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JEL classification: G12, G17, D84

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Mass Psychology in Action: Identification of Social Interaction Effects in the German Stock Market

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April 23, 2009

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1 Introduction

Opinion dynamics in financial markets have been modeled by Kirman (1993), Lux (1995, 1998) and Alfarano, Lux and Wagner (2008) among others. These models make use of epidemic processes of information transmission between agents that allow for an endogenous formation of expectations. Markets with such interacting speculators give easily rise to speculative bubbles, crashes and excess volatility and, therefore, provide an avenue towards an explanation of these ubiquitous phenomena. Perhaps even more important, a certain number of these agent-based models has also been shown to exhibit more fundamental statistical properties of financial returns: Models like the ones proposed by Lux and Marchesi (1999, 2000), Iori (2002) or Pape (2007) generate time series that replicate the well-known stylized facts like fat tails and clustered volatility, even up to close numerical proximity of key empirical statistics of financial data (cf. Lux, 2009b, for an overview of this literature).

While this literature has become quite sizable over the last decade, empirical implementations of these models are relatively sparse. This lack of empirical work concurs more broadly with a dearth of validations of agent-based models. While a few attempts at an empirical implementation have been put forth recently (cf. Amilon, 2008; Franke, 2008; Lux, 2009a), there is certainly a lack of an established general toolbox in this area. Our goal in this paper is to go one (or two) steps beyond a previous paper (Lux, 2009a) that introduced a method for identification of the parameters of microscopic opinion processes from aggregate data. This paper, however, was confined to estimation of the parameters of a model for a univariate time series, namely the diffusion index form (number of optimistic individuals minus number of pessimistic individuals) of a business climate survey. While the same model and estimation methodology could be applied for financial sentiment data (which often share the format of diffusion indices), a uni-variate model would only allow to cover one of the building blocks of the above asset pricing models. As a minimum requirement, however, for an empirical validation of a stochastic behavioral asset pricing model one would like to study the joint dynamics of asset prices and sentiment. We will, therefore, extend our previous model into this direction and provide parameter estimates for a simple version of a simultaneous system. Since our underlying time series cover two sentiment variables, one for the short horizon and one for the medium-term horizon, we can even go one step further and study two interacting opinion processes together with the time development of the asset price. Since this amounts to studying the dynamics of a tri-variate series, we proceed in this paper from the 1D case of Lux (2009a) to the 2D and 3D cases. As in the previous paper, the methodology presented below could be applied to a wide variety of hypothesized opinion dynamics interacting with objective economic variables. In order to demonstrate the practical use of estimated agent-based models, we also perform an out-of-sample forecasting experiment based on our estimated models.

The rest of the paper is structured as follows: In sec. 2 we introduce our stochastic framework of sentiment dynamics and simultaneous price changes. Sec. 3 provides details on our estimation methodology, maximum likelihood estimation based on numerical approximation of the transient density of the underlying stochastic process. Sec. 4 gives details on the sentiment data we use as well as an overview on previous findings on the interaction between sentiment and returns within a non-behavioral VAR framework. In sec. 5 we present results for univariate population dynamics and diffusion processes for each one of our three time series. In sec. 6 and 7 we proceed to various combinations of 2D models and the full-fletched model of three simultaneous stochastic processes. Sec. 8 summarizes our findings and concludes. An Appendix provides details on the numerical approximation schemes for the dynamics of the transient density.

2 The Joint Dynamics of Asset Prices and Sentiment

For the group dynamics that govern the time development of traders' mood, we adopt the Weidlich model of opinion formation (Weidlich and Haag, 1983, Weidlich, 2000, Lux, 1995): agents have the choice of voicing one of two opinions, denoted here by "+" and "-" (optimistic, pessimistic). Agents change their beliefs in continuous time, with a Poisson process formalizing the switches from the "+" to the "-" group and *vice versa* within the next instant. The pertinent transition rates are denoted by $\omega_\uparrow$ and $\omega_\downarrow$ and assume an exponential functional form:

$$\omega_\uparrow = \nu \exp(U); \omega_\downarrow = \nu \exp(-U). \tag{1}$$

Here ν is a scale parameter that determines the frequency of transitions, and the function U covers those determinants that might exert an influence on agents' decisions to change their belief. In our present application we assume that our two sentiment variables, say x (short-run sentiment) and y (medium-run sentiment) are both determined via a similar epidemic opinion process. We denote by $w_\uparrow^s(w_\downarrow^s)$ and $\omega_\uparrow^m(\omega_\downarrow^m)$ the transition rates for short-run and medium-run sentiment, respectively. Allowing for cross-dependencies between both processes as well as for dependency on returns (ret), we specify the pertinent rates of eq. (1) as:

$$\omega_\uparrow^s = \nu_s \exp(U_1), \omega_\downarrow^s = \nu_s \exp(-U_1) \tag{2}$$

$$\text{with } U_1 = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 ret,$$

for the dynamics governing short-run sentiment and

$$\omega_\uparrow^m = \nu_m \exp(U_2), \omega_\downarrow^m = \nu_m \exp(-U_2) \tag{3}$$

$$\text{with } U_2 = \beta_0 + \beta_1 y + \beta_2 x + \beta_3 ret.$$

for the dynamic evolution of medium-run sentiment.We denote the number of optimistic (pessimistic) individuals in the short-run index by n_+ and n_-, and the optimists (pessimists) in the medium-term index by m_+ and m_-, respectively. The empirical series, therefore, correspond to:

$$x = \frac{n_+ - n_-}{2N}, y = \frac{m_+ - m_-}{2M} \tag{4}$$

with $2N(2M)$ the total number of respondents.[1] Since we do not have exact information on response rates, N and M are parameters that will also be estimated in our empirical exercise.

Note that our agent-based model consists of $2N + 2M$ coupled Markov processes for agents' belief dynamics in continuous time. In previous work,

[1]The factor 2 is introduced for convenience to make sure that the population size is even allowing for the possibility of a neutral average mood $x = 0$ or $y = 0$.

theoretical results on closely related models have been obtained via approximate dynamics of mean values and higher moments (Lux, 1995, 1998, Alfarano, Lux and Wagner, 2008). For a single opinion process, the aggregate outcome of the process hinges crucially on α_1 (or β_1): if this parameter (that could be labeled the intensity of herding or interaction) is below 1, the stationary distribution has a unique maximum, while it becomes bi-modal under stronger interaction ($\alpha_1 > 1$ or $\beta_1 > 1$). The later case allows for the build-up and breakdown of strongly optimistic or pessimistic majorities and depending on the parameters ν_s and ν_m, a rapid change between one type and the other. With two interacting opinion processes, the picture becomes more complex and also allows for cyclical behavior besides the uni-modal and bi-modal scenarios (cf. Weidlich, 2000, c.4).

Since we found some indication that our baseline opinion process could be overparametrized for the medium-term index, we also tried a model in which this process was replaced by a simple diffusion equation of the Ornstein-Uhlenbeck (OU) type:

$$dy_t = \kappa\left(\bar{y} + \beta_1 x_t + \beta_2 ret_t - y_t\right)dt + \sigma_y dZ_1 \tag{5}$$

with dZ_1 a standard Brownian motion. This variant assumes that medium-term sentiment is a mean-reverting process that is also influenced by the current state of short-term sentiment and contemporaneous returns. We add price dynamics in the form of another diffusion:

$$dp_t = \left(\gamma_0 + \gamma_1 x_t + \gamma_2 y_t\right)dt + \sigma_p dZ_2. \tag{6}$$

Note that for $\gamma_1 = \gamma_2 = 0$, (6) becomes a standard Brownian motion with drift. Significant parameter estimates of γ_1 or γ_2 would indicate an influence of sentiment on price changes that could be exploited for prediction of near-term returns.

3 Estimation Methodology

Lux (2009a) has developed a numerical maximum likelihood approach for a uni-variate population process. For discrete aggregate observations of an opinion index, the conditioned likelihood of each observation, conditional on last period's realization, can be obtained from numerical solutions of the so-called Fokker-Planck equation. The Fokker-Planck equation or forward Kolmogorov equation gives the time evolution of the probability density function of a stochastic system. For diffusion functions, the Fokker-Planck equation is the exact law of motion for the transitional density, while it is obtained as a second-order approximation for many population-based Markov processes (cf. Gardiner, 2004, c.7, Risken, 1996, c.3). Initiating the Fokker-Planck equation with the observation of the configuration of a system at time t, we can infer the likelihood of the subsequent state of the system at $t+1$ from the probability density at $t+1$ conditional on the initial state.

Unfortunately, closed-form solutions to the Focker-Planck equation can only be found for relatively simple cases. The present tri-variate model with one or two highly nonlinear population processes is too complex to derive its transient density in an analytical way. However, since the Fokker-Planck equation is a partial differential equation, we can resort to various well-established numerical integration schemes. Lux (2009a) following an earlier application along similar lines for pure diffusion processes (Poulson, 1999) has resorted to the Crank-Nicolson finite difference scheme. The later has the advantage of unconditional stability and second-order accuracy, at least in applications with only one space dimension. Monte Carlo simulations showed that this estimator did, in general, behave well while an alternative Euler approximation (i.e. approximation of the density over the unit time step between adjacent observations by a Normal distribution)[2] appeared essentially useless. Here we extend this approach to the 2D and 3D case. We consider time series of discrete multi-variate observations, $\{x_t\}_{t=0}^{T} = \{x_{1,t}, x_{2,t}, ..., x_{n,t}\}_{t=0}^{T}$ or subsets thereof. For the Markov population processes and diffusion processes defined above, the joint dynamics of the density $q(x;t)$ is given by the Fokker-Planck equation:

$$\frac{\partial q(x_t;t)}{\partial t} = \sum_i \partial_i \left[A_i(x_t) q(x_t;t)\right] + \sum_{i;j} \partial_i \partial_j \left[B_{ij}(x_t) q(x_t,t)\right] \tag{7}$$

with $-A_i(x_t)$ the drift function associated with variable $x_{i,t}$ and $2B_{ij}(x_t)$

[2]This estimator is also known as quasi maximum likelihood estimator in the literature (e.g. Ait-Sahalia, 1996). It is perhaps quite plausible that a Normal approximation should perform poorly for a potentially bi-modal distribution.

the entries of the matrix of diffusion coefficients. Numerical solutions of this equation over a $(n+1)$-dimensional grid (n "space" dimensions plus time) are used in the maximization of the log likelihood function:

$$\log q_0(x_0|\theta) + \sum_{s=0}^{T-1} \log q(x_{s+1}|x_s, \theta). \tag{8}$$

Since we have no predecessor for the first observation, x_0, its likelihood should in principle be computed on the base of the limiting distribution q_0 (but because of its negligible influence, we will simply discard this observation in practice). The remaining entries are conditional probabilities evaluated numerically with our above approach, and θ is the vector of parameters that we wish to estimate. Note that θ covers all parameters that appear in the pertinent specifications of eqs. (1) to (6). In particular, we also include the numbers of respondents, N and M, in the set of parameters. The reason is that, on the one hand, we only have rough orders of magnitude for the numbers of participants: According to the provider of our data set, average response rates are 20 to 25 percent of a pool of about 2000 subscribers. On the other hand, it might well be that our model does not capture all types of interaction. There might be groups of agents who are exposed to the same factors of influence and might actually exhibit perfectly synchronous behavior, instead of the conditionally independent transition rates of eqs. (2) and (3). Note that our goal is more to explore whether the model provides a good fit of the macroscopic dynamics and not so much whether it is a good formalization of individual behavior.

Before we move on to applications, a few more words on the numerical implementation of our approach are in order. While we keep the Crank-Nicolson approach for 1D applications, we will use other finite difference schemes in 2D and 3D. A number of criteria are relevant when deciding what kind of scheme to use: stability, accuracy and computational efficiency of finite difference schemes are typically crucial features that guide the choice of the applied researchers. In our application to the time development of a transient density, positivity of solutions should also be a concern. Many schemes are conditionally stable, that is, convergence depends on the parameter values and discretization steps. Since our goal is to estimate unknown parameter values, we are interested in unconditional stability. Therefore, we only use schemes in this paper that have been shown to be unconditionally stable for all possible choices of parameters and discretization steps.

Computational efficiency is another crucial concern. Even with an objective-oriented language like C++, the repeated application of finite difference approximations in higher dimension within the maximum likelihood loop becomes very time consuming. In order to economize on computing time, the most efficient difference schemes should be used in 2D and 3D. A high degree of computational efficiency can be achieved with schemes that lead to tri-diagonal systems of equations with zero entries off the main diagonales. In 1D, the implicit and Crank-Nicolson schemes both boil down to the solution of tri-diagonal systems. Since the Crank-Nicolson scheme is of higher accuracy than the purely implicit scheme, it is the method of choice for uni-variate stochastic processes. However, since the Crank-Nicolson scheme cannot be cast into a tri-diagonal form in higher dimensions, we have to resort to alternative algorithms in 2D and 3D, potentially sacrificing accuracy of the discretization scheme. Fortunately, unpublished Monte Carlo simulations indicate that the accuracy of the discretization may not necessarily affect much the quality of parameter estimates.

Positivity of solutions is less of a concern in the available literature, but is also important in our application. There are two aspects that are relevant here: first, positivity (and, in fact, accuracy even of otherwise well-behaved schemes) is problematic in degenerate cases where the diffusion coefficients tend to zero. Various corrections have been developed in the literature for such cases. Luckily, we do not encounter problems of this kind in our present application. However, a second source of violation of the positivity constraint are strong cross-correlations between variables. We, therefore, only consider constrained models in this paper in which the off-diagonal terms of the matrix of diffusion coefficients are assumed to be zero. This means that we assume independent innovations of our diffusion processes and independence between the innovations of the diffusion processes of eqs.(5) and (6) and the stochastic transitions in the Markov population processes. Of course, both, the drift and diffusion functions themselves may depend on the current state of the system. As it turns out, the assuption of independence of the fluctuations of the components of our multi-variate process might be a serious limitation. In the Appendix, we provide details on the drift and diffusion components in the Fokker-Planck equations and the finite difference schemes applied in our empirical study.

4 The Data and Previous Results

4.1 Data

Our data set consists of weekly records of market sentiment for the German stock market. This series have been obtained from *animusX*[3], a provider of technical tools and sentiment data for German investors. Our survey data start in the 29th calendar week of 2004 and extend until the 22nd calendar week of 2008, a total of 202 observations. *animusX* conducts a weekly email survey among about 2000 private and institutional investors who are asked among many other items about their prospects for the German stock market for the next week and the next three months, respectively. The average responses to both questions are reported in the standard format of a diffusion index (corresponding to the definitions of our variables x and y above) and are published each Sunday at 8 p.m. The short-term and medium-term indices (also denoted by S-Sent and M-Sent in the following) together with weekly closing notations of the German share price index DAX constitute our sample under consideration.

In a companion paper (Lux, 2008), vector autoregressive (VAR) models of this trivariate sample showed that this record of sentiment data does have explanatory power for the DAX and that there are subtle interactions between short-term sentiment and medium-term sentiment. Our basis aim in this paper is to explore whether the above simple behavioral model with social interaction can help to explain the findings of the purely statistical VAR model.

[3]Cf. *http://www.animusx.de/* for further information on the structure of the survey and other services provided by this company.

4.2 Previous Results for the Interaction of Sentiment and DAX Returns

In order to compare our subsequent results of the bivariate opinion model with those of Lux (2008), we first review the major findings of this source. Starting with the statistical features of our system composed of short-run sentiment (x), medium-run sentiment (y) and returns (ret), our companion paper finds that : (i) all tree time series appear stationary under a standard ADF test for unit roots, (ii) short-run sentiment exhibits more volatile movements between extremely positive and negative realizations while medium-run sentiment performs more moderate swing in the range of $[-0.5, 0.5]$, (iii) medium run sentiment exhibits more persistence than short-run sentiment (both features are clearly visible in the pertinent series, cf. Fig.1)

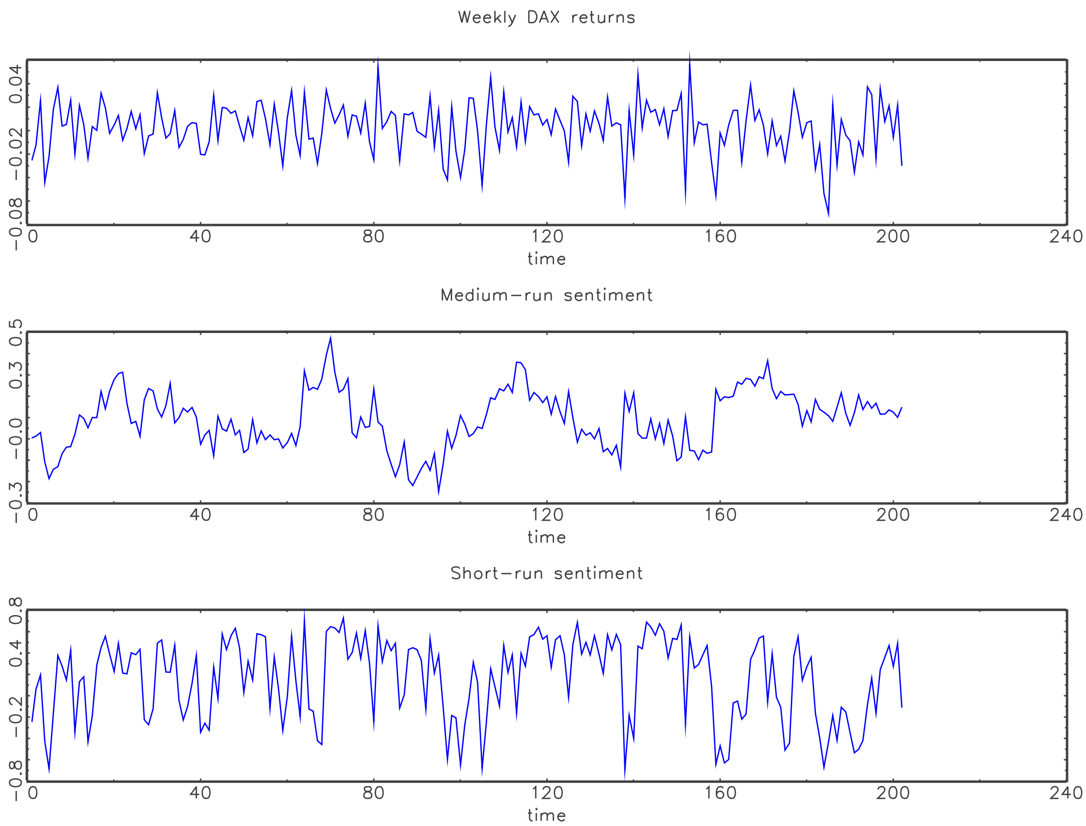

Figure 1: Sentiment and stock market returns. The time horizon is from the $29th$ calendar week of 2006 to the $22nd$ week of 2008.

Estimated VAR models with one or two lags only (favored by the BIC and HQ information criteria) indicate a dominating influence of medium-run sentiment on both short-run sentiment and returns. In total contrast to previous results for sentiment and returns in the U.S. market, Granger causality is found from (medium-run) sentiments on returns but not the other way round[4]. In a richer VAR (5) framework – favored by the AIC criterion – only a few additional parameters enter in a significant way. These appear at lag 5 (while the coefficient matrices at lags 3 and 4 are completely insignificant) and indicate a negative feedback from short-run sentiment and a positive one from returns on medium-run sentiment. M-Sent is, therefore, not exogenous anymore, but dynamic interaction between all variables of the system is found (this, of course, does not change the finding for returns to be endogenous in all specifications).

Subsequent experiments on the forecasting capacity show that the more parsimonious VAR (1) and VAR (2) models only beat the benchmark of the random walk with drift at long horizons (from 6 weeks) while the VAR (5) model provides for a significant forecast improvement at all time horizons and a much more pronounced reduction of mean-squared error of out of sample-forecasts against the benchmark. A trading experiment on the base of VAR signals confirms the economic value of the relatively weak feedbacks from S-Sent and returns on M-Sent: While the VAR (5) achieves an improvement of 16 percent in cumulative returns (from –0.108 p.a. for a buy-and-hold strategy to +0.052 for an actively managed portfolio), the VAR (1) and VAR (2) strategies are not too successful. Bootstrapping tests confirm the significance of the performance of the 'VAR (5) trader'.

The predictive power of sentiment is hard to square with traditional asset pricing models: The finding of Granger causality from sentiment on returns shows that public information can be used to forecast returns (and trading experiments underscore the economic value of these forecasts). This stands in clear contradiction to informational efficiency of the German stock market. However, since the realization of sentiment is public knowledge, its predictive power is also hard to square with a traditional noise trader ap-

[4]Previous studies for the U.S. and Shanghai stock markets found causal influence from returns on sentiment, but not in the other direction (cf. Brown and Cliff, 2004; Wang et al., 2006; Kling and Gao, 2008).

proach. Following DeLong et al. (1990), noise trading would have to be erratic and unpredictable in order for the noise traders to 'create their own space' and to provide limits to arbitrage activity against them. Quite in contrast, our sentiment measure is both highly predictable itself (M-Sent has an autocorrelation coefficient of 0.79) *and* it can be used to predict returns. Although M-Sent might be viewed as a mixture of fundamental information and fad components, it is worthwhile to emphasize that it is neither necessarily *new* information in itself nor is its news component the sole driving force in its predictive performance. Overall, the pattern of results is more in line with a financial market populated by boundedly rational agents whose aggregate behavior could arguably be formalized via a mass-statistical approach as proposed in sec. 3. Our subsequent estimation of this framework will reveal in how far we can recover the results of the simpler VAR models.

In order to follow as closely as possible the approach in our companion paper, we also split our sample into two parts: the first 150 observations are used for in-sample estimation of parameters while the remaining 52 observations serve to assess the out-of-sample forecasting capability of the estimated models. In the following we proceed by estimating various components of our framework. We start with 1D models for univariate series and proceed via estimations of bivariate series to the final case of the complete 3D model.

5 Univariate Dynamics of Sentiment Data and Asset Prices

We start with uni-variate applications estimating separately the parameters of our two hypothesized population processes and the diffusion process for prices. Table 1 exhibits the results of various 1D models. Panels A and B display parameter estimates and goodness-of-fit measures for the opinion process of eqs. (1) to (4) for both the short-run and medium-run sentiment measure. In both cases our list of influence factors in the U-function includes both a cross-dependency on the other index (α_2 and β_2) as well as an additional possible influence of returns on sentiment (α_3 and β_3). The later influence appears quite plausible since many agent-based models allow for positive feedback trading (bandwagon effects, chartist behaviour) that presumably would lead to causation from returns on sentiment. Previous literature on sentiment measures in the U.S. market (Brown and Cliff, 2000) has documented Granger-causality from returns on sentiment, but not *vice versa*. Kling and Gao (2008) found a similar causal structure for the Shanghai stock market. In contrast, our companion paper (Lux, 2008) finds causation from sentiment on returns in the present data but not the other way around. This is consistent with the insignificant parameter estimates for α_3 and β_3 in Table 1 (at least as far as meaningful standard errors could be obtained). Apparently, the influence from returns on sentiment is also minuscule under our hypothesized opinion dynamics. Dropping this entry and reestimating the parameters leads to minor changes only of the significant parameter estimates, while the maximized likelihood changes only very marginally. Typically, the AIC criterion would prefer the more parsimonious model whereas the BIC criterion would still favor inclusion of this apparently insignificant factor.

A closer look at the remaining parameter estimates shows the following: short-run sentiment is characterized by a herding intensity $\alpha_1 > 1$, i.e. within the bi-modal regime. The frequency of agents' revaluations of their mood (ν_s) is very high - about 70 times as high as the corresponding parameter ν_m for medium-run sentiment. The impression from Fig. 1 is in harmony with these findings: short-run sentiment appears to change in a very rapid manner and seems to prefer more extreme positive or negative values. There is also a small positive bias (α_0) as well as a moderate reinforcement from medium-run sentiment (α_2). Our 'effective' number of agents is estimated at about 68, i.e. a model with $2N = 136$ independent agents would get closest to the dynamic structure of the data.

TABLE 1 ABOUT HERE

Medium-run sentiment with its much smoother dynamics leads to a much smaller ν_m. Models I to III in Table 1, Panel B show specifications in which we first estimate the complete set of feedbacks and, then, drop first β_3 and subsequently also β_2 (dependence on x), which both appeared insignificant in the more complete versions of the model. Again, the other parameter estimates as well as the maximized likelihood are hardly affected by these variations. However, panel B reveals a problem in our estimation exercise:

in the case of medium-run sentiment, either the covariance of the parameter estimates failed to invert (models I and III) or, where standard errors could be obtained, none of the estimates appeared to be significant (model II). Close inspection shows that the likelihood function is very flat, in particular, along the M axis. Since standard errors are based on the curvature of the likelihood surface, the estimated standard errors will be extremely large. This situation indicates a multi-collinearity problem: the likelihood may be approximately linear in some parameters (inspection shows that it is not exactly linear).

Further experimentation shows that practically any variation of M (we varied it from 1 to 200) leaves the likelihood almost unchanged. However, the number of agents, M, interacts strongly with the parameters ν_m and β_1 that can change quite remarkably with different M. In the estimates of model IV, we have fixed $M = 68$ (in line with the estimate for short-run sentiment) which leads to an increase of ν_m by two to three times and a change of the estimate of β_1 from 0.046 of model I to 0.629 (indicating a much higher intensity of interactions). Note that fixing M now leads to significant parameters for the key ingredients of the model (this actually happens for any M that we tried). It is, thus, a way to overcome the collinearity problem. It is worthwhile to emphasize, that (approximate) collinearity does not necessarily imply misspecifiation of our model (cf. Kennedy, 2003, c.11). Different specifications might just be observationally nearly equivalent. This actually appears to be the case in our present framework. If interaction is weak, the opinion dynamics is close to a mean-reverting diffusion process in its time series properties. In this case, different combinations of the parameters ν_m, β_1 and M lead to very similar aggregate dynamics, at least for small samples.[5]

Since it appears plausible that the number of agents is the same for both sentiment variables (as they come from the same survey in which respondents are asked to provide both their forecast for the short and medium-term horizon), we proceed in the following mostly with restricted models obeying $M = N$. As an alternative, we have also estimated the Ornstein-

[5] We actually also found the same problem of approximate collinearity in Monte Carlo experiments with data generated from our population process for cases of weak interaction.

Uhlenbeck type model of eq. (5), cf. panel C of Table 2. Although we could have estimated this model via the analytical solution of its transient density, we used the Crank-Nicolson discretization with the same grid as before for better comparability. As with the agent-based model, the parameters for the influence of the short-run sentiment and returns are both insignificant. Medium-run sentiment, therefore, appears to evolve like an exogenous quantity without causal influence from the other components of our data set. This is actually again fully consistent with the results of the VAR analysis in our companion paper (Lux, 2008). Although the agent-based model and the OU diffusion process are not nested, it is tempting to compare their likelihoods.[6] As we can see, the fit is practically the same which underlines our conjecture of the proximity of the agent-based model to a mean reverting diffusion for moderate levels of interaction.

It is interesting to remark that the dynamics of S-Sent cannot be captured to any satisfactory degree by an Ornstein-Uhlenbeck process. Estimates for the short-run sentiment data turned out degenerate with implausible parameter values and goodness-of-fit way below that of the agent-based model. We, thus, see that the practical equivalence between both models only seems to hold for the unimodal case. This is plausible as one can indeed show that the agent-based dynamics converges to an Ornstein-Uhlenbeck process in the limit of a large population (cf. Horst and Rothe, 2008; Lux, 2009a).

We finally turn to estimates for the price dynamics, eq. (6). In the complete model I we find an insignificant parameter for short-run sentiment, but a strongly significant positive influence of medium-run sentiment. Skipping the former, the constant γ_0 also falls below the usual thresholds for significance, so that we finally also tried a model without both γ_0 and γ_1. Here γ_2 increases somewhat while the estimates of the variance, σ_p, remain practically constant over all three specifications. The same applies to the maximized likelihood although both information criteria would favor the more comprehensive model.

[6] Since we have used exactly the same grid in (y, t) space in both estimation algorithms, the likelihoods are comparable in the sense that they result from the assignment of probability mass to the same cells (intervals) on the base of two different estimated models.

6 Bivariate Dynamics

We proceed by estimating bivariate models for each pair of our three variables, x, y and p. As indicated above, we adopt a different finite difference scheme for the 2D case. The main reason is that Crank-Nicolson becomes computationally burdensome for two "space" dimensions. The most versatile and efficient approaches in the 2D case appear to be various implementations of so-called alternative direction implicit schemes (ADI). ADI methods are a device to reduce the two-dimensional problem to a succession of two one-dimensional problems. A unit iteration is split into two half-steps, one in the first space dimension keeping the second variable constant and a second one in which the variables change their roles. As a consequence, both half-steps are similar to an iteration of a 1D system and computation time only increases by a factor roughly equal to 2. From the wide array of ADI schemes we adopt the so-called Peaceman-Rachford algorithm which is simply a combination of two implicit half-steps (cf. Strickwerda, 2004, c.7 and the Appendix).

Our interest is now to explore the robustness of our previous 1D estimates. This means (i) robustness with respect to inclusion of a second simultaneous dynamic process and (ii) robustness with respect to different discretization schemes. Panel A of Table 2 exhibits parameter estimates for a bi-variate simultaneous opinion dynamics for the short-run and medium-run sentiment indices. Model I indicates the unrestricted model whereas Model II stands for a model with the same restriction on effective population sizes as above, i.e. $N = M$. Again, we could not obtain standard errors for the former because of the extremely flat surface of the likelihood function. Taking a population size of 68 as the starting value in the maximization produces the results exhibited for Model I, but with different starting values very different parameters can be obtained with practically equal likelihood. Experimenting with the starting values, we found that the parameters for the short-run sentiment dynamics were pretty unaffected while those for the medium-term components showed more variations. This is completely in line with the behavior of the pertinent 1D models. Apparently, component-wise estimation and simultaneous estimation of both dynamic processes leads to pretty much the same outcome. Fixing $N = M$ allows us to obtain meaningful standard errors for all parameters. These indicate that all parameters are significant although the significance of the biases (α_0, β_0) and cross-dependencies (α_2, β_2) is more or less marginal. The relatively weak interaction between S-Sent and M-Sent is again in line with VAR parameter estimates in our companion paper.

Fig. 2 shows the stationary distribution of our estimated stochastic opinion dynamics. Since an analytical solution is not available for our highly nonlinear bi-variate Fokker-Planck equation, we have simply integrated the transitional density over a very long time horizon until it converged to a stationary distribution in order to obtain Fig. 2. As can be seen, we find a distribution that is at the margin between bi-modality and pronounced skewness. While probability mass is concentrated around zero along the y axis, the stationary distribution has a maximum at about 0.6 for S-Sent together with non-negligible probabilities along the whole range between about -0.7 and $+0.7$.

Table 2 about here

Because of the collinearity in the y component, we also combined the opinion dynamics in x with the OU type diffusion for y (Model III in Panel A). The pertinent parameter estimates are all significant. While κ and σ_y correspond closely to their counterparts in the uni-variate estimation exercises of Table 1, $\bar{y}$ and β_1 show somewhat more variation. Most interestingly, however, the likelihood of this alternative model is practically the same as the one for the simultaneous opinion dynamics of Model I and II. The stationary distribution from Model III is, in fact, *entirely undistinguishable* from the one shown in Fig. 2. Again, this conformity underlines our impression that a population process could be largely equivalent to a simple diffusion model in the presence of relatively weak interaction.

Panels B and C provide parameters estimates for joint bi-variate dynamic processes of each one of our sentiment indices and the share index DAX. For S-Sent, we see that the parameters of the opinion process remain unchanged, but the sentiment-based component (γ_1) of the price dynamics seems highly significant which is in contrast to the previous result of the univariate diffusion for prices. For M-Sent, we have again estimated one model with unrestricted number of agents, one with M fixed at 68 and the

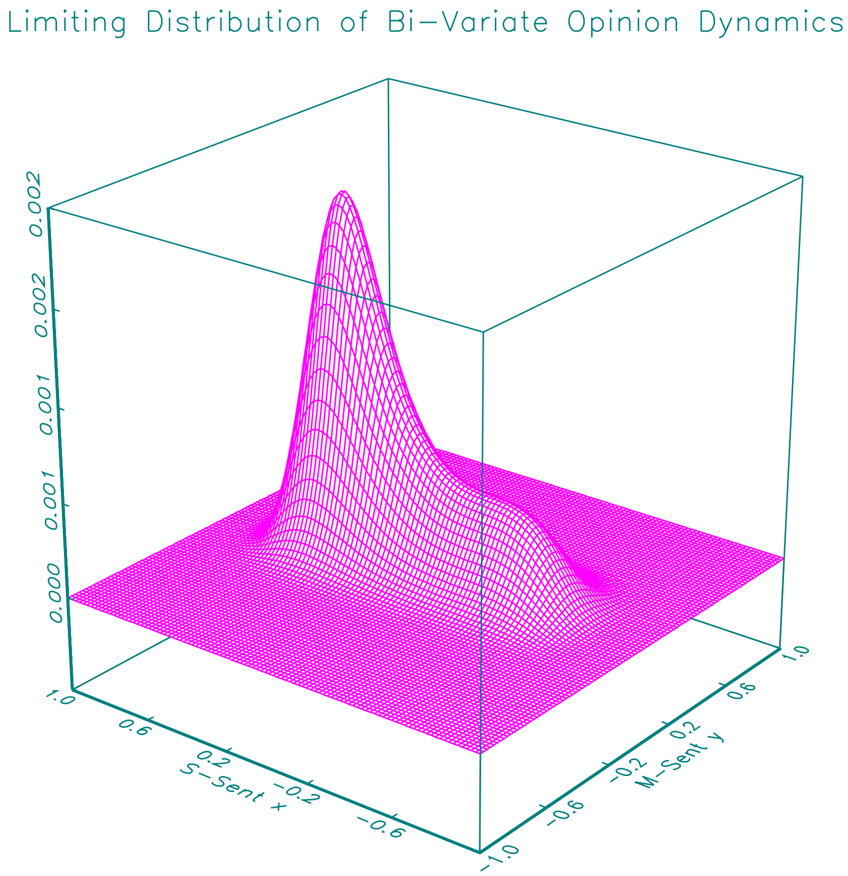

Figure 2: The limiting distribution of the joint opinion dynamics for S-Sent and M-Sent (model II of Table 2, Panel A). Note that a plot of the limiting distribution of model III (opinion dynamics for S-Sent and OU process for M-Sent is undistinguishable from the one displayed here).

OU diffusion. Parameters are close to previous ones with again almost the same quality of the fit, but here we encounter an insignificant parameter for the influence from medium-run sentiment on prices (γ_2) which is in contrast to the results obtained for the 1D price dynamics of sec. 5.

7 The Complete 3D Models

We finally turn to parameter estimates from joint models of all three varables: x, y and p. Unfortunately, with current computational resources, the maximum likelihood estimation in the 3D case approaches the limit of feasible scenarios. Typically, one full estimation takes a couple of days so that no extensive experimentation with different starting values is possible anymore.

Table 3 shows results for two models again using both a second population process or an OU diffusion for medium-run sentiment. Again, we have not been able to obtain meaningful standard errors in Model I that estimates separately the 'effective' numbers of agents for the two population processes governing S-Sent and M-Sent. Fixing $M = N$ a priori, meaningful standard errors are obtained throughout without much variation of parameter estimates and goodness-of-fit. With the Ornstein-Uhlenbeck process replacing the agent-based opinion dynamics of M-Sent (model IV), parameter estimates of the remaining dynamic components and maximized likelihood are in line with those of models I and II. While all parameters of the opinion dynamics are quite homogenous across our various models, the influence from sentiment on prices appears less robust. In models I, II and IV of Table 3, we find significant influence from S-Sent, but insignificant (or only marginally significant) influence from M-Sent on prices. While this appears in harmony with the bivariate models of Table 2, it is in contrast to the univariate diffusion estimated in Panel C of Table 1. It is also in contrast to the very clearcut results of our previous VAR estimates (Lux, 2008) indicating causality from M-Sent on prices, but not so from S-Sent. In order to assess the contribution of γ_1 to the fit of the models, we also estimated restricted versions of the 3D framework without an influence from S-Sent on prices. As it turns out (cf. models III and V in Table 3) this leaves all other parameters more or less unchanged but decreases the likelihood considerably. So, in principle, there seems no reason to skip this term.

TABLE 3 ABOUT HERE

Nevertheless, because of our very different results in the companion paper, we move to an out-of-sample forecasting exercise using both the full 3D versions and the restricted models imposing $\gamma_1 = 0$. Table 4 exhibits root-mean squared errors relative to the benchmark of a random walk with drift for the four models II through V of Table 3. Out-of-sample returns comprise weekly entries for a full year and forecasting horizons of one week up to eight weeks are investigated. Results are depicted for a single-week returns as well as for cumulative returns over the pertinent horizon. Because of the bi-modality of our estimated models (in the S-Sent dimension) we use different predictors: Besides the mean, we also use the nearest mode of two in the bimodal case or the global maximum, i.e. the coordinates with the highest probability mass. This approach follows Creedy et al. (1996) who use a bi-modal error term in an otherwise linear monetary model of the exchange rate. The 'nearest' and 'global' forecast conventions are justified by the observation that the mean might actually be a very unlikely realisation in a bi-modal stochastic process. Because of the persistence of the process, there is a certain chance that the realisation in the near future might remain close to the previously dominant mode (which justifies the 'nearest' convention). On the other hand, one might use the global maximum of the predictive distribution as the most probable single value which also might be quite remote from the mean.

Table 4 about here

Our results in Table 4, nevertheless, show that the mean is mostly a better predictor for longer horizons, while the nearest convention is slightly better than the mean at shorter horizons. The 'global' convention mostly ranks third. What is, however, more remarkable, is the difference between the full and restricted models. While both the full and restricted models start out fairly similar at one and two-weeks horizons, the full models produce catastrophic failures at three-to six-week horizons. Interestingly, these failures are most pronounced in the 'global' convention, but less extreme in the mean forecasts. In the 'nearest' convention, the forecasts are pretty close to the benchmark of the random walk with drift. Apparently, the full model predicts movements to another mode with ensuing large price increases or decreases which are not taking place in the data. The restricted models, in contrast, have much more predictive power – they produce significant improvements against the random walk at most horizons at least at the 95 percent confidence level. Hence, if we disregard the apparently significant influence from S-Sent on returns in our estimated models of Table 3, we arrive at highly significant predictive success of our 3D model. While there is certainly no compelling reasons to simply skip one significant parameter in our analysis, consideration of the VAR results of Lux (2008) could provide a justification to be cautious concerning the influence of near-term sentiment on price changes (it actually was the reason why we tried the restricted version of our model).

Whether we should interpret the very mixed results of Table 4 as positive or negative evidence is a moot point. What is perhaps more interesting is to ask why the behavioral model did pick up the apparently spurious dependency on S-Sent while the VAR estimates have no such channel. One answer could be that the behavioral model picks up instantaneous correlation between S-Sent and returns which indeed is highly significant in the VAR estimates of our companion paper. Of course, instantaneous correlation could not be used to forecast returns. It might, thus, be our negligence of off-diagonal terms in the diffusion matrix (which in our framework would measure instantaneous correlation) that urges the estimate of γ_1 to become significant. An extension of our framework allowing for correlations between innovations would, therefore, be a most worthwhile avenue for future research. It is also remarkable that the explosion of root-mean squared errors in Table 4 occurs at horizons of four to six weeks. In our VAR framework, we found a significantly negative feedback from S-Sent on M-Sent at lag 5 which, of course, is absent in the present continuous-time approach. Including such a feedback (in the sense of a delay factors) might attenuate the overshooting of predicted prices at these horizons. Because of the complexity of stochastic partial differential equations with delays, we also leave this question for future research.

8 Conclusions

Given the long lasting interest in sentiment data in financial economics, it might come as a surprise that there is no empirical work estimating and testing behavioral models for such data. Of course, under an efficient market perspective, such data would represent a relatively unimportant noise component. However, evidence exists for a certain impact of sentiment on prices. While for U.S. data, causality in the short-run seems to run from returns to sentiment (Brown and Cliff, 2004), our German data indicate a causal relationship in the opposite direction (Lux, 2008). Even for the U.S., however, some predictive power of sentiment has been found for longer horizons (Brown and Cliff, 2005).

The significant influence of sentiment on returns in the German data motivated us to adopt a behavioral, agent-based framework for the dynamics of short and medium run sentiment. In order to estimate the parameters of such models, we could take stock of an approach proposed in Lux (2009a) using a numerical maximum likelihood procedure. In terms of methodology, the contribution of this paper is the extension of this estimation technique to higher dimensions. Monte Carlo simulations in Lux (2009a) showed that this method performed well even in relatively small samples like the present one. Choosing appropriate methods from the large range of finite difference approximations for partial differential equations allowed an extension of the univariate approach to 2D and 3D.

Materially, we found evidence for strong social interaction in short-term sentiment and only moderate social influences in medium-run sentiment. With moderate interaction, estimation of the agent-based model appears somewhat cumbersome because of almost collinear behavior of some parameters. As we have seen, in this case, a more parsimonious Ornstein-Uhlenbeck process provides practically the same fit to the data. One could, therefore, use a simple macroscopic equation instead of the full microscopic Markov process. We believe that results like this one are important in that they provide an indication of the necessary degree of complexity of behavioral models in different scenarios (an issue very much neglected in economics where theoretical models are mostly either based on the assumption of a representative agent or an infinite population).

Overall most of our results were in nice coincidence with the previous VAR results of Lux (2008). The social dynamics estimated for S-Sent and M-Sent, therefore, seems a potential candidate for a data generating process of our sample. We only encountered somewhat divergent results in the diffusion equation for the price dynamics. While a simple 1D diffusion produced results in harmony with the previous VAR models, the 2D and 3D dynamics entailed a supposedly spurious influence from S-Sent that distorted our results. Discarding this influence, we found predictive power in line with the VAR model. We conjecture that this divergence is due to a certain incompleteness of the present continuous-time framework in which we did not allow for instantaneous correlation of innovations (which turned out to be highly significant in the VAR model). Obviously, expanding the model in this direction and including off-diagonal terms in the diffusion would be an interesting avenue for future research.

We believe that the present paper and its predecessor (Lux, 2009a) could provide an avenue to empirical estimation of a broad range of agent-based models. While we noted that we reached the limits of current computational power at our 3D applications, we also note that we have only used a small range of numerical schemes so far. Methods using adaptive adjustment of meshes or parallelisation of tasks might allow us to dramatically reduce computation time for in future applications. Further research in this direction should be of high priority.

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Table 1: Parameter estimates for uni-variate models

Panel A: Agent-based model of S-Sent (x)			Panel B: Agent-Based Model of M-Sent (y)				
Param.	Model I	Model II	Param.	Model I	Model II	Model III	Model IV
ν_s	8.851	8.938	ν_m	0.126	0.106	0.111	0.305
	(2.756)	(2.741)		$(\cdot)$	(0.118)	$(\cdot)$	(0.034)
α_0	0.008	0.008	β_0	0.069	0.080	0.073	0.033
	(0.004)	(0.004)		$(\cdot)$	(0.074)	$(\cdot)$	(0.017)
α_1	1.055	1.055	β_1	0.046	-0.118	-0.056	0.629
	(0.014)	(0.013)		$(\cdot)$	(1.211)	$(\cdot)$	(0.096)
α_2	0.062	0.062	β_2	-0.011	-0.013		-0.050
	(0.025)	(0.025)		$(\cdot)$	(0.103)		(0.057)
α_3	-0.014		β_3	-0.036			1.092
	(0.107)			$(\cdot)$			(1.034)
N	68.452	68.402	M	27.935	23.553	24.912	(68)
	(14.411)	(14.376)		$(\cdot)$	(25.640)	$(\cdot)$	
LogL	-694.738	-694.740	LogL	-526.058	-526.071	-526.078	-525.511
AIC	1401.477	1399.481	AIC	1064.116	1062.143	1060.157	1061.022
BIC	1399.416	1399.434	BIC	1062.056	1062.096	1062.124	1060.975

Panel C: Vasicek-type Model for M-Sent (y)			Panel D: Diffusion Model for Prices			
Param.	Model I	Model II	Param.	Model I	Model II	Model III
κ	0.234	0.237	γ_0	21.270	13.858	
	(0.065)	(0.065)		(10.992)	(9.616)	
$\overline{y}$	0.087	0.071	γ_1	-33.995		
	(0.041)	(0.033)		(24.037)		
β_1	-0.129		γ_2	165.636	164.872	208.240
	(0.148)			(62.104)	(62.579)	(55.262)
β_2	2.790		σ_p	102.540	103.241	103.952
	(2.704)			(5.940)	(5.981)	(6.022)
σ_y	0.094	0.095				
	(0.006)	(0.006)				
LogL	-525.419	-526.010	LogL	-895.322	-896.329	-897.345
AIC	1060.839	1058.020	AIC	1798.644	1798.658	1798.689
BIC	1060.792	1062.000	BIC	1800.611	1802.639	1804.683

Note: The models in panels A to C have been estimated via numerical integration of the transitional density, while for the diffusion models in panel D, the exact solution for the transient density could be used. The discretization of the finite difference schemes used steps of $k = \frac{1}{12}$ and $h = 0.01$.

Table 2: Parameter estimates for bi-variate models

Panel A: Interaction of S-Sent and M-Sent			
Param.	Model I	Model II	Model III
ν_s	9.192 (·)	9.191 (2.838)	8.897 (2.717)
α_0	0.010 (·)	0.009 (0.004)	0.009 (0.004)
α_1	1.058 (·)	1.058 (0.013)	1.057 (0.013)
α_2	0.044 (·)	0.044 (0.025)	0.045 (0.025)
N	67.826 (·)	67.809 (14.127)	66.270 (13.872)
υ_m	0.295 (·)	0.294 (0.073)	
β_0	0.053 (·)	0.053 (0.022)	
β_1	0.639 (·)	0.639 (0.127)	
β_2	-0.119 (·)	-0.119 (0.056)	
M	67.983 (·)	M = N	
κ			0.221 (0.063)
$\overline{y}$			0.141 (0.049)
β_1			-0.314 (0.158)
σ_y			0.093 (0.006)
lkl	-1017.309	-1017.308	-1017.119
AIC	2054.617	2052.616	2052.238
BIC	2044.500	2044.513	2044.136

Panel B: S-Sent and Prices	
Param.	Model I
ν_s	8.976 (2.819)
α_0	0.013 (0.004)
α_1	1.056 (0.014)
N	64.128 (13.706)
γ_0	-4.844 (9.478)
γ_1	142.540 (27.741)
σ_p	93.014 (6.097)
lkl	-926.593
AIC	1867.186
BIC	1863.111

Panel C: M-Sent and Prices			
Param.	Model I	Model II	Model III
ν_m	0.372	0.295	
	$(\cdot)$	(0.039)	
β_0	0.023	0.030	
	$(\cdot)$	(0.015)	
β_1	0.693	0.606	
	$(\cdot)$	(0.100)	
M	80.648	(68)	
	$(\cdot)$		
κ			0.240
			(0.066)
$\overline{y}$			0.073
			(0.033)
σ_y			0.096
			(0.006)
γ_0	7.685	25.045	19.677
	$(\cdot)$	(8.639)	(9.831)
γ_2	0.701	3.601	74.530
	$(\cdot)$	(20.804)	(67.139)
σ_p	103.982	103.859	103.479
	$(\cdot)$	(6.280)	(6.26)
lkl	-768.272	-766.121	-765.406
AIC	1550.544	1544.242	1542.811
BIC	1546.469	1542.181	1540.751

Note: The models in panels A to C have been estimated via numerical integration of the transitional density using the ADI (alternative direction implicit) algorithm detailed in the Appendix. The discretization of the finite difference schemes used steps of $k = \frac{1}{12}$ (for time), and $h = 0.02$ (for S-Sent and M-Sent). In Panels B and C, the discretization of the second space dimension (prices) is chosen in a way to generate the same number of grid points as in the x or y dimension, i.e. $N_x = N_y = N_p = 100$. This amounts to roughly 43 basis points of the DAX index.

Table 3: Parameter estimates for tri-variate models

Param.	Model I	Model II	Model III	Model IV	Model V
ν_s	9.133 (·)	8.847 (2.703)	8.112 (2.431)	8.427 (2.600)	8.222 (2.443)
α_0	0.008 (·)	0.008 (0.004)	0.009 (0.004)	0.009 (0.004)	0.009 (0.004)
α_1	1.058 (·)	1.057 (0.013)	1.055 (0.014)	1.055 (0.014)	1.056 (0.014)
α_2	0.054 (·)	0.055 (0.026)	0.055 (0.027)	0.057 (0.027)	0.054 (0.027)
N	70.039 (·)	67.950 (14.135)	64.573 (13.666)	64.832 (13.746)	65.378 (13.677)
ν_m	0.385 (·)	0.273 (0.069)	0.267 (0.068)		
β_0	0.043 (·)	0.062 (0.023)	0.060 (0.024)		
β_1	0.758 (·)	0.647 (0.130)	0.627 (0.136)		
β_2	0.117 (·)	-0.165 (0.065)	-0.148 (0.064)		
M	95.842 (·)	M=N	M=N		
κ				0.200 (0.062)	0.206 (0.063)
$\overline{y}$				0.167 (0.056)	0.154 (0.054)
β_1				-0.445 (0.200)	-0.377 (0.184)
σ_y				0.089 (0.006)	0.091 (0.006)
γ_0	-13.595 (·)	-13.631 (10.229)	17.140 (9.703)	-12.994 (10.055)	18.015 (9.682)
γ_1	150.481 (·)	150.819 (28.322)	- -	149.994 (28.018)	- -
γ_2	106.254 (·)	106.271 (60.118)	113.139 (67.360)	99.982 (59.707)	101.224 (67.118)
σ_p	89.381 (·)	89.426 (6.149)	102.330 (6.404)	89.038 (6.099)	102.293 (6.399)
lkl	-1337.176	-1337.022	-1350.211	-1336.736	-1350.003
AIC	2702.353	2700.044	2724.422	2699.472	2724.005
BIC	2684.178	2683.884	2710.277	2683.312	2709.860

Note: All models have been estimated via numerical integration of the transitional density using a tri-variate ADI (alternative direction implicit) algorithm as detailed in the Appendix. The discretization of the finite difference schemes used steps of $k = \frac{1}{8}$ (for time), and $h = 0.02$ (for S-Sent and M-Sent). The discretization of the second space dimension (prices) is chosen in a way to generate the same number of grid points as in the x or y dimension, i.e. $N_x = N_y = N_p = 100$. This amounts to roughly 43 basis points of the DAX index.

Table 4: RMSEs of Out-of-Sample Forecasts from Trivariate Models

Panel A: Forecasts from models II and III						
Forecasts of single returns						
horizon	near	global	mean	near	global	mean
	full 3D model (II)			restricted model (III)		
1	0.972	0.990	0.982	0.967*	0.964*	0.977*
2	1.097	1.004	0.985	0.994	0.956*	0.981
3	1.001	1.055	0.989	1.026	1.163	0.977
4	0.985	554.427	27.473	0.959	0.987	0.968*
5	1.013	386.010	26.375	0.972	1.009	0.970*
6	1.021	632.376	0.984	1.022	1.015	0.972*
7	1.001	0.970	0.984*	1.002	0.999	0.972*
8	1.017	1.033	0.981*	0.975	0.983	0.971*
Forecasts of cumulative returns						
1	0.972	0.990	0.982	0.967*	0.964*	0.977*
2	1.012	0.990	0.952	0.949*	0.928*	0.960*
3	1.017	1.042	0.952	0.942*	0.979*	0.984*
4	1.018	95.730	5.206	0.930*	1.011	0.933*
5	1.034	80.868	0.952	0.921*	0.952*	0.919*
6	1.024	1.049	0.938	0.910*	0.944*	0.905*
7	1.041	1.055	0.933	0.896**	0.927*	0.894**
8	1.032	1.035	0.927	0.886**	0.912*	0.883**

Panel B: Forecasts from models IV and V						
Forecasts of single returns						
horizon	near	global	mean	near	global	mean
	full 3D model (IV)			restricted model (V)		
1	0.972	0.973	0.983	0.961*	0.957*	0.976*
2	1.097	0.995	0.986	0.985	0.954*	0.981
3	1.001	72.752	0.990	1.040	1.020	0.977
4	1.014	391.874	30.942	0.973	0.978	0.968*
5	1.009	236.388	29.790	1.006	1.006	0.970*
6	1.012	781.600	0.986	1.023	1.032	0.972*
7	1.003	1.064	0.985*	0.976	0.989	0.972*
8	1.028	1.036	0.982*	1.003	1.004	0.971*
Forecasts of cumulative returns						
1	0.972	0.973	0.983	0.961*	0.957*	0.976*
2	1.012	0.990	0.954	0.958	0.928*	0.959*
3	1.017	21.756	0.954	0.950*	0.947*	0.947*
4	1.024	102.791	5.775	0.922*	0.922*	0.932*
5	1.035	96.392	0.957	0.919*	0.919*	0.918*
6	1.026	1.021	0.944	0.907**	0.911*	0.904**
7	1.038	1.038	0.940	0.894**	0.894**	0.893**
8	1.028	1.030	0.934	0.888**	0.889**	0.882**

Note: The table shows relative MSEs of the forecasts under the pertinent convention (i.e., original MSE divided by that of Brownian motion with drift).* and ** identify cases of significant improvement by the behavioral model against the nested alternative of Brownian motion with drift using the one-sided 5 and 1 percent adjusted Diebold-Mariano test statistic for equal predictive accuracy of nested models. To compute this statistics we used Newey-West autocorrelation and heteroscedasticity consistent estimator of the standard deviation with automatic lag selection by Andrew's method.

Appendix: Finite Difference Schemes for the Fokker-Planck Equation

(1) FP Equation in 1D

In the case of a univariate series, the FP equation for the density $q(x;t)$ of a variable x_t reads:

$$\frac{\partial q(x;t)}{\partial t} = \frac{\partial}{\partial x}(A(x,\theta)q(x)) + \frac{\partial^2}{\partial x^2}(B(x,\theta)q(x))$$

with θ the set of parameters we wish to estimate.

The first and second order terms of our candidate processes are:[7]

- for the Weidlich model with transition rates given in eqs. (1) to (3):

$$A(x) = -(\upsilon(1-x)e^{\alpha_0+\alpha_1 x} - \upsilon(1+x)e^{-\alpha_0-\alpha_1 x}),$$

$$B(x) = \frac{1}{2N}(\upsilon(1-x)e^{\alpha_0+\alpha_1 x} + \upsilon(1+x)e^{-\alpha_0-\alpha_1 x}),$$

- for the Ornstein-Uhlenbeck model:

$$A(x) = -k(\bar{y} - y), B(x) = \frac{1}{2}\sigma_y^2,$$

- for a diffusion with constant drift (e.g. as in eq. (6) with $\gamma_1 = \gamma_2 = 0$):

$$A(x) = -\gamma_0, B(x) = \frac{1}{2}\sigma_p^2.$$

The additional influence from exogenous variables in the U function or in the drift of a diffusion process can be easily added without changing the basic steps of our algorithm presented below.

Defining the flux $F(x)$, the FP equation can be more compactly written as:

[7] The drift is $-A(x)$, while the diffusion is $2B(x)$.

$$\frac{\partial q(x;t)}{\partial t} = \frac{\partial F(x,t)}{\partial x} \tag{A1}$$

with

$$F(x) = B(x)\frac{\partial q(x;t)}{\partial x} + (A(x) + \frac{\partial B(x)}{\partial x})q(x;t)$$

Denote by Q_j^i a discrete evaluation of the transient density at space and time coordinates (x_j, t_i), with grid points $x_j = x_0 + jh$; $j = 0, 1, \ldots N_x$ and $t_i = ik$ with $i = 0, \ldots, N_t$. Different possibilities exist to discretize eq. (A1). The Crank-Nicolson scheme approximates the flux at intermediate points $(i + \frac{1}{2})k$ which can be interpreted as a arithmetic average between a forward (explicit) and backward (implicit) approximation. We, thus, obtain the following finite diffence approximation to (A1).

$$\frac{Q_j^{i+1} - Q_j^i}{k} = \frac{1}{h}(\frac{F_{j+\frac{1}{2}}^{i+1} + F_{j+\frac{1}{2}}^i}{2} - \frac{F_{j-\frac{1}{2}}^{i+1} + F_{j-\frac{1}{2}}^i}{2}). \tag{A2}$$

Note that on the right hand side of of (A2), the discrete approximation of the derivative with respect to x has also used the central difference over the cell mid points $x_0 + (j - \frac{1}{2})h$ and $x_0 + (j + \frac{1}{2})h$. Since the discretization of the flux implies :

$$F_j^i = B(x_j)\frac{Q_{j+\frac{1}{2}}^i - Q_{j-\frac{1}{2}}^i}{h} + (A(x_j)Q_j^i + B'(x_j)Q_j^i) \tag{A3}$$

we arrive at:

$$\begin{aligned}
\frac{Q_j^{i+1} - Q_j^i}{k} \quad = \quad & \frac{1}{2h}\{B_{j+\frac{1}{2}}\frac{Q_{j+1}^{i+1} - Q_j^{i+1}}{h} + (A_{j+\frac{1}{2}} + B'_{j+\frac{1}{2}})Q_{j+\frac{1}{2}}^{i+1} \\
& + B_{j+\frac{1}{2}}\frac{Q_{j+1}^i - Q_j^i}{h} + (A_{j+\frac{1}{2}} + B'_{j+\frac{1}{2}})Q_{j+\frac{1}{2}}^i \\
& - B_{j-\frac{1}{2}}\frac{Q_j^{i+1} - Q_{j-1}^{i+1}}{h} - (A_{j-\frac{1}{2}} + B'_{j-\frac{1}{2}})Q_{j-\frac{1}{2}}^{i+1} \\
& - B_{j-\frac{1}{2}}\frac{Q_j^i - Q_{j-1}^i}{h} - (A_{j-\frac{1}{2}} + B'_{j-\frac{1}{2}})Q_{j-\frac{1}{2}}^i\}.
\end{aligned} \tag{A4}$$

Using averages $Q_{j+\frac{1}{2}} \equiv \frac{1}{2}(Q_{j+1}^i + Q_j^i)$ as well as another finite difference approximation to B' (i.e. $B'_{j+\frac{1}{2}} = \frac{B_{j+1} - B_j}{h}$) we can rearrange the system into the form:

$$a_j Q^{i+1}_{j-1} + b_j Q^{i+1}_j + c_j Q^{i+1}_{j+1} = d_j Q^i_{j-1} + e_j Q^i_j + f_j Q^i_{j+1}. \tag{A5}$$

Since this derivation applies to all values $j = 0, 1, \ldots, N_x$ on our grid, we end up with a computationally convenient tri-diagonal system of equations that approximates the continuous-time dynamics of the transient density $q(x;t)$:

$$\mathbf{V}\mathbf{Q}^{i+1} = \mathbf{r}^i \tag{A6}$$

$$\begin{aligned}
\text{with } \mathbf{V} &= \begin{bmatrix} b_0 & c_0 & 0 & \cdots & & & \\ a_1 & b_1 & c_1 & \cdots & & & \\ & & & & & & \\ & & & \cdots & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ & & & \cdots & 0 & a_{N_x} & b_{N_x} \end{bmatrix}, \\
\mathbf{Q}^{i+1} &= (Q_0^{i+1}, Q_1^{i+1}, \ldots, Q_{N_x})' \qquad \text{and} \\
\mathbf{r}^i &= (e_0 Q_0^i + f_0 Q_1^i, d_1 Q_0^i + e_1 Q_1^i + f_1 Q_2^i, \ldots, d_{N_x} Q^i_{N_x-1} + e_{N_x} Q^i_{N_x})'.
\end{aligned}$$

Note that the definition of our grid from x_0 to $x_0 + j \cdot N_x$ implies that at the edges of our system of equations, coefficients outside the $(N_x+1) \times (N_x+1)$ matrix V are indeed equal to zero (e.g. for $j = 0$, the term $a_0 Q^i_{-1}$ vanishes for all i). While this would impose a certain arbitrariness in truncation of the underlying state space in certain applications (e.g. when evaluating the Black-Scholes partial differential equation), in our case of a finite state space for the sentiment data this truncation would be quite natural. In order to conserve overall probability mass (or number of particles) additional 'no-flux' conditions have to be imposed. Going back to (A1), these can be simply defined as:

$$F^i_{-\frac{1}{2}} = F^i_{N_x+\frac{1}{2}} = 0 \qquad \forall i.$$

This keeps all the mass within the confines of the range $[x_0, x_0 + jN_x]$. Keeping track of these boundary conditions, we obtain slightly different definitions of the coefficients a_j, b_j, c_j at $j = 0$ and $j = N_x$ at the edges of the matrix V. The Crank-Nicolson scheme is known to be of accuracy $O(k^2) + O(h^2)$ at the grid points.

(2) FP Equation in 2D

Note, that for bivariate and trivariate systems, the Fokker-Planck equation (7) has drift and diffusion components consisting of the pertinent functions of the various models as we have presented them above. To write down the multivariate Fokker-Planck equations is, therefore, straightforward.

While it is feasible to apply the Crank-Nicoloson scheme to higher dimensions, it is problematic from the computational point of view. The reason is that the resulting system of equations can not be cast into the form of sparse, tridiagonal systems of equations anymore. A popular alternative that preserves the tridiagonal forms for higher dimensions is the class of *alternative direction implicit* schemes (ADI schemes). In our numerical approximation of bivariate and trivariate FP equation, we apply two particular variants from the rich class of ADI schemes.

To set the stage for the 2D and 3D applications, let us first explain the implicit scheme in 1D. This scheme is obtained by approximating the flux in eq.(A1) at points $(i+1)k$ so that instead of (A2) we arrive at:

$$\frac{Q_j^{i+1} - Q_j^i}{k} = \frac{1}{h}(F_{j+\frac{1}{2}}^{i+1} - F_{j-\frac{1}{2}}^{i+1}).$$

Inserting (A3) we derive:

$$\begin{aligned}\frac{Q_j^{i+1} - Q_j^i}{k} &= \frac{1}{h}\{B_{j+\frac{1}{2}}\frac{Q_{j+1}^{i+1} - Q_j^{i+1}}{h} + (A_{j+\frac{1}{2}} + B'_{j+\frac{1}{2}})Q_{j+\frac{1}{2}}^{i+1} \\ &\quad -B_{j-\frac{1}{2}}\frac{Q_j^{i+1} - Q_{j-1}^{i+1}}{h} - (A_{j-\frac{1}{2}} + B'_{j-\frac{1}{2}})Q_{j-\frac{1}{2}}^{i-1}\}.\end{aligned}$$

Rearranging the components of this system, we arrive at a linear system of equations similar to the one obtained for the Crank-Nicolson scheme:

$$a_j Q_{j-1}^{i+1} + b_j Q_j^{i+1} + c_j Q_{j+1}^{i+1} = Q_j^i.$$

Instead of (A6) we now have a tridiagonal system:

$$\mathbf{V}\mathbf{Q}^{i+1} = \mathbf{Q}^{i}$$

with appropriate boundary conditions. Note that in 1D the simplification of the expression on the right-hand side is negligible in terms of computational demand. Since the computational demands of the (fully) implicit and Crank-Nicolson scheme are practically the same, and the implicit method is of first-order accuracy only, we would typically prefer Crank-Nicolson (so that the later was our natural choice for univariate problems).

In 2D computational demands make a difference. Both a direct adaptation of the fully implicit and Crank-Nicolson schemes would lead to broadly banded matrix equations that lack the tridiagonal charm of those derived in 1D. A way to preserve this convenient form is to use an indirect scheme in an alternating way first performing a half (or auxiliary) step into one space dimension and subsequently another half (auxiliary) step into the second space dimension. To provide more details, let us denote the space variables as x_1 and x_2 with grids: $x_{1,j} = x_{1,0} + j \cdot h_1$, $j = 0, 1, \ldots, N_{x_1}$ and $x_{2,l} = x_{2,0} + l \cdot h_2$, $l = 0, 1, \ldots, N_{x_2}$.

The bivariate FP equation can be rewritten using the concept of fluxes in both the x_1 and x_2 direction as:

$$\frac{\partial q(x_1, x_2, t)}{\partial t} = \frac{\partial F_1(x_1, x_2, t)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, t)}{\partial x_2}$$

$$\begin{aligned} \text{with} \quad F_r(x_1, x_2; t) &= \sum_s B_{rs}(x_1, x_2, t) \frac{\partial q(x_1, x_2, t)}{\partial x_s} \\ &+ (A_r(x_1, x_2) + \sum_s \frac{\partial B_{rs}(x_1, x_2)}{\partial x_s}) q(x_1, x_2; t). \end{aligned}$$

As it turns out, inclusion of the off-diagonal terms, $B_{12}(\cdot)$ and $B_{21}(\cdot)$ is potentially problematic for a number of reasons: First, in terms of our bivariate opinion process, we would have to make explicit the dependence between respondents' opinion on the short and medium run. While it is perfectly reasonable to assume some dependency, to come up with a formal framework for this dependent opinion formation would be demanding. We are, therefore, tempted to maintain the assumption of independent Markov processes as data-generating processes for S-Sent and M-Sent. By this assumption, $B_{rs} = 0$ for $r \neq s$ follows as a consequence of the model set-up. In addition, setting the off-diagonal terms of the diffusion matrix equal to zero is computationally convenient as certain results on the accuracy of ADI schemes only apply in the absence of cross-derivatives and a strong off-diagonal influence could also jeopardize positivity of our solutions.

Our method of choice for the 2D case without cross-derivatives is the ADI scheme proposed by Peaceman and Rachford (cf. Strikwerda, c.7; Thomas, c.4). Denoting the finite difference approximation of the transient density in 2D by $Q^i_{j,l}$, i.e. $q(x_1, x_2; t)$ evaluated at $x_{1,j}$, $x_{2,l}$ and t_i, one first evaluates the derivative with respect to x_1 implicitly and that with respect to x_2 explicitly using a time step $k/2$:

$$\frac{Q^{i+\frac{1}{2}}_{j,l} - Q^i_{j,l}}{k/2} = \frac{1}{h_1}(F^{i+\frac{1}{2}}_{1,j+\frac{1}{2},l} - F^{i+\frac{1}{2}}_{1,j-\frac{1}{2},l}) + \frac{1}{h_2}(F^i_{2,j,l+\frac{1}{2}} - F^i_{2,j,l-\frac{1}{2}}).$$

For the next half step, the order is changed leading to:

$$\frac{Q^{i+1}_{j,l} - Q^{i+\frac{1}{2}}_{j,l}}{k/2} = \frac{1}{h_1}(F^{i+\frac{1}{2}}_{1,j+\frac{1}{2},l} - F^{i+\frac{1}{2}}_{1,j-\frac{1}{2},l}) + \frac{1}{h_2}(F^{i+1}_{2,j,l+\frac{1}{2}} - F^{i+1}_{2,j,l-\frac{1}{2}}).$$

As can easily be seen, both half-steps lead to a tri-diagonal system of equations with the explicit expressions all entering into the term on the right-hand side. Boundary conditions are easily imposed by setting $F^i_{1,-\frac{1}{2},l} = F^i_{1,N_x+\frac{1}{2},l} = 0$ for all i, l and $F^i_{2,j,-\frac{1}{2}} = F^i_{2,j,N_y+\frac{1}{2}} = 0$ for all i, j. Like the implicit scheme in 1D, the Peaceman-Rachford scheme is unconditionally stable. It is also second-order accurate in Δt, Δx_1 and Δx_2 (although the 1D implicit scheme is only of first-order accuracy) as long as the cross-derivatives are all equal to zero (Strikwerda, c.7).

(3) FP Equation in 3D

A generalization of the Peaceman-Rachford scheme to three space dimensions (x_i, $i = 1, 2, 3$) can be found in Morton and Mayers (1994, c.3).

With our FP equation expressed in fluxes:

$$\frac{\partial q(x;t)}{\partial t} = \sum_r \frac{\partial F_r(x:t)}{\partial x_r}, \qquad r = 1,2,3$$

and the approximation to the transient density at $(t_i, x_{1,j}, x_{2,l}, x_{3,m})$ denoted by $Q^i_{j,l,m}$, this scheme amounts to the following sequence of systems of equations:

$$\begin{aligned}
\frac{Q^{i+*}_{j,l,m} - Q^i_{j,l,m}}{k} &= \frac{1}{h_1}(F^{i+*}_{1,j+\frac{1}{2},l,m} - F^{i+*}_{1,j-\frac{1}{2},l,m}) \\
&+ \frac{1}{h_2}(F^{i}_{2,j,l+\frac{1}{2},m} - F^{i}_{2,j,l-\frac{1}{2},m}) \\
&+ \frac{1}{h_3}(F^{i}_{3,j,l,m+\frac{1}{2}} - F^{i}_{3,j,l,m-\frac{1}{2}}),
\end{aligned}$$

$$\begin{aligned}
\frac{Q^{i+**}_{j,l,m} - Q^{i+*}_{j,l,m}}{k} &= \frac{1}{h_2}(F^{i+**}_{2,j,l+\frac{1}{2},m} - F^{i+**}_{2,j,l-\frac{1}{2},m}) \\
&- \frac{1}{h_2}(F^{i}_{2,j,l+\frac{1}{2},m} - F^{i}_{2,j,l-\frac{1}{2},m}),
\end{aligned}$$

$$\begin{aligned}
\frac{Q^{i+1}_{j,l,m} - Q^{i+**}_{j,l,m}}{k} &= \frac{1}{h_3}(F^{n+1}_{3,j,l,m+\frac{1}{2}} - F^{n+1}_{3,j,l,m-\frac{1}{2}}) \\
&- \frac{1}{h_3}(F^{n}_{3,j,l,m+\frac{1}{2}} - F^{n}_{3,j,l,m-\frac{1}{2}}).
\end{aligned}$$

Here, the intermediate steps $i + *$ and $i + **$ are better interpreted as auxiliary rather than fractional steps since the second and third system of equations is more of an error-correction form. Unfortunately, Morton and Mayers do not mention the order of accuracy of this scheme. This approach has been used to solve three-dimensional conduction equations. Within this simpler setting, it has been demonstrated to be unconditionally stable and of first-order accuracy in Δt (Marschak, 1990, sec. 28). Most importantly for our purposes it seems to be the method with least computational demands among the many variants of ADI and related finite difference methods for higher dimensions.