteria we proposed. Till this step, we achieve a partial idenfication of this model. Figure (6) compared the impulse responses implied by the remaining solutions to those implied by the true model and by the innovation representation. Both solutions recover the true responses to a positive tax shock in the structural model. One of them (the *"identified model"*) recover the true responses to technology shocks as well. It means our procedure at least pertains the true model. The reason why we can use variance decompositions to identify the right model is that covariance-equivalent representations other than true models are likely to mix different shock together. Therefore, the variance decomposition is distorted in those representations. Such idenfication scheme share the same spirit as the identification methods proposed by Faust (1997) and Uhlig (2005). As long as economic theory gives us enough restrictions on the model, e.g, the variance decompostion, the sign of impulse responses or the sign of magnitude of a particular coefficient, we can always apply them to rule out mis-identified models.

In this example, we cannot uniquely pin down the true model. The reason is that the first solution based on our procedure only mis-specifies the timing or invertibility of the technology shock, but it does disentangle tax shocks and technology shocks effectively. To further refine the result, we might to want to ask for stronger restrictions. For instance,if we have a strong belief that the transmission of technology shocks is fast enough, then the technology shock should explain the bulk of changes in capital and consumption in the short term. Hence, we set up a third criterion: the contribution of technology shocks to the one step forecast error variances in consumption and capital should be higher than 30%. With this extra restriction, we uniquely pin down the model as shown in table 1. In the true model, capital and output fall in response to an anticipated tax shock. Consumption rises on impact but falls in following period. The intial rise is due to the subsitution effect induced by higher tax rate in the future while the following decrease is because of the drop in production capacity. When the model is identified correctly, capital, output and consumption all rise in response to a positive technology shock, while the innovation representation shows capital and output falls in response to it.Adding this third criterion, the true model is uniquely identified. From our perspectivee, criteria three is too strong to be used. Thus, our procedure has not achieved a slam dunk. Nevertheless, using criteria based on variance decompositions are not the only way to impose agnostic restrictions. Other criteria, e.g, based on the sign of responses or even some facts or statistics beyond the time series model could be used to identify models as well. Chances are we can further refine the models with these rich sets of restrictions.

Figure 6: Response To Tax and Technology Shocks (after step four)



Notes: upper panel responses to a positive tax shock; lower panels responses to a positive technology shock

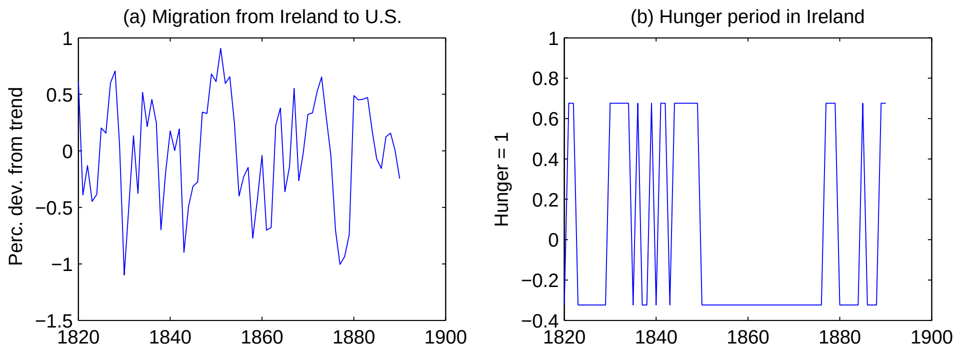
5 Examples with Real Data

5.1 First application: Irish hunger and emigration, 1820-1890

Our first application using actual data is based the extraordinary and tragic experience in 19th century Ireland. A series of famines and hungers occurred over this period, with the largest occurring between 1847 and 1850 which is estimated to have caused over one million deaths. During that century, there was significant immigration from Ireland to many countries, including the U.S.

Figure 7 plots annual data on the immigration from Ireland to the U.S. as well as a dummy variable for whether Ireland experienced a hunger or famine during the year.

Figure 7: Emigration from and hunger in Ireland, 1820-1890



Notes: Migration data is log deviation from HP trend of the number of immigrants and hunger is a binary variable based upon Wikipedia entry on the years of Irish famines and hungers.

This episode provides an interesting application of our procedure. First, the primary cause of most of these hungers and famines was potato diseases. It is reasonable to think about these as exogenous shocks. Second,one might expect to see a delayed response of the type illustrated in section XXX. Third, by considering a bivariate system, there will be only four covariance equivalent structural impulse responses for each variable (i.e. one invertible and three non-invertible).

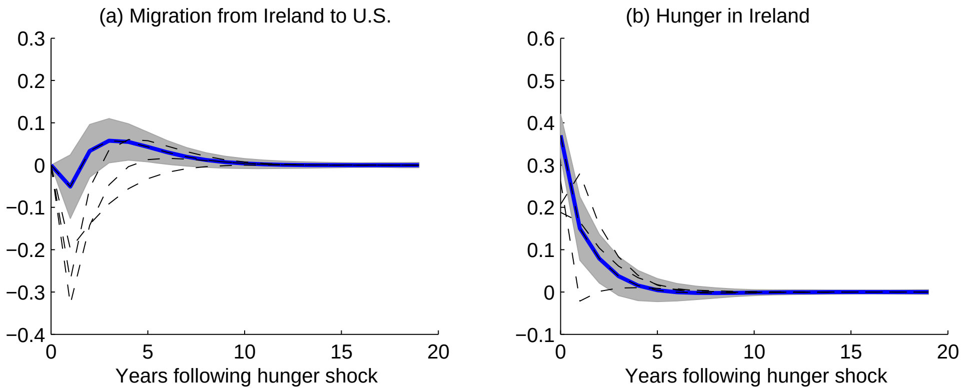
Step one: *Estimate a reduced form VARMA(1,1) for the observables.*

Our observable vector contains two variables, the log number of immigrants to the U.S. and a binary hunger variable. We assume the state system contains two unobserved states and two shock variables.[13]

Step two: *Define the structural shock and select the rotation restriction.*

[13] Migration data is log deviation from HP trend of the number of immigrant annually from Ireland to the U.S. Hunger is a binary variable based upon Wikipedia entries that delineate in which years Irish famines and hungers occurred.

Figure 8: Structural impulse responses to a one-standard deviation positive hunger shock, invertible and non-invertible responses



Notes: The solid line contains the invertible structural response and the shaded region contains the corresponding 95% confidence interval. Each dashed line corresponds to a non-invertible structural response.

A positive famine shock is an exogenous increase in the hunger variable upon the impact of the shock. Second, we impose a rotation restriction via a recursive ordering of the variables so that migration does not respond within the period to the hunger shock. It seems that some people could move within the year. It would be nice to replace this with another restriction, although I am not sure what it would be.

Step Three: *Define the structural shock of interest and impose an SVAR-type restriction.* A positive hunger shock increases hunger on impact. The historical record attributes the start of each famine to poor weather and/or crop disease. We treat these as exogenous. We make the short-run restriction that migration cannot respond within the year to a hunger shock. Booking cross-Atlantic steamers took time and, since most Irish has little income, laying back enough wages to buy these tickets also took time.

Figure 8 plots the four covariance equivalent impulse responses to a one-standard deviation hunger shock. The solid line in each panel represents the invertible response. The invertible one looks very plausible, but the non-invertible ones look less plausible.

Step Four: *Impose an agnostic restriction on each representation, delivered from step three, to* *further rule out potential stuctural responses.*

Here, we impose one restriction: the long-run cumulative migration from Ireland to the U.S. is non-negative in response to a positive hunger shock. The justification for this restriction is self-explanatory. Imposing this restriction, the structural impulse response is uniquely identified. It is the solid line and is also the invertible form.

Although our procedure delivered the invertible representation as the truth, we did not choose this one a priori and ad hoc.

5.2 Second application: long-run identified technology shocks in the U.S., 1955-2000

Fisher (2006) uses a three-variable model to study the effect of technology shocks on the U.S. economy in the second half of the twentieth century. In his exercise, the investment-specific shock, which is captured by suprise changes in the relative price of investment, is important to explain the variation in output and working hours in U.S.

Recently, studies on the effect of ”news shocks”, which is the anticipated component in technology shocks, have drawn more and more attentions of economists, since the seminal work by Beaudry and Portier (2006). They show that technology shocks identified by traditional long run restrictions can be well replicated by another shock originated in the stock index but are orthogonal to contemporaneous technology changes. They argue that this piece of evidence shows technology shocks are anticipated (”news shocks”) and they further show this news shock is important to explain business fluctuations. Jaimovich and Rebelo (2009) show that certain real frictions, including habit persistence in consumption, investment adjustment costs and costly capacity utilization, are important to the propagation of news shocks in a real business cycle model. Christiano et al (2009) estimate a dynamic general equilibrium model featuring norminal and real frictions for the U.S. ecomony and show that news shocks are important sources of business fluctuations. However, Sims (2009) uses traditional SVAR methods to identify news shocks in a large scale VAR model and finds that news shocks fail to generate co-movement in macro variables, so news shocks cannot be a valid candidate for the main driving force of business cycles.

To shed light on the effect of anticipated technology shocks or news shocks on the economy, we estimate a small scale VARMA model similar to Fisher (2006). There are three variables in the model: the growth rate of real equipment price, the growth rate of labor productivity and the log index of average working hours. The rationale behind this exercise is as follows: if there is a significant anticipated component in either the investment specific technology shock or the neutral technology shock, the implied time series becomes non-invertible. With our four-step procedure, we should be able to identify the true model with enough reasonable restrictions, no matter it is non-invertible or not. The application of the four-step procedure is given as follows:

Step one: *Estimate a redued-form VARMA(1,1) on the observables*
First, we estimate a VARMA(1,1) model on the data. In practice, there are at least two advantages of this VARMA(1,1) setup over the traditional long VAR models: (i) the model requires less parameters, which relieves the concern on too many estimated parameters to some extent; (ii) the VARMA(1,1) setting is more consistent with the DSGE models studied in macroeconomics.[14] The VARMA model is estimated in a two-step manner. The first step is estimating a long VAR model to obtain a residual series. In the second step, we estimate a VARMA(1,1) model by adding the residual series from the first step as a regressor and check for convergence.[15] After obtaining the estimated VARMA(1,1) model, we get variance matrix of error terms, $\hat{\Omega}$, which is the estimate of $D_j D_j'$, and the MA coefficient matrix,N, which is the estimate of $B_j D_j^{-1}$. These moment estimates are used in the second step.

Step two: *Calculate all covariance equivalent representations*
Second, we compute all covariance equivalent representations. As we show in section three, all the covariance equivalent representations are solutions of the Potter equation defined by the moments of observable variables, and the true model should be one of

[14] See for example Kehoe (2007).

[15] The efficiency of estimation could be improved by employing a 3SLS procedure or iterated 2SLS procedure. Kascha (2007) gives a good survey on estimation methods of the VARMA models.

them. In the current application, the Potter equation is given by:

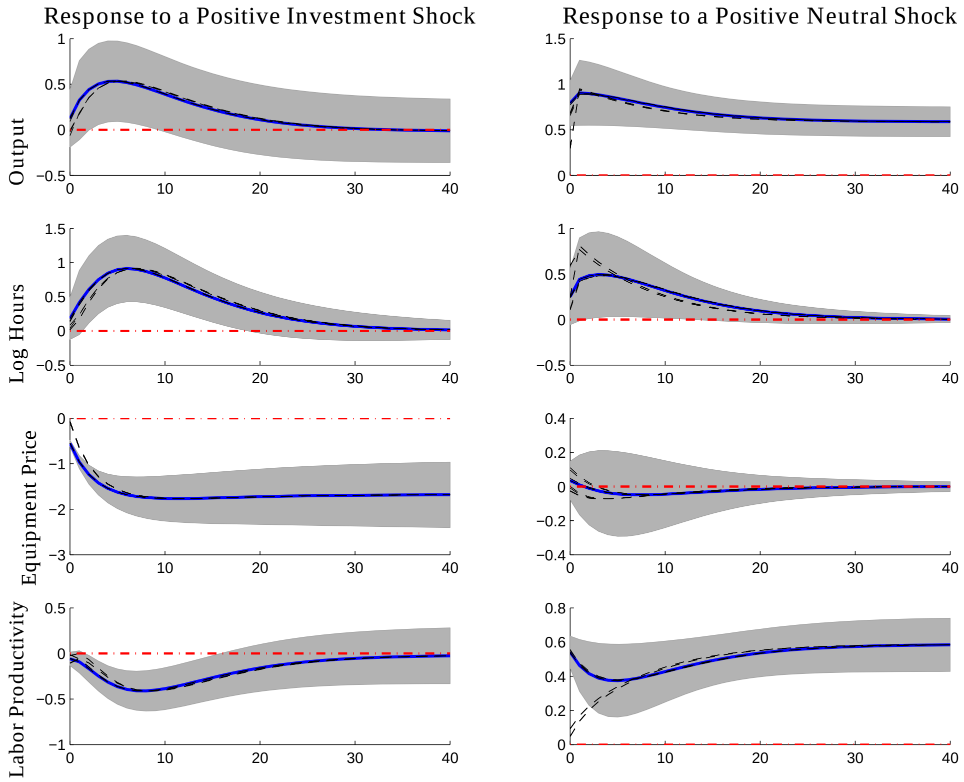
$$\begin{aligned} D_j D_j' + B_j B_j' &= \hat{\Omega} + N\hat{\Omega}N' \\ B_j D_j' &= N\hat{\Omega}. \end{aligned} \tag{48}$$

Step three: *Define the structural shocks of interest and impose an SVAR-type restrictions on each representation.*

Following Fisher (2006) and Altig et al (2009), a positive investment specific shock is defined as the only shock which lowers the real equipment price in the long run, while a positive neutral technology shock is define as the other shock which increases labor productivity in the long run apart from the positive investment specific shock. Based on the definitions, two long run restrictions are imposed on the estimated model to identify the two technology shocks. There are eight structural representations satisfying the Potter equation as well as the two long run restrictions.

Figure 9 shows the impulse reponses of all eight cases along with the point estimate and the confidence interval based on the innovation representation. The latter is the counterpart of the tradtional VAR identification in our VARMA(1,1) setup. In the invertible case, the estimated effect of identified shocks are in line with existing research: in response to a positive investment shock, hours and output increase prominently, however, labor productivity falls for a long period after the shock. Output and labor hours increase less significantly in the case with a positive neutral technology shock. In non-invertible cases, the responses to the investment shocks are similar to those in the invertible case. In response to the neutral technology shock, hours rise faster and stronger in some non-invertible cases, but the response of output on impact becomes weaker. In those cases, labor productivity increases gradually, instead of jumping up as shown in the invertible case. If technology is only disseminated slowly in the economy, we should observe the slow buildup of labor productivity in response to technology shocks as shown here. The strong response of hours in can be readily explained by strong intertemporal substitution effect as in Jaimovich and Rebelo (2009). Up to this step, economic theory cannot distinguish between the invertible and the invertible models. Therefore, we need additional selection criteria to pin down the true model, which is the purpose of the fourth step in our procedure.

Figure 9: Response To Technology Shocks (All Cases)



Notes: solid blue line: the point estimate of impulse responses in the innovation representation; gray area: 90% confidence interval in the innovation representation; dashed black lines: impulse responses from the solutions of the Potter equation

Step four: *Impose agnostic restrictions on each representation, delivered from step three, to further rule out structural responses.*

In this step, we impose agnostic restrictions on variance decompositions: (i) the investment shock should explain the long run variance in the growth of real equipment price at least 10%; (ii) the neutral technology shock contributes the long run variance on the growth of labor productivities at least 10%; (iii) the third shock, with is a combination of other non-technology shocks and measurement errors, should not contribute more then 30% to the long run volatility in either the real equipment price or the labor produtivity.

The result of the variance decomposition is summarized in table 2.

As shown in the table, we successfully rule out some cases. Based on the third criterion, we can rule out case models 1, 3, 5 and 7. In all the four cases, the contribution of other non-technology shocks on the growth of technology in the long run are unreasonablly large. However, we cannot refine the outcome further, in other words, we only achieve a partial identification in this example.

Figure 10 plot the responses of models satisfying the agnostic restrictions based on variance decompositions along with the invertible case. In all the four valid cases, impulse responses are very similar to each other. Furthermore, the invertible case is among the four cases we keep. The variance decomposition analysis also show similar result in all the four cases. Therefore, we can reach the conclusion that the inference based on analysis on an invertible VAR model is valid and reliable. In other words, news or anticipated components in technology shocks does not play important roles when studying the effect of these two types of technology shocks. Between the two technology shocks, the investment specific shock is more important to explain the dynamics in labor hours.

6 Conclusion

Traditional limited information econometric methods, including the widely applied structural VAR apprach, cannot handle non-invertiblility embeded in many business cycle models. However, researchers need not abandon the limited information approach, which is the power and soul of the structural VAR. We show that non-invertible time series can be recovered with its invertible counterpart. That is, there is always an invertible innovation representation corresponding to a non-invertible model. The invertible innovation representation shares the same population moment with the structural model. Therefore, we can recover all the valid models through those consistently estimated moments, regardless of invertibility.

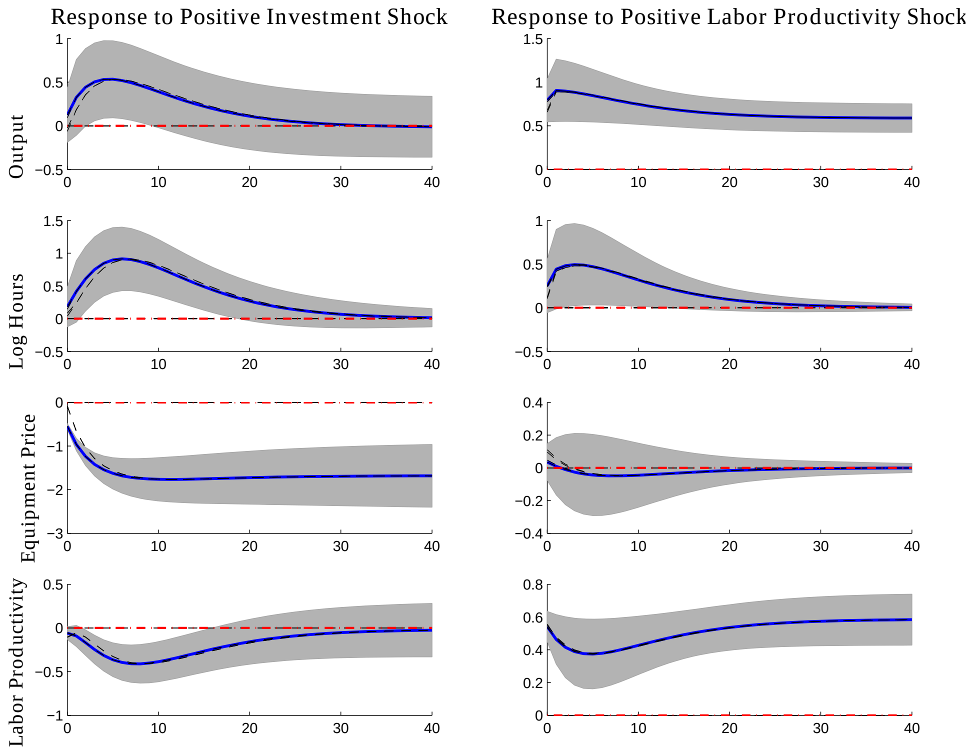
Based on the theory developed in this paper, we propose a four step procedure to handle non-invertibility in practice. This four steps are: (i) estimate a reduced form VARMA(1,1); (ii) compute all VARMA(1,1) models with the same autocovariance struc-

Table 2: Identification Based on Short-Term Variance Decomposition

	Model One	**Model Two**	**Model Three**	**Model Four**	**Model Five**	**Model Six**	**Model Seven**	**Model Eight**
contribution of investment shocks in the long run								
variable 1	97.32	96.46	97.32	96.07	98.24	98.88	98.31	99.22
variable 2	8.87	9.93	8.86	10.10	8.06	7.55	8.08	7.51
variable 3	51.66	51.35	51.66	51.51	51.68	51.69	51.62	51.50
contribution of neutral shocks in the long run								
variable 1	.35	2.84	0.32	2.23	0.40	0.69	0.31	0.46
variable 2	5.69	74.50	6.14	70.08	5.74	76.77	6.14	72.24
variable 3	17.29	13.07	17.23	12.96	17.19	13.31	17.22	13.21
contribution of other shocks in the long run								
variable 1	2.33	0.70	2.35	1.69	1.36	0.43	1.38	0.32
variable 2	85.44	15.57	85.00	19.82	86.21	16.68	85.77	20.24
variable 3	31.04	35.58	31.11	35.99	31.11	35.00	31.18	35.28

variable 1: the growth rate of real equipment price; variable 2: the growth rate of labor productivity; variable 3: labor hours

Figure 10: Response To Technology Shocks (Identified)



Notes: solid blue line: the point estimate of impulse responses in the innovation representation; gray area: 90% confidence interval in the innovation representation; dashed black lines: impulse responses from the solutions of the Potter equation

ture using Potter's (1964) algorithm; (iii) use the outcomes from step two and an SVAR-type restriction to find a finite number of valid structural impulse responses; (iv) use agnostic restriction implied by economic theory to identify, at least partially, the true model.

We then apply this procedure to two model-generated examples. In both the permanent income model FRSW and the anticipated fiscal shock model in LWY, our procedure recovers the true model. We further apply our method to cases with real data. We find that result in Fisher (2006)'s study on technology shocks holds even when we consider possible non-invertibilities in the model. It indicates that anticipated component technology shocks or "news shocks" do not spoil the inference of the transmission mechanism of technology shocks.

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Appendix

A The equivalence between Blaschke Matrices and the Potter Equation

Lippi and Reichlin (1994) show every noninvertible stationary VARMA(p,q) model has one invertible representation by multiplying an approrpiate Blaschke matrix. A Blaschke matrices, $B(z)$, is a special matrix satistify the following property:

$$B(z)B(z^{-1})' = I. \tag{1}$$

As we know, every orthonormal matrix is a Blaschke matrix. In the remaing part of this section, we show how to use Blaschke matrices to get an invertible representation and how this alternative procedure is related to the proposed procedure in the main text.

Lemma *Every covariance-equivalent form can be achieved by multiplying an appropriate Blashke matrix on the original model*

Proof:

$$\begin{aligned} r_{t+1} &= WI - QL^{-1}Ue_t + Ze_{t+1} \\ &= W\sum_{i=0}^{\infty} Q^i Ue_{t-i} \\ &= WQ\bar{W}(r_t - Ze_t) + WUe_t + Ze_{t+1} \\ &= WQ\bar{W}y_t + Ze_{t+1} + (WU - WQ\bar{W}Z)e_t. \end{aligned} \tag{2}$$

For simplicity in notations, define $M = WQ\bar{W}$, $N_0 = Z$ and $N_1 = WU - WQ\bar{W}Z$. Therefore, we have the autocovariance generating function of r_t is given by:

$$G_r(z) = ([I - Mz])^{-1}(N_0 + N_1 z)(N_0 + N_1 z^{-1})'[I - M'^{-1}]^{-1} \tag{3}$$

Equation () is a VARMA(1,1) representation of the structural model, which might be invertible or non-invertible. Next, we show that there is an alternative VARMA(1,1) representation of the same model, and furthermore, this representation is invertible. To this end, we construct a square matrix A(L) of dimension m. This matrix depdends on the matrix lag polynomial $N(L) = N_0 + N_1 L$. More specifically, let $\{\lambda_i\}_{i=1}^m$ be the eigenvalues of $N(L)$. Define a matrix $R(\lambda_i, z)$ as follows:

$$R(\lambda_i, z) = \begin{cases} \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \frac{1-\bar{\lambda}_i z}{1-\lambda_i z} & 0 \\ 0 & 0 & I_{m-i} \end{pmatrix}, & |\lambda_i| > 1 \\ I_m, & \text{otherwise} \end{cases} \tag{4}$$

The matrix $R(\lambda_i, z)$ is known as a Blaschke matrix. It satisfies the property $R(\lambda_i, z)R'(\bar{\lambda}_i, z^{-1}) = I$. Now, we defines another matrix K_i. This matrix is an orthonormal matrix, whose ith column is the normalized solution of $N(\lambda_i)x = 0$.

Firstly, we can construct another lag polynomial $N^i(L) = N_0^i + N_1^i L = (N_0 + N_1 L)K_i R(\lambda_i, L)$. By right multiplying $N(L)$ with K_i, one can move all the entries containing the factor $1 - \lambda_i L$ on the ith column. By further right multiplying $R(\lambda_i, L)$, one replaces $1 - \lambda_i L$ with $\lambda_i - L$ but leave other elements untouched, in other words, "flips" a particular eigenvalue of the lag polinomial. At the same time, we even have:

$$\begin{aligned} G_r^i(z) &= ([I - Mz])^{-1}(N_0^i + N_1^i z)(N_0^i + N_1^i z^{-1})'[I - M'^{-1}]^{-1} \\ &= ([I - Mz])^{-1}(N_0 + N_1 z)K_i R_i(\lambda_i, L)R'(\bar{\lambda}_i, L^{-1})K_i'(N_0 + N_1 z^{-1})'[I - M'^{-1}]^{-1} \\ &= ([I - Mz])^{-1}(N_0 + N_1 z)(N_0 + N_1 z^{-1})'[I - M'^{-1}]^{-1} \\ &= G_r(z) \end{aligned} \tag{5}$$

Therefore, we construct another VARMA(1,1) representation of the structural model:

$$r_{t+1} = Mr_t + N_0^i e_{t+1}^i + N_1^i e_t^i. \tag{6}$$

Compared to the model in equation (A), model (6) has the same variance-covariance structure and the same likelihood. Based on construction, we know that the eigenvalues of the covariance-equivalent forms are either the eigenvalues of the structural form or the reciprocal of them. Therefore, if there are eigenvalues outside the unit circle (non-invertible), there has to be a covariance-equivalent form "flipping" all the explosive eigenvalues while keeping the stable eigenvalues untouched.

Q.E.D

Lemma *The method with Blaschke matrices gives the same result as the procedure based on the Potter equation*

Proof: The proof applies to a general $VARMA(p,q)model$, $M(L)x_t = N(L)w_t$, where $M(L)$ is stable. *(i) Any solution implied by Blaschke matrices is a solution implied by the Potter equation.* This is obvious. Based on construction, a representation generated by using Blaschke matrices have the same covariance structure as the structural form. Hence, it is satisfies conditions (10) to (12)

Any solution satisfying conditions (10) to (12) is a solution by using Blaschke matrices This is based on Theorem 2 in Lippi and Reichlin (1994). Assume the invertible $VARMA(p,q)$ model is given by $M(L)x_t = N(L)u_t$. an arbitrary solution from the potter equation is given by $M(L)x_t = \tilde{N}(L)w_t$. Based on definition, $x_t = M(L)^{-1}\tilde{N}(L)w_t$ is a MA representation of the original VARMA model. Therefore, we have to have $M(L)^{-1}\tilde{N}(L) = M(L)^{-1}N(L)B(L)$, where $B(L)$ is a Blaschke matrix. Thus, $\tilde{N}(L) = N(L)B(L)$.

Q.E.D