February 15, 2009

Uncertainty shocks and financial intermediation in a dynamic general equilibrium model: a Markovian Jump Linear Quadratic Approach[1]

Etsuro Shioji[2]
Hitotsubashi University

---

[1] Key words: uncertainty shock, Markovian Jump Linear Quadratic (MJLQ), dynamic general equilibrium, distortion, financial intermediation. JEL classification codes: E32, E44, C61.

[2] The original version of this paper, Shioji (2009), was written while the author was visiting the Research and Statistics Department of the Bank of Japan. I would like to thank my two research assistants, Vu Tuan Khai and Taisuke Uchino for their superb jobs. I would also like to thank all the members of the Department, especially Kazuo Monma, Munehisa Kasuya, and Hibiki Ichiue for their many valuable comments and warm encouragement. I would also like to thank the participants of the macro lunch workshop at Hitotsubashi University, Thursday workshop at Keio University, and the modern economics workshop at Tohoku University for their comments and suggestions.

Abstract

This paper develops a new framework for analyzing the effects of an uncertainty shock on macroeconomic activities in a dynamic general equilibrium setting. It extends a relatively new solution method called the Markovian Jump Linear Quadratic (MJLQ) approach to solve for a dynamic equilibrium outcome for an economy with imperfect competition and externalities. In the model developed in this paper, increased uncertainty about the firm productivity induces contraction of financial intermediation and thus reduction in productive activities.

1 Introduction

The objective of this paper is to develop a new framework for analyzing the effects of increased uncertainty on macroeconomic activities. It is widely believed that a greater uncertainty is harmful to economic activities, but there have not been many studies that have studied the mechanism behind this effect in a dynamic general equilibrium framework. One of the reasons is that the methods most commonly used to solve for a dynamic general equilibrium, namely the linear quadratic (LQ) approach and the log linearization of the first order conditions and the resource constraints, are not suitable for analyzing this problem, due to the certainty equivalence property. The Markovian Jump Linear Quadratic (MJLQ) approach, which is only beginning to be utilized in macroeconomics in recent years, overcomes this limitation and allows us to incorporate multiplicative uncertainty into our models. Even this approach, however, cannot be directly applied to equilibrium models with distortion such as imperfect competition and/or externalities, which cannot be reduced to a maximization problem of a social planner. This paper extends this approach by making use of the idea of McGrattan (1994), who extended the regular LQ approach to economies with distortions. Using this new methodology, this paper develops a model in which increased uncertainty about firm productivity induces a contraction in financial intermediation and leads to a reduction in economic activities in general.

The motivation behind this paper is related to that of Bloom (2009), who studies how firms' investment and employment respond to a sudden and transitory rise in uncertainty. Unlike Bloom's model which was inherently a partial equilibrium framework, this paper aims to provide a general equilibrium framework usable for macroeconomic research[3].

The rest of the paper is organized as follows. Section 2 reviews the economics literature on MJLQ, as well as the past attempts to incorporate financial intermediation in dynamic general equilibrium models. Section 3 develops the model and Section 4

---

[3] On the other hand, Bloom (2009)'s framework is flexible enough to allow introduction of firm heterogeneity. It should also be mentioned that his objective is to estimate such a model to assess the importance of uncertainty shocks in the real world, while the model in this paper is still highly stylized at the moment. Bloom (2009) also develops a pseudo-general equilibrium version of his model, but the treatment of expectation is somewhat arbitrary there.

explains how an uncertainty shock is introduced into the model. Section 5 performs the impulse response analysis. Section 6 concludes.

2 Overview

In this section, I will first review the MJLQ methodology, starting from the regular LQ methodology, including its recent applications in economics. Then I will review some work that has incorporated financial intermediation into dynamic general equilibrium models.

[1] Linear Quadratic Dynamic Programming

A textbook explanation of this methodology can be found in Chapter 5 of Ljungqvist and Sargent (2004). First, consider a non-stochastic case. Consider an economic agent who faces an intertemporal optimization problem with m state variables and n control variables. The agent's objective function V_t takes a quadratic form in those variables:

$$Max \quad V_t \equiv -\sum_{t=0}^{\infty} \beta^t \cdot \begin{bmatrix} x_t' & u_t' \end{bmatrix} \cdot W \cdot \begin{bmatrix} x_t \\ u_t \end{bmatrix}, \tag{1}$$

where β is the discount factor ($0 < \beta < 1$), x_t is an ($m \times 1$) vector of state variables (whose first entry is "1"), u_t is an ($n \times 1$) vector of control variables, and W is a positive definite symmetric matrix with the following structure:

$$W \equiv \begin{bmatrix} Q & N \\ N' & R \end{bmatrix}. \tag{2}$$

Here, Q, N, and R are ($m \times m$), ($n \times m$), and ($n \times n$) matrices, respectively. The transition equation for the state variables is supposed to take the following linear form:

$$x_{t+1} = A \cdot x_t + B \cdot u_t \tag{3}$$

where A and B are ($m \times m$) and ($m \times n$) matrices, respectively. In practice, we often perform a quadratic approximation of the objective function and a linear approximation of the transition equation around the steady state. It can be shown that the policy function takes a linear form:

$$u_t = P \cdot x_t, \tag{4-1}$$

where P is an ($n \times m$) matrix, and that the value function takes a quadratic form:

$$V_t = x_t' \cdot V \cdot x_t, \tag{4-2}$$

where V is an $(m \times m)$ matrix, which satisfies the following Ricatti equation:

$$V = Q + \beta A'VA + K'JK$$
$$\text{where } J \equiv R + \beta B'VB、K \equiv N' + \beta B'VA \quad (5)$$

which can be solved iteratively.

The above model can be easily extended to a stochastic case in which a mean zero stochastic variable, e_t, enters the transition equation in an additive manner:

$$x_{t+1} = A \cdot x_t + B \cdot u_t + e_t \,. \quad (3')$$

It is known that the certainty equivalence property holds, that is, the policy function in (4-1) is invariant to the inclusion of the additive stochastic term. The policy function is thus also invariant to the level of uncertainty or the variance of the term e_t. This feature makes the methodology not suitable for analyzing the effects of uncertainty on the agent's behavior. The same certainty equivalence is known to hold also for the log linearization (of the first order conditions and the resource constraints) approach. We thus need to move away from the additive specification and look for a way to incorporate uncertainty that enters in a multiplicative manner.

[2] The Markovian Jump Linear Quadratic Approach

In this approach, it is assumed that the economy moves between different states, or "modes", following a Markov process. My overview here is based largely on Svensson and Williams (2007). Suppose there are J possible modes and index each of them by j, where j=1, 2, …, J. Denote the matrix of transition probabilities between these modes as $\left[P_{j,j'}\right]$, where $1 \le j, j', \le J$. Denote the mode the economy is at in period t as jt.

Suppose that the objective function can be written as:

$$Max \quad V_t \equiv -\sum_{t=0}^{\infty} \beta^t \cdot E_0 \left[x_t' \quad u_t'\right] \cdot W_{jt} \cdot \begin{bmatrix} x_t \\ u_t \end{bmatrix}, \quad (6)$$

$$\text{where} \quad W_{jt} \equiv \begin{bmatrix} Q_{jt} & N_{jt} \\ N_{jt}' & R_{jt} \end{bmatrix}. \quad (7)$$

The dimensions of each sub-matrix are the same as the corresponding one in (2), except that the matrices are allowed to vary with mode. Suppose that the transition equation can be written as:

$$x_{t+1} = A_{jt+1} \cdot x_t + B_{jt+1} \cdot u_t \,, \quad (8)$$

where the coefficient matrices, whose dimensions are the same as those in (3), are allowed to depend on the mode. It can be shown that the value function takes the following form:

$$V_t(x_t, jt) = x_t' \cdot V_{jt} \cdot x_t, \tag{9}$$

and that there is the following relationship between the matrices V for different modes:

$$\begin{aligned} &V_{jt} = Q_{jt} + \beta E_t A_{jt+1}' V_{jt+1} A_{jt+1} + K_{jt}' J_{jt} K_{jt}, \\ &\text{where } J_{jt} = R_{jt} + \beta E_t B_{jt+1}' V_{jt+1} B_{jt+1}, \\ &\text{and } K_{jt} = N_{jt}' + \beta E_t B_{jt+1}' V_{jt+1} A_{jt+1}. \end{aligned} \tag{10}$$

Svensson and Williams (2007) propose solving this "intertwined" Ricatti equation by an iterative method.

Note that this approach allows introducing stochastic elements in a multiplicative manner. It is thus possible to study how the degree of uncertainty affects the patterns of the economic agent's behavior. There have not been many studies that have utilized this approach in the economics literature. As of February 2009, only three of such studies have been published, to the best knowledge of the author: do Val and Başar (1999), Zampolli (2006), and Svensson and Williams (2008). In do Val and Başar (1999), a relatively traditional macroeconomic model is extended to incorporate a possibility of stochastic parameter changes. The MJLQ approach is used to derive the optimal policy under such a circumstance. Svensson and Williams (2008) can be considered as an overview and some extensions of Svensson and Williams (2007) which will be mentioned below. Zampolli (2006) develops a model in which the form of the dynamic equation for the exchange rate moves between two modes. He derives the optimal monetary policy under such a circumstance using the MJLQ approach. While Zampolli (2006) included only backward looking variables in the constraint that the central bank faces, Moessner (2005) considered a situation in which the central bank faces a hybrid New Keynesian Phillips Curve, which includes a forward looking element. Svensson and Williams (2007) develops the most comprehensive treatment of the MJLQ approach, including a situation in which the agent cannot observe the current mode. This approach is used to derive the optimal monetary policy under uncertainty about the model's parameters. They also make public their matlab codes for such calculation, from which I have benefited greatly. Note that all of the past studies utilize the MJLQ approach to derive an optimal governmental policy. This paper, to my knowledge, is the first attempt to use this approach to derive a dynamic general equilibrium.

[3] The LQ approach for an economy with distortion

As the LQ (as well as the MJLQ) approach is a solution method for an optimization problem, it cannot be directly pplied to derive an equilibrium path for an economy with distortion, as the equilibrium of such an economy does not coincide with the solution to the social planner's optimization problem. McGrattan (1994) extends the LQ approach to such a circumstance. Assume that the each economic agent (assumed to be homogeneous) has the same kind of the objective function as in the regular LQ problem. On the other hand, the constraint is assumed to take the following form:

$$x_{t+1} = A \cdot x_t + B \cdot u_t + C \cdot z_t \,, \tag{11}$$

where z_t is an $(l \times 1)$ vector of variables that each agent takes as given, and C is an $(m \times l)$ matrix[4]. It is further assumed that, in equilibrium, the following relationship must hold[5]:

$$z_t = \Theta \cdot x_t + \Psi \cdot u_t \,. \tag{12}$$

McGrattan (1994) shows a way to solve this problem by iterating on what can be called the "augmented" Ricatti equation.

[4] Integrating the MJLQ approach and the McGrattan (1994) approach

Consider modifying the MJLQ approach by incorporating the notion of the "z" variables of McGrattan (1994):

$$x_{t+1} = A_{jt+1} \cdot x_t + B_{jt+1} \cdot u_t + C_{jt+1} \cdot z_t \,. \tag{13}$$

We also follow McGrattan (1994) and assume that (12) holds. It can be shown that this problem can be solved by the "intertwined" version of McGrattan (1994)'s "augmented" Ricatti equations, which can be solved iteratively.

[5] Financial intermediation in dynamic general equilibrium

Broadly speaking, efforts to incorporate financial intermediation into a dynamic general

---

[4] As an example, we can think of an economy with many homogeneous firms, in which each firm is making its investment decision taking as given the aggregate investment (which exerts externality on each firm).

[5] In the example in the previous footnote, normalizing the number of firms to be 1, in equilibrium, the level of investment chosen by each firm must equal aggregate investment.

equilibrium setting can be classified into two strands of literature. The first type of models are the ones that incorporate financial market frictions, such as the "financial accelerator" model of Bernanke, Gertler and Gilchrist (1999), and the models of Carlstrom and Fuerst (1997) and Kato (2007). The most noteworthy work in relation to this paper is Christiano, Motto and Rostagno (2003). They study causes of the Great Depression by estimating a New Keynesian model with a banking sector. In their model, banks produce demand deposits by using capital, labor, as well as their excess reserves as inputs. They lend those deposits to firms to be used as working capital. Households derive utility from both cash and demand deposits. On the other hand, time deposits do not affect their utility but yields a higher interest. Banks lend out those time deposits for the physical investment purpose, but this lending is subject to an agency cost. Their model has many shocks, and five of them are related to financial intermediation. They are: (i) shock to the productivity of the banks' excess reserves, (ii) shock to the relative contribution to utility between cash and demand deposits, (iii) shock to total utility derived from cash and demand deposits, (iv) shock to the variance of the idiosyncratic shocks to borrowers of investment loans, and (v) shocks to the rate of destruction of the assets of the lenders of investment loans. Among them, (i), (ii), and (iii) combined can be considered as representing the reaction of the households to an increased uncertainty in the model developed in this paper. Also, (iv) can be considered as representing the banks' reaction to an increased uncertainty about their borrowers. A major difference between their model and the model in this paper is that the latter includes a single and fundamental source of increased uncertainty, namely uncertainty about the firm productivity, and this single shock causes simultaneous reactions from both households and the banks.

The second strand of literature incorporates the banking sector into the New Keynesian model to reexamine the importance of inside money in the conduct of monetary policy. Goodfriend (2005) and Goodfriend and McCallum (2007) specify the bank production function for loans, in which the inputs are the efforts by the bank employees in monitoring the borrowers, and the amounts of collaterals received from the borrowers. The collateral consists of government bonds and capital stock. The loan production is subject to a stochastic productivity shock, and it is shown that monetary policy would be misled if the central bank ignores the presence of this type of shocks. Refer also to

Canzoneri, Cumby, Diba, and López-Salido (2008) for a related model. This paper, like the papers mentioned above, specifies a bank production technology by a functional form (a cost function in the case of this paper). The major difference is that this paper is concerned with a different type of shock, namely an uncertainty shock[6].

3 A Dynamic Model with Financial Intermediation

This section develops a dynamic model with a banking sector which is can be solved by the augmented MJLQ method. Consider a closed economy with fully flexible prices, which consists of households, banks, firms and the government. Time is discrete. Goods are homogeneous and all quantities are expressed in their units. Labor is omitted for simplicity (or it can be considered as fixed). Money does not play a role in this model. Suppose that there are a fixed number of households, and denote the number by J. The numbers of firms and banks are also equal to J. Each household owns one bank. Ownership of each firm, on the other hand, is equally distributed across all households. A household owns a certain amount of goods ("assets") at the beginning of each period but cannot make productive use of them, nor does it have a storage technology to carry them over to the end of the period. Only firms have access to production technology, and they use goods supplied by households as capital to produce new goods. It is, however, impossible for households to directly lend the goods to firms, and they have to be provided through banks. That is, households supply goods to banks in the form of deposits, and banks supply at least a part of them to firms in the form of lending. At the end of the period, goods are distributed back to households, either in the form of firm profit shares, bank profits, or principal and interest payments on deposits. A part of them will be consumed or spent on reinforcing bank capital, and, after paying taxes, the remaining goods will be carried over to the next period. Households derive utility not only from consumption but also from liquidity services provided by bank deposits. Households and banks can also hold government bonds. The government sells those bonds in the market, and consumes the receipt. At the end of the period, it repays the

[6] It should also be noted that the model developed in this paper only considers the real side of the economy, and money is not included. Monetary policy is therefore beyond the scope of this paper.

debt by imposing lump sum taxes on households. Government bonds do not affect the utility of the households directly.

Firm productivity changes in a stochastic manner as will be specified later. Because of that, profitability of bank loans also changes stochastically. As the utility that households derive from bank loans depends positively on bank health, the uncertainty about bank profitability affects their preferences for bank deposits.

[1] Households

I assume that all the J households are homogeneous. Thus, in equilibrium, all of them will take the same behavior. Take one of them, called household i, as an example. Suppose that, at the beginning of period t, this household owns goods whose amount is equal to $Asset_{i,t}$. As it can consume only at the end of the period, and it has no ability to store the goods, it has to "invest" those goods, in the forms of either bank deposits or government bonds. The unit price of a government bond is equal to 1, and its net interest rate is r_t^B, and the gross rate is denoted as $R_t^B \equiv 1+r_t^B$. The value of the interest rate is known at the beginning of the period. Let the amount of government bonds purchased by this household as $B_{i,t}^H$.

Liquidity services produced by bank deposits are differentiated across the banks. As a consequence, the household optimally decides to make positive amounts of deposits to all the banks. Let the amount of bank deposits this household makes to bank j be denoted as $D_{i,j,t}$. The lifetime utility of the household is given by the discounted sum of utility that it obtains in each period, which in turn consists of utility derived from consumption $C_{i,t}$, which is denoted as $U^C(C_{i,t})$, and utility derived from bank deposits, $U^D\left(D_{i,t}\right)$. As will be explained below, $D_{i,t}$ is the total amount of deposits by this household. Specifically, the household utility takes the following form:

$$U_{i,0} = E_0 \sum_{t=0}^{\infty} \beta^t \left[U^C\left(C_{i,t}\right) + U^D\left(D_{i,t}\right) \right], \tag{14-1}$$

where $$U^C(C_{i,t}) = \frac{1}{1-a_H}\left(C_{i,t}\right)^{1-a_H}, \tag{14-2}$$

and $$U^D(D_{i,t}) = b_D \left\{ \left[\sum_{j=1}^{J} \tilde{\theta}_{j,t} \cdot \left(D_{i,j,t}\right)^{(\eta-1)/\eta} \right]^{\eta/(\eta-1)} \right\}^{a_D}. \tag{14-3}$$

The parameters satisfy the following relationships:

$$1 > \beta > 0,\ \eta > 1,\ a_H > 0,\ 1 > a_D > 0,\ b_D > 0\,.$$

In the above, the variable $\tilde{\theta}_{j,t}$ describes how the condition of bank *j* affects the utility this household derives from its deposits to this bank. It is determined as follows:

$$\tilde{\theta}_{j,t} = b_\theta \left(\frac{N_{j,t}}{K_{j,t}}\right)^{a_\theta} \cdot \left[m \cdot \tilde{Y}_{j,t}\right]^{c_\theta}, \tag{14-4}$$

where $1 > a_\theta > 0,\ b_\theta > 0,\ 1 > c_\theta > 0$. Here, $K_{j,t}$ denotes total loans made by bank *j*, and $N_{j,t}$ denotes this bank's capital. The above equation states that a bank with higher capital loans ratio would give higher utility with its deposits. On the other hand, $m \cdot \tilde{Y}_{j,t}$ is the return on loans that this bank obtains, as will be explained in more detail in [4] below. The above equation thus implies that a bank that can earn higher return on its loans can give higher utility to its depositors.

The intratemporal budget constraint of the household is

$$Asset_{i,t} = \sum_{j=1}^{J} D_{i,j,t} + B_{i,t}^{H}, \tag{15-1}$$

and its intertemporal budget constraint is

$$Asset_{i,t+1} = \sum_{j=1}^{J} R_t^j D_{i,j,t} + R_t^B \cdot B_{i,t}^H - tax_t + \tilde{\Pi}_t^F + \tilde{\Pi}_{i,t}^B - C_{i,t} - I_{Ni,t} - ADJ_{Ni,t} - Cost_{i,t}^D\,. \tag{15-2}$$

In the above, R_t^j is the gross interest rate on deposits made with bank *j*, tax_t is lump sum tax imposed at the end of period *t*, $\tilde{\Pi}_t^F$ is distribution of firm profits to each household, and $\tilde{\Pi}_{i,t}^B$ is distribution of profits from bank *i*, which is owned by this household *i*, and $I_{Ni,t}$ is the amount of reinforcement of bank capital that household *i* makes to the bank it owns. On the other hand, $ADJ_{Ni,t}$ is the adjustment cost that is incurred when this household changes the amount of bank capital. Also, $Cost_{i,t}^D$ is the cost of making a certain amount of deposits for this household, which takes the form:

$$Cost_{i,t}^D = \frac{\gamma_D}{2} \cdot \left(D_{i,t}\right)^2, \qquad \gamma_D > 0\,. \tag{15-3}$$

Combining (15-1) with (15-2) yields

$$Asset_{i,t+1} = R_t^B Asset_{i,t} - DX_{i,t} - tax_t + \tilde{\Pi}_t^F + \tilde{\Pi}_{i,t}^B - C_{i,t} - I_{Ni,t} - ADJ_{Ni,t} - Cost_{i,t}^D, \tag{15-4}$$

where $DX_{i,t} \equiv \sum_{j=1}^{J} R_t^{Bj} D_{i,j,t}$ (15-5)

and $R_t^{Bj} \equiv R_t^B - R_t^j$, (15-6)

so $DX_{i,t}$ can be regarded as "total expenditure on bank deposits" by this household, and R_t^{Bj} is the interest rate differential between government bonds and bank deposits (of bank *j*), or the premium paid to the liquidity service provided by this bank's deposits. Note that, at the beginning of period *t*, $\tilde{\theta}_{j,t}, \tilde{\Pi}_t^F$, and $\tilde{\Pi}_{i,t}^B$ are all unknown.

<u>[2] Optimal allocation of deposits within period</u>

Consider the household problem of allocating deposits to each bank, or the problem of choosing $D_{i,j,t}$ ($j = 1, 2, ..., J$) given total expenditure on bank deposits, $DX_{i,t}$. As will be shown later, although $\tilde{\theta}_{j,t}$ has a stochastic element, that element is common across all the banks. In fact, one can write;

$$\tilde{\theta}_{j,t} = \hat{\theta}_t \cdot \theta_{j,t}, \tag{16}$$

where $\hat{\theta}_t$ is the stochastic part common to all the banks, and $\theta_{j,t}$ is the part that can be observed at the beginning of period *t*. In this case, we can write:

$$E_t U(D_{i,t}) = E_t \left(\hat{\theta}_t^{\eta/(\eta-1)}\right)^{a_D} \cdot b_D \left[\sum_{j=1}^{J} \left(\theta_{j,t} \cdot D_{i,j,t}^{(\eta-1)/\eta}\right)^{\eta/(\eta-1)}\right]^{a_D} \tag{17}$$

where E_t denotes the expectation formed at the beginning of period *t* throughout the paper. We can thus formulate the deposit allocation problem as

$$Max \quad u_{it}^D \equiv \left[\sum_{j=1}^{J} \theta_{j,t} \cdot D_{i,j,t}^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)}, s.t. DX_{i,t} \equiv \sum_{j=1}^{J} R_t^{Bj} D_{i,j,t}, \tag{18}$$

which is free from any stochastic element. Solving this problem gives

$$D_{i,j,t} = \theta_{j,t}^{\eta} \left[\frac{R_t^{Bj}}{\hat{R}_t^{BD}}\right]^{-\eta} \frac{DX_{i,t}}{\hat{R}_t^{BD}}, \tag{19-1}$$

where $\hat{R}_t^{BD} \equiv \left[\sum_{j=1}^{J} \theta_{j,t}^{\eta} \cdot \left(R_t^{Bj}\right)^{1-\eta}\right]^{1/(1-\eta)}$. (19-2)

Equation (19-1) states that an individual bank faces a downward sloping demand curve in this differentiated deposit market, which is much like the demand curve we often see in goods market models with differentiated products.

[3] Rewriting the intertemporal optimization problem

Using the above results and the symmetry of the banks, we can rewrite the household intertemporal optimization problem in a form that does not contain the indices for individual banks, j. First, from the symmetry of the banks,

$$\hat{R}_t^{BD} \equiv \left[\sum_{j=1}^{J} \theta_{j,t}^{\eta} \cdot \left(R_t^{Bj} \right)^{1-\eta} \right]^{1/(1-\eta)} = J^{1/(1-\eta)} \theta_t^{\eta/(1-\eta)} R_t^{Bd} , \quad (20)$$

where $\theta_t = \theta_{j,t}$ and $R_t^{Bd} = R_t^{Bj}$ for all j. Also, from the definition of $DX_{i,t}$ and the symmetry of the banks,

$$DX_{i,t} \equiv \sum_{j=1}^{J} R_t^{Bj} D_{i,j,t} = R_t^{Bd} D_{i,t} , \text{ where } D_{i,t} \equiv \sum_{j=1}^{J} D_{i,j,t} . \quad (21)$$

Finally, the utility derived from bank deposits can be rewritten as

$$U^D(D_{i,t}) = b_D \left\{ J^{1/\eta} \cdot \tilde{\theta}_t^{(\eta-1)/\eta} \cdot D_{i,t} \right\}^{a_D} \text{ where } \tilde{\theta}_t = \tilde{\theta}_{j,t} \text{ for all } j. \quad (22)$$

Plug (21) into (15-4) and the intertemporal budget constraint becomes

$$Asset_{i,t+1} = R_t^B Asset_{i,t} - R_t^{Bd} D_{i,t} - tax_t + \tilde{\Pi}_t^F + \tilde{\Pi}_{i,t}^B - C_{i,t} - I_{Ni,t} - ADJ_{Ni,t} - Cost_{i,t}^D . \quad (23)$$

[4] Bank profits

Denote the total amount of deposits that bank i, which is owned by household i, accepts by $X_{i,t}$. Note that, in this paper, I denote the amount of deposits that household i makes by $D_{i,t}$ and the amount of deposits that bank i accepts by $X_{i,t}$ to distinguish the two (in equilibrium they will be equal). From (19-1), the demand curve this bank faces takes the following form:

$$X_{i,t} = \theta_{i,t}^{\eta} \left[\frac{R_t^{Bi}}{\hat{R}_t^{BD}} \right]^{-\eta} \frac{J \cdot DX_t}{\hat{R}_t^{BD}} , \text{ where } DX_t \equiv \sum_{j=1}^{J} DX_{j,t} \Big/ J , \quad (19\text{-}1')$$

which would be a downward sloping curve if we put the interest rate differential between government bonds and deposits on the vertical axis, and is also affected by the bank's own choices through the term $\theta_{i,t}$. The bank allocates the deposits $X_{i,t}$ between loans, $K_{i,t}$, and purchases of government bonds, $B_{i,t}^{B}$. I specify the production function of the firm that receives loans from this bank, which will be called firm *i*, as follows:

$$\tilde{Y}_{i,t} = \tilde{A}_t \cdot \left(K_{i,t}\right)^{\alpha}, \qquad 0 < \alpha < 1. \tag{24}$$

In the above, $\tilde{Y}_{i,t}$ denotes this firm's output (net of the amount lent by the bank), and $\tilde{A}_t$ is the stochastic productivity term that is common across all firms (there is no idiosyncratic shock in this model), whose value is not known by certainty at the beginning of period *t*. The bank receives a fixed fraction of the firm's output plus the principal of loans minus capital depreciation as its profit. This fraction is denoted as *m*, where $0<m<1$[7]. The rest becomes the firm's profits. Denoting the bank's profit as $\tilde{\Pi}_{i,t}^{B}$, it can be written as

$$\tilde{\Pi}_{i,t}^{B} = m \cdot \left[\tilde{A}_t \cdot \left(K_{i,t}\right)^{\alpha} + \left(1-\delta_K\right) K_{i,t}\right] + R_t^{B} \cdot B_{i,t}^{B} - R_t^{i} \cdot X_{i,t} - Cost_{i,t}^{B}, \tag{25-1}$$

where $0 < \delta_K < 1$ is the depreciation rate of capital in the hands of the firm. Also, in (25-1), $Cost_{i,t}^{B}$ summarizes the bank's cost of maintaining a balance of deposits and the cost of lending, and is specified as follows:

$$Cost_{i,t}^{B} = b_C \cdot X_{i,t}^{\;1+a_C} + b_K \left(\frac{K_{i,t}}{X_{i,t}}\right)^{a_K} \cdot K_{i,t}, \qquad a_C>0,\ b_C>0,\ a_K>0,\ b_K>0. \tag{25-2}$$

The first term on the right hand side of (25-2) states that the cost of maintaining deposits is increasing in the amount of deposits that this bank accepts. The second term implies that the cost of lending is increasing in the amount of loans but is decreasing in the amount of deposits that the bank accepts. The idea behind this specification is that deposits are inputs into production of loans: the more inputs are available to the bank, the easier it is to make loans. Because of the relationship $X_{i,t} = K_{i,t} + B_{i,t}^{B}$, (25-1) becomes:

[7] This fraction *m* can be interpreted in two ways. The first is to consider a Nash bargaining between the bank and the firm. In this view, once a bank and a firm are randomly matched, neither side can bargain with other counterparts any more, and they will bargain with each other and split the proceeds. In such a case, *m* would represent the bargaining power of the bank. The second interpretation considers an informational friction in the financial market. That is, it is assumed that the bank can demonstrate to a third party existence of only a fraction of the firm's output, and thus can claim only that fraction of output. In this case, *m* would correspond to the fraction that is verifiable.

$$\tilde{\Pi}_{i,t}^{B} = R_t^{Bi} \cdot X_{i,t} + \left\{ m \cdot \left[\tilde{A}_t \cdot \left(K_{i,t} \right)^{\alpha} + \left(1 - \delta_K \right) K_{i,t} \right] - R_t^{B} \cdot K_{i,t} \right\} - Cost_{i,t}^{B},$$

where $R_t^{Bi} \equiv R_t^{B} - R_t^{i}$. (25-1')

On the other hand, firm *i*'s profit, denoted by $\tilde{\Pi}_{i,t}^{F}$, is

$$\tilde{\Pi}_{i,t}^{F} = (1-m) \cdot \left[\tilde{A}_t \cdot \left(K_{i,t} \right)^{\alpha} + \left(1 - \delta_K \right) K_{i,t} \right]. \quad (26)$$

[5] Evolution of bank capital

Denote bank *i*'s capital at the beginning of period *t* as $N_{i,t}$. Its evolution over time is expressed by the following equation:

$$N_{i,t+1} = \left(1 - \delta_N \right) N_{i,t} + I_{N,i,t}, \quad (27)$$

where δ_N is the depreciation rate of bank capital, which takes a value between 0 and 1.The adjustment cost of changing the level of bank capital is specified as a quadratic function as follows:

$$ADJ_{N,i,t} = \frac{b_{ADJ}}{2} \left(\frac{N_{i,t+1} - N_{i,t}}{N_{i,t}} \right)^2. \quad (28)$$

[6] Government

The government fixes the amount of debt repayment at the end of each period, $\overline{B}$. Then, at the beginning of each period, the interest rate on government bonds is determined competitively, which in turn determines the beginning-of-period issuance of government debt, B_t^S:

$$B_t^S = \overline{B} / R_t^B. \quad (29\text{-}1)$$

The entire revenue from the issuance of bonds will be spent on government consumption, which is assumed to be a pure waste. The entire debt repayment at the end of the period is financed by lump sum taxes. Thus,

$$G_t = B_t^S, \quad (29\text{-}2)$$

and $$tax_t = \overline{B}. \quad (29\text{-}3)$$

According to (29-2), when the demand for government bonds increases, that pushes up bond prices (and thus pushes down the interest rate on those bonds), and thus more resources are wasted in government consumption.[8]

[7] Market equilibrium

In equilibrium, all the households take the same behavior, and so do all the banks and all the firms. Each household, in maximizing its utility, takes as given the following variables: the interest rate on government bonds, R_t^B, the average expenditure on deposits of all households DX_t, the differential between the interest rate on government bonds and the interest rate on deposits offered by each bank, $\hat{R}_t^{BD}$ (note that this variable is inclusive of the term $\theta_{j,t}$: it contains information about bank capital and loans etc. of the banks owned by the other households). The equilibrium conditions for the goods market and the bonds market are

$$\tilde{Y}_t = \tilde{A}_t \cdot (K_t)^{\alpha} = C_t + I_{Nt} + G_t + ADJ_{Nt} + Cost_t^D + Cost_t^B, \quad (30\text{-}1)$$

and $$B_t^S = B_t^H + B_t^B, \quad (30\text{-}2)$$

respectively. Here, variables without subscripts *i*, such as $\tilde{Y}_t, K_t, Cost_t^B$, and B_t^B, express averages across all the banks.

4 Stochastic process for productivity

To make concrete the idea that the economy goes back and forth between regimes with higher and lower uncertainty, we assume that the productivity $\tilde{A}_t$ follows the following Markov process. Assume that there are four states or "modes", called mode 1 through 4. Denote the mode which the economy is in in period *t* as s_t (=1, 2, 3, or 4). Mode1 is also called "Regime I", and modes 2-4 combined will be called "Regime II". For reasons that will be made clear soon, Regime I will be also called the "Calm", while Regime II will also be called the "Storm". In modes 1 and 3, the productivity takes an intermediate value. Mode 2 is characterized by a low productivity, while mode 4 is a high productivity state. At the beginning of each period, it is not known for certainty which mode the economy is in: it will be made clear only at the end of each period. Table 1

[8] This model formulation represents the idea that, when the interest rate on government bonds decreases, that makes it easier for the government to raise funds by issuing more bonds, and thus it weakens the budgetary discipline, resulting in greater social waste.

summarizes the transition probabilities across the modes.

Table 1: Transition matrix between the modes

		"mode" next period			
		1: Calm, Intermediate productivity	2: Storm, Low productivity	3: Storm, Intermediate productivity	4: Storm, high productivity
"mode" this period	1: Calm, intermediate productivity	1-P	P*p/2	P*(1-p)	P*p/2
	2: Storm, low productivity	P	(1-P)*(1-p)	(1-P)*p	(1-P)*0
	3: Storm, intermediate productivity	P	(1-P)*p/2	(1-P)*(1-p)	(1-P)*p/2
	4: Storm, high productivity	P	(1-P)*0	(1-P)*p	(1-P)*(1-p)

In Table 1, the capital letter P denotes the transition probability from Regime I to Regime II and vice versa. It is supposed to be a fairly small number. In the simulation reported in the next section, this value is set at 0.001. That is, as long as the economy is in Regime I (or mode 1), people do not have to worry so much about the possibility of productivity going down suddenly at the end of the period. That is the reason why this regime is named the "Calm". On the other hand, once the economy enters Regime II, chances are that it will stay there for a long time.

On the other hand, the lower case letter p determines the transition probabilities within Regime II, and is supposed to be greater than P. In the simulation in the next section, its value is set at 0.05. For example, once the economy is in mode 3, even though the productivity last period was at the intermediate level, the probability that it will change to either the low or the high level is much higher. In that sense, Regime II is characterized by a greater uncertainty, and is thus named the Storm regime.

This paper is concerned with what happens when the level of uncertainty goes up, *holding constant the current level of productivity*. To investigate this question, let us assume that the economy was originally in Regime I. At a certain point in time, the economy moves to Regime II. For the moment, it is in mode 3, which means that the productivity level itself is the same as before. But people have to take into account the increased uncertainty about the productivity at the end of the period. Responses of various economic variables, including deposits, loans, and production, to this increased uncertainty will be studied using a numerical simulation.

5 Simulation

In this section, we run a numerical simulation of the model presented in the previous section, using the augmented Markovian Jump Linear Quadratic approach. This simulation is not meant to replicate detailed characteristics of observed data (i.e., it is not a calibration). Rather, the purpose here is to investigate properties of the model solution. I set values of certain parameters exogenously, while for others, choose their values so that the non-stochastic steady state (with the intermediate level of productivity) levels of certain variables will be at their pre-specified values.

Following Svensson and Williams (2007), for each of the four modes that exist in this model, the non-stochastic steady state that corresponds to the specified productivity level is derived (note that this implies that the non-stochastic steady state for modes 1 and 3 are the same). Then, for each mode, I take the quadratic approximation of the objective function and the linear approximation of the constraint around the mode specific non-stochastic steady state. As Svensson and Williams (2007) argue, such a procedure is justifiable if, once the economy is in a certain mode, it tends to stay there for a fairly long time. Then the augmented MJLQ approach is applied. As already mentioned in the previous section, the numbers that govern the transition probabilities are set as P=0.001 and p=0.05. Denoting the productivity level in mode s as A_s, I set

$$A_1 = A_3, \qquad A_2 = 0.9 \cdot A_3, \qquad \text{and } A_4 = 1.1 \cdot A_3 .$$

That is, supposing that the economy is in mode 3 (that is, it is in the “Storm” but the productivity level is intermediate), there is a 2.5% chance that the productivity will be 10% higher and the equal chance that it will be 10% lower. The rest of the parameter values are summarized in Table 2.

Table 2-1 Parameters whose values are set directly

	Parameter	Meaning	Value
Households	β	Discount factor	0.95
	a_H	Elasticity of utility with respect to consumption	0.5
	a_D	Elasticity of utility with respect to deposits	0.5
	η	Elasticity of substitution between deposits of different banks	2
	a_θ	Determines the elasticity of utility with respect to bank capital	Set to be equal to $\alpha \cdot c_\theta$.
	b_θ	Strength of influence of bank health on utility	1
	c_θ	Elasticity of utility from deposits with respect to the productivity level	0.5
	γ_D	Level of deposit maintenance cost	0.01
Firms and Banks	α	Elasticity of output with respect to capital	0.35
	δ_K	Firm capital depreciation rate	0.05
	δ_N	Bank capital depreciation rate	0.05
	m	Profit shares for banks	0.5
	a_C	Cost elasticity of accepting deposits (-1))	1
	b_{ADJ}	Size of bank capital adjustment cost	0.01
	a_K	Elasticity of lending cost	1
	b_K	Size of lending cost	0.05
Government	$\bar{B}$	End-of-period repayment of government debt	0.20

Table 2-2 Parameters whose values are set from the steady state values of endogenous variables

Endogenous variables		Parameter	
	Steady state		meaning
R^{Bd}	0.02	b_C	Size of the cost of accepting deposits
K	0.9	A_1=A_3	Productivity level (intermediate)
X	1	b_D	Weight on utility from deposits

Under those parameter values[9], I start from mode 1 (the "Calm") and run the simulation, generating the productivity level each period according to the Markov process specified in Table 1. In the simulation reported, the economy moved from mode 1 to 3 in period 991; that is, the economy moved into the "Storm" but the productivity remained the same. This situation continued until period 1052. In Figures 1 and 2, I report evolution of various variables between period 981 (that is, ten periods before the transition) and 1011 (twenty periods after the transition).

Starting from deposits in Figure 2, in the upper-left panel, as soon as the economy enters the "Storm", deposits go down, because of the heightened uncertainty about the profitability of bank loans. Loans in the upper-right panel of the same figure also shrink, due to the increased uncertainty about returns to loans. This is also partly because the decrease in deposits increases the cost of lending, as in (25-2). The decrease in loans directly affects output, and, as a consequence, total household asset in the upper-left panel of Figure 1 gradually decreases. In the upper-right panel of Figure 1, bank capital decreases. This is because, as the household's incentive to make deposits weakens, it makes less sense for banks to pay the cost to maintain a high level of capital, as it would not attract deposits as effectively as before. In the lower-right panel of Figure 1, the interest rate on government bonds decreases, because the uncertainty causes both households and banks to make a "flight to quality", that is, their demand for government bonds increases.

[9] As in Table 2-1, parameter a_θ is set to be equal to $\alpha \cdot c_\theta$. This is for the sake of simplifying the computation: note that, with this specification, two terms involving loans, *K*, in the household utility function are offset with each other, and this variable drops out.

Figure 1: Simulation results (1)

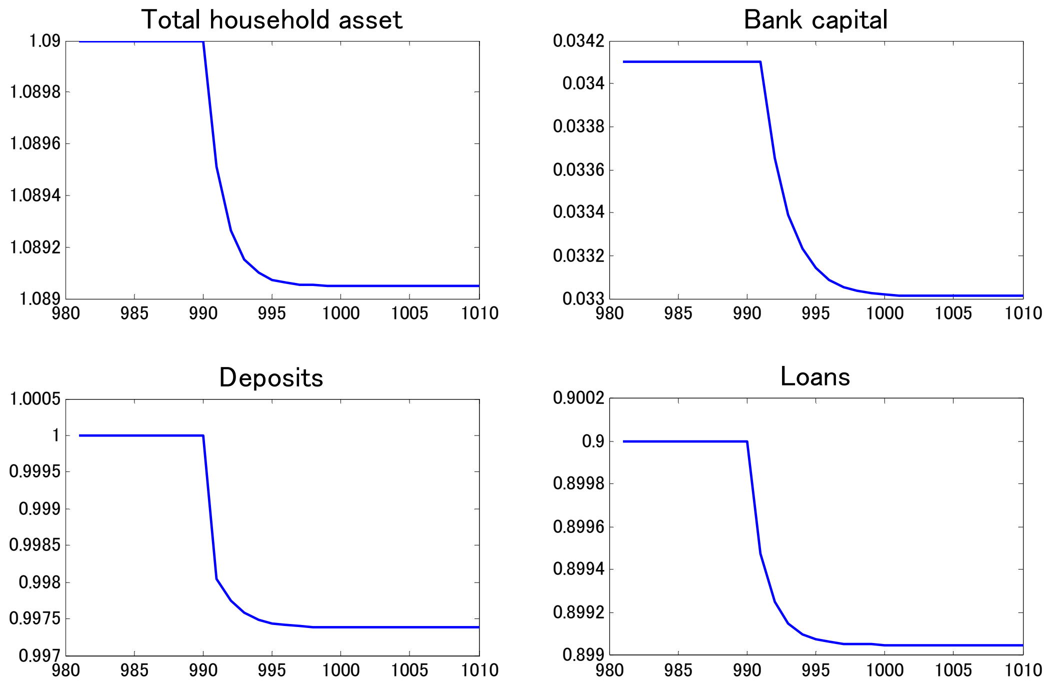
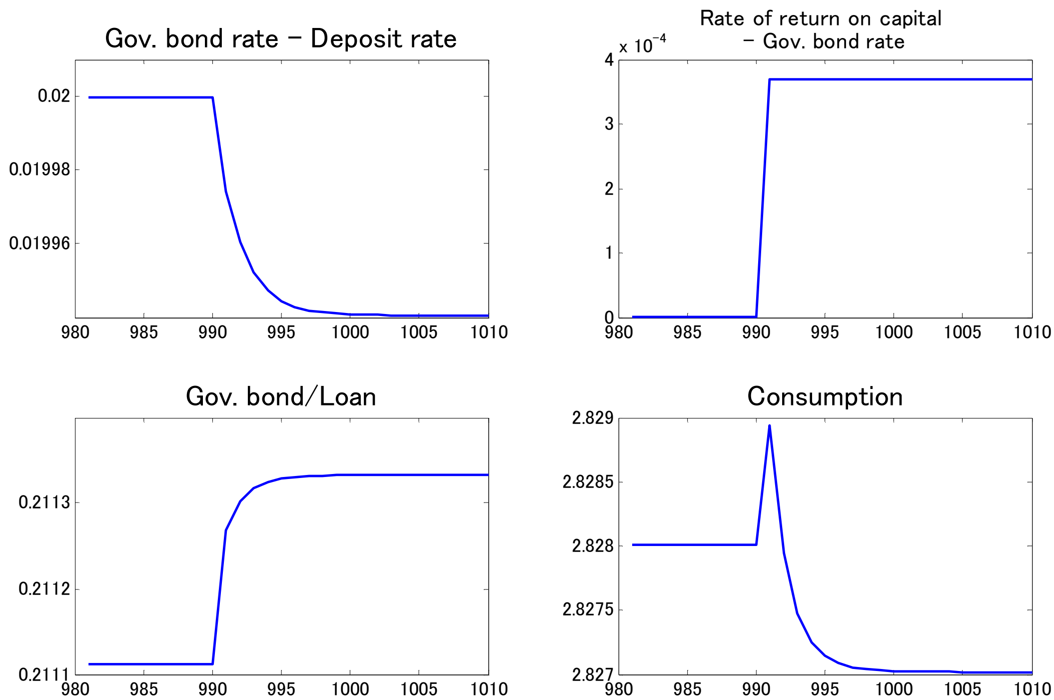


Figure 2: Simulation results (2)



In the lower-left panel of Figure 2, the interest rate differential between government bonds and deposits, or the premium for liquidity services provided by deposits, decreases. This is partly because of the decrease in the government bonds rate that we saw above, and also partly because households now wish to hold less deposits and thus the interest rate on deposits is higher. The lower-right panel of Figure 2 shows that, as loans decrease and less firm capital is invested, the expected rate of return on loans is higher due to diminishing returns to capital. The difference between this and the government bonds rate, or the risk premium charged on bank loans, increases.

Finally, the lower-left panel of Figure 1 reports the evolution of consumption. It jumps up as soon as the economy enters the “Storm” and then starts to decrease from the next period onwards. This may seem unrealistic, though the intuition behind the result is straightforward. As the increased uncertainty discourages households from holding assets, they prefer to consume more in the current period. As this gradually erodes household assets, consumption also decreases gradually. Thus, the consumption boom occurs in this model because households have no other means to reduce assets but to consume more. This result may be weakened when the model is extended to incorporate more realistic features, such as endogenous labor supply.

6 Conclusions

This paper has developed a new approach to analyze the effects of an uncertainty shock in a dynamic general equilibrium model with distortions. This approach has been used to study how an increased uncertainty influences output, loans, deposits, and so on, in a model with a banking sector. The paper has been successful in finding a case in which all of the three variables mentioned above decrease simultaneously. Shioji (2009) develops an extended version of this model in which there are two types of deposits, liquid deposits and less liquid deposits, as well as two types of loans, short term loans and long term loans.

Two important topics are left for future research. First, quantitatively speaking, the effects of an uncertainty shock presented in this model are small. This problem might be resolved by incorporating stronger form of non-linearity in the model, such as presence of bankruptcy which would make the downside risk of financial transaction greater.

Also, by making the governmental supply of safe assets more elastic (note that, in this paper, the supply of government bond is nearly fixed by treating $\overline{B}$ as a constant), the quantitative effects of an uncertainty shock could be made greater. The second remaining problem is that, in this paper, depositors' aversion to making deposits with banks with a low capital loans ratio and low profitability is incorporated directly by the specification of the utility function, rather than derived from a model with a deeper micro-foundation. It would be an important extension to give such a foundation to the current model.

References

Bernanke, Ben S., Mark Gertler, and Simon Gilchrist, "The financial accelerator in a quantitative business cycle framework" *Handbook of macroeconomics. Volume 1C,* (1999), 1341-93. Amsterdam; New York and Oxford: Elsevier Science, North-Holland.

Bloom, Nick, "The impact of uncertainty shocks", NBER Working paper 13385, forthcoming in *Econometrica* (2009).

Canzoneri, Matthew, Robert Cumby, Behzad Diba, and David López-Salido, "Monetary aggregates and liquidity in a neo-Wicksellian framework", *Journal of Money, Credit, and Banking* (2008), 40, 1667-1698.

Carlstrom, Charles T., and Timothy S. Fuerst, "Agency costs, net worth, and business cycle fluctuations: a computable general equilibrium approach" *American Economic Review* (1997), 87, 893-910.

Christiano, Lawrence, Roberto Motto, and Massimo Rostagno, "The Great Depression and the Friedman-Schwartz Hypothesis" *Journal of Money, Credit, and Banking*, Part 2 December (2003), v. 35, iss. 6, 1119-97.

do Val, Joao B.R., and Tamer Başar, "Receding horizontal control of jump linear systems and a macroeconomic policy problem", *Journal of Economic Dynamics and Control*, August (1999), 23(8), 1099-1131.

Goodfriend, Marvin, "Narrow money, broad money, and the transmission of monetary policy", In: Faust, J., Orphanides, A., Reifschneider, D. (Eds.), *Models and Monetary Policy: Research in the Tradition of Dale Henderson*, Richard Porter, and Peter Tinsley. Board of Governors of the Federal Reserve System, (2005).

Goodfriend, Marvin, and Bennett T. McCallum, "Banking and Interest Rates in Monetary Policy Analysis: A Quantitative Exploration" *Journal of Monetary Economics*, July (2007), 54(5), 1480-1507.

Kato, Ryo, "Liquidity, infinite horizons and macroeconomic fluctuations" *European Economic Review* (2007), 50, 1105-1130.

Ljungqvist, Lars and Thomas J. Sargent, *Recursive Macroeconomic Theory* 2nd ed., MIT Press (2004).

McGrattan, Ellen R. "A note on computing competitive equilibria in linear models", *Journal of Economic Dynamics and Control* (1994), 18, 149-160.

Moessner, Richhild, "Optimal discretionary policy in rational expectations models with regime switching", Bank of England Working Paper 299 (2006).

Shioji, Etsuro, "Increased uncertainty and the demand for liquid assets: A dynamic general equilibrium analysis" (in Japanese), Bank of Japan Working Paper Series 09-J-1 (2009).

Svensson, Lars E.O. and Williams, Noah, "Monetary policy with model uncertainty: distribution forecast targeting", mimeo, (2007), available from Noah Williams' web site at http://www.ssc.wisc.edu/~nwilliam/.

Svensson, Lars E.O. and Williams, Noah, "Optimal monetary policy under uncertainty: a Markov jump-linear-quadratic approach", Federal Reserve Bank of St. Louis *Review*, July/August (2008), 275-293.

Zampolli, Fabrizio, "Optimal monetary policy in a regime-switching economy: the response to abrupt shifts in exchange rate dynamics", *Journal of Economic Dynamics and Control* (2006) 30 1527-1567.