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Introduction <to be written>

II. Basic Structure of the Model

Consider a simple macro growth model of two economies, Country I and Country II, whose levels of labor endowment at time t are denoted $L_I(t)$ and $L_{II}(t)$, each of which is growing at rate n_I and n_{II}. Let $K_I(t)$, $K_{II}(t)$ and $C_I(t), C_{II}(t)$ denote the capital stock and consumption level in country I and country II at time t. $B(t)$ denotes the level of the debt of country I to country II at time t. Let us assume that the identical (well behaved) linear homogeneous production function.

$$P_j(t) = F(K_j(t), L_j(t)),\ \ j = I, \text{and } II, \qquad \#$$

where P_j is the amount of output for country j (j=I and II). We neglect the rate of depreciation for both countries. Capital moves freely between the countries so that the marginal productivity of capital between the two countries be equated at the level of

$$r(t) = f'(k_I(t)) = f'(k_{II}(t)), \qquad \#$$

where $f(k_j(t))$ is a per capita production function defined by $f(k_j(t)) \equiv F(k_j(t), 1) \equiv F(K_j(t)/L_j(t), 1) = F(K_j(t), L_j(t))/L_j(t)$. Let us denote the saving ratio out of net income in the two countries as s_I and s_{II}, whichi are assumed to be constant for the time being, and $s_I < s_{II}$.

This is the framework analyzed in Hamada (1966), where capital movements are explaining solely from by saving-investment process and trade flows are assumed to accommodate the inter-temporal choice. In the actural world economy, both this kind of inter-temporal choice and trade activities are taking place in a consistent manner. Let us denote by $B(t)$ and $\frac{d}{dt}B(t)$ the stock of foreign indebtedness of country I to country II and its increase, namely, the net capital inflow to country I from country II. The amount of the capital flow from country II to country I that is required to keep the equality of the rates of returns in the two countries is then:

$$\mathbf{D}B(t) = (s_{II} - s_I)\frac{K_{II}}{K}P_I(t) + (n_I - n_{II})\frac{K_I K_{II}}{K} + s^* r B(t), \qquad \#$$

where $\mathbf{D}$ is the operator of taking derivatives such that $\mathbf{D} = (d/dt)$. and

$$K = K_I + K_{II,} \quad s^* = (s_I K_{II} + s_{II} K_I)/K, \text{ and, for later reference, } s = (s_I K_I + s_{II} K_{II})/K. \qquad \#$$

The third term is the effect of savings on the renumeration to lending and could be neglected. Thus *capital flow moves form a higher saving country to a lower saving country, and from a country with a lower labor growth rate to a country with a higher growth rate.*

Next consider the simpler case where the natural growth rates are identical across the countries, namely,

$n_I = n_{II} = n$. In this case, reminding the definition of s as the average saving ratio defined in eq. ( ) above, we can trace the stationary state where the system converges and calculate the debt capital ratio at the stationary state.

$$B/K_I = n(s_{II} - s_I)\ell_{II}/(sn - s_I s_{II} r), \qquad \#$$

which can be approximated, when saving ratios are small and r is close to n, by neglecting the second term of the denominator, as

$$B/K_I = \frac{(s_{II} - s_I)}{s}\ell_{II}. \qquad \#$$

Thus, the converging debt capital ratio is approximately proportional to the difference in savings rates and to the relative size of the neighboring country.

Modern readers will be unsatisfied because the difference in saving ratios are ad

hoc, given exogenously. Let us assume two countries have the identical size, the identical technology and both stationary population. In country I agents have a higher rate of time preference ρ_I than those in country II who has the rate of time preference ρ_{II} so that $\rho_I > \rho_{II}$.

Suppose international world market operate under free capital mobility in such a way that the rate of returns in the two countries are equalized and that the international debt, $B(t)$, which denotes the borrowing of country I from country II, will yield the rate of return $r(t)$. The agents in both countries maximizes the discounted sum of the future consumption streams.

$$U_i = \int_{t=0}^{\infty} u(C_j(t)) \exp\{-\rho_i t\} dt, \quad i = I, II. \qquad \#$$

Also, the movement of $K_I(t), K_{II}(t)$ and $B(t)$ are described by:

$$\mathbf{D}K_I(t) = F(K_I(t), L_I(t)) - r(t)B(t) + \mathbf{D}B(t) - C_I(t), \qquad \#$$

$$\mathbf{D}K_{II}(t) = F(K_{II}(t), L_{II}(t)) + r(t)B(t) - \mathbf{D}B(t) - C_{II}(t), \qquad \#$$

and the rate of return equality, by normalizing the labor in both countries as unity is defined as

$$r(t) \equiv f'(K_I(t)) = f'(K_{II}(t)). \qquad \#$$

Since the labor endowment is identical between the countries, we can normalize

them as unity. Since technology is identical, the equality of the returns to capital means the equality of capital used in both countries. That is,

$$K_I(t) = K_{II}(t). \qquad \#$$

Define the net wealth of both countries as

$$X_I(t) = K_I(t) - B(t), \quad X_{II}(t) = K_{II}(t) + B(t), \qquad \#$$

then

$$K_I(t) = K_{II}(t) = (X_I(t) + X_{II}(t))/2, \text{ and} \qquad \#$$

$$B(t) = (X_{II}(t) - X_I(t))/2 \qquad \#$$

Then the system is described as the following differential equations in terms of $X_I(t)$

and $X_{II}(t)$.

$$\mathbf{D}X_I(t) = f[(X_I(t) + X_{II}(t))/2] - r(t)[(X_{II}(t) - X_I(t))/2] - C_I(t), \quad \#$$

$$\mathbf{D}X_{II}(t) = f[(X_I(t) + X_{II}(t))/2] + r(t)[(X_{II}(t) - X_I(t))/2] - C_{II}(t), \quad \#$$

Then the problem is reduced to the simultaneous dynamic control problem over time to mazimize U_I and $U_{II.}$. This is a standard simultaneous optimization problem and the Euler equation becomes for the two country, denoting the (again assumed identical) elasticity of substitution of utility as θ.

$$\theta\frac{\mathbf{D}C_I(t)}{C_I(t)} = r(t) - f''(t)[(X_{II}(t) - X_I(t))/2] - \rho_I, \quad \#$$

$$\theta\frac{\mathbf{D}C_{II}(t)}{C_{II}(t)} = r(t) + f''(t)[(X_{II}(t) - X_I(t))/2] - \rho_{II}, \quad \#$$

where

$$f''(t) \equiv f''(K_I(t)) = f''(K_{II}(t)). \quad \#$$

This is the differential game formulation of the choice of path $X_I(t)$, given the initial wealth $X_I(0)$ and given the path of $X_{II}(t)$ combined by the choice of the path $X_{II}(t)$, given $X_{II}(0)$ and given the path of $X_I(t)$.

In terms of variables $K(t) \equiv [X_I(t) + X_{II}(t)]/2 = [K_I(t) + K_{II}(t)]/2$, $B(t) \equiv (X_{II}(t) - X_I(t))/2$, one can rewrite the system,

$$\mathbf{D}X_I(t) = f[K(t)] - r(t)B(t) - C_I(t), \quad \#$$

$$\mathbf{D}X_{II}(t) = f[K(t)] + r(t)B(t) - C_{II}(t), \quad \#$$

or

$$\mathbf{D}K(t) = f[K(t)] - [C_I(t) + C_{II}(t)]/2, \quad \#$$

$$\mathbf{D}B(t) = r(t)B(t) + [C_I(t) - C_{II}(t)]/2, \quad \#$$

combined with the Euler equations

$$\theta\frac{\mathbf{D}C_I(t)}{C_I(t)} = r(t) - f''(t)B(t) - \rho_I, \quad \#$$

$$\theta\frac{\mathbf{D}C_{II}(t)}{C_{II}(t)} = r(t) + f''(t)B(t) - \rho_{II}, \quad \#$$

The system of the last four equations can be studied for characterizing the stationary point of the optimal control problem. From the latter equations, by equating the left hand sides to zero, one obtains the rate of returns and the level of debt at the stationary point.

At the stationary point, the level of per-capita K is defined by the average of rate of time preference, by the addition of the two equations,

$$r \equiv f'(K) = (\rho_I + \rho_{II})/2, \quad \#$$

and the level of per capita debt B that country I owes to country II is determined by the difference between the rates of time preference, as obtained by the subtraction of two equations

$$B = -(\rho_I - \rho_{II})/(2f''(K)). \qquad \#$$

<Xu, as notations, I would like to put upper bar on the stationary K and B, in vain> The stationary values of consumption levels are given by

$$C_I = f(K) - rB = f(K) + (\rho_I^2 - \rho_{II}^2)/(4f''(K)) \qquad \#$$

$$C_{II} = f(K) + rB = f(K) - (\rho_I^2 - \rho_{II}^2)/(4f''(K)) \qquad \#$$

Since $f''(K)$ is negative, a country with more patient attitude towards future will end up a creditor country with a higher level of stationary consumption. A country with less patient attitude will end up as a debtor with a lower level of consumption.

<Xu, Please merge your manuscript of the system and phase diagram>

In order to have asimpler view on the paths of consumption, capital accumulation and international debl along the optimal growth behavior defined above, we follow Masanao Aoki and the formulation with the sum and difference of consumption levels such that

$$C(t) \equiv [C_I(t) + C_{II}(t)]/2, \quad V(t) \equiv [C_I(t) - C_{II}(t)]/2. \qquad \#$$

Heuristically, in the end the more patient country II (assuming $\rho_{II} < \rho_I$) is increasing its consumption and country II will reduce consumption. Does $B(t)$ approach zero? No it doesnot so that the chronical debt is not necessarily abnormal as long as the rates of time preference is different.

Suppose next $\rho_I = \rho_{II}$, but the government of country I always spend $G(t) = G$. The equation ( ) becomes

$$\mathbf{D}K_I(t) = F(K_I(t)) - r(t)B(t) + \mathbf{D}B(t) - G - C_I(t), \qquad \#$$

Along with this change, the growth path certainly changes. The stationary equilibrium relations remain unchanged, and the stationary value of B will be zero.. Therefore we can state the

Proposition: Unless rates of population growth, technological progress or the rates of time preference are different, there will be no stationary long run equilibrium with a non-zero per-capita debt in this Nash formulation of two country optimal growth.

Sketch of proof: Check the stationary forms of two Euler equations and take the dirrerence. Then B cannot be different from zero.

<Do you agree? How can we explain or demonstrate? more systematically?> <Any hint, examples, or calibration, interpretaions, or intutive criticism of the model will be welcome. I am presenting the analytical part at a lunch seminar at Tokyo University and Graduate Institute of Policy Studies. Expository statements in Introduction can be

added (by me) with little problems since the messages are very clear.

<here continues Xu's formulation with sums and differences — technique advocated by Masanao Aoki of UCLA. Your formulation is completely fine. Yet be reminded always that the model assume perfect capital market but that the choice of time path $x_i(t)$ is the Nash equilibrium reaction to the time path $x_j(t)$.

I appreciate that you resolved the identity question of alternative expression of stationary B, and I appreciate again if you could translate the machine language to the expository solution.>

Differential equations in Page I of my Note on K, B. (now see the above definition) C and V are almost right, but, as you mentioned, special attention should be given the quotient form by C. Because of that, and an important omission of B in the third equation (my mistake in the handwritten note.) the characteristic equation of Page II of the memo should read < I believe the derivatives with x (here C) in those equations cancel because of the fact that they are evaluated at the equilibrium. Note that the Jacobian of the differential equation is evaluated at the stationary point.>

Now your system (6), (7), (8) and (9) can be written as,by rearranging the variables

<Xu, please define $\alpha_1 \equiv C_1/(C_1 + C_2)$ and $\alpha_2 \equiv C_2/(C_1 + C_2)$and noting that $(\alpha_2 - \alpha_1) > 0$.

$$\begin{bmatrix} \mathbf{D}K(t) \\ \mathbf{D}C(t) \\ \mathbf{D}B(t) \\ \mathbf{D}V(t) \end{bmatrix} = \begin{bmatrix} f[K(t)] \; - C(t) \\ \{r[K(t)] \; + (\alpha_2 - \alpha_1)f''[K(t)]B(t) - (\alpha_1\rho_I + \alpha_2\rho_{II})\}C(t) \\ r[K(t)]B(t) + V(t) \\ \{-(\alpha_2 - \alpha_1)r[K(t)] \; + f''[K(t)]B(t) - (\alpha_1\rho_I - \alpha_2\rho_{II})\}C(t) \end{bmatrix}$$

This system consists of the fundamental differential equations describing the Nash behavior of this (open loop) differential game. (As already shown by Xu, if the value of B(t) is hypothetically given, then we can write a phase diagram for K(t) and C(t) that depicts the saddle point behavior; given the value of K(t) is hypothetically given, a phase diagram for B(t) and V(t) has also a saddle property.

The optimal nature of this game is guaranteed by the saddle point nature of the local property of this differential equations (Buiter, note in Econometrica, <Xu, check and send the article>

The Jacobian of the differential equations near the equilibrium point is obtained and

accordingly the characteristic equation of the system in terms of z is written as

$$\begin{vmatrix} r - z & -1 & 0 & 0 \\ [f' + (\alpha_2 - \alpha_1)f'' B]C & -z & (\alpha_2 - \alpha_1)f' C & 0 \\ f' \mathbf{B} & 0 & r - z & 1 \\ [-(\alpha_2 - \alpha_1)f' - f'' B]C & 0 & -f' C & -z \end{vmatrix} = 0 \qquad \#$$

By the Laplace expansion of the determinant, we can rewritten the determinant equation as

$$\begin{vmatrix} r-z & -1 \\ [f'+(\alpha_2-\alpha_1)f''B]C & -z \end{vmatrix}\begin{vmatrix} r-z & 1 \\ -f'C & -z \end{vmatrix}+\begin{vmatrix} -1 & 0 \\ -z & (\alpha_2-\alpha_1)f'C \end{vmatrix}\begin{vmatrix} f'\mathbf{B} & 1 \\ [-(\alpha_2-\alpha_1)f'-f''B]C & -z \end{vmatrix}=0. \qquad \#$$

and by Rouché’s theorem (<a theorem to predict what is going within a contour by knowing the property on the contour> See Alfors, Complex Analysis.), one can prove the saddle point property of the 4 equation system (See Buiter for the requirement for signs for optimality, Econometrica, please send it to my new mail address) by checking the comparison of two absolute values of the expressions with substitution of [absolute values are $\surd(a^2+b^2)$ for $a+bi$] when θ moves from zero to plus minus infinity (i is the imaginary unit). <Assume: (Assumption2)

$$f'+(\alpha_2-\alpha_1)f''B<0 \qquad \#$$

This assumption is quite easily justified unless there is a big gap in the rates of time preference.

Under this assumption, one can easily ascertain,

$$\begin{vmatrix} r & -1 \\ [f'+(\alpha_2-\alpha_1)f''B]C & 0 \end{vmatrix}<0,$$

and

$$\begin{vmatrix} r & 1 \\ -f'C & 0 \end{vmatrix}<0 \qquad \#$$

<*** TO WORK ON ROUCHE7S THEOREM.>

Define $\Phi(z)=[-(r-z)z+Cf'+(\alpha_2-\alpha_1)Cf''B][-(r-z)z+Cf'] \equiv \Phi_I(z)\Phi_{II}(z)$, and $\Psi(z)=C(\alpha_2-\alpha_1)f'[f''Bz-(\alpha_2-\alpha_1)Cf'-Cf''B] \equiv. \Psi_I(z)C(\alpha_2-\alpha_1)f'$, where

$\Psi_I(z) \equiv [f''Bz-(\alpha_2-\alpha_1)Cf'-Cf''B]$. Then,

The following Lemma ensures that, under condition ( ), the number of positive or negative real parts of the latent roots of equation ( ) coincides those of eqwuation ( ). Accordingly, by the Rouche’s theorem, the system will have the same saddle point property with $\Phi(z)=0$.

Lemma, On the imaginary axis $z=\theta i$, where $-\infty<\theta<\infty$, the absolute value of $\Phi(z)$ is always larger than that of $\Psi(z)$ if Assumption

$$f'+(\alpha_2-\alpha_1)f''B<0 \qquad \#$$

is satisfied..

Sketch of proof.

First, $|\Phi(z)| \equiv |\Phi_I(z)||\Phi_{II}(z)|$, $|\Psi(z)| \equiv |\Psi_I(z)||C(\alpha_2-\alpha_1)f'|$.

With substitution of z by θi,
$|\Phi_I(z)|^2 = (r\theta)^2 + [-\theta^2 + Cf'' + (\alpha_2 - \alpha_1)Cf'''B]^2, |\Phi_{II}(z)|^2 = (r\theta)^2 + [-\theta^2 + Cf'']^2$

$|\Psi_I(z)|^2 = (f''B\theta)^2 + [(\alpha_2 - \alpha_1)Cf'' + Cf'''B]^2.$

$|\Phi_I(z)|^2 - |\Psi_I(z)|^2 = [r^2 - (f''B)^2]\theta^2 + \theta^4 + C^2[1 - (\alpha_2 - \alpha_1)^2][(f'')^2 - (f'''B)^2] + 2\theta^2C^2[f'' + (\alpha_2 -$
long as the assumption is satisfied. The first term is positive without further assumptions because of the relationship between B and r (difference and sum of $\theta' s$) already mentioned. It is easy to see $|\Phi_{II}(z)|^2 > [C(\alpha_2 - \alpha_1)f'']^2$. Roughly QED because the outer edge of the disk (counter) always the fourth degree polynominal dominates a single degree in absolute value.)

Assume: (Assumption2)

$$f'' + (\alpha_2 - \alpha_1)f'''B < 0 \qquad \#$$

This assumption is quite easily justified unless there is aextreme gap between the rates of time preference.

Under this assumption, one can easily ascertain,

$$\begin{vmatrix} r & -1 \\ [f'' + (\alpha_2 - \alpha_1)f'''B]C & 0 \end{vmatrix} < 0, \qquad \#$$

and

$$\begin{vmatrix} r & 1 \\ -f''C & 0 \end{vmatrix} < 0 \qquad \#$$

Thus, both sub systems have a positive and a negative latent root. Figuratively speaking, the first two equations with a constant B, or the two equations with a constant K will both have saddle points in their trancated systems. <Please merge the equaations and the phase diagrams to this manuscript you wrote>

Since this proof goes through, the saddle point (local) properties are fulfilled.

Mini conclusion: Choronical debt is a norm rather than exception if and only if the rates of time preference are different.

Other (labor growth, technological growth) rates can be different, but the stationary state is rather odd. We need nonlinear simulation.

<Remaining problems

What happens if rate of technical progress or birth rate diverge for a limited time but then return to the normal level? (More difficult, if rate of time preference returns to normal after consumption or income level approach a certain level. One <not me!> could simulate these transient paths by connecting phase diagrams. This will create a series of important exercises of calibrating capital movement paths. (I can hire students and friends to do the detail. You are more important to analyze the theoretical implications.)

One can connect, by at least taking the assumption of one traded good, this to the trade model with non-traded goods. Can the manipulation of real exchange rate achieve a state coming from the different time preference? This relate to Obstfeld Rogoff's calibration with Japans Balance of Payments.

More generally, can we say something about the solvency issues, or the debt criterion issues in the develpoing countries? HIPPIC issues when to allow a country to be excused of its debt, Current criteria lack sufficient dynamic (micro) foundation. It could be an interesting as an econometric problem. Observe a path of debt and growth of a poor country and tell in a dynamic sense that it is insolvent. Can it be too difficult or too trivial for a chapter in your thesis?

When a country is in debt, are they violating transversality conditions or too myopic? Can the debt relief revive a country as tonic (campfer) to increase the saving incentives to make it return on a growth path from stagnant path? This leads to the connection to the behavioral economics.>

My immediate e-mail number is <koichihamada@grips.ac.jp>. This yale^edu mail is also working.

Call me 81-3-5815-8585 from Louise or Kathy's desk at either 9.15 or 4.45, if you have any problems.