Mittag-Leffler Distributions and Long-run Behavior of Macro-economic models

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Basic Setup

- As time progresses, a stream of innovations hits a model composed of sectors or clusters.
- An innovation either hits sector i of size n_i with rate $(n_i - \alpha)/(n + \theta)$, $0<\alpha<1$, $\theta > 0$,or create a new sector of size 1 with rate
- $1-\sum (n_i - \alpha)/(n+\theta)=(\theta + k\alpha)/(n+\theta)$, where
- $n=\sum n_i$, and k is the number of sectors existing at that time.

Basic Facts of M-L function

- Definition: $E_\alpha(x)=\sum_n^\infty x^n / \Gamma(1+n\alpha)$, $\alpha>0$.(H.Pollard)
- $E_\alpha(-x)=\int_0^\infty e^{-xt}dG_\alpha(t)$, with distribution function G_α. From the moment generating function we see:
- Moment condition:
- $\int_0^\infty x^p g_\alpha(x)\,dx = \Gamma(p+1) / \Gamma(\alpha p+1)$, $p=1, 2, \ldots$
- where
- Moment generating function
- $\int_0^\infty e^{xt} g_\alpha(t)dt = E_\alpha(t)$, and
- Laplace transform: $\int_0^\infty e^{-xt} g_\alpha(t)dt = \sum_n (-x)^n / \Gamma(1+n\alpha)$.

Remarks

- Mittag-Leffler function is a generalization of the exponential function, E_1 (t)=e^{-t}.
- It is uniquely determined by the moments.
- (it satisfies Careleman's condition:
- $\sum_n m_n^{-1/n} = \infty$.) (Feller vol.2, p. 224).
- Its extenstion: $E_{\alpha, \beta} = \Gamma(\beta\} \sum_n (-t)^n / \Gamma(\alpha n + \beta)$.

Some questions and answers

Question:

How are the M-L functions useful in long-run analysis of macro-models?

Answers:

They are generic $\cdots$ they generically characterize long-run behavior; tail of M-L distributions are power laws.

Examples of long-run clusters of heterogeneous collections of agents:

such as Pitman's chinese restaurant processes, grow patterns of sectors of economies (Markov branching processes, such as

Feng-Hoppe analysis of branching model), and

Extension of the one parameter Poisson-Dirichlet (Ewens) model to two-parameter version by J. Pitman

Long-run behavior of both classes of models have M-L distributions

Some Facts and Applications

- Mittag-Leffler distributions are uniquely determined by their moments.$\cdots$ Method of moments applies:
- g_α has $\Gamma(p+1)/\ \Gamma(\alpha\ p+1)$ as its p-th moment, $0<\alpha<1$
- Fractional master equations $\cdots$ mean-first passage times, waiting distributions in finance, and possibly others.

Two-parameter Extensions

- $g_{\alpha, \theta}(x) = B\, x^{\theta/\alpha}\, g_{\alpha}(x)$
- where

$B = \Gamma(\theta + 1)/\Gamma(\theta/\alpha + 1)$

It is known that as $n \to \infty$

$E(K_n/n^{\alpha}) \to \theta\, \Gamma(\theta)/\, [\alpha\, \Gamma\, (\theta + \alpha)]$.

This is the same as the mean of $Bx^{\theta/\alpha} g_{\alpha}(x)$.

Two-parameter Poisson-Dirichlet distribution (extension of the Ewens distribution)

- Let K_n denote the number of clusters formed by n agents.
- Suppose
- $P(K_{n+1}=k|K_n =k)=(n-k\alpha)/(n+\theta)$
- $P(K_{n+1}=k+1|K_n=k)=(\theta +k\alpha)/(n+\theta)$
- Then
- K_n/n^α converges a.s. to M-L distrbiution

Feng-Hoppe Model

- A simple branching process due to Karlin and McGregor: Let I(t) be a stream of new types of agents (resources) arriving stochastically.
- Arrival rate $=\theta + k\,\alpha$, where $k=|I(t)|$, $\theta=\beta-\alpha$
- Each new arrival (innovation) starts its own group that grow stochastically.
- Let N(t) be the total size of the economy
- Its growth rate $=\alpha\,(k-1)+\beta+\sum_{j=1}^{k}(n_j-\alpha)=n+\theta$ where the i-th arrival (innovation) grows at rate $n_i-\alpha$.

- I(t)/N(t) converges to a ratio of two dependent Gamma random variables, which is M-L distributed

Analysis of Long-run Behavior: Simple Cases

- Let F(s) be the Laplace transforms of some f(t).
- In control or system theory, the final value theorem says
- $\mathrm{Lim}_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$.
- Tauberian theorm of Karamata slowly varying function

Darling-Kac Theorem

- Under a set of conditions, Mittag-Leffler distributions are the only possible limit laws. (Regular Variations, Bingham, Goldie, Teugels, CUP 1998, pp.388)

Some Asymptotic Differences in one- and two-parameter Ewens models

- K_n / n^α in the two-parameter Ewens model is not self-averaging.

Question

- Does simulations of models with non-self averaging properties yield estimate of α and θ ?
- Moments of M-L distributions given earlier can be used for that purpose?