Rational Expectations and the Stability of Balanced Monetary Development*

by

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Abstract

The expanding/contracting behavior of monetary economies under perfect foresight or rational expectations is primarily driven by government deficits. Their effects on inflation and monetary growth determine the real value of money which guarantees stationary positive levels of output and employment in the long run only if the stationary real value of money is *positive*. For a class of *nonlinear monetary macroeconomic models* of the AS–AD type derived from a microeconomic structure with OLG consumers, it is shown that such economies generically have no stationary equilibria under perfect foresight/rational expectations when tax revenue is income dependent (no lump sum taxes) and government consumption is autonomous.
Proportional deficit rules induce constant proportional monetary growth. However, in such cases, all positive orbits with balanced monetary expansion and rational expectations are unstable.

When government consumption is autonomous, nonrandom, and not too large, there typically exist two positive balanced equilibria under perfect foresight, a stable and an unstable one, both with constant positive real money balances, plus a stable nonmonetary one under hyperinflation (or a monetary bubble). The results of the paper imply that these properties are true for a large class of deterministic AS-AD models.

Under (Hicks neutral) stochastic production shocks, such economies may have positive stable balanced stationary equilibria under rational expectations dynamics if the government policy has a small strictly positive *nonrandom autonomous* demand component in all cases of uncertainty. There exists a stochastic monetary trap inducing a threshold dividing the possible long run behavior into two mutually exclusive regimes of excessive money creation, outgrowing prices and price expectations, with diverging allocations *or* positive balanced monetary expansion which may be converging or diverging depending on parameters and the perturbations.
The results are derived using techniques from the theory of random dynamical systems which allows a complete theoretical and numerical analysis of the *dynamics of random expanding time series and their stability* of the *nonlinear stochastic model.*

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Contents

1 Introduction

Permanent government deficits in closed monetary economies induce expanding quantities of money or government debt. While the necessary conditions for stability and sustainability of such expanding orbits of temporary equilibria seem to be well understood, sufficient conditions are often not considered or examined. Sustainability here means that the nominally expanding orbits of prices, money, and/or debt (tending to infinity) support positive stationary real allocations associated with monetary equilibria. Specifically,

in monetary economies

- stationary policies induce endogenous nominal government deficits
- which vary in size and sign at each temporary equilibrium

$\Longrightarrow$ making monetary expansion/contraction endogenous

$\Longrightarrow$ that the quantity of fiat money held by consumers as savings in each period is endogenous

$\Longrightarrow$ inducing explicit endogenous dynamics of the quantity of money.

Therefore, the issue becomes

what are the consequences of such permanent and expanding deficits in the long run?

- What are the possibilities for a stable monetary expansion?
- Under rational expectations and perfect competition?
- Which stationary fiscal/monetary policies guarantee are sustainable to guarantee positive stationary real allocations?
- Are there *threshold levels* of policy parameters[1] destroying sustainability.

To investigate these question in a properly defined monetary macroeconomic model one needs to have a rich enough model for which the sequential development of

- general temporary equilibrium of closed monetary systems
- with essential heterogeneity of agents
- with fiscal/monetary policy

can be described.

History and Literature: of sustainability

- The monetarist tradition: Monetary Rules,
  "The optimal quantity of money" (for example Friedman, 1968, 1969) seem to take the existence and the stability (sustainability) issue for granted;
- some classical papers in the linear tradition of rational expectations (using the notion from Muth, 1961) rarely discuss sustainability;
- Lucas (1972) on neutrality of money, nonlinear but nothing on sustainability or stability;

[1]as in the debate between Reinhart-Rogoff and Krugman; see http://www.bloomberg.com/news/2013-05-28/krugman-feud-with-reinhart-rogoff-escalates-as-austerity-debated.html

- Blanchard (1979), Blanchard & Kahn (1980) and others raise some doubts about stability in linear environments;
- few contributions discuss link between real sector and money;
- few contributions use a nonlinear time series approach.

2 The AS–AD Model

Consider a version of the AS–AD Model based on micro foundations (as in Böhm, 2010)

– with heterogeneous OLG consumers, endogenous labor supply

– with government demand/consumption and a proportional income tax

to find a general structural answer to the stability question

– for nonlinear dynamic macroeconomic models

– in deterministic and stochastic environments

to examine the conditions whether

– stable balanced orbits exist under rational expectations

– supporting positive stationary real allocations in the long run.

It is shown that

there exist two fiscal policies

– one is always unstable and one sometimes stable

– there exist critical thresholds of government policies beyond which sustainability no longer holds

Basics of The Model

Aggregate Supply is defined as an *equilibrium consistent* description of the labor market under technological conditions, i.e.

$$\underbrace{n_f h\left(\frac{w}{p}\right)}_{\text{labor demand}} \overset{!}{=} \underbrace{N\left(\frac{w}{p} V\left(\frac{p^e}{p}\right)\right)}_{\text{labor supply}} \qquad \Longrightarrow \qquad y^S = AS\left(\frac{p^e}{p}\right)$$

with n_f identical competitive firms with smooth concave production functions satisfying Inada conditions and globally invertible labor demand function $h(w/p)$. N is the aggregate labor supply function of a finite number of worker-consumers with two period lives, homothetic intertemporal preferences in consumption and time separable utility function with respect to disutility of labor (see Blanchard & Fischer, 1989; Böhm, 2010). This implies a strictly decreasing indirect utility of income $V(p^e/p)$ as a function of the expected inflation rate p^e/p and a labor supply function N of the given form. Under these assumptions aggregate supply AS is a strictly decreasing function of the expected rate of inflation.

Aggregate Demand is taken as the *income consistent* description of the equivalence of total income and aggregate commodity expenditure, i.e.

$$Y = py = pg + M + c_w \text{Net Wages} + c_\pi \text{Net Profits}$$

$$\Longrightarrow \quad y^D = AD\left(\frac{M}{p}\right) = \frac{M/p + g}{1 - c_w(1-\tau_w)\frac{\text{Wages}}{py} - c_\pi(1-\tau_\pi)\frac{\text{Profits}}{py}}.$$

Temporary Equilibrium is given by a price level $p > 0$ which solves

$$AD(M/p) \stackrel{!}{=} AS(p^e/p) \quad \Longrightarrow \quad p = \mathcal{P}(M, p^e) \quad \text{Price Law}.$$

As a consequence the equilibrium price level is given by a mapping $p = \mathcal{P}(M, p^e)$ called the price law which is homogeneous of degree one.

3 Fiscal Deficit Rules and Monetary Expansion

Consider first the case where, in each period government spending differs from tax revenues by a constant fraction $\rho > -1$. For simplicity it is assumed that consumers are identical in their net consumption propensities, i.e. $c = c_w = c_\pi$ and $\tau = \tau_w = \tau_\pi$, so that the income distribution among the consumers has no influence on aggregate demand. Then, government spending in every period satisfies the budget equation

$$pg \stackrel{!}{=} (1+\rho)\tau py, \quad \rho > -1 \tag{3.1}$$

for the proportional tax rate $0 < \tau < 1$ on overall income. As a consequence, this rule implies that government real consumption $g = (1+\rho)\tau y$ is a constant fraction $(1+\rho)\tau$ of output, defining a constant fiscal share $g/y = \tau(1+\rho)$. Nonnegativity of aggregate income imposes an upper bound on ρ given by

$$\rho < \rho_{\max} := \frac{1-\tau}{\tau}(1-c).$$

The deficit rule and aggregate income consistency yield an aggregate demand function

$$y = AD(M/p) := \frac{M/p}{(1-c)(1-\tau) - \tau\rho} \tag{3.2}$$

which is unit elastic in the price level. Together with aggregate supply $y^s = AS(p^e/p)$, an equilibrium price level $p > 0$ solves

$$AS\left(\frac{p^e}{p}\right) \stackrel{!}{=} AD\left(\frac{M}{p}\right).$$

If AS is globally invertible, p is unique and globally defined for all $(M, p^e) \gg 0$. Thus, the price law $p = \mathcal{P}(M, p^e)$ is globally defined. Given the assumptions on aggregate demand $\mathcal{P}$ is strictly increasing, homogeneous of degree one, and concave in (M, p^e).

3.1 Perfect Foresight Dynamics

Perfect foresight in prices prevails if $p^e_{t,t+1} \equiv p^e = p_1 \equiv p_{t+1}$. Therefore, the price dynamics with perfect foresight are defined by the inverse of the price law, given explicitly by

$$p_1 = \mathcal{P}^e(M,p) := pAS^{-1}\left(\frac{1}{(1-c)(1-\tau)-\tau\rho}\frac{M}{p}\right). \tag{3.3}$$

One finds that $\mathcal{P}^e(M,p)$ is strictly increasing, homogeneous of degree one in (M,p), satisfying the so-called convex Inada conditions in p, i.e. $\lim_{p\to 0}\mathcal{P}^e(M,p)/p = 0$ and $\lim_{p\to\infty}\mathcal{P}^e(M,p)/p = \infty$.

The dynamics of money balances are given by

$$M_1 = M + pg - \tau py = M + \rho\tau py = M\left(\frac{\tau\rho}{(1-c)(1-\tau)-\tau\rho} + 1\right) \tag{3.4}$$

which implies a constant growth rate of money equal to

$$\rho_M := \frac{M_1}{M} - 1 = \frac{\tau\rho}{(1-c)(1-\tau)-\tau\rho}$$

which is independent of prices, incomes, and output. This confirms that the proportional deficit rule ρ of a government is equivalent to a constant monetary policy ρ_M in the sense of Friedman with $\rho_M \neq 0$ if and only if $\rho \neq 0$.

Therefore, the two equations (3.3) and (3.4) induce a **two-dimensional dynamical system** under perfect foresight

$$\begin{aligned} M_1 &= M + pg - \tau py = \frac{(1-c)(1-\tau)}{(1-c)(1-\tau)-\tau\rho}M \\ p_1 &= \mathcal{P}^e(M,p) := pAS^{-1}\left(\frac{1}{(1-c)(1-\tau)-\tau\rho}\frac{M}{p}\right). \end{aligned} \tag{3.5}$$

This system is homogeneous of degree one which has no positive fixed points for $\rho \neq 0$ since the point $(p,M) = (0,0)$ is the only fixed point. As a consequence, balanced orbits/paths of monetary growth and of inflation are the objects which characterize the behavior of the economy in the long run.

Definition 3.1 *An orbit* $\{(p_t, M_t)\}_{t=0}^{\infty}$ *is called balanced if and only if there exists* $\rho_M > -1$ *such that*

$$M_{t+1} = (1+\rho_M)M_t \quad \text{and} \quad p_{t+1} = (1+\rho_M)p_t.$$

Balanced orbits induce constant allocations. To discuss their existence and stability, one considers the dynamics in intensive form. Here it is useful to define $\tilde{p} := (p/M)$ (the inverse of real money balances!) and consider the one-dimensional dynamical system in $\tilde{p}$ defined by the time-one map

$$\tilde{p}_1 := \frac{p_1}{M_1} = \frac{1}{1+\rho_M}\mathcal{P}^e(1, p/M) = \frac{1}{1+\rho_M}\mathcal{P}^e(1,\tilde{p}). \tag{3.6}$$

This reduced time-one map inherits all the properties of $\mathcal{P}^e$, namely strict monotonicity plus the so called convex weak Inada conditions[2]. Therefore, (3.6) is a strictly convex map which has exactly one positive fixed point $\bar{p} > 0$, plus two further fixed points $p = 0$ and $p \to \infty$. Since the inverse of the price law $\mathcal{P}^e$ satisfies

$$\mathcal{P}^e(0, M) = 0 \qquad \forall M > 0,$$

there exists a unique positive fixed point $\bar{p}$ which defines a unique positive level $\bar{m} \equiv 1/\bar{p} > 0$ of real money balances. However, zero is a globally stable fixed point while $\bar{p}$ is unstable due to convexity.

Only positive fixed points of (3.6) define positive balanced paths of (3.5), therefore no orbit $\{(p_t, M_t)\}$ of (3.5) with $p_0/M_0 \neq \bar{p}$ converges to the positive balanced path $\bar{p} > 0$.

In summary,

– only three types of long run behavior under a constant deficit rule $\rho > 0$ with perfect foresight are possible, whose characteristics depend on the initial condition.

(1) If $p_0/M_0 = \bar{p}$, balanced monetary growth is possible.

– The real allocation in the economy remains constant over time

– while prices and the quantity of money grow at the same rate ρ_M.

However, this is a zero probability event!

(2) If $p_0/M_0 > \bar{p}$,
prices grow at a faster rate than the quantity of money, which implies $\lim_{t\to\infty} \frac{M_t}{p_t} = 0$,

– all trades, output, and employment tend to zero since $\lim_{t\to\infty} y_t = \lim_{t\to\infty} AD(M_t/p_t) = 0$,

– i.e. in the limit money has no value and all trades converge to zero.

(3) If $p_0/M_0 < \bar{p}$,
money balances grow at a faster rate than prices implying that real money balances grow beyond bound.

– the model predicts perpetual growth of output and employment, with $\lim_{t\to\infty} y_t = \infty$ as one possibility if there are no capacity restrictions in production or labor supply.

– If the production function is bounded above or if labor supply is bounded, the aggregate supply function becomes bounded above as well and output grows to its upper bound.

However, all orbits are not balanced, output and employment are not stationary.

– These results extend to situations under rational expectations if the government follows a random stationary deficit rule $\rho(\omega)$ even in cases when the deficit equals zero on average, i.e. when for example $\mathbb{E}\rho(\omega) = 0$ holds (see Chapter 6 in Böhm, 2010).

3.2 Random Deficits and Monetary Expansion

– Instead of a fixed deterministic fiscal rule, a government may consider a

random deficit rule defined by a stochastic process $\{\rho_t\}$, $\rho_t : \Omega \to (-1, \rho_{max})$,

[2] A mapping $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to satisfy the convex weak Inada conditions if $f(x)/x$ is strictly increasing with $\lim_{x\to 0} f(x)/x = 0$ and $\lim_{x\to +\infty} f(x)/x = +\infty$. This is the convex analogue to the concave weak Inada conditions which are often useful in growth theory.

which implies a random monetary growth rate

$$1+\rho_M(\omega) := \frac{(1-c)(1-\tau)}{(1-c)(1-\tau)-\tau\rho(\omega)}. \tag{3.7}$$

For example

- $\{\rho_t\}$ could be i. i. d. with zero mean on an interval $\Sigma := [-\epsilon, \epsilon] \subset (-1, \rho_{\max})$
- with a measure $\lambda_t = \lambda$ with $\lambda_t \in Prob(\Sigma)$, the set of probability measures on Σ.
- implying a balanced budget and zero money growth in expectations

$\Longrightarrow$ stochastic aggregate demand function

$$AD(M/p, \rho(\omega)) = \frac{M/p}{(1-c)(1-\tau)-\tau\rho_M(\omega)}$$

– which is unit elastic in the current price with a random multiplier.

$\Longrightarrow$ Market clearing $AS = AD$ implies a random price law $\mathcal{P}(M, p^e, \rho(\omega))$

– which is monotonically increasing in ρ, which

– preserves all the properties of the price law with a deterministic policy rule for every (M, p^e, ρ)

- homogeneity, monotonicity, concavity, and the Inada conditions in p^e
- with explicit random inverse

$$p^e = pAS^{-1}\left(\frac{M/p}{(1-c)(1-\tau)-\tau\rho_M(\omega)}\right) \tag{3.8}$$

- satisfying homogeneity, monotonicity, and the convex Inada conditions in p

3.3 Random Deficits with Rational Expectations

– Let $p^e_t \equiv p^e_{t,t+1}$ denote the conditional mean forecast in t for prices in $t+1$

– while ρ_t has distribution $\lambda_t \in Prob(\Sigma)$, the set of probability measures on $\Sigma \subset (-1, \rho_{\max})$.

– every mean prediction p^e_t induces a prediction error (a deviation from the mean)

$$\mathrm{err}(M_t, p^e_t, \rho_t, p^e_{t-1}) := p_t - p^e_{t-1} = \mathcal{P}(M_t, p^e_t, \rho_t) - p^e_{t-1} \tag{3.9}$$

which is a random variable.

– A mean prediction[3] p^e_t is called unbiased, or equivalently,

rational expectations prevail, if the mean of the prediction error is equal to zero, i.e.

[3] Following the same reasoning and methodology as in the deterministic case, a predictor ψ is a mapping taking past and current data (prior to the realization of the production shock) to make a point prediction for the mean of the future price level conditional on the information available in this period (see Böhm & Wenzelburger (2002, 2004) for details).

– if $\mathbb{E}_{\lambda_t}\{\mathrm{err}(M_t, p^e_t, \rho_t, p^e_{t-1})\} = 0$ taken with respect to the true measure λ_t of the budget shock, or equivalently if

$$(\mathbb{E}\mathcal{P})(M_t, p^e_t, \lambda_t) := \mathbb{E}_{\lambda_t}\{\mathcal{P}(M_t, p^e_t, \rho)\} := \int \mathcal{P}(M_t, p^e_t, \rho)\lambda_t(d\rho) \stackrel{!}{=} p^e_{t-1}.$$

Definition 3.2 *A mean value predictor $\psi^* : \mathbb{R}^2_+ \times Prob(\Sigma) \to \mathbb{R}$ is called unbiased if it solves*

$$(\mathbb{E}\mathcal{P})(M, \psi^*(M, p^e_{-1}, \lambda), \lambda) = p^e_{-1} \tag{3.10}$$

for all $(M, p^e_{-1}, \lambda) \in \mathbb{R}^2_+ \times Prob(\Sigma)$

An unbiased predictor is a time invariant mapping on a fixed information set $\mathbb{R}^2_+ \times Prob(\Sigma)$.

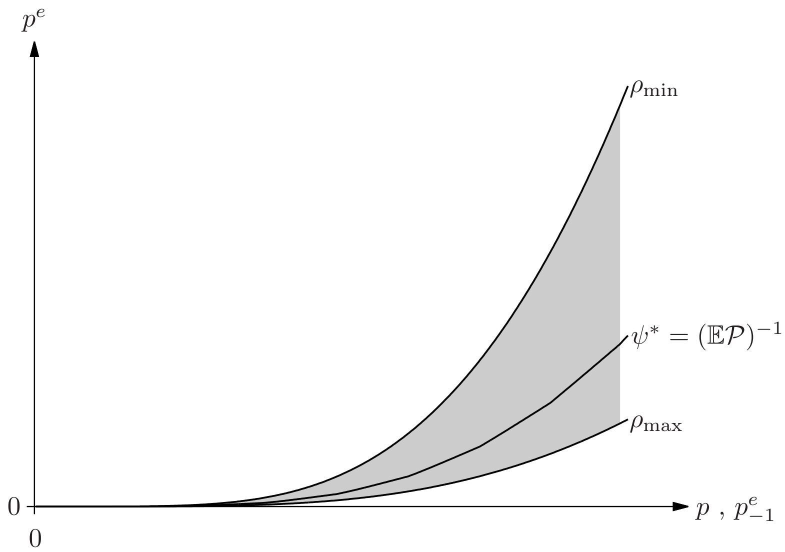


Figure 3.1: Existence of an unbiased predictor $p^e = \psi^*(M, p^e_{-1}, \lambda) := (\mathbb{E}\mathcal{P})^{-1}(M, p^e_{-1}, \lambda)$

Lemma 3.1
There exists a unique globally defined unbiased predictor $\psi^* : \mathbb{R}^2_+ \times Prob(\Sigma) \to \mathbb{R}_+$ given by

$$\psi^*(M, p^e_{-1}, \lambda) := (\mathbb{E}\mathcal{P})^{-1}(M, p^e_{-1}, \lambda) \tag{3.11}$$

which is homogeneous of degree one in (M, p^e_{-1}), and which satisfies

$$\lim_{p^e_{-1} \to 0} \frac{\psi^*(M, p^e_{-1}, \lambda)}{p^e_{-1}} = 0 \qquad \lim_{p^e_{-1} \to \infty} \frac{\psi^*(M, p^e_{-1}, \lambda)}{p^e_{-1}} = +\infty. \tag{3.12}$$

Therefore, rational expectations dynamics are governed by

– two homogeneous stochastic difference equations

$$\begin{aligned} p^e &= \psi^*((1+\rho_M(\omega))M_{-1}, p^e_{-1}, \lambda) = M_{-1}\psi^*\left(1+\rho_M(\omega), \frac{p^e_{-1}}{M_{-1}}, \lambda\right) \\ M &= (1+\rho_M(\omega))M_{-1}, \end{aligned} \tag{3.13}$$

– where the random growth rate of money $\rho_M(\omega)$ enters into both equations.

– The unbiased predictor ψ^* is a deterministic function[4]

[4]assuming from now on an i.i.d. fiscal policy with constant λ

– (3.13) induces the one-dimensional system in intensity form

$$q^e := \frac{p^e}{M} = \frac{1}{1+\rho_M(\omega)}\,\psi^*\left(1+\rho_M(\omega), q^e_{-1}\right). \tag{3.14}$$

– which has two random fixed points, a positive one which is asymptotically unstable and

– zero which is asymptotically stable with non-degenerate positive basin of attraction.

Thus, there exist positive stochastic balanced orbits

– associated with the unstable random fixed point,

– which are empirically unobservable.

Therefore, balanced expansion with

– stationary positive levels of employment, output, and real money balances

– is a zero probability event under the random budget policy.

4 Stochastic Production and Rational Expectations

Consider the same economy as in Section 3, but now with a multiplicative Hicks-neutral productivity shock Z of the form

$$y = G(Z,L) := ZF(L), \qquad Z \geq 1.$$

With competitive labor demand by producers, the associated random aggregate supply functionis of the special form

$$y^S = ZAS\left(\frac{p^e}{pZ}\right).$$

4.1 Deterministic Deficit Policy and Rational Expectations

For a given deterministic deficit rule ρ and mean prediction p^e

Temporary equilibrium for each (M,p^e,Z) is defined by a price $p > 0$ solving

$$\frac{M/p}{(1-c)(1-\tau)-\tau\rho_M} = ZAS\left(\frac{p^e}{pZ}\right) \tag{4.1}$$

implying a

Random price law $\mathcal{P}(M,p^e,Z)$ which is homogeneous in (M,p^e) and strictly concave in p^e and

– whose inverse with respect to expectations preserves the main properties of the deterministic case (3.8) for each (M,Z),

$\Longrightarrow$ implying a concave mean price law $\mathbb{E}\mathcal{P}(M,p^e)$ with global inverse

– defining an unbiased predictor $\psi^* = (\mathbb{E}\mathcal{P}(M,p^e))^{-1}$ which is strictly convex in p^e.

$\Longrightarrow$ Rational expectations dynamics are given by

$$
\begin{aligned}
p^e &= \psi^*((1+\rho_M)M_{-1}, p^e_{-1}, \mu) = M_{-1}\psi^*\left(1+\rho_M, \frac{p^e_{-1}}{M_{-1}}, \mu\right) \\
M &= (1+\rho_M)M_{-1}.
\end{aligned}
\tag{4.2}
$$

– If the production shocks are i.i.d., the system is *deterministic* being driven exclusively by the predictor, as in the case without production shocks.

– In more general situations, with Markovian production shocks, expectations updating and money balances will be random variables.

– Nevertheless, expectations and money balances induce stochastic orbits of random prices, wages, output, and employment in all cases of a non-degenerate shock.

In summary:

under i.i.d. noise

- Rational expectations dynamics are deterministic
- with a unique unstable positive fixed point.

under Markovian noise

- Rational expectations dynamics are stochastic
- with one unstable positive random fixed point.

Therefore, as in the previous cases, balanced expansion with

– stationary positive levels of employment, output, and real money balances

– is a zero probability event under fixed budget policies.

4.2 Fiscal Policy with Autonomous Government Demand

Given the negative results on the possibility of stable and observable rational expectations dynamics under fixed budget rules of the previous sections, this section examines the possibilities of sustainability of positive balanced expansion under rational expansion with random production when government demand is autonomous and exogenously given. Let the stationary fiscal policy consist of a choice of the two parameters (g, τ) without a restriction for the budget to be followed in each period. Therefore, provided government demand is not too large and tax rates are appropriately set allowing feasible equilibrium outcomes, the impact of such policies will appear through two channels, i.e. first, through an effect on the price level and on output via the government multiplier in each temporary equilibrium and second, on changes in money holdings through the government deficit. Both effects operate simultaneously and interactively for any given level of policy parameters (g, τ) in each period.

The exogenous government demand g and the income tax rate τ imply an aggregate demand function of the form

$$
y^D = AD\left(\frac{M}{p}, g\right) := \frac{M/p + g}{\tilde{c}} = \frac{M/p + g}{1 - c(1-\tau)} \tag{4.3}
$$

where $\tilde{c} := 1 - c(1-\tau)$ is the demand multiplier. Then, temporary equilibrium is defined by a price level $p > 0$ solving

$$ZAS\left(\frac{p^e}{pZ}\right) \overset{!}{=} AD\left(\frac{M}{p}, g\right)$$

inducing the random price law

$$p = \mathcal{P}(M, p^e, Z).$$

Figure 4.1 portrays the impact of the random productivity on aggregate supply and on the

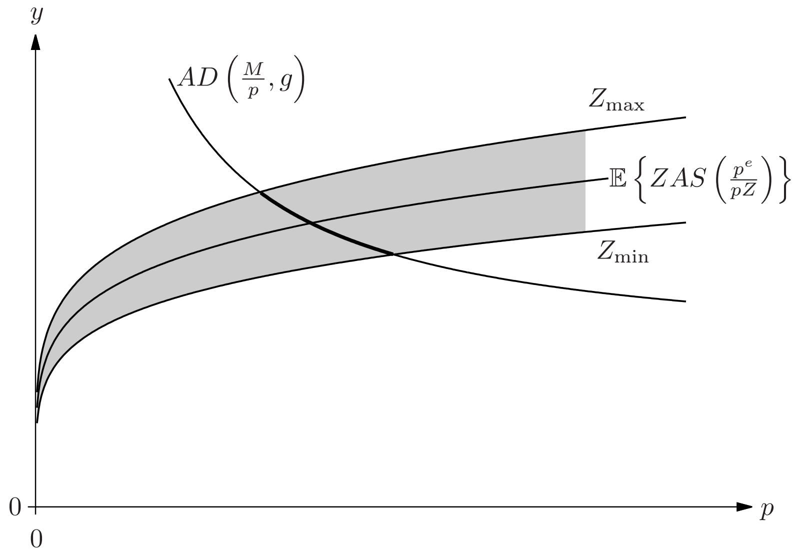


Figure 4.1: The Role of Random Productivity on Prices and Output

equilibrium price. Since a positive productivity shock implies an upward shift of the aggregate supply function AS, one obtains the random equilibrium configurations as displayed in Figure 4.1

4.3 Rational Expectations Dynamics

– Let $p^e_t \equiv p^e_{t,t+1}$ denote the predicted mean price in t for prices in $t+1$ (or the mean-value prediction) made by consumers,

– the production shocks are distributed on $\Sigma := [Z_{\min}, Z_{\max}] \subset \mathbb{R}_{++}$

– by a measure $\mu_t \in Prob(\Sigma)$, the set of probability measures on Σ

– a mean-value prediction p^e_t in period t is called unbiased[5], i.e. "rational expectations prevail",

– if the prediction error[6] defined as

$$\text{err}(M_t, p^e_t, Z_t, p^e_{t-1}) := p_t - p^e_{t-1} = \mathcal{P}(M_t, p^e_t, Z_t) - p^e_{t-1}$$

– has zero mean taken with respect to the true measure μ_t,

– implying equality of the mean prediction with the next mean price, i.e.

[5] proceed as in Section 3.3, Böhm & Wenzelburger (2002)

[6] or deviation of the price p_t from the mean prediction p^e_{t-1}, a random variable of period t

$$(\mathbb{E}\mathcal{P})(M_t, p^e_t, \mu_t) := \mathbb{E}_{\mu_t}\{\mathcal{P}(M_t, p^e_t, Z_t)\} := \int \mathcal{P}(M_t, p^e_t, Z)\mu_t(dZ) \overset{!}{=} p^e_{t-1}. \tag{4.4}$$

Definition 4.1 *A mean value predictor $\psi^* : \mathbb{R}^2_+ \times Prob(\Sigma) \to \mathbb{R}$ is called unbiased if it solves*

$$\mathbb{E}_{\mu_t}\mathcal{P}(M_t, \psi^*(M_t, p^e_{t-1}, \mu_t), Z_t) = p^e_{t-1} \tag{4.5}$$

for all $(M_t, p^e_{t-1}, \mu_t) \in \mathbb{R}^2 \times Prob(\Sigma)$, where $Prob(\Sigma)$ is the set of probability measures on Σ.

Observe,

– for any random price law $\mathcal{P}$ the mean error function is also a time invariant mapping. Its zero contour is a time invariant subset of $\mathbb{R}^2_+ \times Prob(\Sigma)$ and defines the set of all unbiased predictions.

Thus, it is appropriate to define an unbiased predictor also as a time invariant mapping on the same fixed information set $\psi^* : \mathbb{R}^2_+ \times Prob(\Sigma) \to \mathbb{R}_+$, satisfying $p^e = \psi^*(M, p^e_{-1}, \mu)$.

Therefore, unbiased mean predictors are the appropriate forecasting rules in every period, which, by construction, induce rational expectations along random orbits in the usual sense.

Finally, ψ^* is unbiased if it solves $(\mathbb{E}\mathcal{P})(M, \psi^*(M, p^e_{-1}, \mu), \mu) = p^e_{-1}$, for every (M, p^e_{-1}, μ).

In other words, an unbiased predictor must be an inverse of the mean price law with respect to the previous expected price.

– Analogous to Lemma 3.1 one has

Lemma 4.1 *Let the random price law $\mathcal{P}$ be continuous and globally invertible with respect to p^e for every $(M, Z) \gg 0$ and let $\mu \in Prob(\Sigma)$ denote the distribution of the production shock. There exists a unique globally defined unbiased predictor $\psi^* : \mathbb{R}^2_+ \times Prob(\Sigma) \to \mathbb{R}_+$*

$$\psi^*(M, p^e_{-1}, \mu) := (\mathbb{E}\mathcal{P})^{-1}(M, p^e_{-1}, \mu) \tag{4.6}$$

which is homogeneous of degree one in (M, p^e_{-1}), and which satisfies

$$\lim_{p^e_{-1} \to 0} \frac{\psi^*(M, p^e_{-1}, \mu)}{p^e_{-1}} = 0, \tag{4.7}$$

$$\lim_{p^e_{-1} \to \infty} \frac{\psi^*(M, p^e_{-1}, \mu)}{p^e_{-1}} \leq Z_{max} A S^{-1} \left(\frac{g/Z_{max}}{1 - c(1-\tau)} \right). \tag{4.8}$$

Figure 4.2 displays the graph of the mean price law $\mathbb{E}\mathcal{P}$ for fixed M and the basic geometric features of the range of the random price law for the isoelastic case with production shocks distributed on a compact interval $\Sigma = [Z_{\min}, Z_{\max}]$. The mean price law is a concave strictly increasing function with a global inverse which is the unbiased predictor ψ^* whose existence is essentially guaranteed under those assumptions which guarantee the uniqueness of temporary equilibrium.

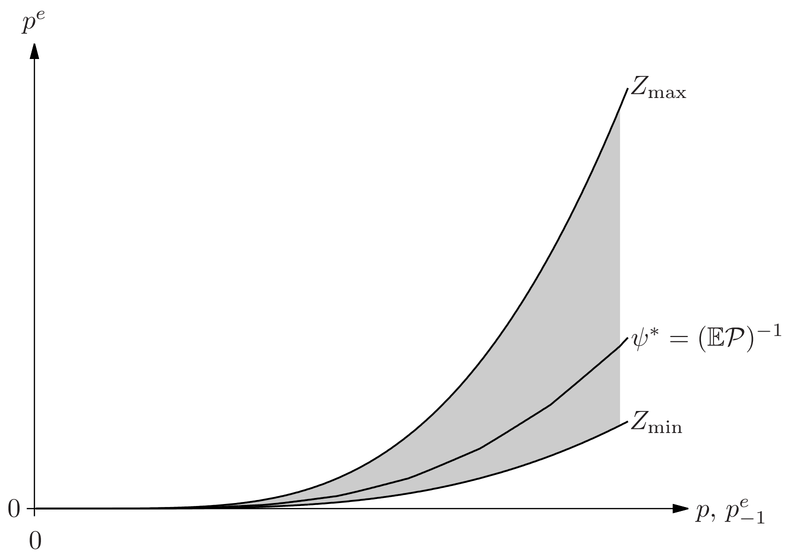


Figure 4.2: Existence of an unbiased predictor $\psi^*(M, p^e_{-1}, \mu) := (\mathbb{E}\mathcal{P})^{-1}(M, p^e_{-1}, \mu)$

Assume i.i.d. productivity shocks with constant measure μ in every period t. Then, the unbiased predictor is the same *deterministic* function of money balances and expectations alone in each period. Together with the dynamics for money balances this implies a system of two stochastic difference equations

$$\begin{pmatrix} p^e_{t,t+1} \\ M_{t+1} \end{pmatrix} = \begin{pmatrix} M_t\, \psi^*(1, p^e_{t-1,t}/M_t) \\ M_t\, \dfrac{\tilde{c}-\tau}{\tilde{c}} \big(1 + g\mathcal{P}(1, \psi^*(1, p^e_{t-1,t}/M_t), Z_t)\big) \end{pmatrix} \tag{4.9}$$

– which generate the dynamics under rational expectations, where $\tilde{c} := 1 - c(1-\tau)$.

– Since ψ^* is a time invariant mapping satisfying (4.6)

$\Longrightarrow$ **Rational expectations dynamics** are generated by

a system of two stochastic difference equations

– homogeneous of degree one in (M, p^e),

– without hyperbolic fixed points generically, for fixed Z.

– without stationary solutions for stationarity production shocks $\{Z_t\}$.[7]

observable orbits are those inducing stationary real allocations

– with expectations and money balances expanding/contracting uniformly and

positive, stationary, and bounded real expected money balances and real allocations.

Therefore, one considers the one-dimensional dynamics of expected real money balances in intensive form. As in the deterministic case, the dynamics of the economy "in real terms" (or in intensive form) is well defined. Obviously, the ratio of the two maps of the system (4.9) defines the time-one shift $q^e_t := p^e_{t-1,t}/M_t \mapsto q^e_{t+1} = p^e_{t,t+1}/M_{t+1}$ through the first order stochastic

[7] The failure is a consequence of the fact that the system (4.9) generically fails to have deterministic fixed points for each level Z (see Böhm, 2010).

difference equation given by

$$q^e_{t+1} = S(Z_t, q^e_t) := \frac{\tilde{c}}{\tilde{c} - \tau} \frac{\psi^*(1, q^e_t)}{1 + g\mathcal{P}(1, \psi^*(1, q^e_t), Z_t)}. \tag{4.10}$$

It is the inverse of (ex post) expected real money balances in every period t which is an empirically observable state variable of the random dynamics with respect to the information of $t-1$. Therefore, $q^e_{t+1} = S(Z_t, q^e_t)$ becomes t-measurable. For technical reasons it is more convenient to consider q^e_t as a state variable of the dynamical system rather than its inverse $M_t/p^e_{t-1,t}$. For lack of a better term at the moment, q^e_t will be dubbed "real expected prices" or simply "real expectations". One may of course, in a loose sense, speak of the dynamics of expected real money balances being described by (4.10).

Since μ is time invariant, only the denominator of the mapping is random while the numerator is deterministic. As a consequence, for $g = 0$, the mapping becomes deterministic. Nevertheless, the orbits of prices, output and employment are stochastic even in the case with zero government consumption. The main features of the time one mapping S which guarantee the properties of the real dynamics are summarized in the following lemma.

Lemma 4.2
Assume $g > 0$ and $Z \in \Sigma = [Z_{min}, Z_{max}]$. Then, the mapping S is monotonically increasing in q^e and satisfies for every $Z \in \Sigma = [Z_{min}, Z_{max}]$

$$\lim_{q^e \to 0} \frac{S(Z, q^e)}{q^e} = \lim_{q^e \to \infty} \frac{S(Z, q^e)}{q^e} = 0 \tag{4.11}$$

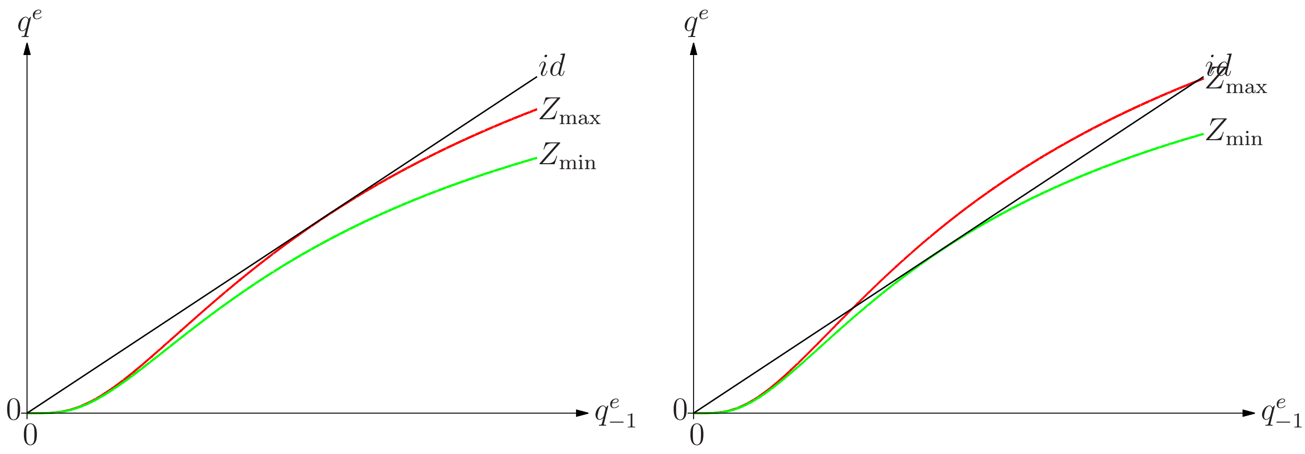
$$\frac{\partial}{\partial g} S(Z, q^e) < 0 \qquad \text{and} \qquad \frac{\partial}{\partial Z} S(Z, q^e) > 0. \tag{4.12}$$

There exist positive levels of government demand $g^{\star\star} > g^{\star} > 0$ such that

$$S(Z, \cdot) \quad \text{has no positive fixed point for } g > g^{\star\star}, \tag{4.13}$$

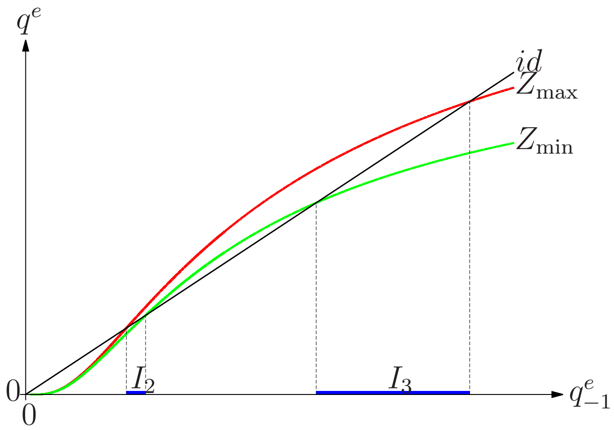
$$S(Z, \cdot) \quad \text{has at least two positive fixed points for } g < g^{\star}. \tag{4.14}$$

Figures 4.3 and 4.4 portray the role of government demand $g \geq 0$ and $Z \in \Sigma = [Z_{\min}, Z_{\max}]$ as stated in the lemma.

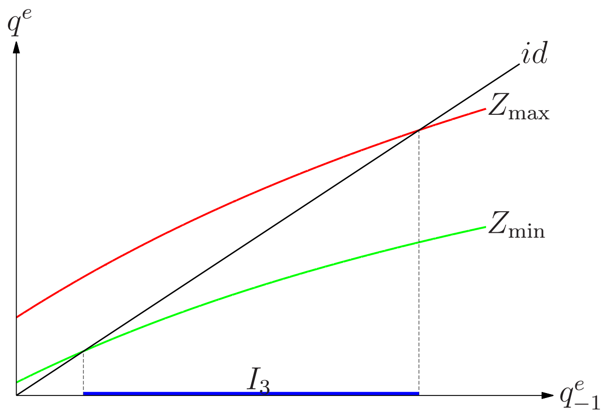


(a) $g = g^{\star\star}$ (b) $g = g^{\star}$

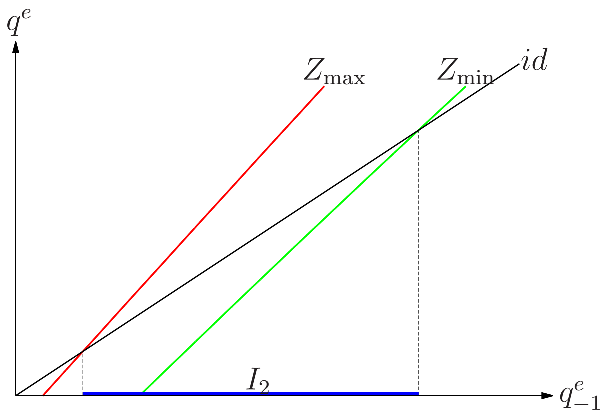
Figure 4.3: The role of g



(a) $0 < g < g^\star$



(b) I_3



(c) I_2

Figure 4.4: Multiple fixed points and stationary intervals for $0 < g < g^\star$

Figure 4.3 displays the critical role of g for the existence and number of fixed points for $g \geq g^{**}$ and $g = g^*$. Since S is monotonically decreasing in g, only the origin will remain as a fixed point when g is sufficiently large.

Figure 4.4 portrays the implications of Lemma 4.2 for $0 < g < g^*$ when the mapping S is S-shaped with exactly three fixed points for each $Z \in \Sigma$, which occurs when the unbiased mean predictor ψ^* is derived from isoelastic production and utility functions[8]. In this case, the associated fixed points for $Z_{\min}$ and $Z_{\max}$, (see subfigure (a)) can be ordered $0 < q_2(Z_{\max}) < q_2(Z_{\min}) < q_3(Z_{\min}) < q_3(Z_{\max})$ and they identify two coexisting non-overlapping intervals I_2 and I_3, see enlargements (b) an (c) of Figure 4.4. These intervals degenerate or become empty when g is large[9].

The lemma shows that the random dynamics of the economy under rational expectations will be governed by two fundamental structural features embodied in the stochastic difference equation S: (1) the fact that this time one map is S-shaped, and (2) a monotonic effect of government demand on the location of the function. In deterministic systems, the first causes typically multiple equilibria and threshold effects similar to poverty traps (as in Azariadis & Drazen, 1990) inducing a similar effect in the stochastic case. The monotonicity with respect to government demand shows that positive fixed points of the deterministic map S at any level of production shocks no longer exist for large government demand implying that positive stationary equilibria will also not exist.

[8] or the aggregate supply function is isoelastic

[9] In the general nonisoelastic case, if there are more than three fixed points for a given value of Z, q_2 and q_3 are chosen to be the ones with the lowest values to show the existence of two such intervals .

The consequences of these structural features for balanced monetary expansion are discussed in the next sections, after some additional mathematical concepts are introduced. Theorem 4.1 in Section 4.4 provides the main structural details for both critical effects and their role on *existence and stationarity* of balanced stochastic monetary orbits while Theorem 4.2 in Section 4.7 derives conditions for the *convergence to balanced stochastic monetary orbits* in the nominal state space $\mathbb{R}^2_+$. Section 4.4 discusses and characterizes the stationary orbits. Some typical qualitative properties of the main macroeconomic variables are portrayed in Section 4.6 for a numerical example.

4.4 Balanced Monetary Expansion

Balanced Stochastic Orbits

In deterministic two dimensional homogeneous systems fixed points of the one dimensional system in intensity form induce balanced growth paths of the two dimensional system. A similar relationship exists for stochastic two-dimensional homogeneous systems, namely so-called random fixed points of the one-dimensional system (4.10) in intensive form induce balanced stochastic orbits of the two dimensional system[10]. Due to the homogeneity, real money balances, output, employment, and inflation are bounded and stationary along random fixed points of the model in intensive form while stochastic orbits in nominal monetary terms are expanding and diverging.

Theorem 4.1 states the main result for the dynamics of expected real money balances as given by the system for (4.10). In order to justify such a statement for the monetary economy, to discuss balanced stochastic expansion properly and investigate its stability some additional mathematical concepts have to be introduced which link the process of random production shocks to the stochastic difference equations discussed in Lemma 4.2.

Consider a stationary Markov process for Z on Σ in its canonical representation with associated probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

– Let $\omega := (\ldots, Z_{-1}, Z_0, Z_1, \ldots) \in \Omega$ denote a realization (a sample path).

– Define $\vartheta : \Omega \to \Omega$ to be the so-called left shift on Ω, i.e. $(\vartheta\omega)(s) := \omega(s+1)$ for all $s \in \mathbb{Z}$ and

– denote by ϑ^t the $t-$th iterate of ϑ.

Then: $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\})$ is an ergodic dynamical system (see Arnold, 1998).

Next, consider the stochastic difference equation $S(Z, q^e)$ of the previous section as a stochastic mapping $S(\omega) : \mathbb{R} \to \mathbb{R}$ given by

$$S(\vartheta^t\omega)q^e_t := S(Z_t, q^e_t) = q^e_{t+1}. \tag{4.15}$$

– Repeated applications of S under the perturbation ω induce the measurable mapping $\phi : \mathbb{N} \times \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t, \omega, q^e_0) := \begin{cases} S(\vartheta^{t-1}\omega) \circ \ldots \circ S(\omega)\, q^e_0 & \text{if} \quad t > 0 \\ q^e_0 & \text{if} \quad t = 0 \end{cases} \tag{4.16}$$

such that $q^e_t = \phi(t, \omega, q^e_0)$ is the state of the system at time t.

[10]The notion of balanced expansion or contraction of money and expectations will be made precise in the definition of a *stable balanced* stochastic path in Section 4.7.

– The mapping (4.16) defines a *random dynamical system* (in the sense of Arnold, 1998, Chapters 1 & 2).

– In other words, the stochastic law of motion $S(q_t^e, Z_t)$ (or equivalently (4.15)) induces the mapping (4.16) which replaces the standard flow properties of a deterministic dynamical system.

– For any initial value q_0^e and perturbation $\omega \in \Omega$, the sequence $\gamma(q_0^e) := \{\phi(t, \omega, q_0^e)\}_{t=0}^{\infty}$, $t \in \mathbb{N}$, defines a stochastic orbit of (4.16) (and a solution of (4.15)).

Definition 4.2
A ***random fixed point*** *of the random dynamical system ϕ is a random variable $q^* : \Omega \longrightarrow \mathbb{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}-$almost surely*

$$q^*(\vartheta\omega) = \varphi(1, \omega, q^*(\omega)) \equiv S(\omega)q^*(\omega).$$

It is called stable ($\mathbb{P}$-almost surely) if there exists a random neighborhood $U(\omega) \subset \mathbb{R}$ with $q^*(\omega) \in U(\omega)$) such that

$$\lim_{t\to\infty} |\varphi(t, \omega, q_0) - q^*(\vartheta^t\omega)| = 0 \qquad \textit{for all} \quad q_0 \in U(\omega).$$

In an analogous way to the deterministic case, random fixed points of the intensive form system (4.10) are related in a natural way to balanced orbits of the monetary system (4.9) using the following definition.

Definition 4.3
Given $\omega \in \Omega$ and (M_0, p_0^e). A random orbit $\left\{(\bar{M}_t, \bar{p}_t^e)\right\} = \{(M(t, \omega, (M_0, p_0^e)), p^e(t, \omega, (M_0, p_0^e))\}$ of (4.9) is called ***balanced****, if there exists a random fixed point $q^\star : \Omega \to \mathbb{R}_+$ of (4.10)*

$$q_{t+1}^e = S(\vartheta^t\omega)q_t^e = \frac{\tilde{c}}{\tilde{c} - \tau} \frac{\psi^*(1, q_t^e)}{1 + g\mathcal{P}(1, \psi^*(1, q_t^e), Z(\vartheta^t\omega))}$$

such that

$$\bar{p}_t^e = \bar{M}_t q^\star(\vartheta^t\omega) \qquad \textit{for all } t \geq 0, \quad \mathbb{P}\textit{-a.s.}$$

Thus, *balanced* expansion or contraction of $\left\{(\bar{M}_t, \bar{p}_t^e)\right\} = \{(M(t, \omega, (M_0, p_0^e)), p^e(t, \omega, (M_0, p_0^e))\}$ in absolute terms occur along random paths induced by the stationary fluctuation of a random fixed point $q^* : \Omega \to \mathbb{R}_+$ in intensive form. If its orbits are contained in a positive compact interval the associated orbits of real money balances, of output, and of employment are positive and given by stationary random variables, see Section 4.5.

It is now possible to formulate the first part of the main result describing the role of fiscal policy for the *possibility of random balanced expansion*, some of its properties, and the thresholds of fiscal policy for their viability.

Theorem 4.1 *Let S be given by (4.10) with $g \geq 0$, $\Omega = [Z_{min}, Z_{max}]^{\mathbb{Z}}$, and $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\})$. Define the associated random dynamical system (using (4.16)) as*

$$q^e(t, \omega, q_0^e) := S(\vartheta^{t-1}\omega) \circ \cdots \circ S(\vartheta\omega) \circ S(\omega)q_0^e.$$

Then:

(1) *$q^e(\omega) \equiv 0$ is an asymptotically stable random fixed point.*

(2) *There exists $g^{\star\star} > 0$ such that for $g > g^{\star\star}$:*

- $q^e = 0$ *is the unique asymptotically stable random fixed point*
- *whose basin of attraction contains a non-degenerate compact interval $[0, \bar{q}] \subset \mathbb{R}$ such that for all $(q_0^e, \omega) \in [0, \bar{q}] \times [Z_{min}, Z_{max}]^{\mathbb{Z}}$*

$$\lim_{t\to\infty} q_t^e = \lim_{t\to\infty} q^e(t, \omega, q_0^e) = 0$$

(3) *There exists $0 < g^\star < g^{\star\star}$ such that for $0 < g < g^\star$:*

- *there exist two positive random fixed points*

$$q_2^\star : \Omega \to \mathbb{R}_+ \quad \textit{and} \quad q_3^\star : \Omega \to \mathbb{R}_+ \quad \textit{satisfying} \quad q_2^\star(\omega) < q_3^\star(\omega)$$

- *with invariant measures $q_2^\star\mathbb{P}$ and $q_3^\star\mathbb{P}$*
- *whose supports are nondegenerate disjoint intervals*

$$I_2 := [q_2^e(Z_{min}), q_2^e(Z_{max})] \qquad \textit{and} \qquad I_3 := [q_3^e(Z_{max}), q_3^e(Z_{min})].$$

(4) $q_3^\star$ *is globally stable on I_3 while $q_2^\star$ is unstable.*

The statements in the theorem indicate the implications of the two main features of the stochastic difference equation already referred to in Lemma 4.2. Their implications for long run observable balanced expansion are as follows:

[1] Large government consumption financed by deficits and money creation excludes the possibility of balanced monetary expansion. In other words, g^{**} defines an endogenously determined level of government demand beyond which *balanced* monetary expansion under rational expectations with bounded positive stationary levels of output and employment is impossible. Nevertheless, for all initial conditions $q_0^e > 0$, all stochastic orbits satisfy rational expectations and are positive since $q^e(t, \omega, q_0^e) > 0$ for all t, but $\lim_{t\to\infty} q^e(t, \omega, q_0^e) q^e = 0$, $\mathbb{P}$-a.s.. This proves the existence of a global critical level of government demand depending on the structural parameters of the economy, beyond which all deficit spending fails to support positive stationary allocations. For $g^* < g < g^{**}$, the economy may exhibit positive balanced expansion whose existence depends on specific features of the distribution of the noise process.

[2] For $0 < g < g^*$ balanced monetary expansion exists and is stable (in the intensity dynamics!) with a basin of attraction including all initial conditions $q_0^e \geq q_2^e(Z_{\max})$, which is larger than the interval I_3, the support of the random fixed point. The fact that S is monotonic but not a contraction with a fixed point zero for all production shocks implies that positive stationary solutions (if they exist) coexist with the degenerate boundary solution where real expectations become zero and money balances outgrow expectations and prices along rational expectations orbits. In other words, there exists the phenomenon of a *stochastic monetary trap*[11]. This occurs for sufficiently low levels of real expectations (high levels of expected real money balances) causing divergence of orbits with money balances accelerating to outgrow expectations without the possibility of a recovery. In other words, the economy can only exhibit two mutually exclusive observable regimes in the long run: positive stationary balanced monetary expansion or degenerate (zero) real expectations.

[3] The associated critical threshold level of real expectations separating the two regimes is determined by the unstable random fixed point $q_2^\star$, a random variable which is not observable

[11]akin to the poverty trap in growth theory (see Azariadis & Drazen, 1990; Schenk-Hoppé, 2005)

but has a nondegenerate support. Thus, the basins of attraction of either regime are random sets, implying that the probability of convergence to the positive fixed point of a particular subinterval of initial conditions in I_2 at time t' for any ω depends on the future realizations $\vartheta^t\omega$, $t > t'$. As a consequence, for any $q_0^e \in I_2$, probabilities of success of convergence to the positive fixed point depend on the statistical properties of the measure $q_2^\star\mathbb{P}$.

[4] The features above indicate that the success of policy measures (e.g. of a once and for all discretionary decrease of government demand at a given initial position q_0^e) depends in a complex way on properties of the mapping S and on the unstable random fixed point q_2^* (after the policy change!), but also on the future realizations (and the statistical properties) of the production shocks which cannot be estimated from historical data along an orbit with rational expectations. Thus, there are empirical and theoretical difficulties to understand (after observing the history of part of an orbit under rational expectations) from what level of government activity on money financed deficits are no longer viable, in order to ascertain and determine the limits before thresholds are surpassed beyond which only drastic interferences may halt or avoid catastrophes[12]. How such policies could be designed to escape or leave the monetary trap is a difficult and intricate issue.

[5] When the production shocks are independently and identically distributed the stable random fixed point $q_3^\star$ defines a Markov equilibrium of the intensive form model. In other words, the unique stationary measure $\nu \equiv q_3^\star\mathbb{P}$ induced by the random fixed point is a Markov equilibrium with $\nu(B) = \mathbb{P}\{\omega \mid q_3^\star(\omega) \in B\}$ and transition probability $P(q^e, B) = \mathbb{P}\{\omega \mid S(\omega)q^e \in B\}$. The same property is true for the degenerate fixed point $q^e \equiv 0$ with Dirac measure at zero[13].

4.5 The Real Economy along Balanced Stochastic Orbits

Under the monotonicity condition of the mapping S the balanced invariant behavior of the real stochastic economy is in many ways similar and comparable to the results derivable in the deterministic case.

If government demand is not too large the invariant behavior of the economy may display three distinct stationary scenarios, two of which are asymptotically stable phenomena and observable. Convergence to the positive stationary solution $q^\star$ implies the existence of associated stationary paths of output and employment, of real money balances and inflation rates on compact sets. These are random variables defined by the associated equilibrium mappings inducing associated stationary distributions. Because of ergodicity, their properties can be obtained from the limiting properties of numerical simulation studies for classes of parameterized models. This also allows to establish many invariant statistical properties on a macroeconomic level, for example tradeoffs between inflation and employment/output, the role of government demand, taxes, deficits, or monetary growth.

Real Money Balances, Output, Employment, and Inflation

Many of the properties of the positive random fixed point $q_3^\star$ can be derived using the stochastic difference equation under the fixed point property. Let $q_3^\star \equiv q^\star : \Omega \to I_3$ denote the stable

[12] The role of such stochastic monetary (or debt) traps seem to be the real issue to be investigated also in the debate between Reinhart-Rogoff and Krugman.

[13] However, the unstable random fixed point is not a Markov equilibrium for S, see Schenk-Hoppé (2005).

random fixed point which satisfies

$$q^\star(\vartheta\omega) = S(\omega)q^\star(\omega) := \frac{\psi^*(1, q^\star(\omega))}{1 + g\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))} \quad \mathbb{P}\text{-a.s.}.$$

In other words, the random variable $q^\star$ has a representation under the one-step time shift given by the right hand side of the equation, a fact which can be exploited numerically to determine its statistical properties. In addition, for all initial values $q_3^e(Z_{\min}) < q_0^e < q_3^e(Z_{\max})$ orbits converge to the sample path of $q^\star$.

Let $q^\star\mathbb{P}$ denote its stationary measure which has the compact support I_3. The monotonicity of the time one map in government demand g indicates that $q_3^\star$ and its support I_3 undergo a left shift if government demand g increases. For the properties of the stationary solutions of the real variables of the economy one finds the following relationships.

stationary real money balances $m^\star : \Omega \to \mathbb{R}_+$ are defined by

$$m^\star(\omega) := \frac{1}{\mathcal{P}(1, \psi^*(1, q^\star(\omega)), Z(\omega))}.$$

stationary output $y^\star : \Omega \to \mathbb{R}_+$

$$y^\star(\omega) = AD(m^\star(\omega), g) = \frac{1}{\tilde{c}}(m^\star(\omega) + g)$$

$\Longrightarrow$ The linearity of the aggregate demand function implies that real balances and output are identical random variables which differ by a mean shift

$\Longrightarrow$ real balances and output are perfectly correlated

stationary employment $L^\star : \Omega \to \mathbb{R}_+$

$$L^\star(\omega) = F^{-1}\left(\frac{AD(m^\star(\omega), g)}{Z(\omega)}\right).$$

$\Longrightarrow$ Government demand has a positive effect on stationary output and employment.

Output–employment correlation

$$y^\star(\omega) = Z(\omega)F(L^\star(\omega))$$

– makes random output a product of two stationary random variables.

$\Longrightarrow$ comovement of output and employment.

– When Z is a discrete random variable with two values (Z_{min}, Z_{max}), the support of $L^\star\mathbb{P}$ satisfies

$$\text{supp}\,(L^\star\mathbb{P}) \subseteq F^{-1}\left(\frac{1}{Z_{min}}\text{supp}\ y^\star\mathbb{P}\right) \cup F^{-1}\left(\frac{1}{Z_{max}}\text{supp}\ y^\star\mathbb{P}\right).$$

real wage and employment correlation

– real wage and employment are governed by the marginal product rule,

$$\alpha^\star(\omega) = Z(\omega)F'(L^\star(\omega)), \tag{4.17}$$

– is the product of two random variables.

$\Longrightarrow$ they must be negatively but not perfectly correlated.

stationary rate of inflation: from the definition

$$\theta_t := \frac{p_t}{p_{t-1}} = \frac{M_t}{M_{t-1}} \frac{\mathcal{P}(1, \psi^*(1, q_t^e), Z_t)}{\mathcal{P}(1, \psi^*(1, q_{t-1}^e), Z_{t-1})}$$

one obtains using (4.9)

$$\begin{aligned} \theta^\star(\omega) &:= \frac{\psi^*(1, q^\star(\vartheta^{-1}\omega))}{S(\vartheta^{-1}\omega)} \frac{\mathcal{P}(1, \psi^*(1, q^\star(\omega), Z(\omega)))}{\mathcal{P}(1, \psi^*(1, q^\star(\vartheta^{-1}\omega), Z(\vartheta^{-1}\omega)))} \\ &= \hat{m}^\star(\omega) \frac{m^\star(\vartheta^{-1}\omega)}{m^\star(\omega)} \end{aligned} \tag{4.18}$$

$\Longrightarrow$ the stationary rate of inflation equals
the stationary rate of monetary growth times
the ratio of stationary real money balances at two successive dates,

– i.e. it is the product of three stationary random variables

$\Longrightarrow$ inflation rates must show significant serial correlation,

4.6 Numerical Results: Stability and Stationarity

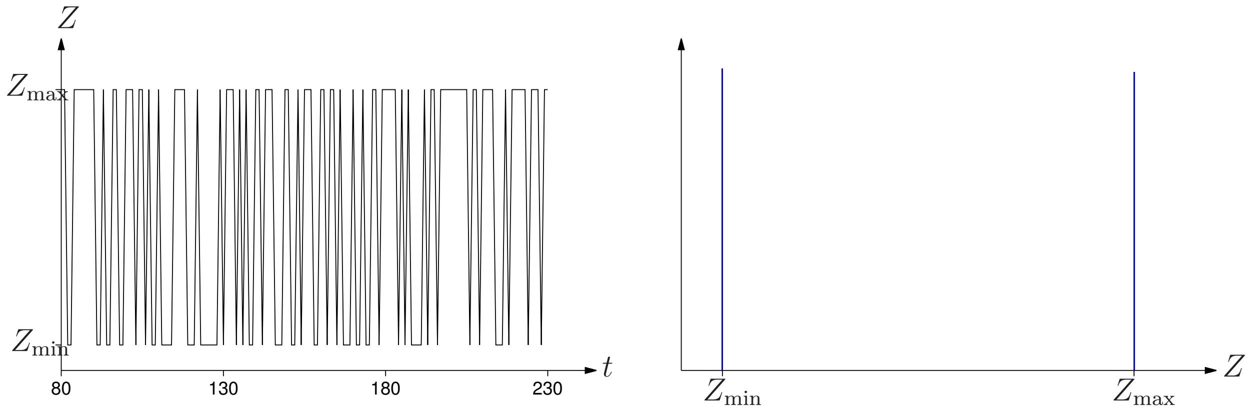
The following diagrams portray the characteristics of convergence of the random fixed point $q_3^\star$ to and of the stationary solutions of the economy under a discrete two point production shock for the values of the parameters given in the table when the production function and the labor supply functions are isoelastic. In this case the integral equation (Definition 3.2) defining the unbiased predictor implicitly can be solved numerically, which allows to calculate the associated stochastic orbits under rational expectations for a given sample path of the discrete noise. The parameter $0 < B < 1$ denotes the elasticity of the production function F while $1 + C$ is the elasticity of the labor supply function N[14].

With the assumption of a discrete i.i.d. noise process, the random dynamical system q^e becomes a so-called *iterated function system* (IFS) (see Barnsley, 1988). These have been studied widely. The conditions for existence and stability of random fixed points are significantly weaker than those imposed in Theorem 4.1 (see Arnold & Crauel, 1992).

Z_{min}	Z_{max}	B	C	c	τ	g	g^*	g^{**}
1.0	1.01	0.6	0.6	0.5	0.7	0.8240	0.8285	0.8328

Table 1: Standard parametrization a

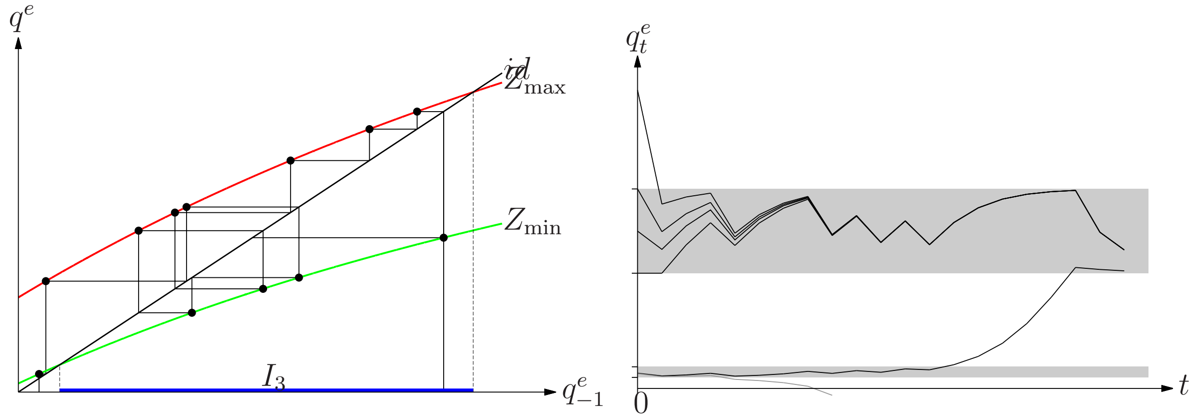
[14] I am indebted to Oliver Claas for providing the numerical results.



(a) Time profile of $Z(\omega)$ (b) histogram of $Z(\omega)$ for $T = 2 \cdot 10^4$

Figure 4.5: $Z \sim \mathcal{U}\{Z_{\min}, Z_{\max}\}$ with equal probability

The convergence of orbits of the random dynamical system (4.10) for the stochastic difference equation S to I_3 and to the stationary solution are shown in Figure 4.6, which displays in panel (a) the associated graph of the two time one maps and an orbit starting outside of I_3. Panel (b) displays the time profile of six different orbits, five of which converge to I_3 represented by the large shaded interval. Two initial conditions start in I_2 (the small shaded interval) both of which leave this interval in finite time, one diverging the other one converging to I_3. Figure



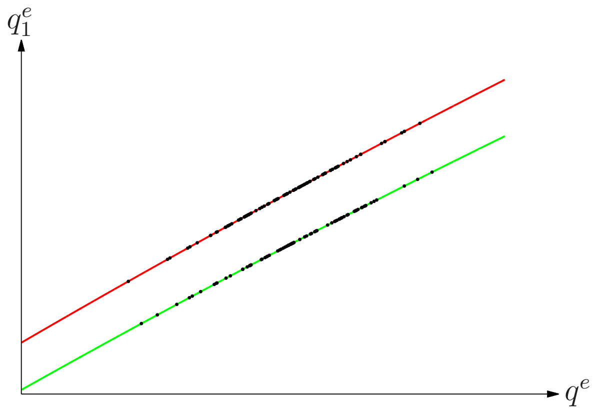
(a) Convergence to attracting set I_3 (b) Convergence from I_2 and I_3 to $q^\star_3$

Figure 4.6: Convergence to $q^\star$; $B = C = .6$, $c = 0.5$; $Z_{min} = 1.0$, $Z_{max} = 1.05$

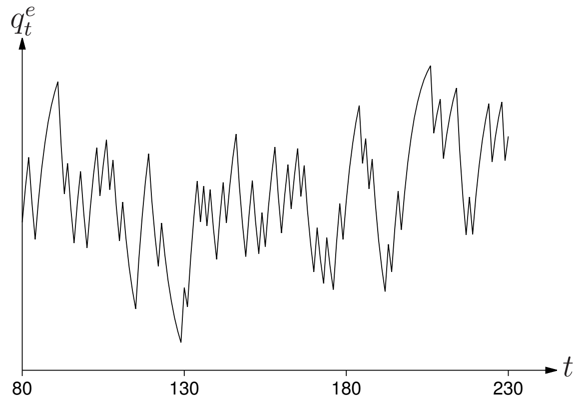
4.7 displays the main properties of the random fixed point $q^\star : \Omega \to I_3$ for the stationary solution associated with $Z(\omega)$. Subfigure (a) shows a typical phase plot (an attractor in the space $(q^e_t, q^e_{t+1}) \in I_3 \times I_3$) when there are discrete production shocks. All orbit pairs lie on the graphs of two nonlinear time one maps associated with the two values of Z which are disjoint sets. This reveals in particular that the attractor (or support[15] of the joint distribution) is not a rectangle in $\mathbb{R}^2$. This also shows that the dynamics of real expected money balances cannot be approximated well by a one dimensional AR1 system in q^e. Nevertheless, because of stationarity, the two piece attractor implies a well defined and structurally simple autocorrelation such that the two marginal distributions (of the projections onto the two axis) must be identical, as shown in subfigure (c). The raggedness of the histogram is a typical feature for an IFS, which often does not decrease or become more smooth as the number of iterations becomes large.

Note, however, that in many situations, both fixed points $q^\star_i$, $i = 2, 3$ induce nondegenerate

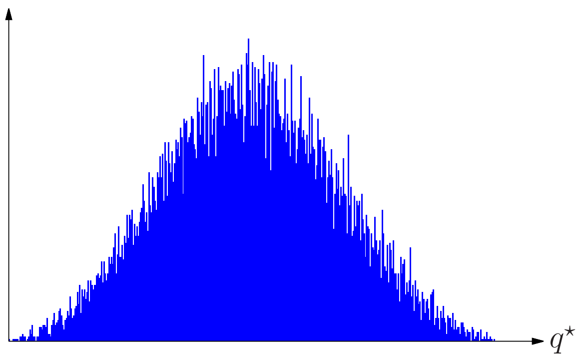
[15] i.e. supp $\nu \in \mathcal{F}$ of a measure ν is the smallest closed subset with full measure, which satisfies $\nu(\text{supp } \nu) = 1$



(a) Phase plot of $q_3^\star(\omega)$ and autocorrelation



(b) Time profile of $q_3^\star(\omega)$

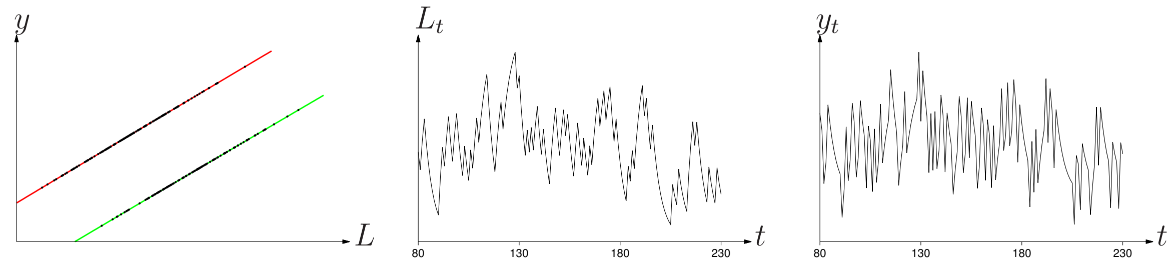


(c) Histogram of $q_3^\star(\omega)$; $T = 2 \cdot 10^4$

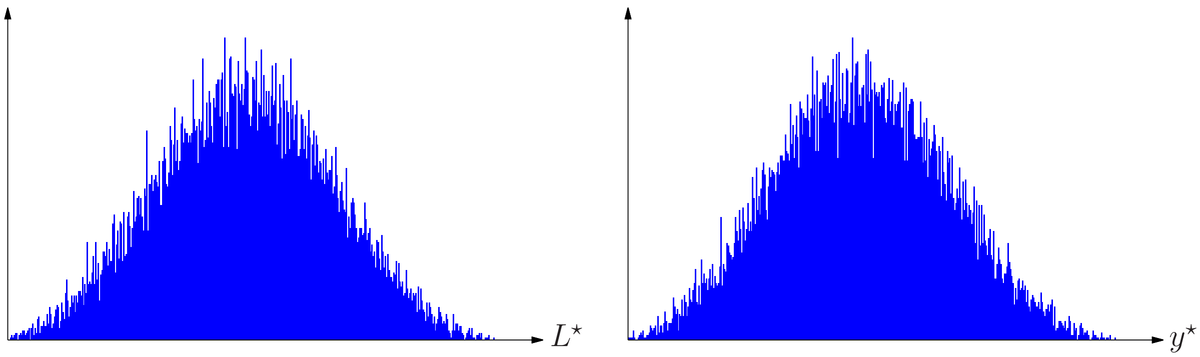
Figure 4.7: Stationary solution $q^\star$; $Z_{min} = 1.0$, $Z_{max} = 1.01$

invariant measures $q_i^\star\mathbb{P}$, $i = 2, 3$, whose supports are the full respective intervals I_i, even though the production shocks are concentrated on discrete points and the state space representation indicates attractors as subsets of two disjoint graphs. This nondegeneracy is essentially a consequence of the continuity of the time one map S, preventing discrete stationary solutions to occur under discrete shocks, a feature which occurs in particular for the parameters in Table 3. However, for some parameters of the economy determining the slope of the mapping S and the size of the production shock corresponding to each other in a particular way, the invariant measure may have a very complicated structure with no density and the support may be a Cantor set with Lebesgue measure zero (see Barnsley, 1988, for examples).

Figure 4.8 displays the features of the joint distribution of employment and output (subfigure (a)) together with the two time profiles and their histograms which display the typical features of a discrete noise process. The two point distribution of the production shock implies that the joint support of the empirical distribution must be a subset of the two graphs of he production function. At the the same time, subfigure (a) displays the typical comovement of employment and output as one would expect.



(a) Employment - output correlation (b) Time profile of employment $L^{\star}$ (c) Time profile of output $y^{\star}$

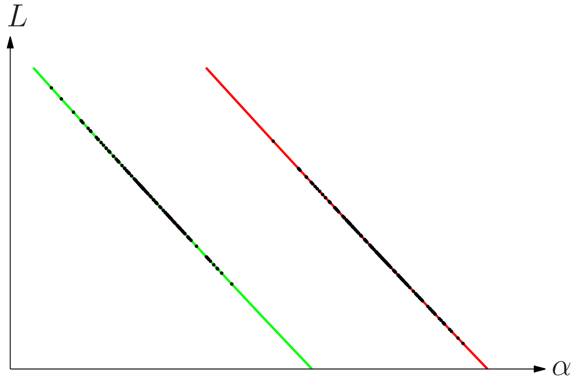


(d) Stationary employment; $T = 2 \cdot 10^4$ (e) Stationary output; $T = 2 \cdot 10^4$

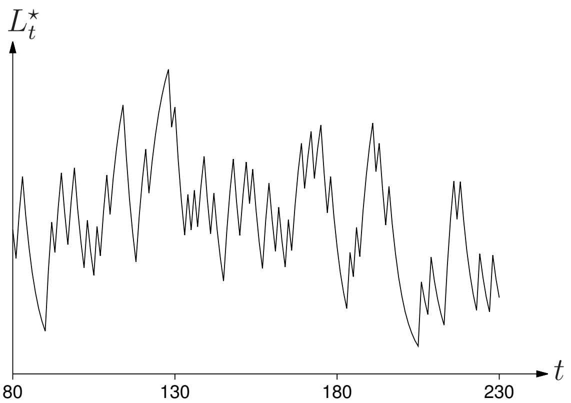
Figure 4.8: Stationary Output and Employment

4.6.1 Employment and the real wage

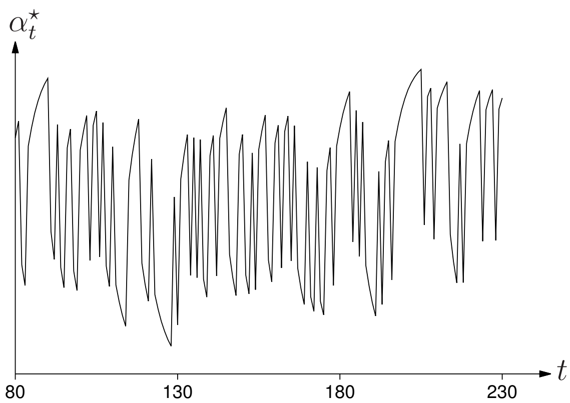
In a similar fashion one obtains the corresponding diagrams of the statistics between employment and the real wage which shows the distinctive properties of an IFS. The two point distribution of the production shock combined with the marginal product rule of profit maximization implies that the joint support in $\mathbb{R}^2_+$ must be a subset of the two associated graphs of the marginal product curves (subfigure (a)). In other words, Hicks neutral production shocks induce a negative correlation and not a (positive) empirical comovement of employment and the real wage. Most interestingly, however, one finds that the distribution of the real wage is bimodal and not necessarily symmetric, and that its support consists of two (almost) disjoint intervals . The gap in the support arises jointly because of the size of the production shocks and the slope of the marginal product curve. In other words, the long run dynamics of the real wage fluctuates between the two intervals in a stochastic (nonperiodic) way.



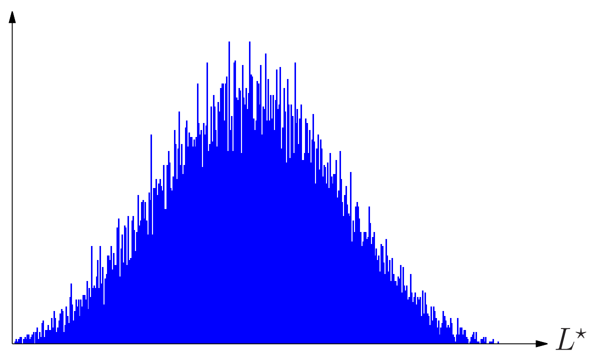
(a) Real wage – employment tradeoff



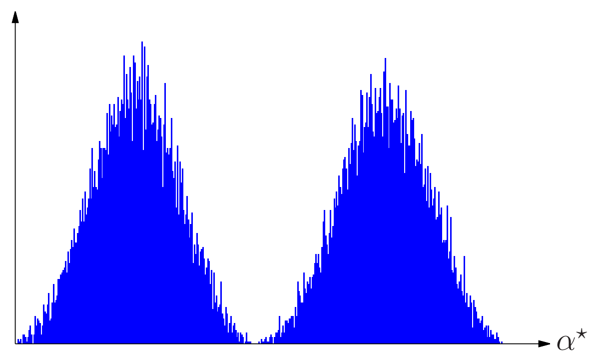
(b) Stationary employment



(c) Stationary real wage



(d) Stationary employment; $T = 2 \cdot 10^4$

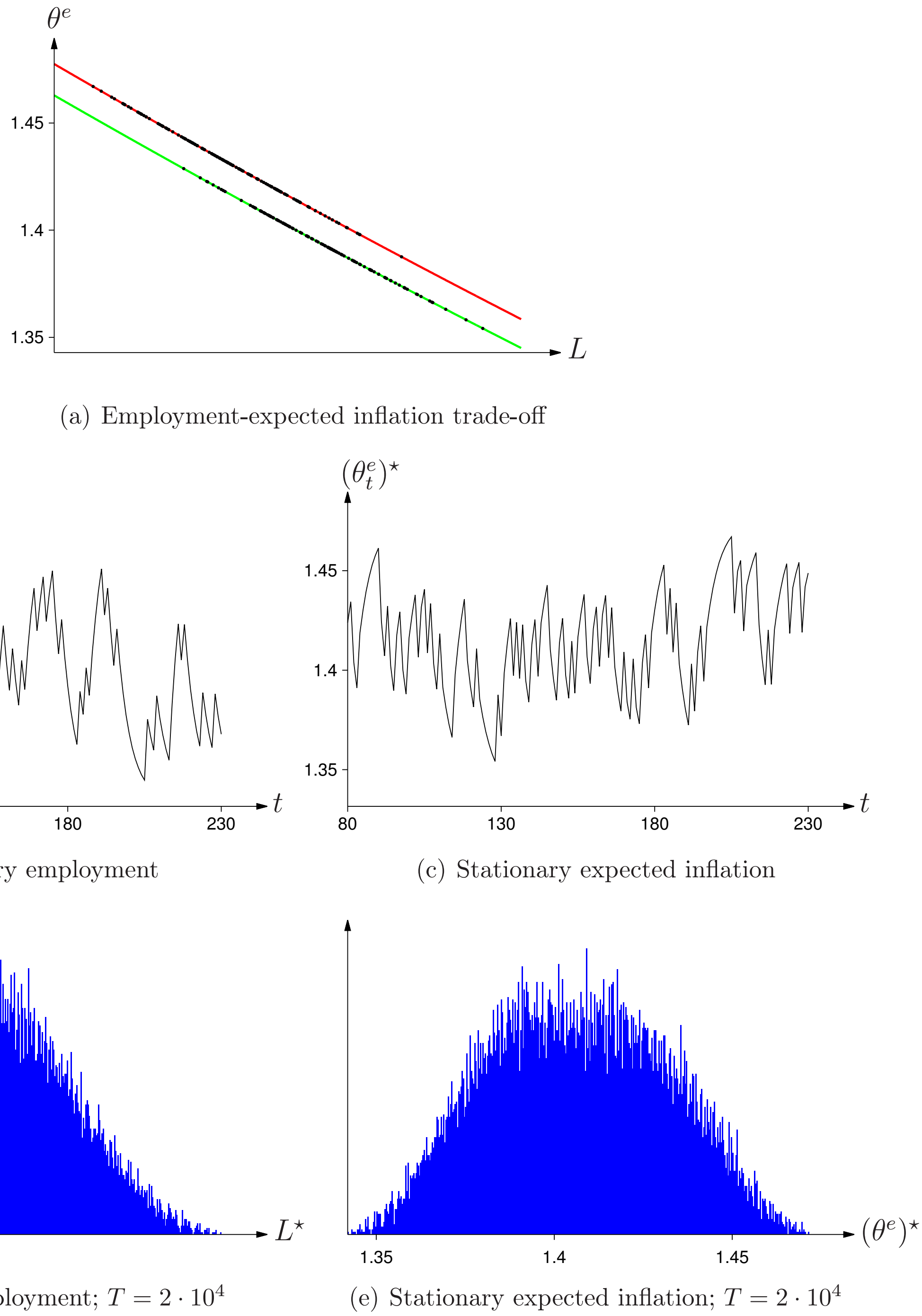


(e) Stationary real wage; $T = 2 \cdot 10^4$

Figure 4.9: Stationary employment and real wage

4.6.2 Employment and expected inflation

A similar phenomenon occurs the correlation between employment and the expected rate of inflation $\theta^e = p^e/p_t$. Since the stationary levels of θ^e belong to two small disjoint intervals, its stationary distribution is clearly bimodal and asymmetric.



(a) Employment-expected inflation trade-off

(b) Stationary employment

(c) Stationary expected inflation

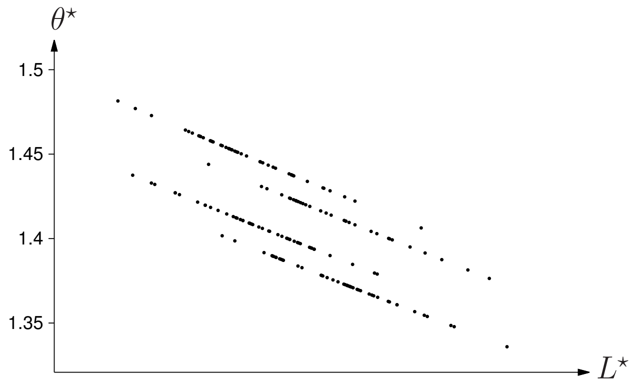
(d) Stationary employment; $T = 2 \cdot 10^4$

(e) Stationary expected inflation; $T = 2 \cdot 10^4$

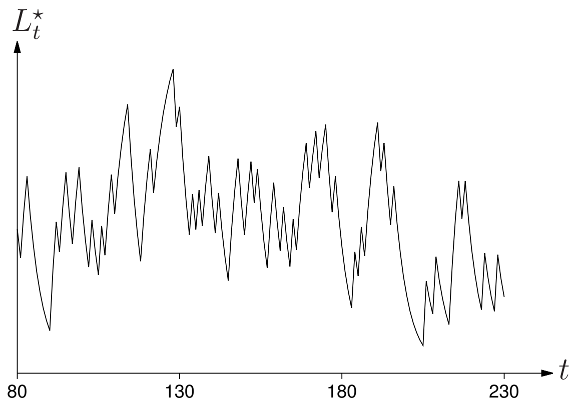
Figure 4.10: Stationary employment and expected inflation

4.6.3 Employment and inflation

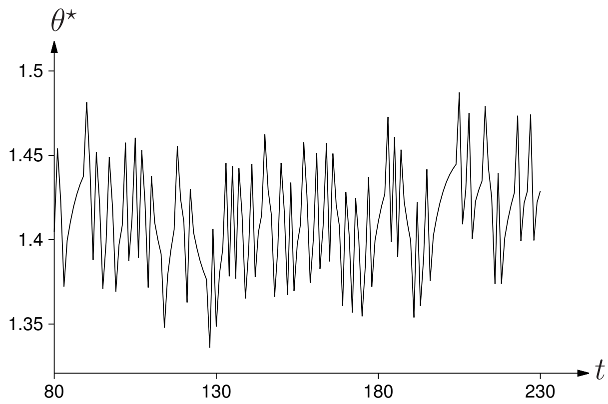
The long run tradeoff between inflation and employment exhibits the typical Markovian structure in the correlation diagram (the Phillips curve) which is an outcome of the central features of the stationary competitive equilibrium under rational expectations.



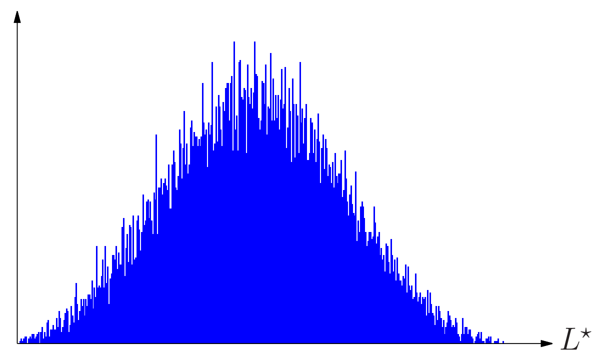
(a) Employment-inflation tradeoff



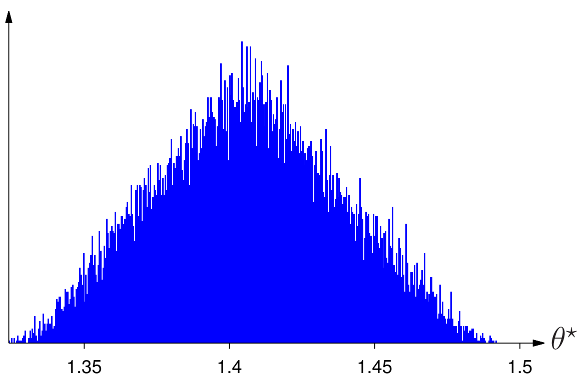
(b) Stationary employment



(c) Stationary inflation



(d) Stationary employment; $T = 2 \cdot 10^4$



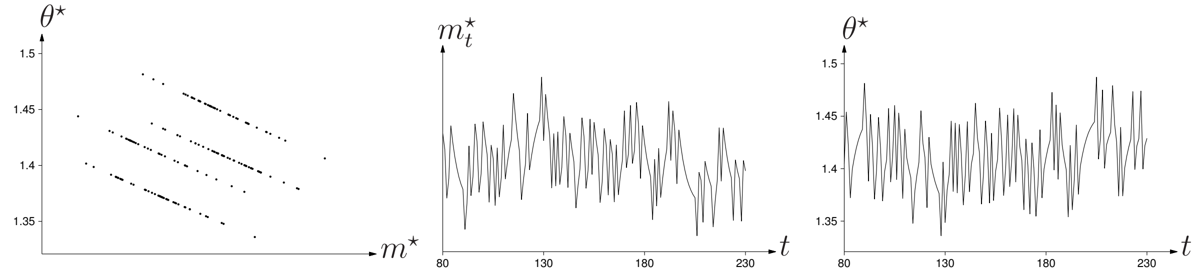
(e) Stationary inflation; $T = 2 \cdot 10^4$

Figure 4.11: Stationary employment and inflation

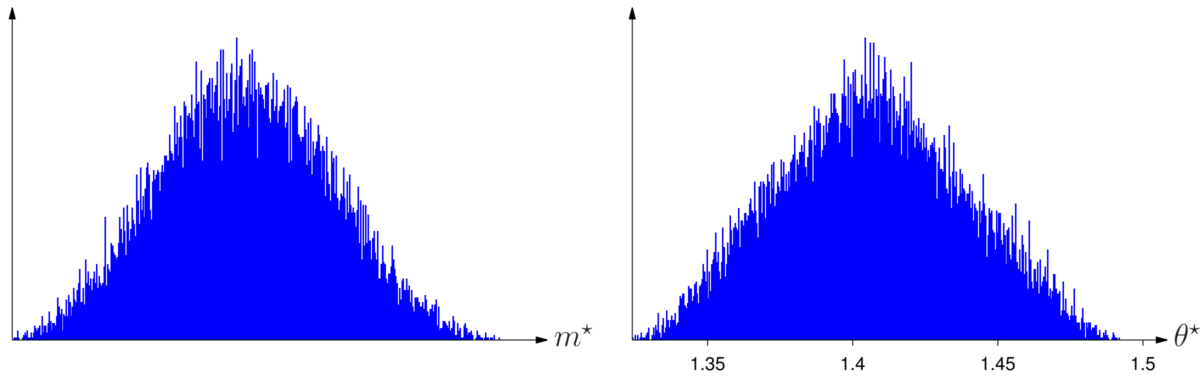
4.6.4 Real money balances and inflation

In an analogous way, one obtains the correlation between real balances and inflation as derived in (4.18) which also shows the typical Markovian structure in the correlation diagram (Figure 4.12 (a)). Since output and real balances have the same stationary distribution, the corresponding diagram between $y^\star$ and $\theta^\star$ would show a similar negative correlation.

Subfigure (e) in both diagrams also reveals the consequences of the autocorrelation of prices/inflation rates as compared to employment or money balances by an additional nonlinear influence on the shape of the histogram which is less curved (and more triangular).



(a) Real balances-inflation tradeoff (b) Stationary real balances (c) Stationary inflation



(d) Stationary real balances; $T = 2 \cdot 10^4$ (e) Stationary inflation; $T = 2 \cdot 10^4$

Figure 4.12: Stationary real balances and inflation

4.7 Convergence to Balanced Stochastic Orbits

An analysis of the convergence properties of orbits of the two-dimensional system (4.9) of stochastic difference equations requires some additional considerations, since the two mappings are homogeneous of degree one in (M, p^e) without contractive properties and without fixed points in $\mathbb{R}^2_+$ for each Z. As in deterministic homogeneous two-dimensional systems, the stability conditions for the dynamics of real and nominal variables do not match because of the different dimensions of the two systems and the lack of fixed points in the latter one. Therefore, distinct convergence criteria are mathematically necessary to define convergence to balanced paths.

The balanced random orbits of (4.9) in $\mathbb{R}^2_+$ are unbounded for $t \to \infty$ without random fixed points. Since both variables grow beyond bounds a stability/convergence criterion in the stochastic case for an auxiliary dynamical system has to be defined appropriately which is introduced in Definition 4.4. Fortunately this can be done in an analogous way to the deterministic case[16]. Moreover, stability and convergence to random fixed points in ratios (or in intensive form) becomes a necessary condition for convergence to balanced random orbits of money and expectations in $\mathbb{R}^2_+$.

Therefore, for any $\omega \in \Omega$ and (M_0, p^e_0), consider again the two dimensional system of random difference equations (4.9)

$$M_{t+1} = M_t \, \frac{\tilde{c} - \tau}{\tilde{c}} \left(1 + g\mathcal{P}(1, \psi^*(1, p^e_t/M_t), Z(\vartheta^t\omega))\right)$$
$$p^e_{t+1} = M_t \, \psi^*(1, p^e_t/M_t)$$

which induces the two dimensional random dynamical system (using (4.16))

[16](see Deardorff, 1970; Böhm, Pampel & Wenzelburger, 2005; Pampel, 2009; Böhm, 2010)

$$\begin{aligned} M_t &= M(t,\omega,(M_0,p_0^e)) \\ p_t^e &= p^e(t,\omega,(M_0,p_0^e)). \end{aligned} \tag{4.19}$$

This generates random orbits $\{(M_t,p_t^e)\}_{t=0}^\infty$ in $\mathbb{R}_+^2$. According to Definition 4.3 they are called **balanced**, if there exists a random fixed point $q^\star:\Omega\to\mathbb{R}_+$ of (4.10)

$$q_{t+1}^e = S(\vartheta^t\omega)q_t^e = \frac{\tilde{c}}{\tilde{c}-\tau}\frac{\psi^*(1,q_t^e)}{1+g\mathcal{P}(1,\psi^*(1,q_t^e),Z(\vartheta^t\omega))}$$

such that

$$\bar{p}_t^e = \bar{M}_t q^\star(\vartheta^t\omega) \qquad \text{for all } t\geq 0, \quad \mathbb{P}\text{-a.s.}$$

For any initial state $(M_0,p_0^e)\in\mathbb{R}_+^2$, define the distance of its orbit generated by (4.19) from the balanced path associated with the random fixed point $q^\star$ as $\Delta_t := p_t^e - q^\star(\vartheta^t\omega)M_t$. Then one finds that the evolution of the distance

$$\begin{aligned} \Delta_{t+1} := p_{t+1}^e - q^\star(\vartheta^{t+1}\omega)M_{t+1} &= \frac{M_{t+1}}{M_t}\frac{q_{t+1}^e - q^\star(\vartheta^{t+1}\omega)}{q_t^e - q^\star(\vartheta^t\omega)}\Delta_t \\ &= \frac{\psi^*(1,q_t^e)}{S(\vartheta^t\omega)q_t^e}\cdot\frac{S(\vartheta^t\omega)q_t^e - S(\vartheta^t\omega)q^\star(\vartheta^t\omega)}{q_t^e - q^\star(\vartheta^t\omega)}\cdot\Delta_t \end{aligned} \tag{4.20}$$

is described by a stochastic difference equation in (q^e,Δ). Together with (4.10) the definition induces a two dimensional auxiliary system of stochastic difference equations in (q^e,Δ) given by

$$\begin{aligned} q_{t+1}^e &= S(\vartheta^t\omega)q_t^e \\ \Delta_{t+1} &= \frac{\psi^*(1,q_t^e)}{S(\vartheta^t\omega)q_t^e}\cdot\frac{S(\vartheta^t\omega)q_t^e - S(\vartheta^t\omega)q^\star(\vartheta^t\omega)}{q_t^e - q^\star(\vartheta^t\omega)}\cdot\Delta_t \end{aligned} \tag{4.21}$$

implying a two dimensional random dynamical system $\phi:\mathbb{N}\times\Omega\times I_3\times\mathbb{R}\to I_3\times\mathbb{R}$ with

$$\phi(t,\omega,(q_0^e,\Delta_0)) := \begin{pmatrix} q^e(t,\omega,q_0^e) \\ \Delta(t,\omega,q_0^e,\Delta_0)\end{pmatrix}.$$

with unique random fixed point $(q^\star,0)$. This allows for the following definition of asymptotic stability of balanced orbits.

Definition 4.4
A balanced orbit $\{(M(t,\omega,(M_0,p_0^e)),p^e(t,\omega,(M_0,p_0^e))\}$ of (4.19) associated with the random fixed point $q^\star:\Omega\to I_3$ is called ***asymptotically stable*** *if there exists a random neighborhood $N(\omega)\subset I_3\times\mathbb{R}$ with $(q^\star(\omega),0)\in N(\omega)$ such that for all $(q_0^e,\Delta_0)\in N(\omega)$ with $q_0^e = p_0^e/M_0$ one has $\mathbb{P}$-a.s.*

$$\lim_{t\to\infty}|q^e(t,\omega,q_0^e) - q^\star(\vartheta^t\omega))| = 0, \tag{4.22}$$

$$\lim_{t\to\infty}|\Delta(t,\omega,q_0^e,\Delta_0)| = 0. \tag{4.23}$$

With these preliminaries it is now possible to state the theorem on the stability of balances stochastic expansion (analogous to the result by Pampel, 2009).

Theorem 4.2 *Let S be differentiable and increasing with respect to q^e and let $q^\star$ be an asymptotically stable random fixed point of system* (4.10)

$$q^e_{t+1} = S(\vartheta^t\omega)q^e_t = \frac{\tilde{c}}{\tilde{c}-\tau}\frac{\psi^*(1,q^e_t)}{1+g\mathcal{P}(1,\psi^*(1,q^e_t),Z(\vartheta^t\omega))}.$$

Let $(p^e_0, M_0) >> 0$ *and* $p^e_0/M_0 = q^e_0 \in I_3$, *and* $q^e_0 \neq q^\star(\omega)$ *and* $\lim_{t\to\infty}|q^e(t,\omega,q^e_0) - q^\star(\vartheta^t\omega)| = 0$, $\mathbb{P}$*-a.s.. Then, for almost all* $\omega\in\Omega$ *the distance* $\Delta_t = p^e(t,\omega,(M_0,p^e_0)) - q^\star(\vartheta^t\omega)\,M(t,\omega,(M_0,p^e_0))$ *satisfies*

$$\lim_{t\to\infty}|\Delta_t| = 0 \quad \textit{if} \qquad \mathbb{E}\log(S'(\omega,q^\star(\omega))) + \mathbb{E}\log\frac{\tilde{c}-\tau}{\tilde{c}}(1+g\mathcal{P}(1,\psi^*(1,q^\star(\omega)),Z(\omega))) < 0 \tag{4.24}$$

$$\lim_{t\to\infty}|\Delta_t| = \infty \quad \textit{if} \qquad \mathbb{E}\log(S'(\omega,q^\star(\omega))) + \mathbb{E}\log\frac{\tilde{c}-\tau}{\tilde{c}}(1+g\mathcal{P}(1,\psi^*(1,q^\star(\omega)),Z(\omega))) > 0. \tag{4.25}$$

4.8 Convergence and Growth Rates of Monetary Expansion

In order to understand the conditions (4.24) and (4.25) for stability/instability it is useful to consider the two central equations of the auxiliary system (4.21)

$$q^e_{t+1} = S(\vartheta^t\omega)q^e_t = \frac{\tilde{c}}{\tilde{c}-\tau}\frac{\psi^*(1,q^e_t)}{1+g\mathcal{P}(1,\psi^*(1,q^e_t),Z(\vartheta^t\omega))}$$

$$\Delta_{t+1} = \frac{\tilde{c}-\tau}{\tilde{c}}\left(1+g\mathcal{P}(1,\psi^*(1,q^e_t),Z(\vartheta^t\omega))\right)\cdot\frac{S(\vartheta^t\omega)q^e_t - S(\vartheta^t\omega)q^\star(\vartheta^t\omega)}{q^e_t - q^\star(\vartheta^t\omega)}\cdot\Delta_t$$

which is a **skewed** system of stochastic difference equations in (q^e,Δ) and **linear** in Δ, which also makes the random dynamical system also skewed and linear in Δ.

Observe that the coefficient of Δ consists of a product of two random variables,

– the first converges to the growth rate of money $\hat{m}_t := M_t/M_{t-1}$ along $q^\star_3$ while

– the second term converges to the derivative of S since $\lim_{t\to\infty}|q^e_t(t,\omega,q^e_0) - q^\star(\vartheta^t\omega)| = 0, \mathbb{P}$-a.s.

Thus, convergence of the distance $\Delta \to 0$ occurs if the growth factor $\hat{m}^\star(\vartheta^t\omega)\cdot S'(q^\star_3(\vartheta^t\omega))$ of the linear system is sufficiently contracting, i.e.

if and only if $\mathbb{E}\left\{\hat{m}^\star(\vartheta^t\omega)\cdot S'(q^\star_3(\vartheta^t\omega))\right\} < 1$, a condition imposing

average contractivity only (see Böhm, Pampel & Wenzelburger, 2005; Pampel, 2009)

– which are essentially the conditions (4.24) and (4.25) stated in Theorem 4.2.

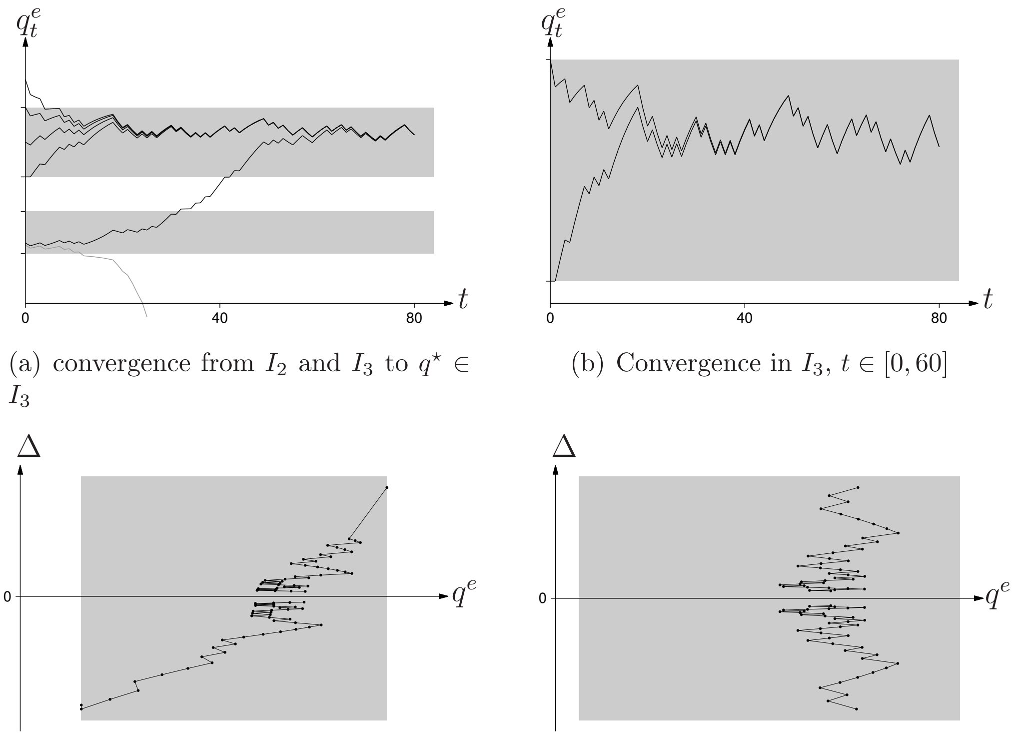
– Given the assumption that S is contracting on I_3, i.e.

$$S'(q^\star_3(\vartheta^t\omega)) < 1, \quad \mathbb{P}\text{-a.s.},$$

this allows **permanently positive growth rates of money** along observable random orbits with rational expectations.

Therefore,

- nominal orbits with permanent monetary growth converging to balanced random orbits exist
- where the rate of monetary expansion is larger than one along ω, $\mathbb{P}$-a.s. or most of the time,
- but it should not make the product with S' larger than one too often.



(a) convergence from I_2 and I_3 to $q^\star \in I_3$

(b) Convergence in I_3, $t \in [0, 60]$

(c) convergence in I_3: $t \in [0, 30]$

(d) Symmetry of orbits for $t > 30$

Figure 4.13: Convergence in (q^e, Δ)–space for values in Table 2

Figure 4.13 displays the convergence features for the numerical example of Section 4.6 and when the balanced orbit associated with $q_3^\star$ is asymptotically stable, a situation which occurs for the parameters given in Table 2. For these values – with a small production shock – the time one map S becomes almost linear on I_3, subfigure (a). Panel (a) and (b) show the convergence for six different initial conditions in the space of real expectations while (c) and (d) display the convergence in (q^e, Δ)–space for the same ω. Notice the difference in scale between the subfigures (c) and (d). For the numerical experiment the orbits are calculated for the same ω with six different initial conditions.

For the numerical example, there is asymptotic convergence for the values of he paameters in Table 2 and

Z_{min}	Z_{max}	B	C	c	τ	g	g^*	g^{**}
1.0	1.01	0.6	0.6	0.5	0.75	0.8392	0.8400	0.8449

Table 2: Standard parametrization b

divergence for those in Table 3.

Z_{min}	Z_{max}	B	C	c	τ	g	g^*	g^{**}
1.0	1.01	0.6	0.6	0.5	0.7	0.8240	0.8285	0.8328

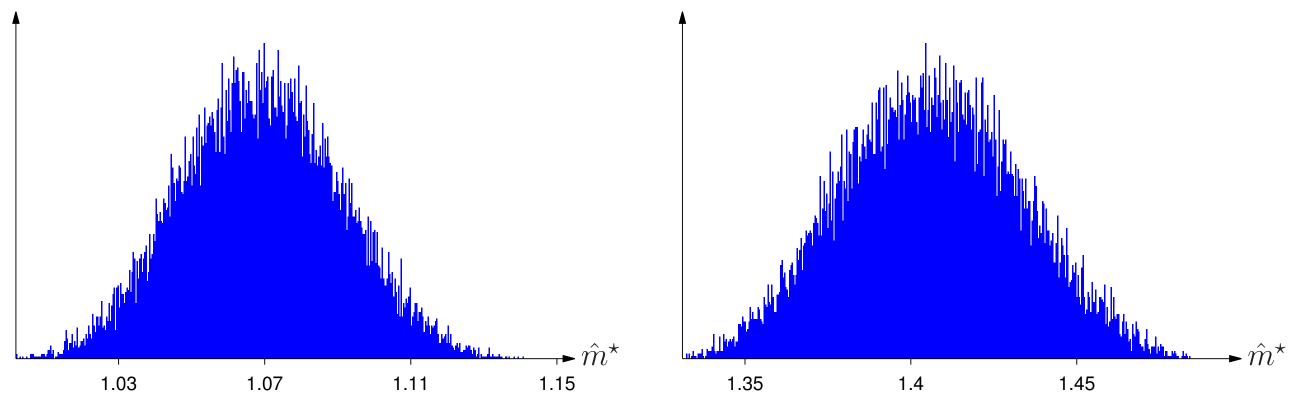
Table 3: Standard parametrization a

The Main Difference between the values in Table 3 and 2 consists of a

lower government demand and a higher tax rate.

$\Longrightarrow$ lower deficits and thus lower rates of inflation at any one time and

$\Longrightarrow$ lower growth rates of money at any one time.



(a) Money growth stable: $\mathbb{E}\hat{m}^\star = 1.0690$ (b) Money growth unstable: $\mathbb{E}\hat{m}^\star = 1.4063$

Figure 4.14: Stationary growth rates of money: $T = 2 \cdot 10^4$

Figure 4.14 displays the histograms of the rates of monetary expansion in the two cases with their respective means, showing clearly the reason for the instability of the balanced orbit in the case of the parameters of Table 3.

5 Summary and Conclusions

1. Mathematics

– The stability of balanced random evolution of expanding orbits under rational expectations can be analyzed successfully using

– the theory of random dynamical systems in the sense of Arnold (1998)[17]

– taking full account of the random perturbations, of the homogeneity, and of all nonlinearities.

– Convergence properties in intensity form as well as in the state space can be successfully demonstrated in the same way as in the deterministic case (see Böhm, 2010).

– There is no need

[17] These methods have been applied successfully in other areas, for example in growth theory (see Schenk-Hoppé & Schmalfuss, 2001; Böhm & Wenzelburger, 2002) and in mathematical finance (see Böhm & Wenzelburger, 2005; Böhm & Chiarella, 2005).

- to linearize the random difference equation near a deterministic steady state and analyze approximating stochastic systems[18],
- to suppress stochastic orbits altogether and analyze stationary Markov equilibria only[19],
- to describe stable scenarios by convergence in distributions only[20].

$\Longrightarrow$ The same tools analyze the theoretical and empirical/observable objects: time series or orbits.

2. Economics

Stable balanced monetary expansion can be sustained along stable and empirically observable orbits with **rational expectations**

– within a general class of monetary models of the AS–AD type

– supporting positive stationary levels of output and employment

– under a set of economically challenging conditions.

For small levels of autonomous government demand rational expectations orbits may exhibit

– two mutually exclusive regimes in the long run in real terms which are observable:

- positive stationary balanced real monetary expansion or
- degenerate (zero) real expectations.

– These are separated by a

stochastic monetary trap or random threshold levels

– of real expectations (or real money balances) defined by an empirically unobservable random variable of real money balances depending on the future realization of the perturbations.

As in the deterministic case, for small government demand,

- monetary rational expectations orbits converge to the positive balanced path
- under additional conditions which are not universally satisfied;
- otherwise, monetary orbits with hyperinflation and hyperdeflation may occur,

implying a decisive role to autonomous and nonstochastic government policy and

making policy decisions difficult to halt diverging orbits under rational expectations.

3. Extensions and Implications

– **more assets:** government bonds/debt, stocks, shares, inventory

– **noncompetitive markets:**

[18] a procedure used widely in macroeconomic applications, see Taylor & Uhlig (1990); Marimon & Scott (1999)
[19] as in Wang (1993); Duffie, Geanakoplos, Mas-Colell & McLennan (1994)
[20] which is a weaker and empirically unobservable concept (as by Bhattacharya & Majumdar, 2004, and others)

- monopolistic competition in commodity markets
- noncompetitive labor markets: monopolies and bargaining,

– **adaptive expectations:** forecasting rules matter! naive, learning, statistical updating which change the dynamic features significantly

– disequilibrium trading: sequential opening of markets and rationing

– **more micro structure:** agent based?

$\Longrightarrow$ which seem to be doable within the same nonlinear and stochastic framework!

A Appendix: On Random Dynamical Systems

In order to understand the *dynamic* behavior of a system defined by a stochastic difference equation like (4.10), it is necessary to introduce some additional mathematical concepts not commonly used in dynamic macroeconomics. They supply the necessary tools in order to analyze, from a time series perspective, the simultaneous interaction of the *dynamic* forces of a mapping combined with the *stochastic implications of ongoing randomness*. They are also necessary to formulate the main results of this paper.

The classical theory of stochastic processes and the theory of random dynamical systems are two closely related mathematical tools to analyze the evolution of a dynamical system subjected to regular and ongoing exogenous stochastic perturbations. The mathematical literature provides different approaches, depending on the type of questions or characterizations for which an answer is sought. Even the use of the term random dynamical system is not used uniformly in the literature[21]. Depending on the perspective and the objective of the desired properties and results one may be more appropriate than the other. For many economic applications, it seems most natural to use an approach which uses stochastic orbits as the primitive object of investigations, as proposed by Arnold (1998), since these are the typical observable objects in economic empirical work, rather than distributions or Markov kernels which are theoretical concepts typically unobservable empirically as well as generically.

Mathematical tools

Following Arnold (1998), let $\vartheta : \Omega \to \Omega$ be a measurable mapping on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with measurable inverse ϑ^{-1} and assume that

– ϑ is measure preserving with respect to $\mathbb{P}$ (i.e. $\mathbb{P}(E) = \mathbb{P}(\vartheta^{-1}(E))$, for all $E \in \mathcal{F}$) and

– $\mathbb{P}$ is ergodic with respect to ϑ (i.e. $\vartheta^{-1}(E) = E$ implies $\mathbb{P}(E) = 0$ or $\mathbb{P}(E) = 1$).

– Denote by ϑ^t the t-th iterate of the map ϑ.

The collection $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\})$ is called an *ergodic metric dynamical system.*

To connect these probabilistic structures with the dynamic properties of the mapping $S : \Sigma \times X \to X$, $\Sigma \subset \mathbb{R}^d$, $X \subset \mathbb{R}^k$, let $\xi : \Omega \to \mathbb{R}^d$ denote a measurable map such that the

[21]For example, Bhattacharya & Majumdar (2004) use the term random dynamical system in a different way than the one adopted here from Arnold (1998)

stochastic process $\{\xi_t\}_{t\in\mathbb{Z}}$ has the so called canonical representation

$$\xi_t(\omega) \equiv \xi(\vartheta^t\omega). \tag{A.1}$$

Finally, using the evaluation map $\xi : \Omega \to \Sigma$, $\xi(\omega) := \omega(0) \equiv Z(\omega)$ the stochastic difference equation (4.10) can now be rewritten (using the same symbol S) as

$$x_{t+1} = S(\vartheta^t\omega)x_t := S(\xi(\vartheta^t\omega), x_t). \tag{A.2}$$

Therefore, for any initial point x_0, repeated applications of S under the perturbation ω induce the measurable mapping $\phi : \mathbb{N} \times \Omega \times X \to X$ defined by

$$\phi(t, \omega, x_0) := \begin{cases} S(\vartheta^{t-1}\omega) \circ \ldots \circ S(\omega)x_0 & \text{if} \quad t > 0 \\ x_0 & \text{if} \quad t = 0 \end{cases} \tag{A.3}$$

such that $x_t = \phi(t, \omega, x_0)$ is the state of the system at time t. The map (A.3) (or equivalently (A.2)) defines a *random dynamical system* time in the sense of Arnold(1998, Chapters 1 & 2) in forward time. For any initial value x_0 and perturbation $\omega \in \Omega$, the sequence $\gamma(x_0) := \{x_t\}_{t=0}^{\infty}$ with $x_t = \phi(t, \omega, x_0)$, $t \in \mathbb{N}$ defines a stochastic orbit of the system S.

Random fixed points are the respective extension of the concept of a fixed point of deterministic systems to the random case, see Schmalfuß (1996, 1998), Schenk-Hoppé & Schmalfuss (2001). Random fixed points induce stationary orbits (or stationary solutions) of the random dynamical system.

Definition A.1
*A **random fixed point** of the random dynamical system φ generated by $S(\omega) : X \to X$ is a random variable $x^* : \Omega \longrightarrow X$ such that $\mathbb{P}$-almost surely*

$$x^*(\vartheta\omega) = \varphi(1, \omega, x^*(\omega)) \equiv S(\omega)x^*(\omega). \tag{A.4}$$

It is called stable if there exists an open set (a random neighborhood) $U(\omega) \subset X$ with $x^*(\omega) \in N(\omega)$ such that $\mathbb{P}$-almost surely

$$\lim_{t\to\infty} ||\varphi(t, \omega, x_0) - x^*(\vartheta^t\omega)|| = 0 \qquad \textit{for all} \quad x_0 \in U(\omega). \tag{A.5}$$

A random fixed point $x^*(\omega)$ defines a sample path in the state space X which depends on the perturbation only. The first part of the definition implies that $x^*(\vartheta^{t+1}\omega) = S(\vartheta^t\omega)x^*(\vartheta^t\omega)$. In other words, the random fixed point $x^* : \Omega \to X$ generates orbits $\{x^*(\vartheta^t)\}_{t\in\mathbb{N}} = \{x^* \circ \vartheta^t\}_{t\in\mathbb{N}}$, $\omega \in \Omega$ which solve the random difference equation (A.3). In addition, the process $\{x^* \circ \vartheta^t\}_{t\in\mathbb{N}}$ is stationary and ergodic since ϑ is measure preserving and ergodic.

Let $x^*\mathbb{P}$ denote the probability distribution of x^* defined by

$$x^*\mathbb{P}(B) := \mathbb{P} \circ (x^*)^{-1}(B)) = \mathbb{P}\{\omega \in \Omega \,|\, x^*(\omega) \in B\} \tag{A.6}$$

which is invariant, since $\mathbb{P}$ is invariant under ϑ. Thus, $((x^*\vartheta)\mathbb{P})(B) = (x^*\mathbb{P})(B)$. If $\mathbb{E}||x^*|| < \infty$, stability and ergodicity together imply that for any $B \in \mathcal{B}(X)$,

$$\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} 1_B(\varphi(t, \omega, x_0(\omega))) = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} 1_B(x^*(\vartheta^t\omega)) = x^*\mathbb{P}(B) \tag{A.7}$$

for all $x_0(\omega) \in U(\omega)$ $\mathbb{P}$-almost surely, where $1_B(x) \in \{0, 1\}$ is the indicator function with value 1 if and only if $x \in B$. In other words, the empirical law of an asymptotically converging orbit induces the true stationary probability distribution.

An Example

It is straightforward to develop an intuitive understanding of these concepts and of the simultaneous interaction of the dynamic forces and the random perturbations from generic one dimensional examples. Let the stochastic difference equation of the type (4.10) be given by a *parameterized* dynamical system $G : \mathbb{R}^m \times \ \mathbb{R} \to \mathbb{R}$ defining a family of mappings

$$G(\xi, \cdot) : \mathbb{R} \to \mathbb{R} \qquad x \mapsto G(\xi, x). \tag{A.8}$$

Here $\xi \in \mathbb{R}^m$ is a vector of parameters of the system while $x \in X$ is the vector of endogenous variables defining the state of the system at any one time. Then, the one step time change of x for a *fixed* value of the parameter $\xi \in \mathbb{R}^m$ is given by

$$x_{t+1} = G_\xi(x_t) \qquad G_\xi \equiv G(\xi, \cdot), \tag{A.9}$$

i.e. the *dynamics* follows the rules and the description of a deterministic dynamical system once the value of a *particular* ξ is given. Assume that ξ is driven by a stochastic process with a

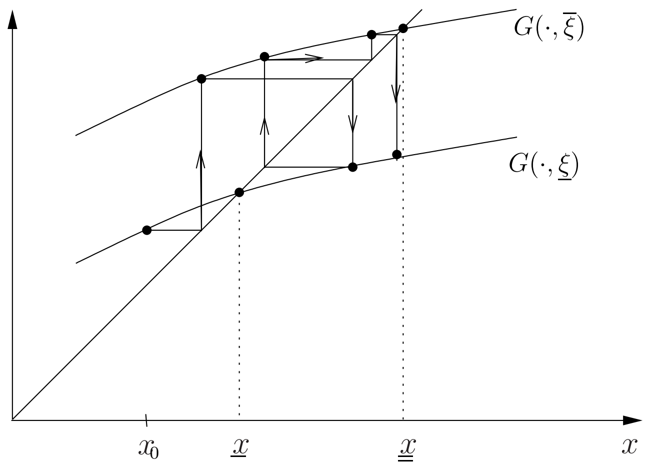


Figure A.1: Orbit for $(x_0, \underline{\xi})$ converging to the trapping set $[\underline{x}, \underline{\underline{x}}]$ for $\omega = (\ldots, \underline{\xi}, \overline{\xi}, \underline{\xi}, \overline{\xi}, \overline{\xi}, \underline{\xi}, \ldots)$

given random path of the perturbation described by $\omega := (\ldots, \xi_{s-2}, \xi_{s-1}, \xi_s, \xi_{s+1}, \ldots)$. Then, the change of ξ over time implies at each iteration t the application of a *different mapping* $G(\xi_t, \cdot)$ for the determination of x_t. If, for example, G is a contraction for all ξ assuming only two values $\{\underline{\xi}, \underline{\underline{\xi}}\}$, then for any path ω of the perturbation the associated evolution of x can be visualized as in Figure A.1 for any given initial condition x_0. In this case, the orbit will eventually be trapped in some compact interval $[\underline{x}, \underline{\underline{x}}]$, suggesting that the limiting behavior may be stationary if the perturbation is stationary as well.

Stability of a random fixed point

In the case of contractions $G(\xi, \cdot)$, random fixed points exist. But the literature shows that a weaker condition called average contractivity is often sufficient (Arnold & Crauel, 1992; Schenk-Hoppé & Schmalfuss, 2001). Then, converging sample paths of the system φ can be described by forward iteration for all initial values $x_0 \in U(\omega)$ as in Figure A.2 and the limiting orbits provide a good approximation of the random fixed point in time domain, while the statistical characteristics of the stationary distribution can be computed from long enough time series.

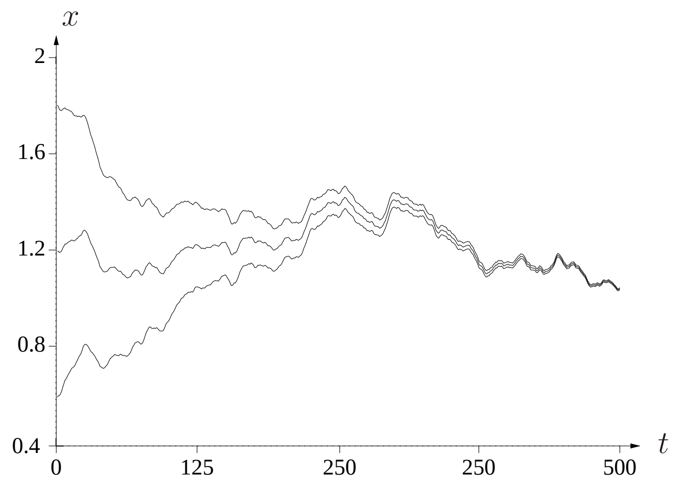


Figure A.2: Convergence of orbits to random fixed point for three initial values x_0 and the same ω

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