経済リスクの統計学の新展開2013

稀な事象と再起的事象[1]

国友直人[2] & 川崎能典[3]
(共編)

2014年2月

[1] この報告書は文部科学省・科学研究費プロジェクト「経済リスクの統計学の新展開：稀な事象と再起的事象」(2013年度〜) が2013年12月26日に東京大学経済学研究科において開催した研究集会における講演の内容をまとめたものである。

[2] 東京大学大学院経済学研究科
[3] 統計数理研究所

概要

日本では2011年3月に発生した東日本大震災を一つの契機に「通常の常識では起こりにくいとされる事象」についてのリスク解析や対策の重要性についての認識が高まっている。経済・社会における近年の現象でも2008年に起きたリーマンショック・経済危機、2011年から経験しているヨーロッパ諸国の金融危機なども我々が暮らしている国際的な経済社会においては、従来の議論ではほとんど考慮されていない経済変動の例である。こうした事前には予想が困難で無視されてきた事象、自然災害、経済変動の中でも実際に起きると大きな影響のある不確実な事象を科学的に理解し、有効な対策を考察する研究が必要であり重要である。本研究プロジェクトでは近年の日本など現代の経済・社会の理解にとって重要になっている「きわめて稀に起きる事象」と「しばしば起きる事象」の評価・分析法について研究する予定である。「稀な事象」に関わる経済リスクの分析という課題について理論的・実証的な観点から分析することにより、科学的根拠にもとづいた経済・社会における「経済リスクの分散化」という方策、公共的政策のあり方の提案することが目標である。近年に特に関心が高まっている「従来の常識では希にしか起きない、無視できると見なされる事象」と「ときどき経済・社会では起きると見なされる現象」の科学的解析を柱に、確率論・統計学と経済学・金融（ファイナンス・保険）における既存の理論と現実の乖離、新しい数理的理論の構築と応用、新しい数理的理論を踏まえた「経済リスクの解析と分散化の方策」について研究活動を行う予定である。本研究プロジェクトでは経済リスクを (i) 社会・人口リスク, (ii) 自然災害と極端な事象のリスク, (iii) 経済・金融・保険の対象となるリスク、に関連した３つの領域の経済リスクに分類し、リスクに係わる問題と相互に関わる総合的問題という二つの方向から問題を理論的に解明し、総合的な研究をふまえた経済リスクの科学的制御・管理の方策を提言することを目指す。さらに、経済統計学における研究・研究者と確率論・統計学など数理科学の関係者、さらに金融（ファイナンス）の関係者を交え、現代の社会・経済においては重要ではあるが、既存の研究分野では十分に取り上げられなかった研究課題を研究するとともに、経済リスクの分析と科学的制御・統計的管理法についての共同研究を行う計画である。

今回の研究集会では、経済リスクの統計学を巡るさまざまなトピックについて報告を行う機会であった。このような情報交換が関係者の知的刺激となり、経済リスクの統計学の今後の展開の一助になることを期待する次第である。

研究集会・プログラム

＜セッション I：リスク尺度と統計分析＞
Chair： 一場知之 (カリフォルニア大学サンタバーバラ校)

9:50〜10:30「リスク尺度と法則不変性」楠岡成雄 (東京大学)
10:35〜11:15「Backtesting distortion risk measure and its backtestability”」Hideatsu Tsukahara(塚原英敦, 成城大学)
11:20〜12:00「Decreasing Trends in Stock-Bond Correlations」Tatsuyoshi Okimoto(沖本竜義, 一橋大学)

(休憩)

＜セッション II：保険市場と統計分析＞
Chair： 松井宗也 (南山大学)

14:00〜14:40「公表データにもとづく損保リスクモデル」 田中周二 (日本大学)
14:45〜15:25「On a generalization from ruin to default in Levy insurance risks」Yasutaka Shimizu(清水泰隆, 大阪大学)

＜セッション III：高頻度金融データと統計分析＞
Chair： 一場知之 (カリフォルニア大学サンタバーバラ校)

15:30〜16:10「先物市場の高頻度データ」川崎能典 (統計数理研究所)
16:15〜16:45「高頻度金融データ分析とシグナル・ノイズ」国友直人 (東京大学)

リスク尺度と法則不変性

楠岡成雄
東京大学大学院数理科学研究科

金融リスクの計量化

考え方： Föllmer の考え方が代表的
１期間モデルを基礎にする

良いレビュー

Hans Föllmer, Thomas Knispel

Convex Risk Measures:
Basic Facts, Law-invariance and beyond,
Asymptotics for Large Portfolios

この中では中心極限定理との関係（アクチュアリー的な問題）
も述べられている

Backtesting Distortion Risk Measures

Hideatsu Tsukahara

(tsukahar@seijo.ac.jp)

Dept of Economics, Seijo University

Contents

1. Introduction to Distortion Risk Measures (DRMs)

2. Statistical Estimation
   - Asymptotic results
   - Estimation of asymptotic variance
   - Bias correction

3. Backtesting DRMs
   - Unconditional & Conditional approaches
   - Backtestability & Elicitability

1. Distrotion Risk Measures

A random variable X represents a loss of some financial position

DRM

Any coherent risk measure satisfying law invariance and comonotonic additivity is a **distortion risk measure**:

$$\rho(X) = \rho(F) := \int_{[0,1]} F^{-1}(u)\,\mathrm{d}D(u) = \int_{\mathbb{R}} x\,\mathrm{d}D \circ F(x).$$

where F is the df of X, F^{-1} is the quantile of X, and D is a convex distortion, i.e., a df on $[0,1]$.

►► a.k.a. spectral risk measure (Acerbi), weighted V@R (Cherny)

Example: *Expected Shortfall (ES)*

The expected loss that is incurred when VaR is exceeded:

$$\mathrm{ES}_\theta(X) := \frac{1}{\theta}\int_{1-\theta}^{1} F^{-1}(u)\,\mathrm{d}u \doteq \mathrm{E}(X \,|\, X \geq \mathrm{VaR}_\theta(X))$$

Taking distortion of the form

$$D_\theta^{\mathrm{ES}}(u) = \frac{1}{\theta}\big[u - (1-\theta)\big]_+, \quad 0 < \theta < 1$$

yields ES as a distortion risk measure.

►► Typical values for θ are: 0.05, 0.01, ...

Other Examples of DRM:

- *Proportional Hazards*:

$$D_\theta^{\mathsf{PH}}(u) = 1 - (1-u)^\theta,$$

- *Proportional Odds*:

$$D_\theta^{\mathsf{PO}}(u) = \frac{\theta u}{1-(1-\theta)u}$$

- *Gaussian* (Wang transform):

$$D_\theta^{\mathsf{GA}}(u) = \Phi(\Phi^{-1}(u) + \log\theta)$$

★ See Tsukahara (2009) *Mathematical Finance*, vol. 19.

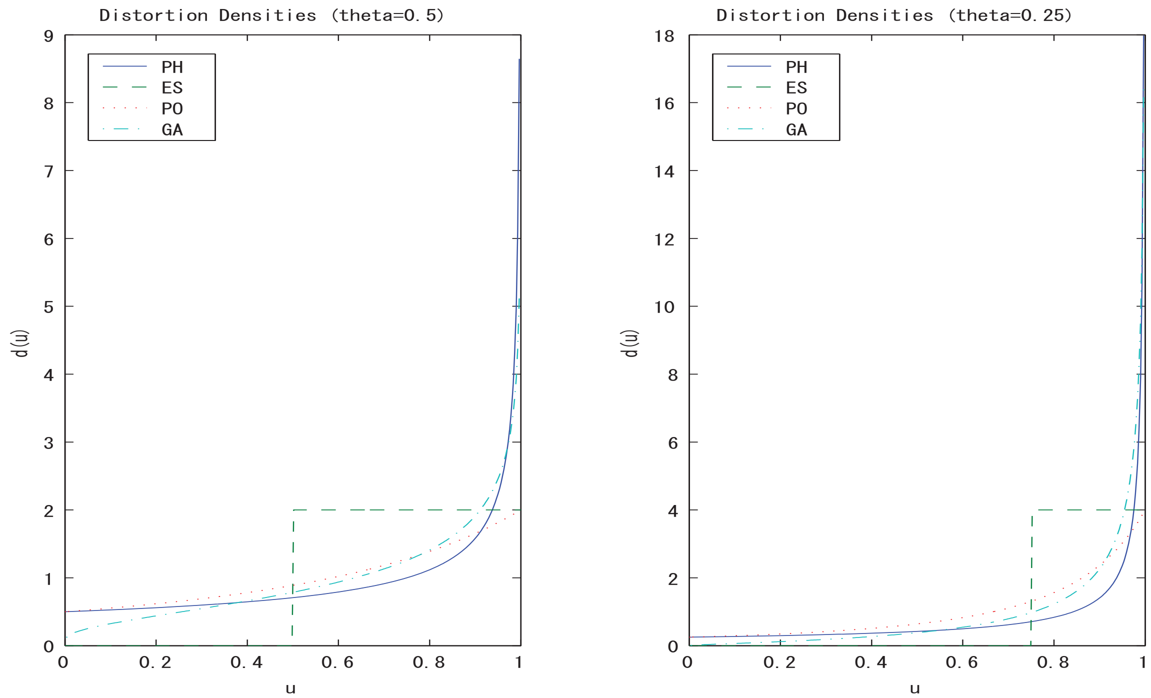


Figure 1: Distortion densities ($\theta = 0.5$, $\theta = 0.25$)

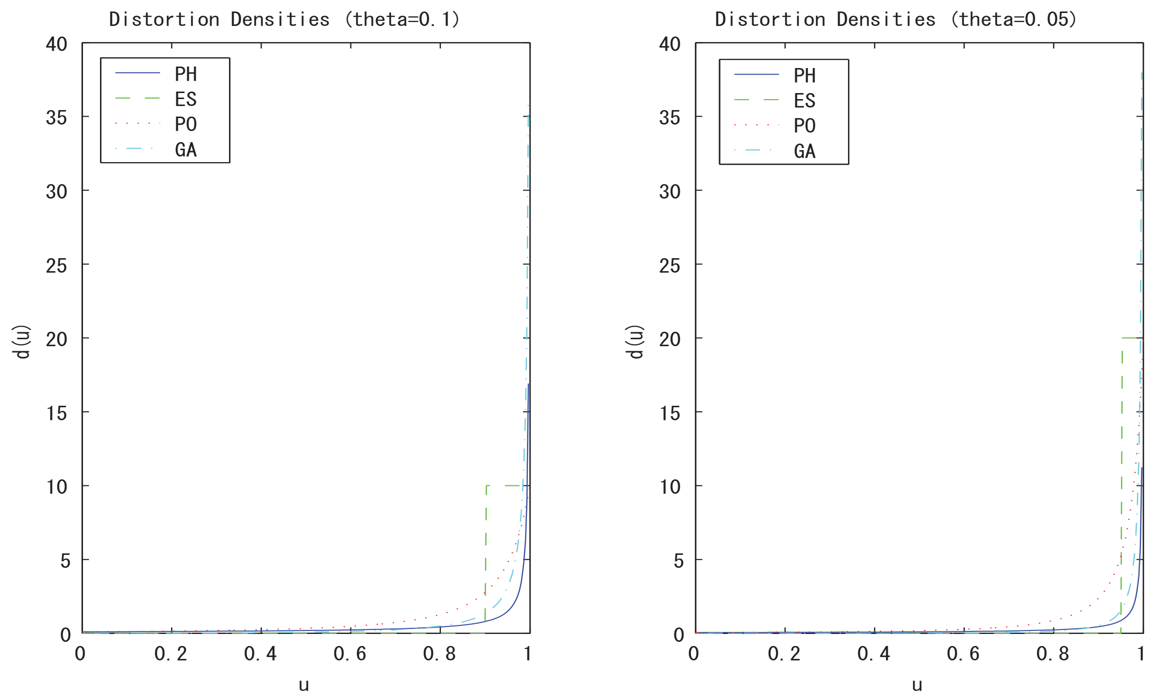


Figure 2: Distortion densities ($\theta = 0.1$, $\theta = 0.05$)

2. Statistical Estimation

$(X_n)_{n\in\mathbb{N}}$: strictly stationary process with $X_n \sim F$

$\mathbb{F}_n$: empirical df based on the sample $X_1, \ldots, X_n$

A natural estimator of $\rho(F)$ is

$$\widehat{\rho}_n = \int_0^1 \mathbb{F}_n^{-1}(u)\, dD(u)$$

$$= \sum_{i=1}^{n} c_{ni} X_{n:i}, \qquad c_{ni} := D\left(\frac{i-1}{n}, \frac{i}{n}\right]$$

This type of statistics is called L-statistics

Strong consistency

Let $d(u) = \frac{\mathrm{d}}{\mathrm{d}u} D(u)$ for a convex distortion D, and $1 \leq p \leq \infty$, $1/p + 1/q = 1$. Suppose

- $(X_n)_{n\in\mathbb{N}}$ is an ergodic stationary sequence
- $d \in L^p(0,1)$ and $F^{-1} \in L^q(0,1)$

Then

$$\widehat{\rho}_n \longrightarrow \rho(F), \quad \text{a.s.}$$

For a proof, see van Zwet (1980, AP)

[All we need is SLLN and Glivenko-Cantelli Theorem].

Assumptions for asymptotic normality:

- $(X_n)_{n\in\mathbb{N}}$ is strongly mixing with rate
$$\alpha(n) = O(n^{-\theta-\eta}) \quad \text{for some } \theta \geq 1+\sqrt{2},\ \eta > 0$$

- For F^{-1}-almost all u, d is continuous at u

- $|d| \leq B, \quad B(u) := Mu^{-b_1}(1-u)^{-b_2}$,

- $|F^{-1}| \leq H, \quad H(u) := Mu^{-d_1}(1-u)^{-d_2}$

Assume b_i, d_i & θ satisfy $b_i + d_i + \dfrac{2b_i+1}{2\theta} < \dfrac{1}{2}, \quad i = 1, 2$

Set

$$\sigma(u,v) := [u \wedge v - uv] + \sum_{j=1}^{\infty}[C_j(u,v) - uv] + \sum_{j=1}^{\infty}[C_j(v,u) - uv],$$

$$C_j(u,v) := \mathrm{P}(X_1 \le F^{-1}(u), X_{j+1} \le F^{-1}(v))$$

Theorem (Asymptotic Normality)

Under the above assumptions, we have

$$\sqrt{n}(\widehat{\rho}_n - \rho(F)) \xrightarrow{\mathcal{L}} N(0, \sigma^2),$$

where

$$\sigma^2 := \int_0^1 \int_0^1 \sigma(u,v)d(u)d(v)\,\mathrm{d}F^{-1}(u)\mathrm{d}F^{-1}(v) < \infty$$

- GARCH model:

  $X_t = \sigma_t Z_t, \quad (Z_t):$ i.i.d.

  $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$

  ►► If the stationary distribution has a positive density around 0, then GARCH is strongly mixing with exponentially decaying $\alpha(n)$

- Stochastic Volatility model:

  $X_t = \sigma_t Z_t, \quad (Z_t):$ i.i.d., $\quad (\sigma_t):$ strictly stationary positive

  (Z_t) and (σ_t) are assumed to be independent

  ►► The mixing rate of (X_t) is the same as that of (σ_t)

Estimation of Asymptotic Variance

Let

$$Y_n := \int [\mathbf{1}\{X_n \le x\} - F(x)] d(F(x))\, \mathrm{d}x, \quad n \in \mathbb{Z}.$$

Then Y_n is also a strictly stationary and strongly mixing sequence with the same mixing coefficient as X_n. Furthermore

$$\mathrm{E}(Y_n) = 0, \qquad \sigma^2 = \sum_{h=-\infty}^{\infty} \gamma(h) < \infty,$$

where $\gamma(h) := \mathrm{E}(Y_n Y_{n+h})$.

Let f be the spectral density of (Y_n). Then

$$\sum_{h=-\infty}^{\infty} \gamma(h) = 2\pi f(0)$$

$\Longrightarrow$ Use a consistent estimator of $f(0)$ (JHB approach)

The lag window estimator is defined by

$$\widehat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{|k|<K_n} w(k/K_n) \widehat{\gamma}_n(k) \cos k\lambda$$

where w is a "lag window", and $\widehat{\gamma}_n(k) := \frac{1}{n} \sum_{i=1}^{n-k} Y_i Y_{i+k}$

▶▶ F in the expression of Y_n is unknown, so we replace it with the empirical df. That is, we use

$$Y_{i,n} := \int [\mathbf{1}\{X_i \le x\} - \mathbb{F}_n(x)] d(\mathbb{F}_n(x)) \, \mathrm{d}x, \quad i = 1, \dots, n$$

Let

$$\widetilde{\gamma}_n(k) := \frac{1}{n} \sum_{i=1}^{n-k} Y_{i,n} Y_{i+k,n} \quad \text{and} \quad \widetilde{f}_n(0) := \frac{1}{2\pi} \sum_{|k|<K_n} w(k/K_n) \widetilde{\gamma}_n(k)$$

Then $2\pi \widetilde{f}_n(0)$ should give a consistent estimator of the asymptotic variance σ^2

Theorem

In addition to the conditions assumed in the above theorem, suppose that J is Lipschitz, w is a bounded even function which is continuous in $[-1, 1]$ with $w(0) = 1$ and equals 0 outside $[-1, 1]$. Also assume $\mathrm{E}|Y_n|^4 < \infty$ and the fourth-order cumulants

$$\begin{aligned} \kappa(h, i, j) := &\mathrm{E}(Y_1 Y_{1+h} Y_{1+i} Y_{1+j}) - \gamma(h)\gamma(i - j) \\ &- \gamma(i)\gamma(h - j) - \gamma(j)\gamma(h - i) \end{aligned}$$

are summable: $\sum_{h,i,j=-\infty}^{\infty} |\kappa(h, i, j)| < \infty$.

Let K_n be a sequence of integers such that $K_n \to \infty$ and $K_n/\sqrt{n} \to 0$ as $n \to \infty$. Then we have

$$2\pi \widetilde{f}_n(0) \xrightarrow{L_1} \sigma^2, \quad n \to \infty$$

Bias of L-statistics

By Fubini, for any df F and any distortion D,

$$\int_{[0,1]} F^{-1}(u)\,\mathrm{d}D(u) = -\int_{\infty}^{0} D(F(x))\,\mathrm{d}x + \int_0^{\infty} [1 - D(F(x))]\,\mathrm{d}x$$

By Fubini and Jensen, for convex D,

$$\begin{aligned}
&\mathrm{E}\left[\int_{[0,1]} \mathbb{F}_n^{-1}(u)\,\mathrm{d}D(u)\right] \\
&= \int_{\infty}^{0} \mathrm{E}(-D(\mathbb{F}_n(x)))\,\mathrm{d}x + \int_0^{\infty} \mathrm{E}[1 - D(\mathbb{F}_n(x))]\,\mathrm{d}x \\
&\leq \int_{\infty}^{0} -D(\mathrm{E}(\mathbb{F}_n(x)))\,\mathrm{d}x + \int_0^{\infty} [1 - D(\mathrm{E}(\mathbb{F}_n(x)))]\,\mathrm{d}x \\
&= \int_{[0,1]} F^{-1}(u)\,\mathrm{d}D(u)
\end{aligned}$$

Therefore

$$\mathrm{E}(\widehat{\rho}_n) - \rho(F) \leq 0$$

$\Longrightarrow$ $\widehat{\rho}_n$ has a negative bias

Need bias correction methods. For the i.i.d. case,

- Xiang (1995): Modify the form of L-statistics
- Kim (2010): Bootstrap-based method

►► The bootstrap methodology is still available in the dependent case (see Lahiri (2003), Example 4.8).

Moving Block Bootstrap (MBB)

- Data: $X_1, \ldots, X_n$
- Block size: ℓ, $\quad \#$ of blocks: $N := n - \ell + 1$
- Blocks: $\mathcal{B}_i = (X_i, \ldots, X_{i+\ell-1}),\ \ i = 1, \ldots, N$

Resample $k = [n/\ell]$ blocks from $\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ with replacement to get $\mathcal{B}_1^*, \ldots, \mathcal{B}_k^*$

Write $\mathcal{B}_i^* = (X^*_{(i-1)\ell+1}, \ldots, X^*_{i\ell})$

$\Longrightarrow X_1^*, \ldots, X_{k\ell}^*$: MBB sample

MBB version of $\widehat{\rho}_n$ is

$$\widehat{\rho}_n^* = \frac{1}{n}\sum_{i=1}^{n} c_{ni} X^*_{n:i}, \qquad c_{ni} := D\left(\frac{i-1}{n}, \frac{i}{n}\right]$$

Validity of MBB follows from an argument specific to our case.

►► The approach based on Hadamard differentiability of L-functional

$$T(F) := \int_0^1 h(F^{-1}(u))J(u)\,\mathrm{d}u$$

is not convenient. See Boos (1979, AS), Lahiri (2003), Section 12.3.5.

Simulation example: inverse-gamma SV model

$$X_t = \sigma_t Z_t$$

Z_t i.i.d. N(0,1) and $V_t = 1/\sigma_t^2$ satisfies

$$V_t = \rho V_{t-1} + \varepsilon_t,$$

where $V_t \sim$ Gamma(a, b) for each t, (ε_t) i.i.d. rv's, and $0 \leq \rho < 1$

$\Rightarrow$ X_t has scaled t-distribution with $\nu = 2a$, $\sigma^2 = b/a$

►► Lawrance (1982): the distribution of ε_t is compound Poisson

►► Can be shown that (X_t) is geometrically ergodic

Simulation results for estimating VaR, ES & PO risk measures with inverse-gamma SV observations ($n = 500$, # of replications $= 1000$)

$X_t = \sigma_t Z_t$, where $V_t = 1/\sigma_t^2$ follows AR(1)
with gamma(2,16000) marginal & $\rho = 0.5$, Z_t i.i.d. N(0,1)

		VaR		ES		PO	
	θ	bias	RMSE	bias	RMSE	bias	RMSE
	0.1	0.0692	10.9303	−2.2629	22.1361	−1.7739	17.5522
SV	0.05	2.5666	17.6755	−1.2168	37.2719	−2.0200	28.5053
	0.01	14.9577	61.2290	−11.9600	103.9269	−15.7888	73.7147
	0.1	0.7976	10.5893	−1.2914	19.5756	−1.3574	15.3271
i.i.d.	0.05	0.7974	16.1815	−2.6346	31.3166	−2.8342	23.9933
	0.01	10.6838	53.2567	−12.9355	95.9070	−15.8086	69.5425

Simulation results for estimating variance and bias of PO risk measure

($n = 500$, $K_n = 5$, Parzen kernel $w(x) = 1 - x^2$, block size$= 5$, # of bootstrap replicates $= 800$, # of replications $= 10000$)

	ρ	θ	MC bias	MC s.e.	$\widehat{\text{A-s.e.}}$	BS bias	BS s.e.
IG-SV		0.1	−0.8328	15.4456	14.0956	−0.8151	13.9829
$\alpha = 2$	0.1	0.05	−2.0580	24.6961	20.9719	−1.8170	20.6863
$\beta = 16000$		0.01	−13.3608	68.9197	46.6943	−10.2030	46.0788
IG-SV		0.1	−0.3345	10.7979	10.4231	−0.6812	10.3933
$\alpha = 4$	0.1	0.05	−1.3663	15.1946	14.0623	−1.3511	13.9725
$\beta = 48000$		0.01	−6.8659	34.4725	26.4183	−6.0749	26.4446
IG-SV		0.1	−0.5432	9.0853	8.8370	−0.6048	8.8281
$\alpha = 10$	0.1	0.05	−1.1786	11.7923	11.2289	−1.1263	11.2003
$\beta = 144000$		0.01	−5.8673	22.9686	18.7767	−4.4474	18.9614

	ρ	θ	MC bias	MC s.e.	$\widehat{\text{A-s.e.}}$	BS bias	BS s.e.
IG-SV		0.1	−1.0054	17.5469	15.0711	−0.8793	14.6925
$\alpha = 2$	0.5	0.05	−2.2714	27.1465	22.0852	−1.9450	21.4374
$\beta = 16000$		0.01	−13.9208	74.8887	47.7943	−10.6541	46.8379
IG-SV		0.1	−0.5791	11.4856	10.7162	−0.6957	10.5906
$\alpha = 4$	0.5	0.05	−1.3472	15.7116	14.4718	−1.3994	14.2658
$\beta = 48000$		0.01	−7.4680	35.1014	26.7575	−6.1939	26.7115
IG-SV		0.1	−0.8213	9.2632	8.9299	−0.6062	8.8957
$\alpha = 10$	0.5	0.05	−1.0663	11.9443	11.3608	−1.1368	11.2996
$\beta = 144000$		0.01	−5.7987	23.1130	18.8147	−4.4769	18.9896

	ρ	θ	MC bias	MC s.e.	$\widehat{\text{A-s.e.}}$	BS bias	BS s.e.
IG-SV		0.1	−2.0408	28.2224	15.5015	−0.9609	14.7212
$\alpha = 2$	0.9	0.05	−4.8204	42.1005	22.1388	−2.0483	20.9685
$\beta = 16000$		0.01	−23.5844	106.4374	43.6402	−10.1556	42.4681
IG-SV		0.1	−1.1973	14.9586	11.1112	−0.7274	10.8092
$\alpha = 4$	0.9	0.05	−2.2346	20.8199	14.8937	−1.4366	14.4566
$\beta = 48000$		0.01	−10.2968	42.5085	26.3137	−6.1439	26.0855
IG-SV		0.1	−0.5956	10.3666	9.1248	−0.6262	9.0293
$\alpha = 10$	0.9	0.05	−1.4212	13.6534	11.5934	−1.1609	11.4494
$\beta = 144000$		0.01	−6.3827	25.2688	18.8986	−4.4824	19.0079

	ρ	θ	MC bias	MC s.e.	$\widehat{\text{A-s.e.}}$	BS bias	BS s.e.
		0.1	−0.5734	8.2886	8.0638	−0.5619	8.0667
$N(0, 126.5^2)$	iid	0.05	−1.1557	10.1327	9.8175	−1.0116	9.8117
		0.01	−4.4730	18.1714	14.9659	−3.6136	15.2192
		0.1	−0.9038	15.3536	13.9544	−0.8121	13.8815
$t_4(0, 126.5^2)$	iid	0.05	−1.8468	24.3247	20.8781	−1.7928	20.6468
		0.01	−12.5608	73.3170	46.9313	−10.2243	46.3147
		0.1	−0.5538	10.7575	10.3154	−0.6687	10.2909
$t_8(0, 126.5^2)$	iid	0.05	−1.4518	14.9271	13.9883	−1.3379	13.9033
		0.01	−6.8385	34.8496	26.4076	−6.8385	26.4531
		0.1	−0.5470	9.0123	8.8209	−0.5985	8.8127
$t_{20}(0, 126.5^2)$	iid	0.05	−1.1266	11.6915	11.2178	−1.1176	11.1965
		0.01	−5.5631	22.9298	18.7808	−4.4588	18.9697

3. Backtesting

Purpose of Backtesting:

1. Monitor the performance of the model and estimation methods for risk measurement
2. Compare relative performance of the models and methods

Idea

ex ante risk measure forecasts from the model

vs.

ex post realized portfolio loss

Setup

Entire observations: $X_1, \dots, X_T$

Estimation window size $= n$, $\quad m := T - n$

	data	estimand	realized loss
1.	$X_1, \dots, X_n$	$\rho(X_{n+1})$	X_{n+1}
2.	$X_2, \dots, X_{n+1}$	$\rho(X_{n+2})$	X_{n+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m.$	$X_{T-n}, \dots, X_{T-1}$	$\rho(X_T)$	X_T

Two approaches to risk measurement

Assume that the loss process $(X_t)_{t\in\mathbb{Z}}$ is a stationary time series with stationary df F. At time t, we have two options:

I. Unconditional Approach

Look at the risk measure associated with $F(x) = \mathrm{P}(X_{t+1} \le x)$
(For a large time horizon; credit risk and insurance)

II. Conditional Approach

For a given filtration $\mathscr{F}_t$, look at the risk measure associated with the conditional df $F_t(x) := \mathrm{P}(X_{t+1} \le x \,|\, \mathscr{F}_t)$,
(For a short time horizon; market risk)

In the case of VaR

- Unconditional VaR, denoted by VaR_α, satisfies

$$\mathrm{E}\big(\mathbf{1}\{X_{t+1} \ge \mathrm{VaR}_\alpha\}\big) = \alpha$$

  But $\mathbf{1}\{X_{t+1} \ge \mathrm{VaR}_\alpha\}$'s might not be independent

- Conditional VaR, denoted by VaR_α^t, satisfies

$$\mathrm{E}\big(\mathbf{1}\{X_{t+1} \ge \mathrm{VaR}_\alpha^t\} \,|\, \mathscr{F}_t\big) = \alpha$$

By Lemma 4.29 of MFE, if (Y_t) is a sequence of Bernoulli rv's adapted to $(\mathscr{F}_t)$ and if $\mathrm{E}(Y_{t+1} \,|\, \mathscr{F}_t) = p > 0$, then (Y_t) must be i.i.d.

Therefore $\mathbf{1}\{X_{t+1} \geq \mathrm{VaR}_\alpha^t\}$, $t = n, \ldots, T-1$ are i.i.d. Bernoulli rv's.

$$\Downarrow$$

This gives the grounds for backtesting using $\mathbf{1}\{X_{t+1} \geq \widehat{\mathrm{VaR}}_\alpha^t\}$, where $\widehat{\mathrm{VaR}}_\alpha^t$ is an estimate of the VaR associated with the conditional df $F_t(x) := \mathrm{P}(X_{t+1} \leq x \mid \mathscr{F}_t)$. Namely,

(i) Test $\displaystyle\sum_{t=n}^{T-1} \mathbf{1}\{X_{t+1} \geq \widehat{\mathrm{VaR}}_\alpha^t\} \sim \mathrm{Bin}(m, \alpha)$

(ii) Test independence of $\mathbf{1}\{X_{t+1} \geq \widehat{\mathrm{VaR}}_\alpha^t\}$, $t = n, \ldots, T-1$ (e.g., runs test)

Backtesting DRMs

Note that, with $d(u) = \frac{\mathrm{d}}{\mathrm{d}u} D(u)$ and $X \sim F$,

$$\rho(X) = \int_{-\infty}^{\infty} x \, \mathrm{d}D \circ F(x) = \int_{-\infty}^{\infty} x d(F(x)) \, \mathrm{d}F(x)$$

$$= \mathrm{E}[X d(F(X))]$$

Thus $X d(F(X)) - \rho(X)$ has mean 0 unconditionally.

►► In the conditional case, $\mathrm{E}[X_{t+1} d(F_t(X_{t+1})) \mid \mathscr{F}_t] = \rho_t(X_{t+1})$, but this does not help much.

I.I.D. case (rough-and-ready)

If $X_1, \dots, X_T$ are i.i.d. with df F, then we can base the backtesting of our method/model on

$$X_{n+1} d(\widehat{\mathbb{F}}_{1:n}(X_{n+1})) - \widehat{\rho}_{(1:n)},$$

$$\vdots$$

$$X_T d(\widehat{\mathbb{F}}_{T-n:T-1}(X_T)) - \widehat{\rho}_{(T-n:T-1)}$$

where $\widehat{\mathbb{F}}_{k:l}$ and $\widehat{\rho}_{(k:l)}$ are estimates based on the sample $X_k, \dots, X_l$

►► If we have dependent data or we use the conditional approach, it is necessary to introduce more explicit time series models.

Conditional Approach

Write $\rho_t(X_{t+1})$ for a distortion risk measure with a distortion D for the conditional df $F_t(x) := \mathrm{P}(X_{t+1} \le x \mid \mathscr{F}_t)$, $\mathscr{F}_t := \sigma(X_s \colon s \le t)$:

$$\rho_t(X_{t+1}) := \int_{[0,1]} F_t^{-1}(u)\, \mathrm{d}D(u)$$

Assumption

Suppose that for $\mathscr{F}_{t-1}$-measurable μ_t and σ_t,

$$X_t = \mu_t + \sigma_t Z_t,$$

where (Z_t) is i.i.d. with finite 2nd moment.

Example: ARMA(p_1, q_1) with GARCH(p_2, q_2) errors

Let (Z_t) be i.i.d. with finite 2nd moment.

$$X_t = \mu_t + \sigma_t Z_t,$$
$$\mu_t = \mu + \sum_{i=1}^{p_1} \phi_i (X_{t-i} - \mu) + \sum_{j=1}^{q_1} \theta_j (X_{t-j} - \mu_{t-j}),$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p_2} \alpha_i (X_{t-i} - \mu_{t-i})^2 + \sum_{j=1}^{q_2} \beta_j \sigma_{t-j}^2,$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, p_2$, $\beta_j \geq 0$, $j = 1, \ldots, q_2$.

Usually, it is assumed that (X_t) is covariance stationary, and $\sum_{i=1}^{p_2} \alpha_i + \sum_{j=1}^{q_2} \beta_j < 1$.

By (conditional) translation equivariance and positive homogeneity,

$$\rho_t(X_{t+1}) = \mu_{t+1} + \sigma_{t+1} \rho(Z)$$

where Z is a generic rv with the same df G as Z_t's.

(i) If G is a known df, $\rho(Z)$ is a known number.
We need to estimate μ_{t+1} and σ_{t+1} based on $X_{t-n+1}, \ldots, X_t$ using some specific model and method (e.g., ARMA with GARCH errors using QML). Then the risk measure estimate is given by

$$\widehat{\rho}_t(X_{t+1}) := \widehat{\mu}_{t+1} + \widehat{\sigma}_{t+1} \rho(Z)$$

Observe that

$$\rho(Z) = \mathrm{E}\left[Z_{t+1} d(G(Z_{t+1}))\right]$$

$$\Downarrow$$

$$\mathrm{E}\left[(Z_{t+1} - \rho(Z)) d(G(Z_{t+1}))\right] = 0$$

Defining

$$R_{t+1} := Z_{t+1} - \rho(Z) = \frac{X_{t+1} - \rho_t(X_{t+1})}{\sigma_{t+1}}$$

one sees that $(R_t d(G(Z_t)))_{t \in \mathbb{Z}}$ is i.i.d.

This suggests that in practice, we may perform backtesting by examining mean-zero behavior of $\widehat{R}_{t+1} d(G(\widehat{Z}_{t+1}))$, $t = n, \ldots, T-1$, where

$$\widehat{R}_{t+1} := \frac{X_{t+1} - \widehat{\rho}_t(X_{t+1})}{\widehat{\sigma}_{t+1}}$$

and

$$\widehat{Z}_{t+1} = \frac{X_{t+1} - \widehat{\mu}_{t+1}}{\widehat{\sigma}_{t+1}} = \widehat{R}_{t+1} + \rho(Z)$$

►► Bootstrap test can be used

(ii) When G is unknown, we need to estimate G in addition to μ_{t+1} and σ_{t+1}.

In ARMA with GARCH errors model, we could use the empirical df based on the residuals $\widetilde{Z}_s$'s: for $s = t-n+1, \ldots, t$,

$$\widetilde{Z}_s = \widetilde{\varepsilon}_s / \widetilde{\sigma}_s, \quad \widetilde{\varepsilon}_s \text{ : residual from ARMA part}$$

and

$$\widetilde{\sigma}_s^2 = \widehat{\alpha}_0 + \sum_{i=1}^{p_2} \widehat{\alpha}_i \widetilde{\varepsilon}_{s-i}^2 + \sum_{j=1}^{q_2} \widehat{\beta}_j \widehat{\sigma}_{s-j}^2,$$

Then

$$\widetilde{G}_t(z) = \frac{1}{n} \sum_{s=t-n+1}^{t} \mathbf{1}\{\widetilde{Z}_s \le z\},$$

Simulation study

Simulate GARCH(1,1) process:

$$Y_t = \sigma_t Z_t, \quad Z_t \sim N(0,1) \text{ i.i.d.}$$

$$\sigma_t^2 = 0.01 + 0.9\sigma_{t-1}^2 + 0.08 Y_{t-1}^2$$

Set $T = 1000$, $n = 500$ and $\theta = 0.05$

For $t = n+1, \ldots, T$, plot

(i) $X_t d(\widehat{\mathbb{F}}_{t-n:t-1}(X_t)) - \widehat{\rho}_{(t-n:t-1)}$ (historical, unconditional)

(ii) $\widehat{R}_t d(G(\widehat{Z}_t))$ (normal-GARCH based, conditional)

(i) mean $= -0.0286$, std $= 2.073$

(ii) mean $= -0.0185$, std $= 1.019$

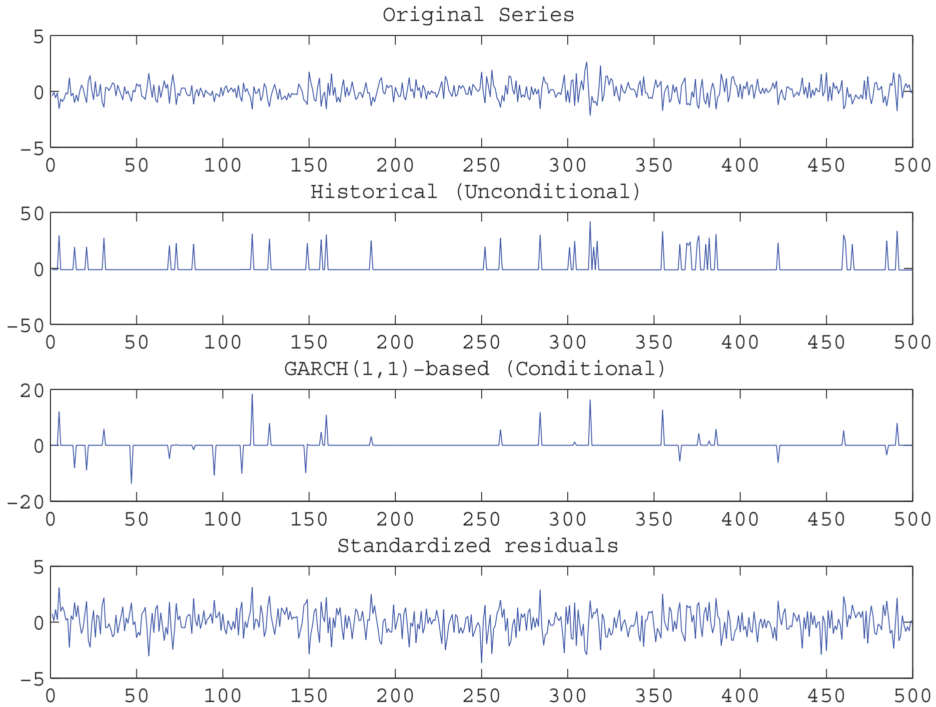


Figure 3: Backtesting results for expected shortfall ($\theta = 0.05$)

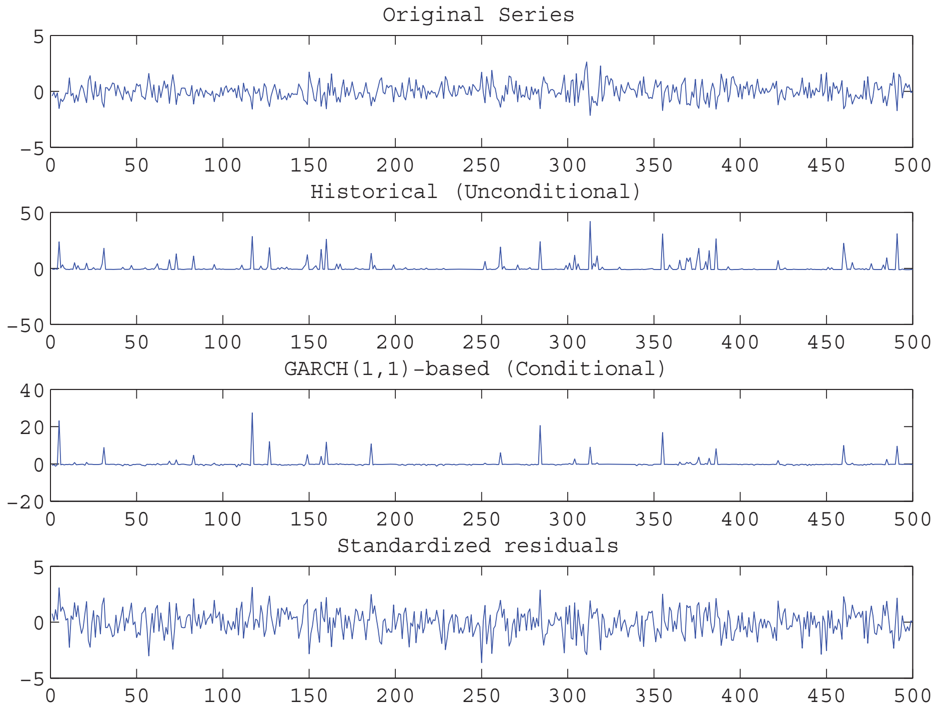


Figure 4: Backtesting results for proportional odds distortion ($\theta = 0.05$)

Issue: Backtestability

"*It is more difficult to backtest a procedure for calculating expected shortfall than it is to backtest a procedure for calculating VaR*" (Yamai & Yoshiba, Hull, Daníelsson, among others)

1. Because the existing tests for ES are based on
   - parametric assumptions for the null distribution
   - asymptotic approximation for the null distribution
2. Because testing an expectation is harder than testing a single quantile.

Elicitability

"*Expected shortfall (and spectral risk measures) cannot be backtested because it fails to satisfy* **elicitability** *condition*" (Paul Embrechts, Mar 2013, *Risk Magazine*)

Def (Osband 1985; Gneiting 2011, JASA)

A statistical functional $T(F)$ is called **elicitable** r.t. $\mathscr{F}$ if $T(F)$ is a unique minimizer of $t \mapsto E^F[S(t, Y)]$ for some scoring function S, $\forall F \in \mathscr{F}$.

Examples

- $\mathrm{VaR}_\theta(F) = F^{-1}(1-\theta)$ is the unique minimizer for

$$S(t,y) = [\mathbf{1}\{t \le y\} - \theta](y-t)$$
$$= \begin{cases} \theta|y-t| & \text{if } t > y \\ (1-\theta)|y-t| & \text{if } t \le y \end{cases}$$

  $\mathscr{F} = \{F\colon$ absolutely continuous, $\int |y| \,\mathrm{d}F(y) < \infty\}$.

- Mean functional $T(F) = \int y \,\mathrm{d}F(y)$ is the unique minimizer for

$$S(t,y) = (y-t)^2$$

  $\mathscr{F} = \{F\colon \int y^2 \,\mathrm{d}F(y) < \infty\}$.

It is useful when one wants to compare and rank several estimation procedures: With forecasts x_i and realizations y_i, use

$$\frac{1}{n}\sum_{i=1}^{n} S(x_i, y_i)$$

as a performance evaluation criterion.

►► But there seems to be no clear connection with backtestability

e.g., mean cannot be backtested nonparametrically based on the sum of squared errors without invoking asymptotic approximation or assuming parametric distribution.

Basel Committee on Banking Supervision: Consultative Document (October 2013)

> "Move from Value-at-Risk (VaR) to Expected Shortfall (ES):
> A number of weaknesses have been identified with using VaR for determining regulatory capital requirements, including its inability to capture "tail risk". For this reason, the Committee proposed in May 2012 to replace VaR with ES. ES measures the riskiness of a position by considering both the size and the likelihood of losses above a certain confidence level. The Committee has agreed to use a 97.5% ES for the internal models-based approach and has also used that approach to calibrate capital requirements under the revised market risk standardised approach"

Basel Committee on Banking Supervision: Consultative Document (October 2013)

Backtesting assessment (in *Revised Models-based Approach*):

> "In addition to P&L attribution, the performance of a trading desk's risk management models will be evaluated through daily backtesting. Backtesting requirements would be based on comparing each desk's 1-day static value-at-risk measure at both the 97.5th percentile and the 99th percentile to actual P&L outcomes, using at least one year of current observations of the desk's one-day actual and theoretical P&L. The backtesting assessment would be run at each trading desk as well as for the global (bank-wide) level."

Concluding Remarks

- Estimation of DRMs is possible with time series data, but for some DRMs, we do not get nice asymptotic properties.

- Backtesting procedure can be performed with DRMs. May need more rigorous/effective procedures.

- Euler capital allocation based on DRMs are easy to compute and widely applicable (with importance sampling)

- Most of the estimation part is published in *Journal of Financial Econometrics* (2013, online)

設定

Ω ：シナリオの集合（今後考えられる経済変動等のシナリオすべて）

$\mathcal{F}$: Ω 上の σ-加法族　（数学技術の理由から設定）

$X(\omega)$: シナリオ ω が起きた時の (事後の) 会社資産価値

$X \in m(\mathcal{F})$ ：$\mathcal{F}$-可測関数　（数学技術の理由からの仮定）

$X(\omega)$ は事前のポートフォリオにより決まる

許容できる事後の資産状況 $\mathcal{A} \subset m(\mathcal{F})$

$\mathcal{A}$ の満たすべき性質

(1) $0 \in \mathcal{A}$

(2) $X \in \mathcal{A},\, X \leqq Y \Rightarrow Y \in \mathcal{A}$

付加的な仮定

(凸性の仮定) $X, Y \in \mathcal{A},\, \lambda \in (0,1) \Rightarrow \lambda X + (1-\lambda)Y \in \mathcal{A}$

(正 1 次同次性) $X \in \mathcal{A},\, \lambda > 0 \Rightarrow \lambda X \in \mathcal{A}$

［計量化］

(1) 対象となる $m(\mathcal{F})$ の部分ベクトル空間 $\mathcal{X}$ を設定

(2) $\rho : \mathcal{X} \to [-\infty, \infty]$ を以下で定義

$$\rho(X) = \inf\{a \in \mathbf{R};\ a + X \in \mathcal{A}\}, \qquad X \in \mathcal{X}$$

（通常は $\rho(\mathcal{X}) \subset \mathbf{R}$ となるように設定される）

$\rho(X)$ はポートフォリオを変えないならば第 0 期に必要な資本と解釈

（金利はここでは無視する）

$\mathcal{A}_\rho = \{X \in \mathcal{X};\ \rho(X) \geqq 0\}$ ：これも許容できるものの集合

$\rho : \mathcal{X} \to \mathbf{R}$ の性質

(0)（ゼロルール）$\rho(0) = 0$

(1) (平行移動不変性）$\rho(X + c) = \rho(X) - c, \quad X \in \mathcal{X},\, c \in \mathbf{R}$

(2)（単調性）$X, Y \in \mathcal{X},\, X \leqq Y \Rightarrow \rho(X) \geqq \rho(Y)$

上記の (0), (1),(2) を満たすものを　リスク尺度と呼ぶ

(凸性の仮定)

$X, Y \in \mathcal{X},\, \lambda \in (0, 1) \Rightarrow \rho(\lambda X + (1 - \lambda)Y) \leqq \lambda\rho(X) + (1 - \lambda)\rho(Y)$

(正一次同次性の仮定)

$X \in \mathcal{X},\, \lambda > 0 \Rightarrow \rho(\lambda X) = \lambda\rho(X)$

凸性、正一次同次性を持つものリスク尺度： coherent risk measure

凸性、正一次同次性の仮定は妥当か？

凸性は自然な面がある（後述）

Föllmer は同次性にはあまり重きを置いていない

正一次同次性は危険という実務家もいる

過去データをどのように考えるべきか！

確率の概念の導入

P : $(\Omega, \mathcal{F})$ 上の確率測度

(法則不変性 (law invariant) の仮定)

$X, Y \in \mathcal{X},\, X, Y$ の分布が P の下で等しい $\Rightarrow \rho(X) = \rho(Y)$

法則不変性は問題を単純化するという利点がある

これまで提案された具体的なリスク尺度は ほぼすべて法則不変性をもつ！

Value at Risk

Average Value at Risk,

(Conditinal Tail Expectation, Expected Shortfall)

shortfall risk measures

divergence risk measures

Haezendonck risk measures

the entropic risk measures

VaR　一次同次性も持つ、　凸性は持たない

AVaR　coherent risk measure

法則不変性、凸性を持つリスク尺度

基本 AVaR

AVaR_λ, $\lambda \in [0,1]$

$$\mathrm{AVaR}_\lambda(X) = \frac{1}{\lambda}\inf_{z\in\mathbf{R}} E^P[(z-X)^+ - \lambda z], \qquad \lambda \in (0,1]$$

$$\mathrm{AVaR}_0(X) = ess.sup(-X)$$

分布関数 F に対して

$$R_\lambda(F) = \frac{1}{\lambda}\inf_{z\in\mathbf{R}} \int_{\mathbf{R}} (z-x)^+ dF - \lambda z), \qquad \lambda \in (0,1]$$

$$R_0(F) = \sup\{x\in\mathbf{R};\ F(-x) > 0\}$$

とおけば

$$\mathrm{AVaR}_\lambda(X) = R_\lambda(F_X^P), \qquad \lambda \in [0,1]$$

AVaR は Building Block

$\mathcal{M}_1([0,1])$ $[0,1]$ 上の確率測度全体

定理 1 (基本定理) 法則不変性、凸性を持つリスク尺度 ρ に対して $\beta^{\min} : \mathcal{M}_1([0,1]) \to [0,\infty]$ が存在して

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1([0,1])} (\int_{[0,1]} \mathrm{AVaR}_\lambda(X)\mu(d\lambda) - \beta^{\min}(\mu)).$$

$$R_\rho(F) = \sup_{\mu \in \mathcal{M}_1([0,1])} (\int_{[0,1]} R_\lambda(F)\mu(d\lambda) - \beta^{\min}(\mu))$$

とおけば

$$\rho(X) = R_\rho(F_X^P).$$

「リーマンショックに対する反省」

リスク規制が役に立たなかった

100 年に一度の出来事だから仕方ない（100 年に一度とは本当か？）

P が所与（あるいは知ることが出来る）と考えたことが誤りではないか

モデルリスク

(1) 統計推測が誤っているリスク（通常はこれを指す）

(2) モデルが根本的に間違っているというリスク

(1) のタイプのモデルリスク

データをある程度信頼している　（平時を想定してのリスク管理）

(2) のタイプのモデルリスク

データを信頼しない　（金融危機を想定してのリスク管理）

Föllmer の Beyond Law-Invariance

(1) のタイプのモデルリスクを想定していると見える（かつネイマン的）

例えば　ρ が法則不変なリスク尺度とする時、
それは X の P の下での分布 F_X^P の関数となる、則ち

$$\rho(X) = R(F_X^P), \qquad X \in \mathcal{X}$$

であるが $\mathcal{Q}$ を P を含む　$(\Omega, \mathcal{F})$ 上の確率測度よりなる族とした時

$$\tilde{\rho}(X) = \sup\{R(F_X^Q);\ Q \in \mathcal{Q}\}$$

を考えるという発想

P を推定値とすると
$\mathcal{Q}$: いわば信頼区間のようなもの

ベイズ的やり方も考え得る（一例）

$$\rho^P(X) = R(F_X^P), \qquad X \in \mathcal{X}$$

元々考えている法則不変なリスク尺度

$\mathcal{P}$ を $(\Omega, \mathcal{F})$ 上の確率測度よりなる族とし、
ν を $\mathcal{P}$ 上の確率測度（事後分布）とした時

$$\tilde{\rho}(X) = \tilde{R}(G)$$

ただし、 G は $-R(F_X^Q)$ $(= -\rho^Q(X))$ の $\nu(dQ)$ の下での分布関数
$\tilde{\rho}(X)$ は凸性を持つリスク尺度
$-\rho^Q(\lambda X + (1-\lambda)Y) \geqq -\lambda\rho^Q(X) - (1-\lambda)\rho^Q(Y)$

(2) のタイプのモデルリスクに対してはどうすればよいか

基本的にはストレスシナリオによるストレステストの考え方

となるのではないか

(1)　ストレスシナリオとは何か？

(2)　合格基準は何か

通常のストレステスト

$\omega_1, \omega_2, \ldots, \omega_n \in \Omega$　ストレスシナリオ

$X \in \mathcal{A}$ の条件：$X(\omega_k) \geqq c_k,\ k = 1, \ldots, n$ (合格基準)

シナリオが限定的

$X(\omega)$ が ω について「ロバスト」（連続）でないと意味がない

ストレステストの一般化（以下は一例）

基本定理をもう一度見てみる

Q：$(\Omega, \mathcal{F})$ 上の確率測度とする

$\mathrm{AVaR}^Q_\lambda,\ \lambda \in [0,1]$ を

$$\mathrm{AVaR}^Q_\lambda = \frac{1}{\lambda} \inf_{z \in \mathbf{R}} E^Q[(z - X)^+ - \lambda z], \qquad \lambda \in (0,1]$$

$$\mathrm{AVaR}^Q_0 = Q\text{-}ess.sup(-X)$$

で定めると

$\mathrm{AVaR}^Q_\lambda,\ \lambda \in [0,1]$, は coherent risk measure

$\beta : \mathcal{M}_1([0,1]) \to [0,\infty]$　に対して

$$\rho^{Q,\beta}(X) = \sup_{\mu \in \mathcal{M}_1([0,1])} \left(\int_{[0,1]} \mathrm{AVaR}^Q_\lambda(X)\mu(d\lambda) - \beta(\mu)\right).$$

と定める

$\rho^{Q,\beta}$ は平行移動不変性、単調性、凸性、Q-法則不変性　を持ち

$$\rho^{Q,\beta}(0) \leqq 0$$

Q_k, $k = 1, \ldots, n$, $(\Omega, \mathcal{F})$ 上の確率測度（リスクシナリオ）
$\beta_k : \mathcal{M}_1([0,1]) \to [0,\infty]$, $k = 1, \ldots, n$, （合格基準）

$$\tilde{\rho}(X) = \max\{\rho^{Q_k,\beta_k}(X) - a_k;\ k = 1, \ldots, n\}$$

とおくと $\tilde{\rho}$ 平行移動不変性、単調性、凸性を持ち
$\tilde{\rho}(0) \leqq 0$
平時の凸性を持つリスク尺度 ρ_0
$\rho = \tilde{\rho} \vee \rho_0$ は凸性を持つリスク尺度

課題

統計的推測の誤りを考慮したリスク尺度として何がよいか

ストレステストの一般化として適切なものは何か

まったく異種のリスクが組み合わさった時のリスクの考え方

$(\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, 2$,
$(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \otimes P_2)$
ρ_i, $i = 1, 2$ $(\Omega_i, \mathcal{F}_i, P_i)$ 上のリスク
「　$\rho_1 \otimes \rho_2$　」 をどう考えるべきか

変額保険：マーケットリスク・アクチュアリーリスク　まったく異なる

Decreasing Trends in Stock-Bond Correlations*

Harumi Ohmi†

Mizuho-DL Financial Technology Co., Ltd.

and

Tatsuyoshi Okimoto‡

Graduate School of International Corporate Strategy

Hitotsubashi University

December 17, 2013

*We are indebted to seminar participants at Hitotsubashi University. We thank the Grant-in-Aid for Scientific Research for its financial support.

†Mizuho-DL Financial Technology Co., Ltd., Kojimachi-odori Building 12F, 2-4-1 Kojimachi, Chiyoda-ku, Tokyo 102-0083, JAPAN. Phone: +81-3-4232-2682. Fax: +81-3-3261-1431. E-mail: harumi-ohmi@fintec.co.jp.

‡Corresponding author, Graduate School of International Corporate Strategy (ICS), Hitotsubashi University, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8439, JAPAN. Phone: +81-3-4212-3098. Fax: +81-3-4212-3020. E-mail: tatsuyoshi.okimoto@gmail.com.

Decreasing Trends in Stock-Bond Correlations

Abstract

Previous research documents the existence of long-run trends in comovements among the stock, bond, and commodities markets. Following these findings, this paper examines possible trends in stock-bond return correlations. To this end, we introduce a trend component into a smooth transition regression (STR) model including the multiple transition variables of Aslanidis and Christiansen (2012). The results indicate the existence of significant decreasing trends in stock-bond correlations. In addition, although stock market volatility continues to be an important factor in stock-bond correlations, the short rate and yield spread become only marginally significant once we introduce the trend component. Our out-of-sample analysis also demonstrates that the STR model including the VIX and time trend as the transition variables dominates other models. Our finding of decreasing trends in stock-bond correlations can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior that have taken place in recent years.

JEL classification: C22, G15, G17

Key Words: flight-to-quality; diversification effect; smooth transition regressions

1 Introduction

Understanding time variations in stock-bond return correlations is one of the most important issues in finance because it has profound implications for asset allocation and risk management. Naturally, a number of studies examine the dynamics of stock-bond correlations and identify the economic factors driving their time series behavior. For instance, Li (2002) conducts a regression analysis to investigate the relationship between stock-bond correlations and macroeconomic variables, showing that unexpected inflation is the most important determinant of stock-bond correlations. Similarly, Ilmanen (2003) argues that stock-bond correlations are more likely to be negative when inflation is low and stock market volatility is high. Yang, Zhou, and Wang (2009) examine stock-bond correlations over the past 150 years using the smooth transition conditional correlation (STCC) model and find that higher stock-bond correlations tend to follow higher short rates and (to a lesser extent) higher inflation rates. In addition, Connolly, Stivers, and Sun (2005, 2007) identify the VIX stock market volatility index as an important determinant of stock-bond correlations. Furthermore, Aslanidis and Christiansen (2010, 2012) demonstrate that stock-bond correlations are explained mostly by short rates, yield spreads, and the VIX. On the other hand, Pastor and Stambaugh (2003) note that changes in stock-bond correlations depend on liquidity. Similarly, Baele, Bekaert, and Inghelbrecht (2010) find that macroeconomic fundamentals contribute little to explaining stock-bond correlations but that liquidity plays a more important role. Other related studies include Guidolin and Timmermann (2006); Bansal, Connolly, and Stivers (2010); and Viceira (2012).

A number of recent studies also investigate long-run trends in international financial markets. For instance, Christoffersen et al. (2012) examine copula correlations in international stock markets and find a significant increasing trend that can be explained by neither volatility nor other financial and macroeconomic variables. Similarly, Berben and Jansen (2005) and Okimoto (2011) report increasing dependence in major equity markets. In international bond markets, Kumar and Okimoto (2011) find an increasing trend in correlations among international long-term government bonds and a decreasing trend in correlations between short- and long-term government bonds within single countries. Existing trends in comovements are also documented in commodities markets. For example, Tang and Xiong (2012) show that the prices of non-energy commodity futures in the US have become increasingly correlated with oil prices. In addition, Ohashi and Okimoto (2013) find increasing trends in the excess comovements of commodities prices. Other related studies include Longin and Solnik (1995), Silvennoinen and Teräsvirta (2009), and Silvennoinen and Thorp (2013).

The main contribution of this paper is to provide new evidence of long-run decreasing trends in stock-bond correlations by extending the smooth transition regression (STR) model of Aslanidis and Christiansen (2012). Although a growing number of studies exploring long-run trends in international financial markets suggest that it is of interest to analyze possible trends in stock-bond correlations, none of the previously mentioned studies consider these types of trends. Thus, it is very instructive to investigate long-run trends in stock-bond correlations. Indeed, our results indicate that there is a significant decreasing trend in realized stock-bond correlations. More importantly, although stock market volatility continues to be an important factor for stock-bond correlations, other important financial variables, namely the short rates and spreads between long- and short-term interest rates, become only marginally significant once we introduce the decreasing trend. Our out-of-sample analysis also indicates that the STR model including the VIX and time trend as the transition variables dominates other models. Our finding of a decreasing trend in stock-bond correlations can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior that have taken place in recent years.

The remainder of the paper is organized as follows: Section 2 presents the model, while Section 3 conducts the empirical analysis and Section 4 provides the conclusion.

2 Smooth Transition Regression Model

The main purpose of this paper is to examine possible long-run trends in realized stock-bond return correlations. To this end, we employ the smooth-transition model that was developed by Teräsvirta (1994) in the AR model framework and later used to analyze the determinants of stock-bond correlations by, among others, Yang, Zhou, and Wang (2009) and Aslanidis and Christiansen (2012). The former authors model correlations as latent variables and analyze them using the STCC model, whereas the latter authors investigate the realized correlation based on the smooth transition regression (STR) model with multiple transition variables. We employ the latter approach in this paper because it considerably facilitates the examination of the determinants of the time series behavior of stock-bond correlations, as emphasized by Aslanidis and Christiansen (2012). In addition, many other studies, including Ilmanen (2003) and Connolly et al. (2005, 2007), examine realized correlations. In particular, we apply the STR model with multiple transition variables to the realized correlations, following Aslanidis and Christiansen (2012).

The STR model used by Aslanidis and Christiansen (2012) is given by

$$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t \tag{1}$$

where FRC_t is the Fisher transformation of the realized correlation, RC_t, namely

$$FRC_t = \frac{1}{2}\log\left(\frac{1 + RC_t}{1 - RC_t}\right), \tag{2}$$

converting the realized correlation into a continuous variable not bounded between -1 and 1.[1] $F(s_{t-1})$ in (1) is the logistic transition function, taking values between 0 and 1. If $F(s_{t-1}) = 0$, the average value of FRC would be ρ_1 and if $F(s_{t-1}) = 1$, the average value of FRC would be ρ_2. In this sense, ρ_1 and ρ_2 in (1) can be considered the average correlations in regimes 1 and 2, respectively.[2] Thus, the conditional mean of FRC_t is modeled as the weighted average of the two correlation extremes; the weight is decided by $F(s_{t-1})$. $s_{t-1} = (s_{1,t-1}\ s_{2,t-1}\ \cdots\ s_{K,t-1})'$ is a $K \times 1$ vector of transition variables,[3] governing the transition between regimes 1 and 2. Specifically,

[1] As a realized correlation, Aslanidis and Christiansen (2012) use the weekly sample correlation calculated from five-minute high frequency stock and bond returns without demeaning, whereas we use monthly sample correlations based on daily data with demeaning.

[2] Specifically, ρ_1 is the average "Fisher-transformed correlation." In what follows, we simply refer to this as "correlation".

[3] In practice, all transition variables are standardized to have a mean of 0 and a variance of 1 as Aslanidis and Christiansen (2012).

$F(s_{t-1})$ is expressed as

$$F(s_{t-1}) = \frac{1}{1+\exp[-\gamma'(s_{t-1}-c)]}$$
$$= \frac{1}{1+\exp[-\gamma_1(s_{1,t-1}-c)+\cdots-\gamma_K(s_{K,t-1}-c)]}, \quad (3)$$

where γ_k is assumed to be positive for at least one k to identify the STR model with multiple transition variables. The location parameter c decides the center of the transition, while the smoothness parameter vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_K)'$ specifies the speed of the transition. More precisely, the transition caused by the transition variable $s_{k,t-1}$ is abrupt for large values of γ_k and gradual for small values of γ_k. One of the main advantages of the STR model is that it can detect detect, from the data, when and how any transitions occur in stock-bond correlations. In addition, the STR model can describe a wide variety of change patterns, depending on the parameters c and γ, which can be estimated from the data. Thus, by estimating the STR model, we can estimate the best transition patterns in stock-bond correlations.

In contrast to Aslanidis and Christiansen (2012), we use time trends as one of the transition variables to capture long-run trends in stock-bond correlations, following Lin and Teräsvirta (1994). In this framework, the time-varying correlation FRC_t changes smoothly from ρ_1 to ρ_2 with time, assuming that γ_k for the time trend is positive. Thus, we can interpret ρ_1 as a correlation around the beginning of the sample and ρ_2 as correlation around the end of the sample. A similar model is applied to conditional correlations by, among others, Berben and Jansen (2005) and Kumar and Okimoto (2011), who examine trends in stock and bond markets, respectively. This paper differs from these studies by investigating possible trends in stock-bond return correlations.

One concern about STR model (1) is possible serial correlation in FRC_t. Aslanidis and Christiansen (2012) address the serial correlation of the error term by calculating the Newey-West standard errors. However, if FRC_t itself has a serial correlation, this results in the inconsistent estimates of the correlation parameters. Indeed, a number of studies based on the dynamic conditional correlation (DCC) model of Engle (2002) suggest that the conditional correlations among financial returns are typically highly serially correlated. To address possible serial correlations in FRC_t, we modify STR model (1) by including the AR(1) term as follows:

$$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \phi FRC_{t-1} + \varepsilon_t. \quad (4)$$

In this STR model, FRC_t can be expressed as the weighted sum of the correlations expected by the economic variables and the previous correlation level. Theoretically, this model is also relevant because economic conditions may not be reflected immediately due, in part, to slow reactions by and imperfect information available to market participants. Therefore, the correlation may be adjusted slowly from the previous level, as in STR model (4).

We estimate STR model (4) using the maximum likelihood estimation (MLE) method, assuming that ε_t follows independently and is identically normally distributed. If the normal distribution assumption is inappropriate, the estimation can be considered to follow the nonlinear least squares method.

3 Empirical Analysis

3.1 Data

Our empirical analysis is based on monthly data, with the sample period lasting from January 1991 to May 2012. All data used in the analysis are obtained from DataStream. The analyzed countries are the United States (US), Germany (GER), and the United Kingdom (UK). Initially, we obtain daily data on futures contracts in the stock and bond markets of these three countries. Using the daily data, we obtain the realized stock-bond return correlations in each country for each month. We use futures on the S&P 500 (US), DAX (GER), and FTSE (UK) stock indices to calculate stock returns and each country's ten-year bond futures to calculate bond returns.

We also obtain the VIX, short rate, and yield spread as transition variables, following Aslanidis and Christiansen (2012), who demonstrate that these three variables are the most important transition variables for determining stock-bond correlation regimes. These three variables are also documented as important determinants of stock-bond correlations by many previous studies. For instance, Aslanidis and Christiansen (2010) find that these three variables are by far the most critical predictors of stock-bond correlations at their low and high quantiles. In addition, Connolly, Stivers, and Sun (2005, 2007) identify the VIX stock market volatility index a factor that influences stock-bond correlations, while Baele, Bekaert, and Inghelbrecht (2010) use the short rate as an important explanatory variable for stock-bond correlations. Furthermore, Viceira (2012) finds that short rates and yield spreads are the two most important predictors of the realized bond CAPM beta and the bond C-CAPM beta.

The VIX (VIX) is the volatility index for the Chicago Board of Options Exchange (CBOE) and is based on the volatility of options on the S&P 500 index. We use the US VIX for all countries due to the limited availability of VIX data for the two other examined countries.[4] The short rate (R) is the three-month Treasury bill rate from the secondary market for the US and the three-

[4] We confirm that the German and UK VIX indices are highly correlated with the US VIX, with a correlation that is greater than 0.8. We also confirm that we can obtain quantitatively similar results even if we use each country's VIX data with a shorter sample period.

month LIBOR rate for Germany and UK, while the yield spread (SPR) is defined as the ten-year constant maturity Treasury bond yield minus the short-rate for each country.

3.2 Benchmark Model Results

Our benchmark model is Aslanidis and Christiansen's (2012) preferred model, namely STR model (4), with $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1})'$. We refer to this model Model 1 and its estimation results are presented in Table 1, in which several items are worth noting. First, the last two rows of the table report the results of a version of Teräsvirta's (1994) linearity test and Eitrheim and Teräsvirta's (1996) additive nonlinearity test. As can be seen, the linearity test rejects the null of linearity in favor of the STR alternative at the 1% significance level for all countries. In contrast, the additive nonlinearity test is not significant, meaning that the proposed model adequately captures all smooth transition regime-switching behavior in the data without additional regimes for all countries.

Second, the AR parameters ϕ are highly significant, with estimated values of 0.38, 0.34, and 0.25 for the US, GER, and the UK, respectively. In other words, our results indicate that stock-bond correlations change from the previous level toward the correlation level expected by economic variables with some serial correlation, which is not captured by Aslanidis and Christiansen's (2012) original model.

Third, the correlation parameters for regime 1 are significantly positive, with estimated values of 0.30, 0.38, and 0.44 for the US, GER, and the UK, respectively, while those for regime 2 are significantly negative, with respective values of -0.32, -0.40, and -0.36. In other words, there are two distinct regimes, one with positive average correlations and the other with negative average correlations. Thus, correlations change smoothly or rapidly from positive to negative or negative to positive, depending on the transition variables.

Finally, all three transition variables, the VIX, short rate, and yield spread, have statistically significant effects on the regime transition at the 5% significance level for all countries. These results are fairly consistent with those of Aslanidis and Christiansen (2012), who demonstrate that stock-bond correlations are explained mostly by these three variables using STR model (1) without the AR term. These three variables are also reported to be important determinants of stock-bond correlations by other studies. For instance, the VIX is identified as a predominant factor for stock-bond correlations by Connoly et al. (2005, 2007) and Bansal et al. (2010). In addition, Baele et al. (2010) use the short rate as an important explanatory variable for stock-bond correlations, while Yang, Zhou, and Wang (2009) find that higher stock-bond correlations tend to follow higher short rates. Furthermore, Viceira (2012) finds that the yield spread and the short rate are important predictors for the realized bond CAPM beta and bond C-CAPM beta, which can be regarded as a transformation of the stock-bond correlation.

To see more detailed information on the regime transitions for each variable, the transition functions of each variable are plotted in Figure 1, holding the other variables constant at their mean values of zero. As can be seen, there is little difference across countries in terms of short rates and yield spreads and the correlation regime changes rather rapidly from the negative regime to the positive regime as these variables get larger. For instance, if the short rate is lower than the average by one standard deviation, the transition function takes a value greater than 0.97, meaning that the weight of the negative correlation regime is greater than 97%. More specifically, if the short rate is lower than the average value by one standard deviation, the average correlation is less than -0.30, -0.39, and -0.35 for the US, GER, and the UK, respectively. On the other hand, if the short rate is higher than the average value plus one standard deviation, the weight of negative regime becomes less than 0.04, making the average correlation more than 0.28 for all countries. Similarly, if the yield spread is lower (larger) than the average value by one standard deviation, the transition function is greater (less) than 0.90 (0.11), with an average correlation of less than -0.26 (greater than 0.18) for all countries. Since larger yield spreads and short rates are usually associated with better macroeconomic conditions, the results indicate that stock-bond correlations tend to be positive when the economy is booming. In other words, when the economy is in recession, stock-bond correlations have a tendency to be negative. This is arguably consistent with flight-to-quality behavior because investors do not want to take many risks when economic conditions are not good.

The VIX transition function also demonstrates flight-to-quality behavior. For the US and GER, the VIX transition function indicates that the correlation regime changes relatively smoothly from the negative regime to the positive regime as the standardized VIX changes from -3 to 3. The UK VIX transition function indicates slower changes in the correlation regime but still suggests that a higher VIX tends to be associated with negative stock-bond correlations. Thus, the results demonstrate that when the VIX is high or there is much uncertainty in the market, investors try to escape from risks, making stock-bond correlations negative.

Finally, the time series of the estimated correlations for Model 1 together with the actual realized correlations for each country are plotted in Panel (a) of Figures 2-4 to indicate goodness of fit. As can be seen, the estimated correlation fits the actual correlation quite well for all countries. More specifically, Model 1 successfully captures the tendency for there to be positive correlations before 2000 and negative correlations after 2000 because the correlation regimes tend to be identified as the positive regime before 2000 and the negative regime after 2000.

In sum, the results of Model 1 indicate that the VIX, short rate, and yield spread are important determinants of stock-bond correlation regimes for all countries, which is consistent with previous studies such as Aslanidis and Christiansen (2012), who estimated a similar model for the US. In addition, we demonstrate the significance of including the AR(1) to allow for smooth adjustments in correlation regimes, in contrast with Aslanidis and Christiansen (2012). Although the performance of Model 1 is quite satisfactory, it is possible to improve Model 1 by including other variables. In particular, recent studies find long-run correlation trends in international financial markets, suggesting that we can modify Model 1 by introducing a time trend component; this is examined in next subsection.

3.3 Introduction of Time Trend Component

The results of Model 1 are fairly consistent with previous studies examining the dynamics of stock-bond correlations. On the other hand, the another previous studies suggest the existence of long-run correlation trends in international financial markets. For instance, Christoffersen et al. (2012) examine copula correlations in international stock markets and find a significant increasing trend in the comovements of international stock returns that can be explained by neither volatility nor other financial and macroeconomic variables. In addition, Kumar and Okimoto (2011) find an increasing trend in correlations between international long-term government bonds and decreasing trends in correlations between the short- and long-term government bonds within single countries. Furthermore, Tang and Xiong (2012) document increasing correlations of commodities returns with crude oil after 2004. It is therefore of interest to analyze possible trends in stock-bond correlations by estimating STR model (4) including time (T) as well the VIX, short-rate, and spread as transition variables (Model 2). Thus, the vector of transition variables for Model 2 is defined as $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1}, T_t)'$.[5]

Table 2 reports the estimation results for Model 2. As can be seen, the results suggest that the basic structure of Model 2 is reasonably similar to that of Model 1. Specifically, the linearity and additive nonlinearity tests documented in the last two rows of Table 2 show that the two-state STR model is preferred to the linear model without regime changes and the three-state STR model with an additional correlation regime. In addition, Model 2 indicates the existence of two distinct correlation regimes, with a negative average correlation for one regime and a positive average correlation for the other, as in Model 1. Furthermore, the AR term is significant at least at the 10% significance level for the US and GER, suggesting smooth adjustments in stock-bond correlations in these countries.

[5]Since T is a non-random predetermined variable, we use T_t instead of T_{t-1} as a transition variable.

Although the basic structures of Models 1 and 2 are quite similar, there are important differences in the determinants of their stock-bond correlation regimes. In particular, the estimation results of Model 2 indicate that the time trend component is highly significant for all countries, suggesting that Model 1 omits an important factor of stock-bond correlations. More specifically, the time trend component coefficient estimates are significantly positive for all countries, meaning that there is a decreasing trend in stock-bond correlations. To see this more clearly, we plot the time trend for the correlations estimated through Model 2 in Panel (a) of Figure 5. As can be seen, the stock-bond correlations for all countries have clear decreasing trends, with a rapid decrease between the late 1990s and the early 2000s, reaching an average of -0.42 by the end of sample period in May 2012. Our finding of the existence of a time trend in correlations between financial assets is completely in line with recent studies. For instance, Berben and Jansen (2005) and Christoffersen (2012) document increasing correlations in the major equity markets. Similarly, Kumar and Okimoto (2011) find an increasing trend in correlations between international long-term government bonds and decreasing trends in correlations between a single country's short- and long-term government bonds.

Another important difference between Models 1 and 2 is the significance of the short rate and yield spread in determining the stock-bond correlation regime. Although the VIX remains an important factor in determining stock-bond correlations, the short rate and yield spread become less important in Model 2. Specifically, neither of these measures are significant for the US, while only one of them is significant for GER and the UK. In addition, the the short rate coefficient for GER is significantly positive instead of negative, making interpretation of the result rather difficult. The results are in contrast with the findings of the previously mentioned studies examining the determinants of stock-bond correlations without a time trend component. Thus, our results demonstrate that some of the important factors suggested by previous studies are not as relevant once we consider possible decreasing trends in stock-bond correlations.

To compare the goodness of fit of Models 1 and 2, we plot the time series of the correlations estimated through Model 2 together with the actual realized correlations for each country in Panel (b) of Figures 2-4. As can be seen, the correlations estimated through Models 1 and 2 are similar to each other and do not differ much over the sample. Thus, they qualitatively have the same power in illustrating the time series behavior of stock-bond correlations.

We can compare the goodness of fit of Models 1 and 2 more formally using the information criteria reported in Table 3, namely the Schwartz information criterion (SIC) and Akaike information criterion (AIC). Although the AIC favors Model 2 for GER and the UK, the SIC prefers Model 1 to Model 2 for all countries. Thus, in terms of the in-sample fit, our results are somewhat inconclusive.

To make a more comprehensive comparison between Models 1 and 2, we conduct an out-of-sample forecast evaluation as follows. First, we estimate both Models 1 and 2 using data from February 1991 to January 2001 and evaluate the terminal one-month-ahead forecast error based on the estimation results. The data are then updated by one month, and the terminal one-month-ahead forecast error is re-calculated from the updated sample (specifically, from March 1991 to February 2001). This procedure is repeated until reaching one month before the end of the sample period, namely April 2012. Finally, we calculate the root-mean-squared forecast errors (RMSE) and mean absolute error (MAE) using the obtained time series of one-month-ahead forecast errors. The third and fourth rows of Table 4 report the RMSE and MAE values for Models 1 and 2. As can be seen, the RMSE and MAE values of Model 2 are smaller than those of Model 1 for GER, while Model 1 exhibits better out-of-sample performance than Model 2 for other two countries.

Overall, our model comparison results show that Model 2 is not necessarily a better model than Model 1, although the time trend component is highly significant. One possible explanation for this result is the weak significance of the short rate and yield spread in Model 2, as mentioned. Indeed, neither of these factors are significant for US, while only one of them is significant for GER and UK. Thus, we might be able to improve the model by excluding these variables. To examine this possibility, we will consider a more parsimonious model in next subsection.

3.4 Results with Selected Transition Variables

Our results for Model 2 indicate that the short rate and yield spread become less important determinants of stock-bond correlations if decreasing trends in stock-bond correlations are taken into consideration. To illustrate this point more clearly, we estimate a more parsimonious STR model (4) that includes only VIX and time as the transition variables (Model 3).

The estimation results for Model 3 are shown in Table 3. As can be seen, the estimation results are essentially same as those of Model 2. The two-state STR model with a negative average correlation for one regime and a positive average correlation for the other regime is preferred to the linear model without regime changes and the three-state STR model. In addition, the AR term is highly significant for the US and GER, suggesting that the stock-bond correlations of these countries change slowly from the previous level toward the correlation level expected by economic variables. Furthermore, the VIX is significantly positive for all countries. Thus, the correlation regime changes from a positive to a negative regime when the VIX is high. Finally, the estimated time trend component is also significantly positive for all countries, meaning that stock-bond correlations tend to be in the negative regime in more recent periods. The decreasing trend can be confirmed visually from the estimated time trend component of stock-bond correlation depicted in Panel (b) of Figure 5. As can be seen from the figure, stock-bond correlations in all countries exhibit clear decreasing trends, with a rapid decrease from an average correlation of over 0.2 in the beginning of 1999 to an average correlation lower than -0.2 at the end of 2003, reaching an average of -0.42 around the end of the sample period in May 2012.

We also plot the time series of the estimated correlation for Model 3 together with the actual realized correlation for each country in Panel (c) of Figures 2-4 to graphically illustrate the performance of Model 3. As can be seen, the estimated correlations of Model 3 are quite similar to those of other models and do not differ much over the sample, suggesting that all models have the same qualitative explanatory power over stock-bond correlation behavior. Given that Model 3 has only two transition variables, this arguably indicates the superiority of Model 3 over the other two models. We can confirm this point more formally using the SIC and AIC reported in Table 3. As can be seen, Model 3 has the smallest SIC and AIC values for all countries, meaning that Model 3 is the best among the three models in terms of in-sample fit.

We additionally compare the out-of-sample performance of Model 3 and the other two models by conducting the same out-of-sample forecast evaluation as before. The results reported in Table 4 indicate that Model 3 exhibits the best out-of-sample performance for all countries, regardless of the employed performance measure.

In sum, our results are clear: Model 3 is the best among the three models, meaning that transitions between correlation regimes can be described sufficiently well by the VIX and time trend components. In other words, we demonstrate the possibility that the short rate and yield spread are not important factors in relation to stock-bond correlation regimes, in great contrast to previous studies such as Aslanidis and Christiansen (2012). Thus, flight-to-quality behavior is not strongly related with economic conditions, measured by short rates and yield spreads, but is associated with market uncertainty, as captured by the VIX. In addition, flight-to-quality behavior has become stronger in more recent years, resulting in decreasing trends in stock-bond correlations.

A possible explanation for this trend in flight-to-quality behavior is the recent increasing trend in correlations in international equity markets, which is documented by Christoffersen, et al. (2012), among others. Specifically, they emphasize that benefits from international diversification have decreased over time and this decrease has been especially drastic among developed markets, such as those examined in this study. In addition, Berben and Jansen (2005) show that correlations among the GER, UK, and US stock markets have doubled between 1980 and 2000. Similarly, Silvennoinen and Teräsvirta (2009) show that stock returns within and across European and Asian markets exhibit a clear upward shift in the level of correlations between 1998 and 2003, which corresponds to the timing of the rapid decrease in the estimated time trend of stock-bond correlations from our models. Thus, benefits from international diversification seem to begin disappearing after 2000. In this case, the investors who allocated their money into the equity markets of those countries have been exposed to higher risks of simultaneous drops in stock prices in recent years. As a consequence, they have more recently needed to make greater use of bond markets to control their risk exposure, producing the decreasing trend in stock-bond correlations. Indeed, the beginning of the integration of international equity markets and the beginning of decreases in stock-bond correlations appear to occur around the same time.

In addition to integration in equity markets, increasing correlations are observed in other markets as well. For instance, Kumar and Okimoto (2011) show that long-term government bond markets have become more integrated since the late 1990s, while Silvennoinen and Thorp (2013) find that correlations among stock, bond, and commodity future returns greatly increased around the early 2000s. Similarly, Tang and Xiong (2012) document increasing correlations of non-energy commodity with crude oil after 2004. These phenomena further diminish the effects of diversification in international financial markets, making investors diversify risks through bond markets. This phenomenon induces a rebalancing, particularly with from stocks to bonds.

Fleming, Kirby, and Ostdiek (1998) and Kodres and Pritsker (2002) study how cross-market hedging theoretically influences asset pricing. Specifically, Fleming, Kriby, and Ostdiek (1998) demonstrate that information linkages in stock and bond markets may be greater if cross-market hedging effects are considered within daily returns. In addition, Kodres and Pritsker (2002) show that a shock in one asset market may generate cross-market rebalancing, which influences prices in non-shocked asset markets. Since the disappearance of diversification effects produces investment behavior involving rebalancing from stocks to bonds, correlations between stocks and bonds tend to be negative, which can be captured by a trend variable, as indicated by our results.

4 Conclusion

In this paper, we investigated the existence of long-run trends in realized stock-bond return correlations. To this end, we introduce a trend component into the smooth transition regression (STR) model with the multiple transition variables of Aslanidis and Christiansen (2012). In addition, we analyzed not only the US, but also Germany and the UK, to conduct a more comprehensive examination. The results indicated the existence of a significant decreasing trend in stock-bond correlations for all countries.

Since a number of studies based on the dynamic conditional correlation (DCC) model of Engle (2002) suggest that conditional correlations between financial returns are typically highly serially correlated, we extended the STR model of Aslanidis and Christiansen (2012) by including the AR(1) term. The AR parameter estimates are highly significant for all countries. Thus, our results demonstrated that stock-bond correlations change slowly from the previous level toward the correlation level expected by economic variables, which is not captured by the original model of Aslanidis and Christiansen (2012).

In the case of transition variables, we examined three variables, namely the VIX, short rate, and yield spread, which have been identified by previous studies as arguably three of the most important factors. All three transition variables have statistically significant effects on regime transitions for all countries in our extended model. The results are fairly consistent with those of previous studies, particularly Aslanidis and Christiansen (2012). However, once we introduce the trend component, although the VIX remains an important factor for stock-bond correlations, the short rate and yield spread become only marginally significant. Indeed, our in-sample analysis suggested that the STR model including the VIX and time trend as the transition variables is the best model based on the SIC and AIC, meaning that the transition of stock-bond correlation regimes can be described sufficiently well by the VIX and time trend components. In addition, our out-of-sample analysis also demonstrated that the STR model with the VIX and time trend as the transition variables dominates other models.

Previous studies document the existence of long-run trends in comovements in the stock, bond, and commodities markets, suggesting that benefits from international diversification have recently been disappearing. Therefore, investors have been exposed to higher risks of simultaneous drops in stock prices in recent years. As a consequence, they have needed to make greater use of bond markets to control their risk exposure, producing the decreasing trend in stock-bond correlations. Interestingly, the beginning of the integration of international equity markets suggested by several previous studies and the beginning of decreases in stock-bond correlations appear to occur around the same time. Thus, our finding of a decreasing trend in stock-bond correlations can be considered a consequence of decreasing diversification effects and more intensive flight-to-quality behavior in recent years.

References

[1] Aslanidis, N. and C. Christiansen (2010), 'Quantiles of the Realized Stock-Bond Correlation,' Working paper, CREATES.

[2] Aslanidis, N. and C. Christiansen (2012), 'Smooth Transition Patterns in the Realized Stock-Bond Correlation,' *Journal of Empirical Finance* 19(4), 454-464.

[3] Baele, L., G. Bekaert and K. Inghelbrecht (2010), 'The Determinants of Stock and Bond Return Comovements,' *Review of Financial Studies* 23(6), 2374-2428.

[4] Bansal, N., R.A. Connolly and C. Stivers (2010), 'Regime-Switching in Stock Index and Treasury Futures Returns and Measures of Stock Market Stress,' *Journal of Futures Markets* 30(8), 753-779.

[5] Berben, R.P. and W.J. Jansen (2005), 'Comovement in International Equity Markets: A Sectoral View,' *Journal of International Money and Finance* 24, 832-857.

[6] Christoffersen, P., V. Errunza, K. Jacobs and H. Langlois (2012), 'Is the Potential for International Diversification Disappearing? A Dynamic Copula Approach,' *Review of Financial Studies* 25(12), 3711-3751.

[7] Connolly, R.A., C. Stivers and L. Sun (2005), 'Stock Market Uncertainty and the Stock-Bond Return Relation,' *Journal of Financial and Quantitative Analysis* 40(1), 161-194.

[8] Connolly, R.A., C. Stivers and L. Sun (2007), 'Commonality in the Time-Variation of Stock-Stock and Stock-Bond Return Comovements,' *Journal of Financial Markets* 10(2), 192-218.

[9] Eitrheim, Ø. and T. Teräsvirta (1996), 'Testing the Adequacy of Smooth Transition Autoregressive Models,' *Journal of Econometrics* 74(1), 59-76.

[10] Engle, R.F. (2002), 'Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models,' *Journal of Business and Economic Statistics* 20(3), 339-350.

[11] Fleming, J., C. Kirby and B. Ostdiek (1998), 'Information and Volatility Linkages in the Stock, Bond, and Money Markets,' *Journal of Financial Economics* 49(1), 111-137.

[12] Guidolin, M. and A. Timmermann (2006), 'An Econometric Model of Nonlinear Dynamics in the Joint Distribution of Stock and Bond Returns,' *Journal of Applied Econometrics* 21(1), 1-22.

[13] Ilmanen, A. (2003), 'Stock-Bond Correlations,' *Journal of Fixed Income* 13(2), 55-66.

[14] Kodres, L., and M. Pritsker. (2002), 'A Rational Expectations Model of Financial Contagion,' *Journal of Finance* 57(2), 769-799.

[15] Kumar, M.S. and T. Okimoto (2011), 'Dynamics of International Integration of Government Securities' Markets,' *Journal of Banking and Finance* 35(1), 142-154.

[16] Li, L. (2002), 'Macroeconomic Factors and the Correlation of Stock and Bond Returns,' Working paper, Yale International Center for Finance.

[17] Lin, C.-F.J. and T. Teräsvirta (1994), 'Testing the Constancy of Regression Parameters Against Continuous Structural Change,' *Journal of Econometrics* 62(2), 211-228.

[18] Longin, F. and B. Solnik (1995), 'Is the Correlation in International Equity Returns Constant: 1960-1990?,' *Journal of International Money and Finance* 14(1), 3-26.

[19] Ohashi, K. and T. Okimoto (2013), 'Dependence Evolution in International Equity Markets,' RIETI Discussion Paper Series 13-E-048.

[20] Okimoto, T. (2011), 'Dependence Evolution in International Equity Markets,' ICS FS Working Paper, Hitotsubashi University.

[21] Pastor, L. and R.F. Stambaugh (2003), 'Liquidity Risk and Expected Stock Returns,' *Journal of Political Economy* 111(3), 642-685.

[22] Silvennoinen, A. and T. Teräsvirta (2009), 'Modelling Multivariate Autoregressive Conditional Heteroskedasticity with the Double Smooth Transition Conditional Correlation GARCH Model,' *Journal of Financial Econometrics* 7(4), 373-411.

[23] Silvennoinen, A. and S. Thorp (2013), 'Financialization, Crisis and Commodity Correlation Dynamics,' *Journal of International Financial Markets, Institutions and Money* 24(1), 42-65.

[24] Tang, K. and W. Xiong (2012), 'Index Investing and the Financialization of Commodities,' *Financial Analysts Journal* 68(6), 54-74.

[25] Teräsvirta, T. (1994), 'Specification, Estimation and Evaluation of Smooth Transition Autoregressive Models,' *Journal of American Statistical Association* 89(425), 208-218.

[26] Viceira, L.M. (2012), 'Bond Risk, Bond Return Volatility, and the Term Structure of Interest Rates,' *International Journal of Forecasting* 28(1), 97-117.

[27] Yang, J., Y. Zhou and Z. Wang (2009), 'The Stock-Bond Correlation and Macroeconomic Conditions: One and a Half Centuries of Evidence,' *Journal of Banking and Finance* 33(4), 670-680.

Table 1: Estimation results of the benchmark model (Model 1)

	US		GER		UK	
	Coef	St. err	Coef	St. err	Coef	St. err
ρ_1	0.298***	0.101	0.378**	0.164	0.437***	0.055
ρ_2	-0.321***	0.129	-0.404***	0.147	-0.360***	0.038
ϕ	0.380***	0.090	0.342**	0.134	0.249***	0.080
VIX	1.370***	0.206	1.308***	0.099	0.537***	0.103
R	-3.414***	1.018	-3.968***	0.528	-3.824***	0.097
SPR	-2.201***	0.673	-2.839***	0.610	-2.476***	0.219
c	0.046	0.095	0.062	0.208	-0.007	0.077
Log-likelihood	-248.86		-250.95		-248.34	
Linearity test	12.3***		24.44***		16.55***	
Additive nonlinearity test	0.22		0.73		0.20	

Note: the table shows the estimation results of the STR Model 1 with transition variables; VIX index (VIX), short rate (R), yield spread (SPR). */**/*** indicates that the variable is significant at the 10%/5%/1% level of significance, respectively. Linearity test reports the LM-type statistic of null of no STR-type nonlinearity. Additive non-linearity shows the LM-Type statistic of null on no remaining STR-type nonlinearity.

Table 2: Estimation results of the model with time trend component (Model 2)

	US		GER		UK	
	Coef	St. err	Coef	St. err	Coef	St. err
ρ_1	0.297**	0.140	0.630***	0.052	0.502***	0.117
ρ_2	-0.368***	0.099	-0.580***	0.027	-0.440***	0.075
ϕ	0.346*	0.192	0.140***	0.028	0.156	0.105
VIX	1.925***	0.616	1.142***	0.083	1.163***	0.354
R	-0.576	0.461	1.323***	0.039	0.159	0.140
SPR	-0.294	0.672	0.051	0.049	-0.450***	0.161
T	2.571***	0.943	2.804***	0.010	2.725***	0.311
c	0.071	0.165	-0.144***	0.054	-0.065	0.158
Log-likelihood	-248.23		-248.25		-247.29	
Linearity test	10.95***		24.26***		21.54***	
Additive nonlinearity tes	1.28		2.55		0.09	

Note: the table shows the estimation results of the STR Model 1 with transition variables; VIX index (VIX), short rate (R), yield spread (SPR), time trend (T). */**/*** indicates that the variable is significant at the 10%/5%/1% level of significance, respectively. Linearity test reports the LM-type statistic of null of no STR-type nonlinearity. Additive non-linearity shows the LM-Type statistic of null on no remaining STR-type nonlinearity.

Table 3: Results of in-sample comparison

	US		GER		UK	
	AIC	SIC	AIC	SIC	AIC	SIC
Model 1	511.72	536.54	515.90	540.71	510.68	535.50
Model 2	512.46	540.82	512.51	540.87	510.58	538.95
Model 3	508.54	529.81	509.30	530.58	507.01	528.28

Note: the table reports the AIC and SIC for STR Models 1-3 to compare in-sample performance.

Table 4: Results of out-of-sample comparison

	US		GER		UK	
	RMSE	MAE	RMSE	MAE	RMSE	MAE
Model 1	0.201	0.155	0.322	0.257	0.259	0.212
Model 2	0.203	0.161	0.297	0.231	0.274	0.221
Model 3	0.174	0.136	0.296	0.231	0.241	0.199

Notes: the table reports the out-of-sample RMSE and MAE for STR Models 1-3. The forecast horizon is 1 month and the forecast period is 2000/12-2012/05.

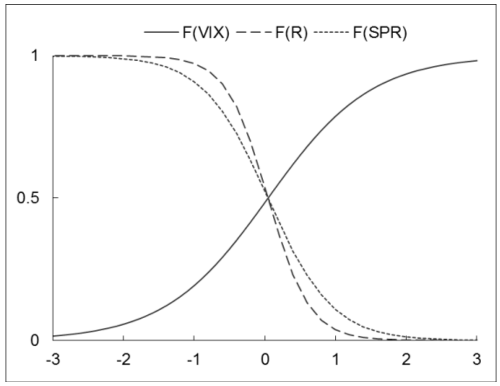
Table 5: Estimation results of the parsimonious model (Model 3)

	US		GER		UK	
	Coef	St. err	Coef	St. err	Coef	St. err
ρ_1	0.289***	0.001	0.459***	0.002	0.483***	0.185
ρ_2	-0.363***	0.002	-0.570***	0.006	-0.419**	0.173
ϕ	0.359***	0.001	0.136***	0.005	0.173	0.192
VIX	1.983***	0.003	1.901***	0.009	1.373***	0.345
T	2.959***	0.003	3.315***	0.095	2.808***	0.675
c	0.068*	0.041	0.005	0.067	-0.106	0.192
LLF	-248.27		-248.65		-247.51	
Linearity test	21.33***		36.88***		38.87***	
Additive nonlinearity test	1.25		0.02		0.61	

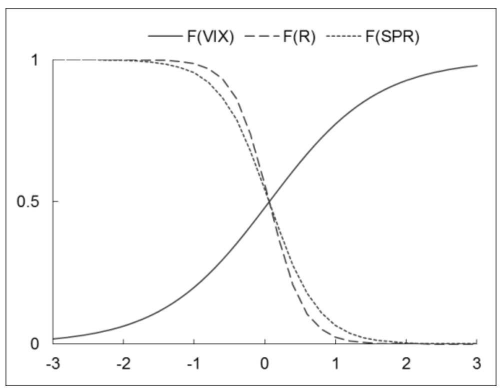
Note: the table shows STR Model 3 with transition variables; VIX index (VIX), Time Trend (T). */**/*** indicates that the variable is significant at the 10%/5%/1% level of significance, respectively. Linearity test reports the LM-type statistic of null of no STR-type nonlinearity. Additive non-linearity shows the LM-Type statistic of null on no remaining STR-type nonlinearity.

Figure 1: Estimated transition function

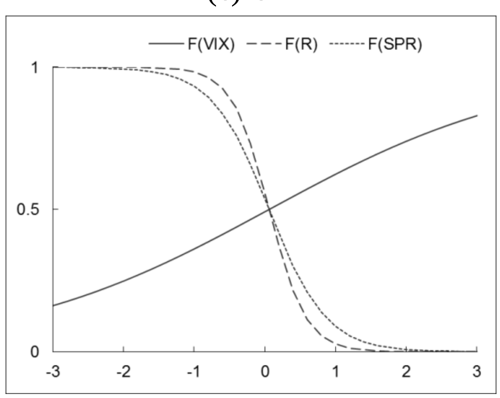
(a) US



(b) GER



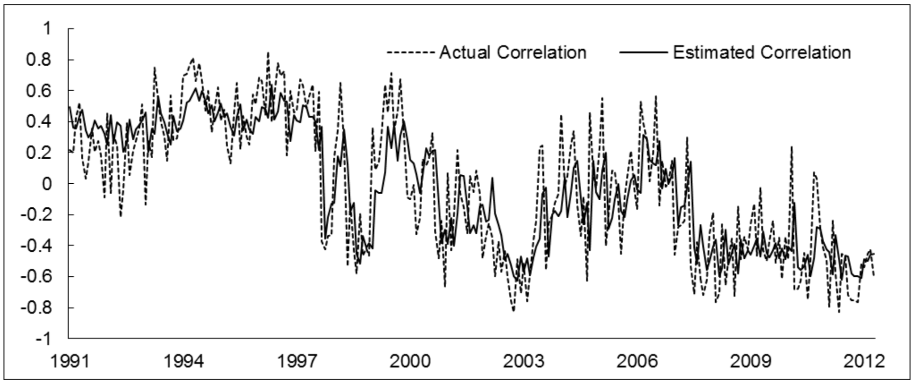
(c) UK



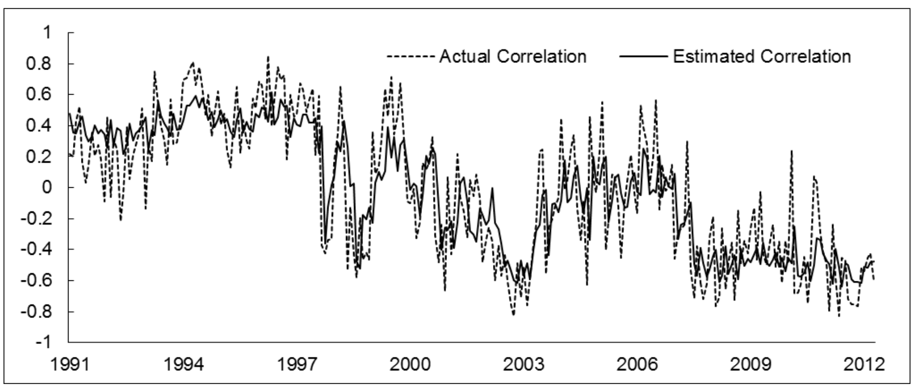
Notes: the graph shows the estimated transition function of model1 against each of the transition variables holding the other transition variables constant at their sample mean. The transition variables are VIX index (VIX), short rate (R) ,and yield spread (SPR).

Figure 2: Estimated stock-bond correlation for US

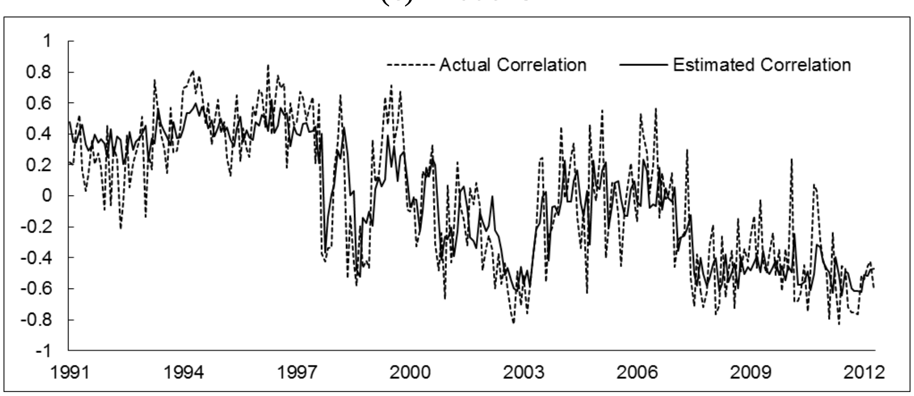
(a) Model 1



(b) Model 2



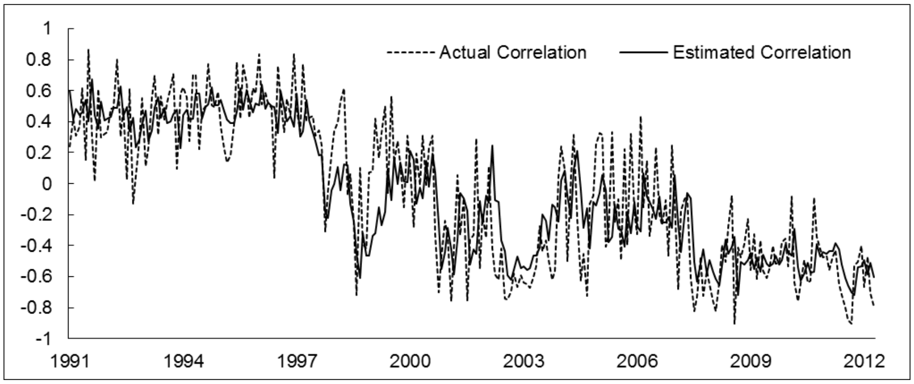
(c) Model 3



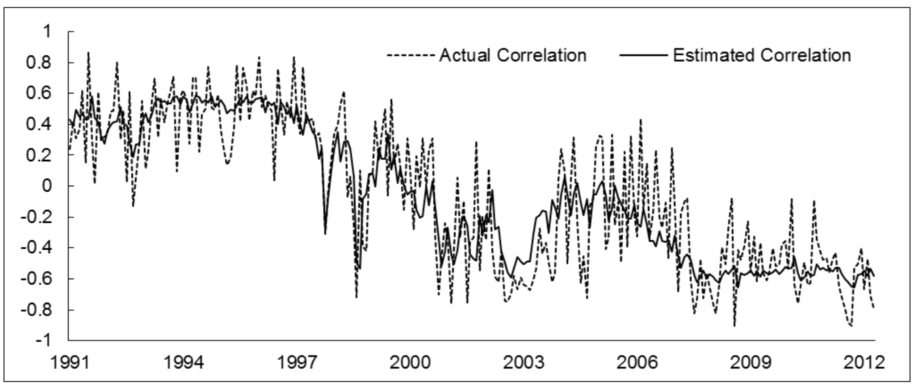
Notes: the graph shows the time series of the actual and estimated stock-bond correlation for Models 1-3 for US.

Figure 3: Estimated stock-bond correlation for GER

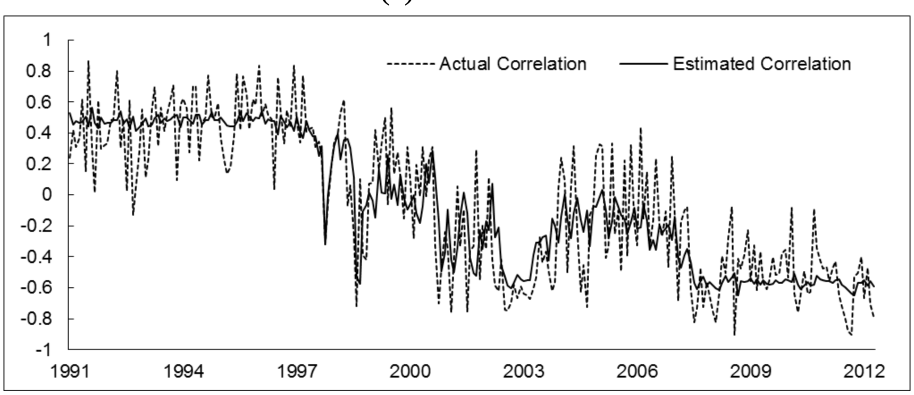
(a) Model 1



(b) Model 2



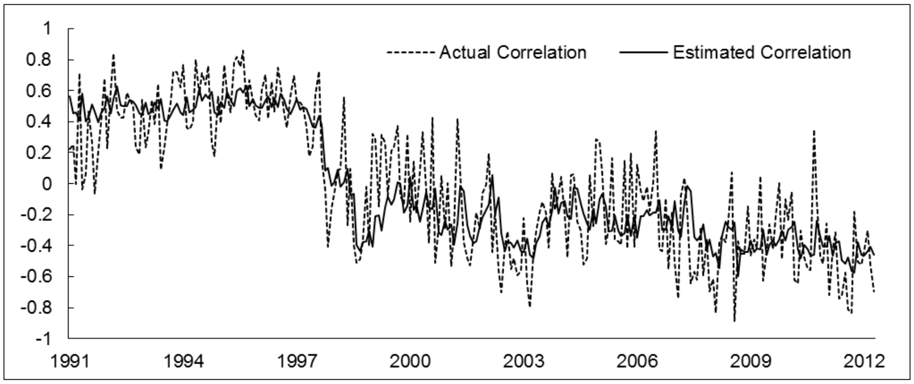
(c) Model 3



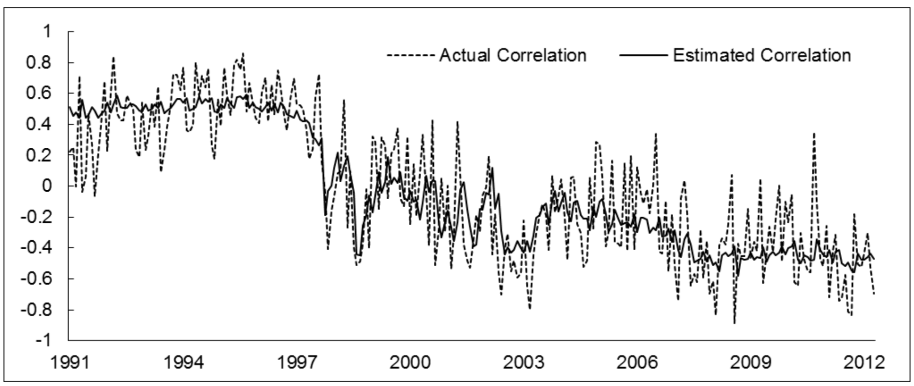
Notes: the graph shows the time series of the actual and estimated stock-bond correlation for Models 1-3 for GER.

Figure 4: Estimated stock-bond correlation for UK

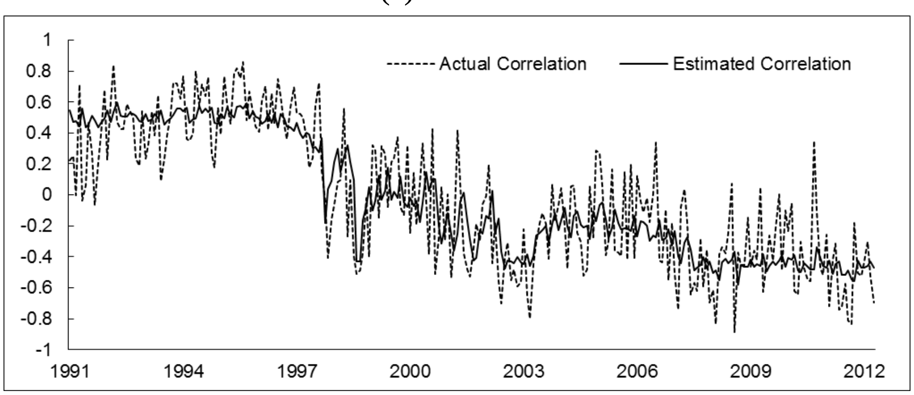
(a) Model 1



(b) Model 2



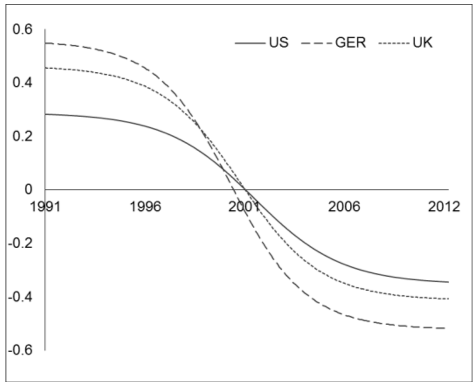
(c) Model 3



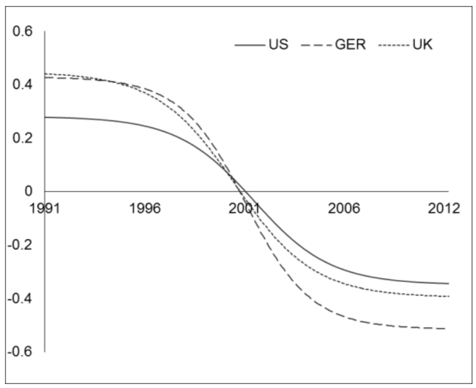
Notes: the graph shows the time series of the actual and estimated stock-bond correlation for Models 1-3 for UK.

Figure 5: Estimated time trend component in the stock-bond correlation

(a) Model 2



(b) Model 3



Note: the graph shows the time series of the estimated time trend component in the stock-bond correlation for Models 2 and 3.

Decreasing Trends in Stock-Bond Correlations

Harumi Ohmi
Mizuho-DL Financial Technology Co., Ltd.

and

Tatsuyoshi Okimoto
GS of International Corporate Strategy (ICS)
Hitotsubashi University

December 2013

Motivations and Main Results

Motivations

1. Stock-bond return correlations have profound implications on
   (a) Asset allocation
   (b) Risk management
2. Understanding the stock-bond correlations might not be easy due to the time variation of the correlation
3. Identifying the economic factors driving its time series behavior is one of the most important issues

4. Identified determinants of stock-bond correlations
   (a) Li (2002): (unexpected) inflation
   (b) Ilmanen (2003): inflation and stock market volatility
   (c) Yang, Zhou, and Wang (2009): short-rate and inflation
   (d) Connolly, Stivers, and Sun (2005, 2007): VIX
   (e) Aslanidis and Christiansen (2010, 2012): short-rate, yields spread, VIX
   (f) Pastor and Stambaugh (2003): liquidity
   (g) Baele, Bekaert, and Inghelbrecht (2010): liquidity

5. Long-run trends in international financial markets
   (a) Christoffersen et al. (2012)
      i. Find a significant increasing trend in correlations in international equity markets
      ii. Trend is much lower for emerging markets
      iii. Confirm that trend can be explained by neither volatility nor other financial and macroeconomic variables
   (b) Berben and Jansen (2005): international equity markets
   (c) Okimoto (2011): international equity markets
   (d) Kumar and Okimoto (2011)
      i. Find an increasing trend in correlations among international long-term government bonds
      ii. Detect a decreasing trend in correlations between short- and long-term government bonds within single countries

(e) Tang and Xiong

i. There was a significant and increasing trend in return correlations of non-energy commodities with oil after 2004

ii. Increasing trend is significantly stronger for indexed commodities (listed in either the SP-GSCI or DJ-UBS index) than for off-indexed commodities

(f) Silvennoinen and Thorp (2013): S&P500 and commodity future returns and returns to the majority commodity futures have increased

(g) Ohashi and Okimoto (2013): Excess comovements of commodities prices

(h) Few studies consider the possible trends in stock-bond correlations

Main Results

1. Examine the possible trend in stock-bond correlation
2. Extend Aslanidis and Christiansen (2012) in several ways
   (a) Treat serial correlations in stock-bond correlations explicitly
   (b) Introduce a time-trend component in stock-bond correlations
   (c) Examine Germany (GER) and UK as well as US
3. Find a significant decreasing trend in stock-bond correlations
4. Short rates and yield spreads become only marginally significant once we introduce the decreasing trend
5. STR model including the VIX and time trend as the transition variables dominates other models
6. Can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior

Related Literature

Aslanidis and Christiansen (2012)

1. Explores the time variation in the stock-bond correlation using high-frequency data
2. Consider the smooth transition regression (STR) model with multiple transition variables

   $$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t,$$

   where FRC_t is the Fisher transformation of the realized correlation
3. Examined transition variables: VIX, short-rate, yield spread, stock return, bond return, inflation, GDP growth
4. Detect one positive and one negative correlation regime systematically related to movements in financial and to a minor extent macroeconomic transition variables
5. Conclude that the short rate, the yield spread, and the VIX are the most important factors

Methodology

STR model

1. Developed by Teräsvirta (1994) in the AR framework
2. STR Model for FRC_t

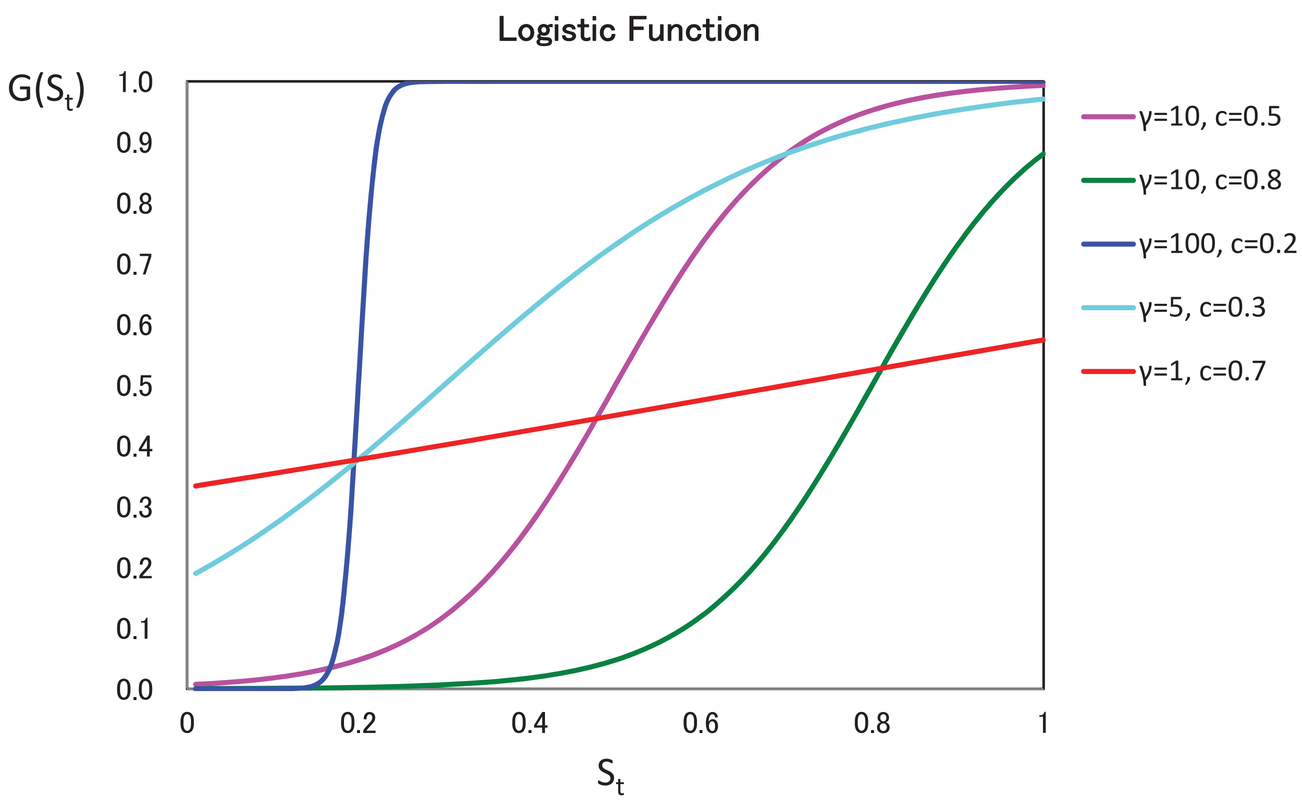
   $$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t$$

3. One of the regime switching models
   (a) Regime 1: $F = 0 \Longrightarrow E(FRC_t) = \rho_1$
   (b) Regime 2: $F = 1 \Longrightarrow E(FRC_t) = \rho_2$
4. Regime transition is modeled by a logistic transition function F

   $$F(s_{t-1}; c, \gamma) = \frac{1}{1 + \exp(-\gamma(s_{t-1} - c))}, \quad \gamma > 0$$

   (a) s_t : Transition variable
   (b) c: Location parameter
   (c) γ: Smoothness parameter

5. F increases monotonically in s_{t-1} from 0 to 1
   (a) ρ_1: conditional mean of FRC when s_{t-1} is small
   (b) ρ_2: conditional mean of FRC when s_{t-1} is large
6. Typical choice of a transition variable
   (a) $s_{t-1} = VIX_{t-1}$
      i. ρ_1: conditional mean of FRC when VIX_{t-1} is small
      ii. ρ_2: conditional mean of FRC when VIX_{t-1} is large
   (b) $s_{t-1} = t/T$
      i. ρ_1: value of FRC around the beginning of the sample
      ii. ρ_2: value of FRC around the end of the sample
7. Can capture dominant long-run trends by adopting $s_t = t/T$ as one the transition variables (Lin and Teräsvirta, 1994)
8. Can describe a wide variety of patterns of change depending on the values of γ, c



9. Transition variable can be a vector of variables

$$F(s_{t-1}) = \frac{1}{1+\exp[-\gamma'(s_{t-1}-c)]}$$
$$= \frac{1}{1+\exp[-\gamma_1(s_{1,t-1}-c)+\cdots-\gamma_K(s_{K,t-1}-c)]}$$

10. All transition variables are standardized to have a mean of 0 and a variance of 1
11. Treat serial correlations in stock-bond correlations explicitly by including the AR term

$$FRC_t = \rho_1\{1-F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \phi FRC_{t-1} + \varepsilon_t.$$

Test of linearity against the STR model

1. STR model: $FRC_t = \rho_1\{1-F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t$
2. Interesting to test linearity or the null of $H_0 : \rho_1 = \rho_2$
3. Cannot use the standard F-test due to the unidentified parameters γ and c under the null
4. Luukkonen, Saikkonen, and Teräsvirta (1988) propose a simple test for the STR model with the logistic transition function
5. Derive auxiliary regression model by replacing F with a first order Taylor expansion around $\gamma = 0$

$$FRC_t = \beta_0 + \beta_1 s_{1,t-1} + \beta_2 s_{2,t-1} + \cdots + \beta_K s_{K,t-1} + e_t$$

6. $H_0 : \rho_1 = \rho_2$ is equivalent to $H_0' : \beta_1 = \cdots = \beta_K$
7. $H_0' : \beta_1 = \cdots = \beta_K$ can be tested by the standard F test
8. Can test the additive nonlinearity (i.e. two state v.s. three state) based on similar idea (Eitrheim and Teräsvirta, 1996)

Empirical Analysis

Data

1. Sample period: from January 1991 to May 2012
2. Analyzed countries: GER, UK, US
3. Collect daily data on futures contracts in the stock and bond markets
4. Stock: S&P 500 (US), DAX (GER), and FTSE (UK) stock index futures
5. Bond: each country's ten-year bond futures
6. Calculate the Fisher transformation of monthly sample stock-bond return correlation
7. Obtain the VIX, short rate, and yield spread as transition variables
8. Use the US VIX for all countries due to the limited availability of VIX data for the two other examined countries

Benchmark model results

1. Model 1: STR model with $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1})'$
2. Aslanidis and Christiansen's (2012) preferred model
3. Linearity test rejects the null of linearity in favor of the STR alternative at the 1% significance level for all countries
4. Additive nonlinearity test is not significant for all countries
5. Two-state model adequately captures all smooth transition regime-switching behavior in the data
6. AR parameters ϕ are highly significant
7. There are two distinct regimes, one with positive average correlations and the other with negative average correlations
8. All three transition variables have statistically significant effects on the regime transition
9. Mostly consistent with Aslanidis and Christiansen's (2012)

Table 1: Estimation results of the benchmark model (Model 1)

	US		GER		UK	
	Coef	St. err	Coef	St. err	Coef	St. err
ρ_1	0.298***	0.101	0.378**	0.164	0.437***	0.055
ρ_2	-0.321***	0.129	-0.404***	0.147	-0.360***	0.038
ϕ	0.380***	0.090	0.342**	0.134	0.249***	0.080
VIX	1.370***	0.206	1.308***	0.099	0.537***	0.103
R	-3.414***	1.018	-3.968***	0.528	-3.824***	0.097
SPR	-2.201***	0.673	-2.839***	0.610	-2.476***	0.219
c	0.046	0.095	0.062	0.208	-0.007	0.077
Log-likelihood	-248.86		-250.95		-248.34	
Linearity test	12.3***		24.44***		16.55***	
Additive nonlinearity test	0.22		0.73		0.20	

10. Plot the transition functions of each variable, holding the other variables constant at their mean values of zero
11. Correlation regime changes rather rapidly from the negative regime to the positive regime as short rates and yield spreads get larger
12. If short rate is lower (larger) than the average value by 1SD, the average correlation is less than -0.30 (more than 0.28) for the US
13. Stock-bond correlations tend to be positive when the economy is booming
14. VIX transition function also demonstrates flight-to-quality behavior
15. Estimated correlation fits the actual correlation quite well
16. Recent studies find long-run correlation trends in international financial markets
17. Instructive to examine whether we can modify Model 1 by introducing a time trend component

Figure 1: Estimated transition function for Model 1

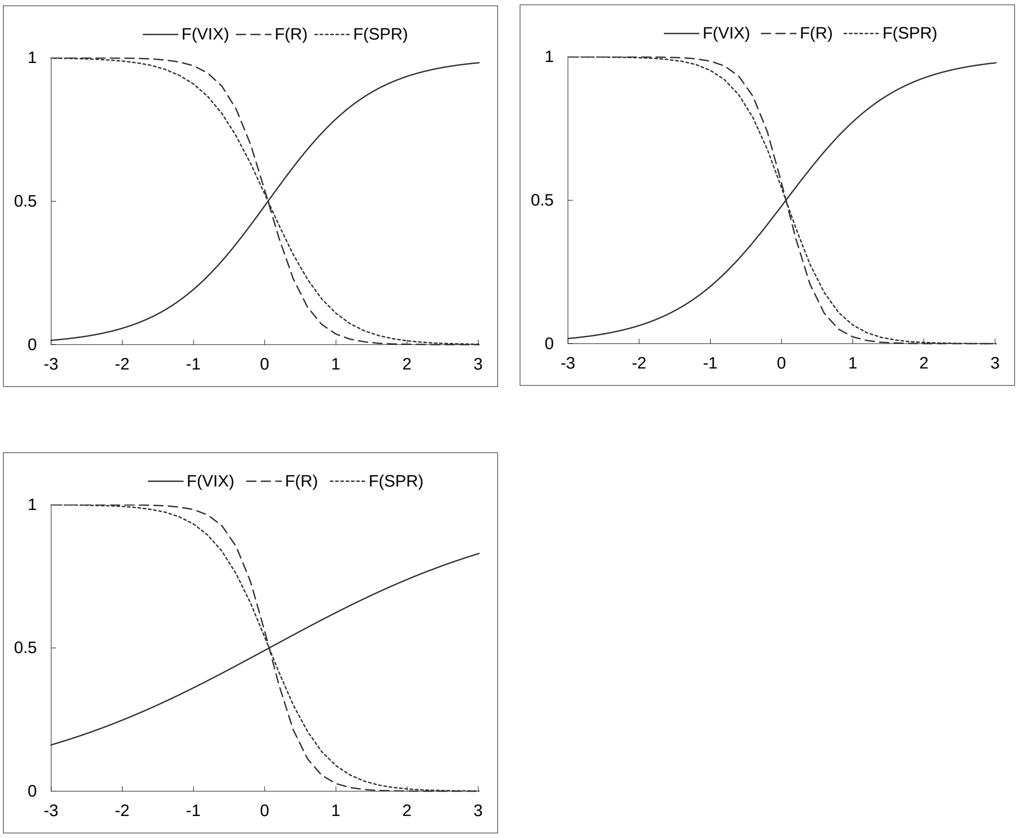
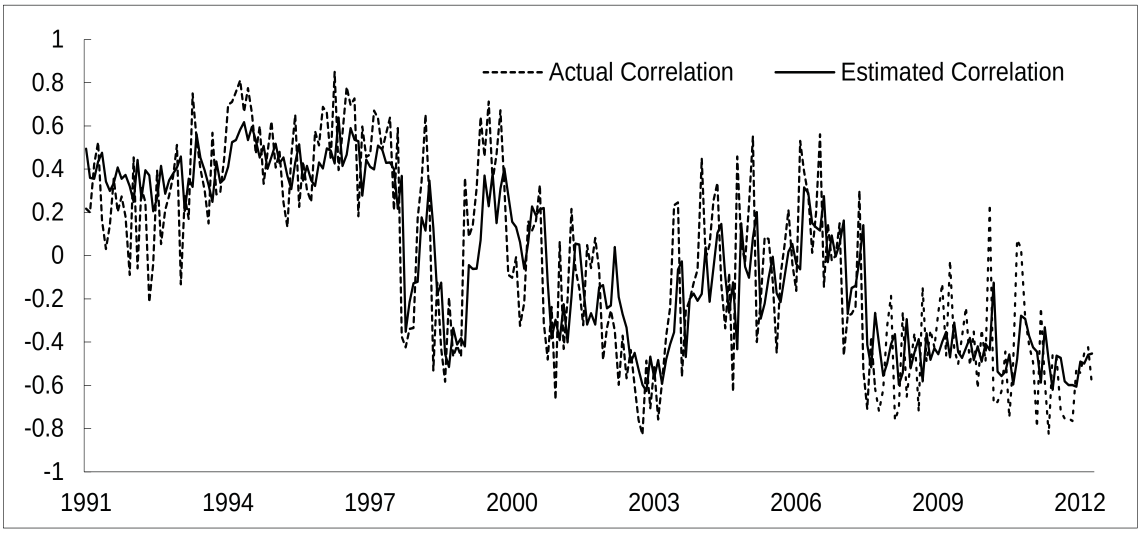


Figure 2: Estimated stock-bond correlation for US (Model 1)



Introduction of time trend component

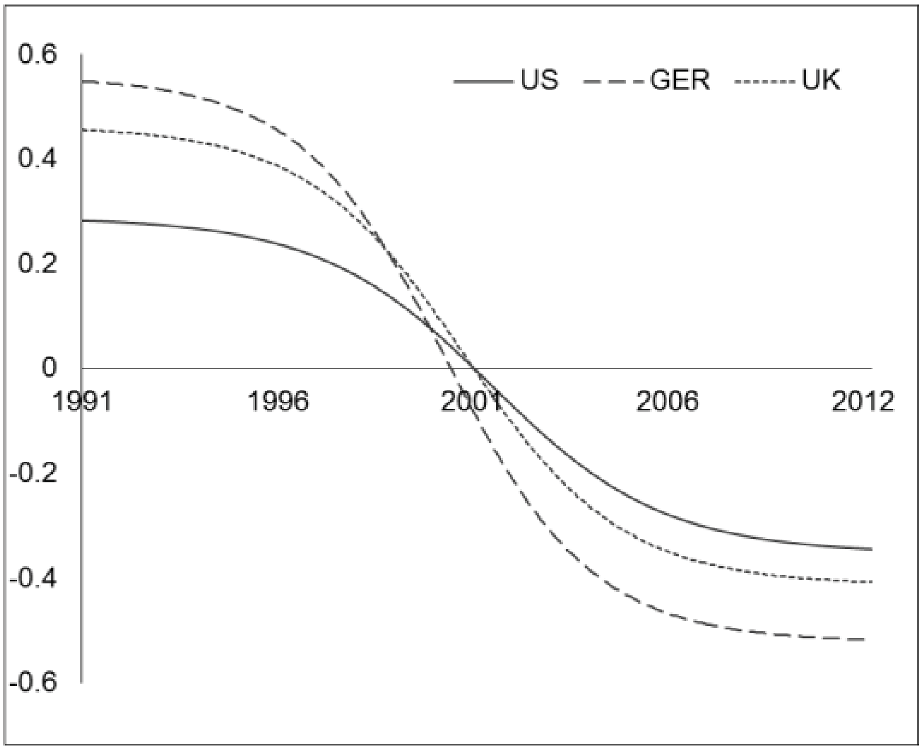
1. Model 2: STR model with $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1}, T_t)'$
2. Two-state model adequately captures all smooth transition regime-switching behavior in the data
3. Two distinct correlation regimes, with a negative average correlation for one regime and a positive average correlation for the other
4. AR term is significant at least at the 10% significance level for the US and GER
5. Time trend component coefficient estimates are significantly positive for all countries
6. There is a decreasing trend in stock-bond correlations
7. Rapid decrease between the late 1990s and the early 2000s, reaching an average of -0.42 by the end of sample period in May 2012

Table 2: Estimation results of the model with time trend component (Model 2)

	US		GER		UK	
	Coef	St. err	Coef	St. err	Coef	St. err
ρ_1	0.297**	0.140	0.630***	0.052	0.502***	0.117
ρ_2	-0.368***	0.099	-0.580***	0.027	-0.440	0.075
ϕ	0.346*	0.192	0.140***	0.028	0.156	0.105
VIX	1.925***	0.616	1.142***	0.083	1.163***	0.354
R	-0.576	0.461	1.323***	0.039	0.159	0.140
SPR	-0.294	0.672	0.051	0.049	-0.450***	0.161
T	2.571***	0.943	2.804***	0.010	2.725***	0.311
c	0.071	0.165	-0.144***	0.054	-0.065	0.158
Log-likelihood	-248.23		-248.25		-247.29	
Linearity test	10.95***		24.26***		21.54***	
Additive nonlinearity test	1.28		2.55		0.09	

Figure 5: Estimated time trend component in the stock-bond correlation

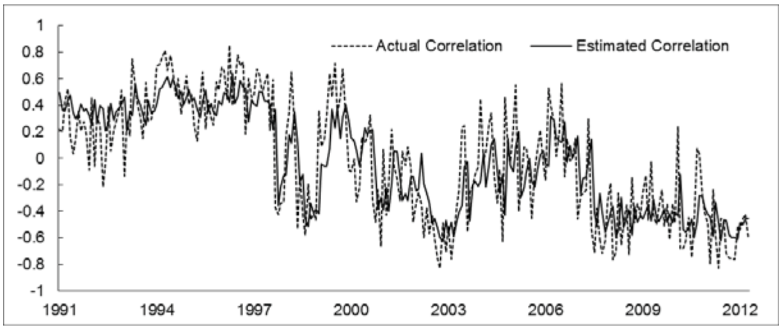
(a) Model 2



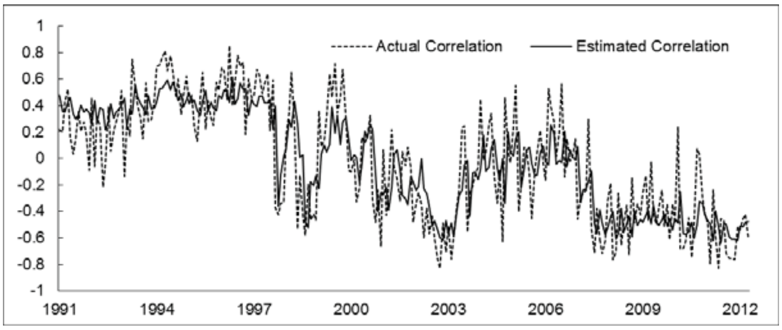
8. VIX remains an important factor in determining stock-bond correlations
9. Short rate and yield spread become less important in Model 2
10. Neither of Short rate and yield spread are significant for the US
11. Only one of them is significant for GER and the UK
12. Correlations estimated through Models 1 and 2 are similar to each other and do not differ much over the sample
13. AIC favors Model 2 for GER and the UK, while the SIC prefers Model 1 to Model 2 for all countries

Figure 2: Estimated stock-bond correlation for US

(a) Model 1



(b) Model 2



Out-of-sample forecast evaluation

1. Conduct an out-of-sample forecast evaluation
   (a) Estimate both Models 1 and 2 using data from February 1991 to January 2001
   (b) Evaluate the terminal one-month-ahead forecast error based on the estimation results
   (c) Data are updated by one month
   (d) Terminal one-month-ahead forecast error is re-calculated from the updated sample
   (e) Repeat (c) and (d) until reaching one month before the end of the sample period
   (f) Calculate the root-mean-squared forecast errors (RMSE) and mean absolute error (MAE)
2. Model 2 performs better than Model 1 for GER
3. Model 1 exhibits better than Model 2 for other two countries

Results with selected transition variables

1. Short rate and yield spread become less important determinants of stock-bond correlations if decreasing trends are accommodated
2. Possible to improve the model by excluding these variables
3. Model 3: STR model with $s_{t-1} = (VIX_{t-1}, T_t)'$
4. Estimation results are essentially same as those of Model 2
5. Stock-bond correlations in all countries exhibit clear decreasing trends, with a rapid decrease between 1999 and 2003
6. Estimated correlations are similar to those of other models
7. Model 3 is the best among the three models in terms of in-sample fit for all countries
8. Model 3 exhibits the best out-of-sample performance for all countries

Table 5: Estimation results of the parsimonious model (Model 3)

	US		GER		UK	
	Coef	St. err	Coef	St. err	Coef	St. err
ρ_1	0.289***	0.001	0.459***	0.002	0.483***	0.185
ρ_2	-0.363***	0.002	-0.570***	0.006	-0.419**	0.173
ϕ	0.359***	0.001	0.136***	0.005	0.173	0.192
VIX	1.983***	0.003	1.901***	0.009	1.373***	0.345
T	2.959***	0.003	3.315***	0.095	2.808***	0.675
c	0.068*	0.041	0.005	0.067	-0.106	0.192
LLF	-248.27		-248.65		-247.51	
Linearity test	21.33***		36.88***		38.87***	
Additive nonlinearity test	1.25		0.02		0.61	

Figure 5: Estimated time trend component in the stock-bond correlation

(b) Model 3

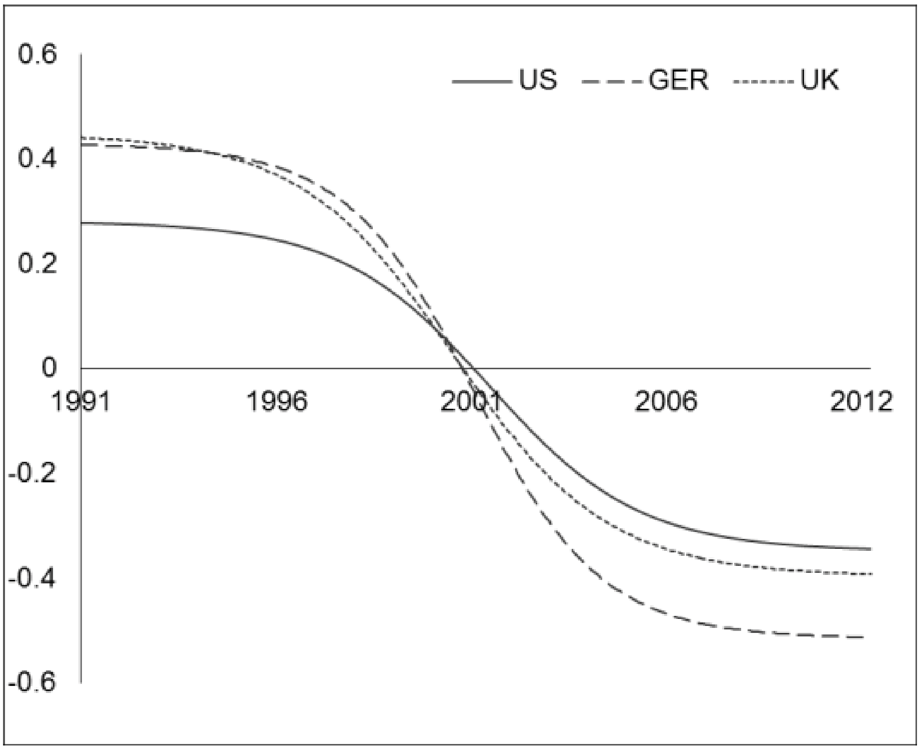
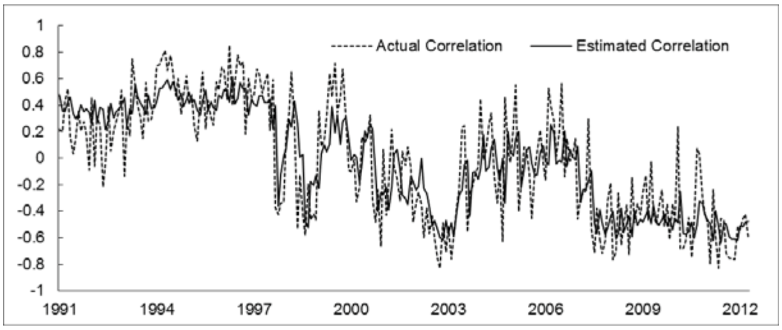


Figure 2: Estimated stock-bond correlation for US

(b) Model 2



(c) Model 3

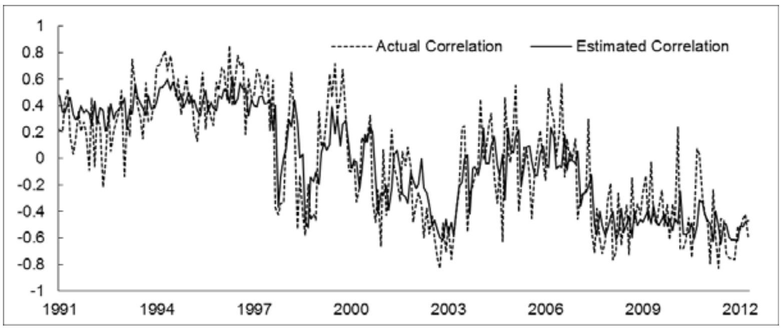


Table 3: Results of in-sample comparison

	US		GER		UK	
	AIC	SIC	AIC	SIC	AIC	SIC
Model 1	511.72	536.54	515.90	540.71	510.68	535.50
Model 2	512.46	540.82	512.51	540.87	510.58	538.95
Model 3	508.54	529.81	509.30	530.58	507.01	528.28

Table 4: Results of out-of-sample comparison

	US		GER		UK	
	RMSE	MAE	RMSE	MAE	RMSE	MAE
Model 1	0.201	0.155	0.322	0.257	0.259	0.212
Model 2	0.203	0.161	0.297	0.231	0.274	0.221
Model 3	0.174	0.136	0.296	0.231	0.241	0.199

Interpretation of the results

1. Short rate and yield spread are not important factors in relation to stock-bond correlation regimes
2. Flight-to-quality behavior is not strongly related with economic conditions, but is associated with market uncertainty
3. Significant decreasing trends in stock-bond correlations
4. Flight-to-quality behavior has become stronger in more recent years
5. Many studies find an increasing trend in correlations in international equity markets as well as other financial markets
6. Diminish the effects of diversification in international financial markets
7. Investors need to make greater use of bond markets to control their risk exposure, producing decreasing trend in stock-bond correlations

Conclusion

1. Examine the possible trend in stock-bond correlation for US, GER, UK
2. Find a significant decreasing trend in stock-bond correlations
3. Short rates and yield spreads become only marginally significant once we introduce the decreasing trend
4. STR model including the VIX and time trend as the transition variables dominates other models
5. Can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior

Future topics

1. High frequency data
2. Model correlation as a latent variable
3. Asymmetric dependence
4. Source of long-run trends in international financial markets

References

[1] Aslanidis, N. and C. Christiansen (2010), 'Quantiles of the Realized Stock-Bond Correlation,' Working paper, CREATES.

[2] Aslanidis, N. and C. Christiansen (2012), 'Smooth Transition Patterns in the Realized Stock-Bond Correlation,' *Journal of Empirical Finance* 19(4), 454-464.

[3] Baele, L., G. Bekaert and K. Inghelbrecht (2010), 'The Determinants of Stock and Bond Return Comovements,' *Review of Financial Studies* 23(6), 2374-2428.

[4] Berben, R.P. and W.J. Jansen (2005), 'Comovement in International Equity Markets: A Sectoral View,' *Journal of International Money and Finance* 24, 832-857.

[5] Christoffersen, P., V. Errunza, K. Jacobs and H. Langlois (2012), 'Is the Potential for International Diversification Disappearing? A Dynamic Copula Approach,' *Review of Financial Studies* 25(12), 3711-3751.

[6] Connolly, R.A., C. Stivers and L. Sun (2005), 'Stock Market Uncertainty and the Stock-Bond Return Relation,' *Journal of Financial and Quantitative Analysis* 40(1), 161-194.

[7] Connolly, R.A., C. Stivers and L. Sun (2007), 'Commonality in the Time-Variation of Stock-Stock and Stock-Bond Return Comovements,' *Journal of Financial Markets* 10(2), 192-218.

[8] Eitrheim, Ø. and T. Teräsvirta (1996), 'Testing the Adequacy of Smooth Transition Autoregressive Models,' *Journal of Econometrics* 74(1), 59-76.

[9] Ilmanen, A. (2003), 'Stock-Bond Correlations,' *Journal of Fixed Income* 13(2), 55-66.

[10] Kumar, M.S. and T. Okimoto (2011), 'Dynamics of International Integration of Government Securities' Markets,' *Journal of Banking and Finance* 35(1), 142-154.

[11] Li, L. (2002), 'Macroeconomic Factors and the Correlation of Stock and Bond Returns,' Working paper, Yale International Center for Finance.

[12] Lin, C.-F.J. and T. Teräsvirta (1994), 'Testing the Constancy of Regression Parameters Against Continuous Structural Change,' *Journal of Econometrics* 62(2), 211-228.

[13] Luukkonen, R., P. Saikkonen and T. Teräsvirta (1988), 'Testing Linearity Against Smooth Transition Autoregressive Models,' *Biometrika* 75, 491-499.

[14] Ohashi, K. and T. Okimoto (2013), 'Dependence Evolution in International Equity Markets,' RIETI Discussion Paper Series 13-E-048.

[15] Okimoto, T. (2011), 'Dependence Evolution in International Equity Markets,' ICS FS Working Paper, Hitotsubashi University.

[16] Pastor, L. and R.F. Stambaugh (2003), 'Liquidity Risk and Expected Stock Returns,' *Journal of Political Economy* 111(3), 642-685.

[17] Silvennoinen, A. and S. Thorp (2013), 'Financialization, Crisis and Commodity Correlation Dynamics,' *Journal of International Financial Markets, Institutions and Money* 24(1), 42-65.

[18] Tang, K. and W. Xiong (2012), 'Index Investing and the Financialization of Commodities,' *Financial Analysts Journal* 68(6), 54-74.

[19] Teräsvirta, T. (1994), 'Specification, Estimation and Evaluation of Smooth Transition Autoregressive Models,' *Journal of American Statistical Association* 89(425), 208-218.

[20] Yang, J., Y. Zhou and Z. Wang (2009), 'The Stock-Bond Correlation and Macroeconomic Conditions: One and a Half Centuries of Evidence,' *Journal of Banking and Finance* 33(4), 670-680.

公表データにもとづく損害保険リスクモデル

田中周二[*] ,栗山 晃[†]

H25.11.16

概 要

損害保険会社の保険引受リスクの計測のために、過去さまざまなモデルが考案され実際に利用されてきた。古典的な集合的危険論に基づく複合分布モデルは理論的にもよく研究されており、現在でも基本的なモデルとなっている。しかし、このモデルは現実の損害保険リスクの一部しか表現できておらず、特に (1) 保険種目間の相関関係を考慮できないこと、(2) 損害率のシステマティックなリスクが反映されていないこと、(3) 特定年度の特殊要因（例えば大規模な台風の上陸）が反映できないこと、などは実用的なリスク計量化のためには克服すべき課題といえよう。

このような現実的なリスクモデルの一例として IAA(2004) は MKL (Meyers,Klinker,Lalonde) モデルを推奨した。このモデルは個社だけのデータではなく損害保険業界の各社の損害率の時系列データを使用することにより、保険種目ごと及び合算した保険ポートフォリオのできるだけ現実的な損害額分布を求めることができる。その特徴は、

- 保険種目間、異なる保険会社間の相関を考慮することができる。
- 損害率のシステマティックな変動をモデルに組み込むことができる。
- モデルのパラメーターの推定方法が開発されている。
- 保険ポートフォリオのできるだけ現実的な損害額分布を求めることによりリスク量の計算が可能。

当論文では日本の損害保険業界のデータを用いて MKL モデルを構築することによりこのモデルの適用可能性を評価することを目的とする。

Keywords: 損保引受けリスク　相関　集合的危険論　経済価値ベースソルベンシー 2　内部モデル MKL(Glenn G. Meyers, Fredrick L. Klinker, and David A. Lalonde) モデル

[*]日本大学大学院総合基礎科学研究科教授

[†]損害保険料率算出機構

1 はじめに

1.1 目的

この論文の目的は、経済価値ベースのソルベンシー評価の導入を視野に入れて、日本の損害保険業界の公表データにもとづき損害保険引受リスクモデルを構築し、そのモデルを利用して現実的なリスク評価プロセスを提案することである。すなわち、

1. 損害保険会社の損害保険引受リスクモデルの構築。MKL モデルを参考にして頻度と強度のパラメーターの不確実性と相関を取り入れる。
2. 期待損害額と相関生成パラメーターの推定方法について新たな方法を検討する。
3. さまざまな前提におけるソルベンシー評価の試算を実施し、その水準や分散効果の程度を見る。

2 最近の損害保険業界の損害率データの概要

これから我が国の損害保険業界の 2001 年度より 2010 度までの損害率の時系列データにもとづいてリスクモデルの構築を行ってゆくがその前に 10 年間の損害保険業の収入と支出の状況を概観しておくことにしよう。

2.1 損害保険業界の収支状況

火災保険は、2004 年度（損害率 71.9 %）は自然災害（台風 16 号、18 号、23 号ほかで 7,274 億円、地震（新潟県中越地震、福岡県西方沖地震）で 305 億円の大口支払があった。2011 年度（148.7 %）の大幅な悪化は前年度末の直前（3 月 11 日）に発生した東日本大震災（12,346 億円）、9 月の台風 15 号（1,123 億円）、タイでの大洪水による保険事故が多額な支払となったことによる。海上保険は、2009 年度（損害率 60.9 %）、2011 年度（62.3 %）が高くなっているが、円高（2009 年度対米ドル為替レート 90 円台、2011 年度年初 80 円台、年央 75 円台、年度末 83 円台）傾向の継続に伴う積荷物流の減少の影響が大きいものと考えられる。

ちなみに、2012 年度は、国内外の自然災害に係る発生保険金の減少と資産運用損益の改善により、多くの会社が黒字転換または増益となったが、主力商品の自動車保険において収支改善の兆しがあるものの依然としてその損害率は高い水準にとどまった。また、2013 年度は、自賠責保険（保険料引き上げ)、火災保険（住宅着工件数の増加)、海上保険（円安の恩恵による輸出荷動きの増加)、賠償責任保険（経済活動の活発化）の収入保険料が伸びた一方、7 月以降の台風や記録的豪

雨、竜巻などの自然災害による被害による保険金支払いの増加が見込まれ損害率水準の行方は依然不透明である。

2.2　損害保険会社の損害率実績の分析

まず対象とする損害保険会社は以下の 12 社 (グループ) とする。損害保険業界はこの 10 年間で合併を繰り返してきたが、ここでは 2012 年時点での合併後の会社 (グループ) を対象とするため過去のデータについては、過去から合併していたものとして合算したデータを用いて分析する。12 社については各社の 2011 年度決算におけるプロフィールを以下に掲げる。また対象とする保険種目は、火災保険、海上保険、運送保険 、傷害保険、自動車保険、自賠責保険、その他の 7 区分とした。

分析の対象となる損害保険会社 (グループ) のプロフィール　(単位：億円, %)

	正味収入保険料	損害率	事業費率	コンバインドレシオ	支払備金
1	17830	76.9	36.7	113.5	8851
2	12811	73.3	40.3	113.6	6200
3	12659	79.0	39.4	118.4	6989
4	10746	74.9	40.0	114.9	4823
5	1366	69.7	41.5	111.2	514
6	2648	75.2	38.8	114.0	1015
7	1592	68.1	40.8	108.9	617
8	385	53.0	42.6	95.6	106
9	381	61.7	50.7	112.4	124
10	6306	78.2	41.5	119.7	3019
11	138	63.0	49.3	112.3	48
12	144	56.0	72.7	128.7	49

（注）損害調査費は損害率には含まず、事業費率に含む方式で算出している。

2.3　会社別損害率の時系列推移

損害率の定義は、損保業界の公表ベースでは「正味支払保険金」を「正味収入保険料」で除した率である。

- 正味収入保険料=元受正味保険料+受再正味保険料-出再正味保険料-収入積立保険料

- 正味支払保険金=元受正味保険金+受再正味保険金-出再正味保険金-回収再保険金

となっており、再保険収支を考慮したその会社の収支を反映した率となっている。元受会社の固有の引受リスクを反映するためには、「元受正味保険金」を「元受正味収入保険料」を除した「元受正味損害率」で測定すべきであるが、本論では再保険も含めた損害率で分析するものとする。なお、「事業費」を「正味収入保険料」で除した率を事業費率と呼んでおり、損害率と事業費率を加えたものがいわゆるコンバインドレシオに当たる。以下は、会社別損害率の推移 (2001 年～2010 年) を表した図である。

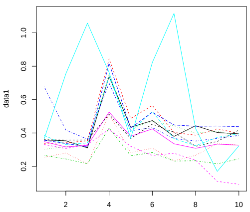


図 1: 火災

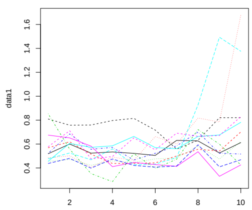


図 2: 海上

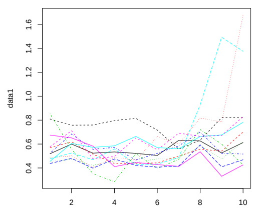


図 3: 運送

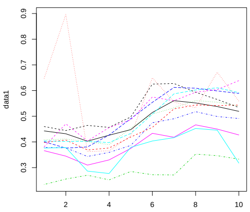


図 4: 傷害

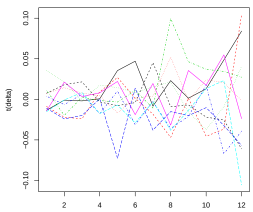


図 5: 自動車

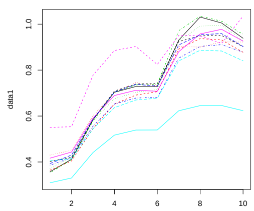


図 6: 自賠責

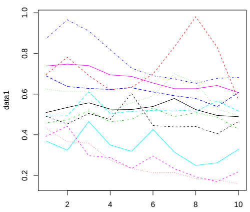


図 7: その他

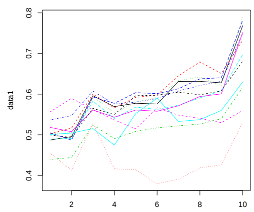


図 8: 合計

ちなみに損害保険会社は、強制保険である地震保険や自賠責保険について法的に地震再保険[1]、自賠責保険共同プール[2]に出再が義務付けられている再保険がある。それ以外の種目においても任意再保険、特約再保険また割合再保険、非割合再保険の形で広く取引されているが、保険種目、会社ごとに再保険政策に相違がみられる。

3 保険損害額モデル

基本的な集合的リスクモデルは以下のような構造を持つ。(Bowers,Gerber,Jones,Hickman and Nesbitt[1997,Ch.13] など)

3.1 集合的リスクモデル

アルゴリズム#1

1. 保険種目 h 内では 1 件あたりの損害発生額は同一の確率変数に従う。
2. 平均 λ_h の分布（計数過程）から発生件数 K_h を選ぶ。
3. 各 h に対し, それぞれの $k = 1.....K_h$ ごとの損害額 Z_{hk} を選ぶ。
4. 保険種目全体 h の損害額は $X_h = \sum_{k=1}^{K_h} Z_{hk}$ である。
5. これから会社全体の損害額は $X = \sum_{h=1}^{H} X_h$ となる。

特に、計数過程がポアソン過程の場合で損害額が対数正規分布やガンマ分布のようによく知られた分布からの標本の場合には複合ポアソン分布となるのでさまざまな計算方法が知られている。ところが、現実には会社間や保険種目間の相関が観察される。この理由は、一部の保険事故の発生や損害額には何らかの共通要因があると考えられるからである。たとえば 1 件あたりの損害額は一般的なインフレ (デフレ) の影響を受けているかもしれないし、台風被害は自動車にも家屋にも損害を与えるため自動車保険と火災保険に何らかの相関が生ずるかもしれない。このような保険種目間の相関構造を組み込む方法の一つが以下のようなアルゴリズム 2 で与えられる。

[1]巨大地震に際して、民間保険会社では支払能力に限度があるため設けられた官民一体の制度で、日本地震再保険会社が一旦全社から再保険を受再し、それを均質化して民間保険会社と政府のそれぞれの責任範囲で再々保険に出すという形をとる。

[2]各損害保険会社の自賠責保険料の純保険料部分が共同プールされ、保険料、保険金とも公平な配分率にもとづいて再配分される。このため各社の収支率の均一化が図られる仕組みとなっている。

3.2 相関構造

アルゴリズム #2

1. 保険種目 h 内では 1 件あたりの損害発生額は同一の確率変数に従う。
2. 平均 λ_h の分布から発生件数 K_h を選ぶ。
3. 各 h に対し, それぞれの $k = 1.....K_h$ ごとの損害額 Z_{hk} を選ぶ。
4. つぎのような確率変数 β を選ぶ：$E[\beta] = 1, Var[\beta] = b > 0$。
5. 会社全体の損害額：$X = \quad \beta \sum_{h=1}^{H} X_h$ とおく。

上に現れた β は、異なる保険種目に対し相関構造を与えることができるパラメーターの不確実性を表す確率変数である。独立な確率変数 X, Y に対し $\beta X, \beta Y$ は相関を持つので、保険種目間の発生の相関を表現できる。

Patrik and Meyers[1980] は、請求強度と請求計数のパラメーターを確率変数として扱うことで集合的危険論のパラメーターに不確実性を入れるモデルを提案した。Heckman and Meyers[1983] は、効率的な計算アルゴリズムを伴うパラメーターの不確実性を考慮するモデルを提案した。

パラメーターの不確実性が分布の分散にどのような影響を与えるかは以下の説明で理解できよう。X をパラメーター θ に依存する確率変数とすると、

$$Var[X] = E_\theta[Var[X|\theta]] + Var_\theta E[X|\theta]$$

もし、パラメーターの不確実性がなければパラメーター分散項はゼロになる。不確実性を入れると、無条件分散は大きくなる。

3.3 MKL モデル

これを拡張して最終的に以下のモデルを得る。ここでは、共分散グループと事故年度の概念が現れる。共分散グループとは、保険種目の相関が 1 とみなせるグループを言う。典型的には保険種目は異なっているが極めて類似した性格を持つ集団は同じ共分散グループとみなすことができる。

次に、事故年度は事故が発生した会計年度を意味し毎年の傾向変化や、ある年度に特有の要因、例えば自然災害の集中や法律改正などの影響を評価するために区分する必要がある。

アルゴリズム #3

1. 確率変数 β を選ぶ：$E[\beta]=1, Var[\beta]=b>0$。
2. 各共分散グループ i に対し、ランダムな分位点 p_i を選ぶ。
3. 各共分散グループ i、各共分散グループ (G_i) 内の各 h、事故年度 y に対し, 損害額 $Z_{hyk}, for\ k=1.....K_h$ を選ぶ。
   - つぎのような確率変数 χ_{hy} の分布の p_i 分位点を選ぶ:$E[\chi_{hy}]=1, Var[\chi_{hy}]c_{hy}>0$。
   - ランダムな発生件数 K_{hy} は平均 $\lambda_{hy}\chi_{hy}$ となるポアソン分布により発生させる。
   - 各 h,y に対し、ランダムに損害額 Z_{hyk} を発生させる。(このようにして損害額の相関は 1 になる)
4. 共分散グループ i の損害額：$X_i=\sum_{k\in G_i}\sum_y\sum_{k=1}^{K_h}Z_{hk}$。
5. 会社全体の損害額：$X=\beta\sum_{i=1}^{n}X_i$。

IAA[2004] ではこのモデルを以下のように修正、特定化したモデルを提案している。

IAA のモデル例

1. 各保険種目 i の損害額について次の操作を行う。
   - ガンマ分布を持つ $E[\chi_i]=1, Var[\chi_i]=c_i>0$ なる確率変数 χ_i を選ぶ：
   - ランダムな発生件数 K_i は平均 $\lambda_i\chi_i$ となるポアソン分布により発生させる。
   - 平均 μ_i, 分散 σ_i^2 の対数正規分布から各 $i, k=1,2,\cdots,K_i$ に対し、ランダムに損害額 Z_{ik} を発生させる。
2. 保険種目 i の損害額：$X_i=\sum_{k=1}^{K_i}Z_{ik}$ とおく。
3. 一様分布 (0,1) から各共分散グループ i に対し共通の分位点 p を選ぶ。各保険種目 i について $E[\beta_i]=1, Var[\beta_i]=b_i>0$ である分布の p 分位点をとる。これによって、各相関係数 $\rho_{ij}=1$ である多変量分布が得られる。
4. 会社全体の損害額：$X=\sum_{i=1}\beta_iX_i$ とおく。

アルゴリズム#3 で β はすべての保険種目で共通としていたが、IAA では保険種目ごとに異なる b_i を仮定していることに注意されたい。Heckman and Meyers[1983] では,b_i を混合パラメーター (mixture parameter),c_i を感染パラメーター (contagion parameter) と呼んでいる。

これから簡単な計算により以下の公式が確認できる。

1. $E[X_i] = \lambda_i \mu_i$
2. $E[X] = \sum_i E[X_i]$
3. $Var[K_i] = \lambda_i + c_i \lambda_i^2$
4. $Var[X_i] = \lambda_i \sigma_i^2 + \mu_i^2(\lambda_i + c_i \lambda_i^2)$
5. $Var[\beta_i X_i] = Cov[\beta_i X_i, \beta_i X_i] = (1 + b_i)Var[X_i] + E[X_i]^2 b_i = (1 + b_i)(\lambda_i \sigma_i^2 + \mu_i^2(\lambda_i + c_i \lambda_i^2)) + b_i \mu_i^2 \lambda_i^2$
6. $i \neq j$ として、$Cov[\beta_i X_i, \beta_j X_j] = \lambda_i \mu_i \lambda_j \mu_j \sqrt{b_i b_j} (\rho_{ij} = 1)$
7. $Var[X] = \sum_i \sum_j Cov[\beta_i X_i, \beta_j X_j]$

必要資本の計算を行うためには、損害額の分布の分位点を計算する必要があり、このためにはモンテカルロシミュレーションを実行するか高速フーリエ変換の方法を使う。前者はプログラミングは容易であるが計算負荷が大きく、後者はその反対の特徴を有す。

しかし、損害額の総額が対数正規分布に従うと仮定するならば、上の公式から平均 mu, 分散σ^2 を求めることができる。対数正規分布の信頼水準 α の $VaR_\alpha(X)$ が求めらと $TVaR_\alpha(X)$ は以下の式により求めることができる。

$$TVaR_\alpha(X) = VaR_\alpha(X) + \frac{E[X] - E[X \wedge VaR_\alpha(X)]}{1 - \alpha}$$

もちろん総額が対数正規分布にはならないため精密評価とは誤差が生ずる。

4 モデルパラメーターの推定プロセス

4.1 パラメーターの不確実性にもとづく相関を考慮するモデル

保険種目 i, 会社 j, 年度 k の損害額 X_{ijk} とすると、前節の公式より以下の式が成立する。ここに,$E_{ijk} = E[X_{ijk}]$ と表記した。

$$Cov[X_{ijk}, X_{i'j'k'}] = \delta_{ii'}\delta_{jj'}(1+b_i)\left\{\left(\frac{\sigma_i^2}{\mu_i} + \mu_i\right)E_{ijk} + c_i E_{ijk}^2\right\} + \delta_{G_i G_i'}\sqrt{b_i b_i'} E_{ijk} E_{i'j'k'} \quad (1)$$

共分散の定義より、

$$Cov[X_{ijk}, X_{i'j'k'}] = E[(X_{ijk} - E[X_{ijk}])(X_{i'j'k'} - E[X_{i'j'k'}])] \tag{2}$$

保険種目 i, 会社 j, 年度 k の損害額 X_{ijk} について、次のように標準化偏差を定義する。

$$\Delta_{ijk} = \frac{X_{ijk} - E[X_{ijk}]}{E[X_{ijk}]} \tag{3}$$

すると、クロネッカーのデルタ δ_{ab}($a = b$ のときは 1、その他では 0) を導入すると、

$$Cov[\Delta_{ijk}, \Delta_{i'j'k'}] = (1 + b_i)\frac{\delta_{ii'}\delta_{jj'}}{E_{ijk}}\left(\frac{\sigma_i^2}{\mu_i} + \mu_i\right) + \delta_{ii'}\delta_{jj'}(1 + b_i)c_i + \delta_{G_iG_{i'}}\sqrt{b_ib_i'} \tag{4}$$

が成立し、特に $j = j'$ のときには、

$$Var[\Delta_{ijk}] = (1 + b_i)\left(\frac{\sigma_i^2}{\mu_i} + \mu_i\right)\frac{1}{E_{ijk}} + c_ib_i + c_i + b_i \tag{5}$$

が成立するので、$Var[\Delta_{ijk}]$ の標本を用いると $1/E$ で回帰することにより $c_i + b_i$ の推定が可能となる。ここで $CV_{i,j,k} = \sqrt{Var[\Delta_{ijk}]}$ とすると、この量は標準偏差を期待値で割ったものであり変動係数を表している。

さらに、$j \neq j'$ のときには、

$$Cov[\Delta_{ijk}, \Delta_{i'j'k'}] = b_i \tag{6}$$

となる。

また、同一会社 ($j = j'$) で保険種目が同じ共分散グループに属さない場合 ($i \neq i'$) には、

$$Cov[\Delta_{ijk}, \Delta_{i'j'k'}] = \delta_{G_iG_{i'}}\sqrt{b_ib_{i'}} \tag{7}$$

となるので、保険種目が同じ共分散グループに属するかの検定に用いることができる。

そこで、パラメーター推定の手順は以下のとおりとなる。

手順

1. 損害額の期待値 E_{ijk} を適切なモデルによって推定する。
2. 標準化変数 Δ_{ijk} を求め、同一保険種目・同一会社の分散 $Var[\Delta_{ijk}]$ を $1/E$ で回帰することにより $c_i + b_i$ の推定を行う。
3. 同一保険種目だが会社の異なる $Cov[\Delta_{ijk}, \Delta_{i'j'k'}]$ の加重平均により b_i の推定を行う。
4. 求めた分散と共分散の推定値により保険種目全体さらに会社全体の分散を求めることができる。

5 損失額期待値の推定

この節では、保険種目別に損失額期待値を推定するモデルの確定とそのモデルにもとづく結果の概要を述べる。MKL 論文では、保険種目別のロスレシオの時系列データをもとに平滑化手法である局所多項式平滑化法 (LOESS) を用いた推定値から損失額期待値を求める方法を紹介しているが、日本のデータを用いて推定を実施したところ殆どのケースでうまくゆかないことが判明したため、今回の推計ではロスレシオの対数値を以下の説明変数による一般化線形モデル (GLM) により、最も適合度の高いモデルを AIC によって選択することにした。なお、回帰計算に当たっては正味保険料による加重を考慮する。ポートフォリオの規模に大きな格差があるため保険データにおいて加重の必要性は MKL 論文でも強調されている。

$$\log(\text{ロスレシオ}) = \beta_1\, size + \beta_2\, trend + \beta_3\, year + \beta_4\, company$$

説明変数

- size(規模)：保険種目のポートフォリオの規模を表す要因であり正味保険料の対数値で表す
- trend(トレンド):10 年を通じて上昇か下降のトレンドがあるか。事業年度で回帰する
- year(年度ごと要因):その年度に特有の事象により生じた要因 (time 1 ∼ 10)
- company(会社固有要因):それぞれのポートフォリオの特性から平均と異なる固有要因 (company1 ∼ 12)

年度要因は、年度のトレンド要因があればそれを優先し、なければ年度ごとの固有要因の有無を調べることにした。会社の規模についても、その要因が効く場合にはそれを優先し、効かない場合には会社固有要因を調べることにした。

この表から分かることは、火災と海上については年度特有の要因により明らかに平年時と異なる損害率を示す年が存在する。火災の場合には巨大台風による風水害、海上の場合には大きな海難事故などが考えられよう。その他の保険種目では年度のトレンドが存在する。

強い規模要因は火災、傷害で観察され、残りの保険種目ではほぼ会社特有の要因に吸収されている。会社特有要因は規模要因を除く地域や市場の偏りによる保険ポートフォリオの異質性を反映するものと考えられるが、保険種目によりまったく異なる傾向を示している。特に火災、傷害とその他は異質性が大きいように見える。

今回は係数に有意性がないものも含めて、このモデルを採用することにした。[3]

[3] 実際には年度要因、会社要因などはグルーピングした方が適切かもしれない。今後の課題としたい。

種目別のモデルと変数の選択結果

	火災	海上	運送	傷害	自動車	自賠責	その他
(切片)	***	***	*	***	.	***	***
size	***		*	***	*		
trend			***	***	***	***	***
year1	***	***					
year2		**					
year3	**						
year4	***						
year5	***	.					
year6	***						
year7		.					
year8		***					
year9							
year10		***					
comp1	***	***	***	***	***	***	***
comp2	***	**	*	**	**		***
comp3	***	***	***	***			***
comp4	***			***	**		***
comp5	***		*	***	*		
comp6	***	*	*	***	*		**
comp7	***	***	**	***	.		.
comp8	***		*	***	.		***
comp9	***		.	***	.		
comp10	***	***		***	*		***
comp11	***		.	***	*		
comp12	***			***	*		*

火災、運送、傷害、自動車では規模要因が効いているので、切片の値が、それ以外に比べて大きくなっている。年度特有要因があるのは火災と海上であり、損失が大きい年度が時々発生することが分かる。火災保険は、台風被害の大きい年度が特徴的に表れている。それ以外の保険種目では損害率のトレンドが観察され、運送、傷害、自動車、自賠責は上昇、その他は下降のトレンドである。

種目別のモデルパラメーター

	火災	海上	運送	傷害	自動車	自賠責	その他
(切片)	**12.9264**	**−0.6174**	**3.2639**	**8.6006**	**2.9744**	**−0.9227**	**−0.5770**
size	**−0.7200**		**−0.2213**	**−0.5084**	**−0.1722**		
trend			**0.0141**	**0.0431**	**0.0142**	**0.0976**	**−0.0114**
year1	0.0000	0.0000					
year2	-0.0633	0.0962					
year3	**0.01291**	-0.0452					
year4	**0.6951**	-0.0186					
year5	**0.1573**	**−0.0566**					
year6	**0.2494**	-0.0455					
year7	0.0588	**0.0601**					
year8	0.0404	**0.1800**					
year9	0.0185	-0.0006					
year10	0.0131	**0.1933**					
comp1	0.0000	0.0000	0.0000	0.0000	0.0000		0.0000
comp2	**−0.3082**	**−0.0677**	**−0.0178**	**−0.1526**	**−0.0648**		**0.3215**
comp3	**−0.2308**	**−0.1397**	**−0.1931**	**−0.0726**	-0.0451		**0.1033**
comp4	**−0.3697**	0.0040	-0.2271	**−0.4562**	**−0.0716**		0.3657
comp5	**−1.6989**	-0.0829	**−0.0889**	**−1.3431**	**−0.4583**		-0.0102
comp6	**−1.3453**	**−0.1655**	**−0.8597**	**−1.0858**	**−0.2987**		**0.2454**
comp7	**−1.6915**	**0.2752**	**−0.7753**	**−0.7789**	**−0.3970**		**−0.1154**
comp8	**−2.4353**	0.0242	**−2.1308**	**−2.3866**	**−0.7317**		**−0.8719**
comp9	**−2.7430**	-0.1395	**−1.1553**	**−2.4761**	**−0.6379**		-0.0847
comp10	**−0.5823**	**−0.2211**	-0.1532	**−0.4645**	**−0.1689**		**0.1494**
comp11	**−3.5954**	0.0682	**−1.1518**	**−2.9910**	**−0.8898**		-0.4452
comp12	**−4.3938**	0.1033	1.5228	**−2.0000**	**−0.7744**		**−0.7522**

(注) 太字は有意水準 10 %以上の要因を示す。

6 損害率の残差分析と b_i と c_i の推定

6.1 損害率の残差の分布

以下に示すグラフは、保険種目別の損害額期待値と実際値の残差の分布を箱髭図で表したものである。

保険種目別の残差の分布

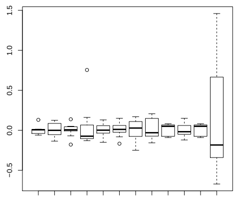


図 9: 火災

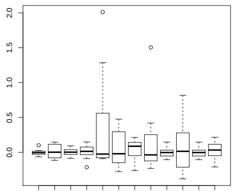


図 10: 海上

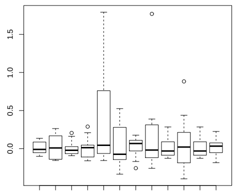


図 11: 運送

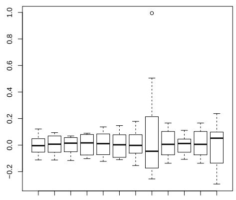


図 12: 傷害

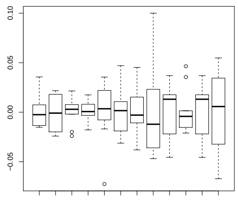


図 13: 自動車

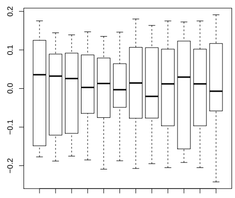


図 14: 自賠責

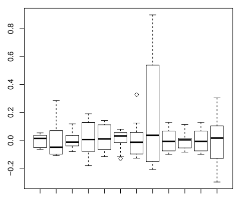


図 15: その他

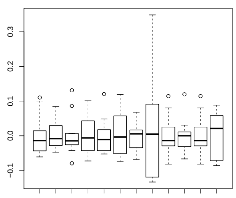


図 16: 合計

6.1.1 $b_i + c_i$の推定

先に述べたように標準化変数 Δ_{ijk} を求め、同一保険種目･同一会社の分散 $Var[\Delta_{ijk}]$ を $1/E$ で線形回帰することにより $c_i + b_i$ の推定を行う。この場合、期待損失額の大きさには大きな差異があるため重み付けをしなければ推定結果に大きなバイアスが生ずる。そこで期待損失額の重み付けによる線形回帰を行い、さらにブートストラップ法を用いて計算精度を上げることにした。ブートストラップ標本としては、重複を許す 12 社の会社標本を 100 通り作成し、また 10 年の年度標本も 100 通り作成し、それぞれの平均値とその標準誤差を計算することにした。推定結果は、保険種目によって精度の高い推定ができるものとそうでないものに分かれた。このブートストラップ標本により、点推定値のほか、95 パーセンタイル信頼区間も求めた。

$b_i + c_i$ の値を求めると同時に、$Var[\Delta_{ijk}]$ の予測値から与えられた会社の変動係数の 2 乗の推定値 $C\hat{V}^2{}_i$ を求めることができる。後述するように、これから対数正規分布の対数標準偏差の推定値を得ることができる。

6.1.2 b_iの推定

次に、同一種目で異なる会社の標本共分散 $Cov[\Delta_{ijk}]$ から、期待損失額の重み付けによる平均値と標準誤差を求めることができる。分散と同様に、ブートストラップ法によって点推定値のほか、95 パーセンタイル信頼区間も求めた。

6.1.3 パラメーターの推定結果

サンプルとして 2010 年度の会社 1 についての推定結果を以下のとおりまとめた。[4]標準誤差 (s.e.) は会社要因と年度要因に分解したものを記載した。保険種目によって標準誤差の格差は非常に大きいことが分かる。

---

[4]他の会社や年度ではもちろん結果は異なる。

保険種目	火災	海上	運送	傷害	自動車	自賠責	その他
期待損失	914.8	388.2	256.8	874.0	5616.0	258.8	1141.2
$b_i + c_i$	0.0309	0.0604	0.0456	0.0112	0.00021	0.016	0.0219
$(s.e.)$	0.0695	0.1348	0.0889	0.0214	0.00003	0.010	0.0273
会社要因	0.0944	0.1739	0.1173	0.0287	0.00003	0.012	0.0360
年度要因	0.0008	0.0008	0.0039	0.0009	0.00003	0.001	0.0034
(95 % VaR)	0.1626	0.3426	0.2091	0.0603	0.00025	0.030	0.0817
b_i	0.0001	0.0012	0.0071	0.0032	0.00006	0.0108	0.0009
$(s.e.)$	0.0008	0.0012	0.0035	0.0007	0.00002	0.010	0.0007
会社要因	0.0001	0.0011	0.0010	0.0005	0.00002	0.013	0.0006
年度要因	0.0010	0.0013	0.0039	0.0009	0.00002	0.003	0.0008
(95 % VaR)	0.0017	0.0032	0.0141	0.0044	0.0001	0.0152	0.0022

7 ソルベンシーマージン

以上得られた推定値より、会社全体のソルベンシーマージンを評価する。保険種目ごとの損害額分布は対数正規分布であることを仮定し、会社全体の損害額分布も近似的に対数正規分布になることを仮定する。

分散 $Var[\Delta_{ijk}]$ の予測値が変動係数の二乗 $\hat{CV^2}_i j$ となることから対数正規分布の対数平均 μ と対数標準偏差 σ を仮定すると、

$$E[X] = e^{\mu + \frac{\sigma^2}{2}}, V[X] = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

となる。従って、変動係数の二乗は、$CV[X]^2 = \dfrac{V[X]}{E[X]^2} = e^{\sigma^2} - 1$ である。

よって,

$$\sigma = \sqrt{\log(1 + CV[X]^2)}, \mu = \log E[X] - 2\log(1 + CV[X]^2)$$

となる。このことから, 対数正規分布が決定され、保険種目ごとの損害額分布の分位点の計算が可能となる。

次に、会社全体の分布を計算するためには、

$$V[X] = \sum_i \sum_j Cov[\beta_i X_i, \beta_j X_j]$$

の公式により、$i \neq j$ のときの $Cov[\beta_i X_i, \beta_j X_j] = \lambda_i \mu_i \lambda_j \mu_j \sqrt{b_i b_j} (\rho_{ij} = 1)$ の計算が必要であるが、b_i の推定値も得られているので計算可能である。

なお、$Var[\Delta_{ijk}]$ や b_i は点推定値だけでなく、信頼区間も推定することができる。従って、パラメーターの推定誤差を考慮したソルベンシーマージンの評価も可能である。ここでは 95 %タイルの上側信頼区間の端点をとり、保守的なソルベンシー評価も行うことにした。

信頼水準 α に対する分位点はバリューアットリスク $VaR_\alpha(X)$ と呼ばれるが、保険ではテールバリューアットリスク $TVaR$ もよく使われている。このリスク尺度はバリューアットリスクに、それを超える分布の平均を加えた量で、公式

$$TVaR_\alpha(X) = VaR_\alpha(X) + \frac{E[X] - E[X \wedge VaR_\alpha(X)]}{1-\alpha}$$

により求めることができる。なお、

$$E[X \wedge VaR_\alpha(X)] = E[X]\Phi(\frac{log(VaR_\alpha(X)) - \mu - \sigma^2}{\sigma}) + VaR_\alpha(X)[1 - F(VaR_\alpha(X))]$$

である。

まず、前提条件となるパラメーターの推定値は以下のとおりであった。

【パラメーターの推定値 (会社 1)】

	期待損害額	最良推定			95 %上側分位点		
	E	CV	σ	μ	CV	σ	μ
火災	914.8	0.04030	0.1987	18.252	0.16268	0.38823	18.030
海上	388.3	0.06041	0.2422	17.353	0.34263	0.54280	16.885
運送	256.9	0.06282	0.2468	16.939	0.40029	0.58024	16.388
傷害	874.0	0.01128	0.1059	18.263	0.06031	0.24199	18.168
自動車	5616.0	0.00021	0.0145	20.145	0.00025	0.016079	20.145
自賠責	2588.0	0.01645	0.1277	19.338	0.03023	0.17258	19.312
その他	1141.3	0.02199	0.1475	18.509	0.08170	0.28025	18.395

信頼水準 99 %タイルと 99.5 %タイルの分位点 VaR と $TVaR$ の試算結果は以下の 3 つの図にまとめられたとおりである。最初の図 17 は、保険種目間の分散効果を考慮しない場合の 2 通りの信頼水準とリスク尺度における結果である。一番左の図は期待損失額となっており、これが基準となって、それぞれの場合の比率が分かるようになっている。1 本の棒は保険種目ごとのリスク量が積み上げられており、下から火災、海上、運送、傷害、自動車、自賠責、その他であるが、自動車、自賠責の期待損失額もリスク量も割合が大きくなっている。期待損失額の 10 %から 30 %程度となっており、現在のソルベンシー基準値と比べても妥当な水準となっている。次の図 18 と図 19 は、それぞれ信頼水準が 95 %タイルと 99.5 %タイルの値となっており、それぞれが保険種目のリスク量の単純和に続いて、VaR の分散効果を考慮した最良推定の水準、同じく信頼区間の上側 95 %タイルの水準となっている。信頼水準が高く、また VaR よりも $TVaR$ の方が大きいのは当然であるが、分散効果によって 4,5 %減少するが推定誤差を考慮するとほぼ分散効果が打ち消されてしまう結果となっている。[5]

[5]損失額の総額の分布も対数正規分布と近似することによる誤差は 0.5~2 %程度であることが試算により判明したので、その誤差も考慮すべきである。

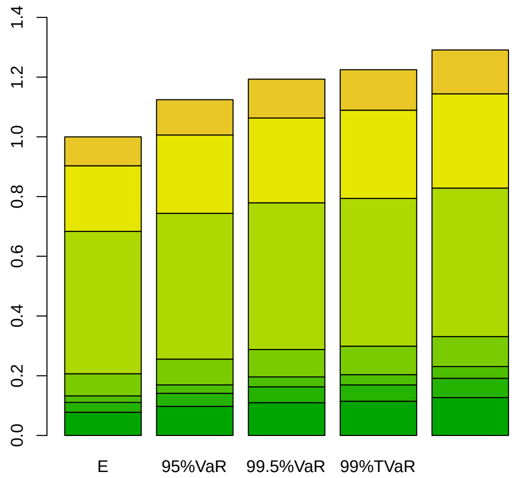


図 17: 保険種目ごとのソルベンシーマージンの総和

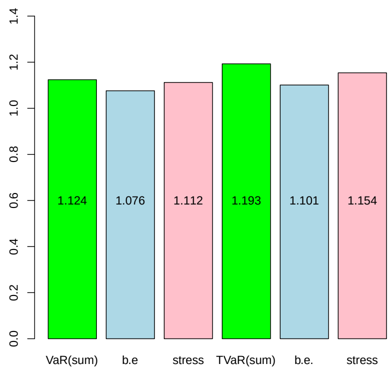


図 18: 信頼水準 95 %の場合

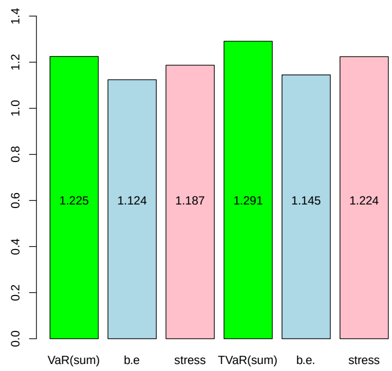


図 19: 信頼水準 99.5 %の場合

8 結論と今後の課題

今回の研究の目的は、MKL モデルの枠組みに沿って、日本の損害保険業界の損害額データを用いた損保引受リスク評価モデルが構築可能かどうか検証することであった。第一段階としては、過去 10 年間の複数の会社ベースの損害率データから当初望んでいたようなリスク計測モデルの構築方法の可能性を得ることができた。しかし、多くの課題が残っている。統計モデルの観点からは、損害率の原データには多くのノイズが含まれているため適当と思われる期待損害額の評価モデルをどのように作り上げるかが鍵となる。単なる平滑化モデルでは良い結果は得られず、今回は一般化線形モデル (GLM) を使ったが、一般化線形混合モデル (GLMM) や一般化加法モデル (GAM) のようなより柔軟な方法の方が良い可能性がある。b_i, c_i の推定方法についてもさらに検討の余地があろう。

積み残した課題の一つは、損害率データの問題である。今回使用したデータは、再保険の収支を考慮した正味損害率であるが、元受保険会社の本来の引受リスクは元受損害率であり、このベースの試算を行う必要がある。また、今回の数値はいわゆる written basis(現金主義) にもとづくが、理論的には inccurred basis(発生主義) さらには収支を正確に対応させる policy-year basis（契約年度主義）へと精度を向上させることが必要である。これにより再保険の効果について検証できるようになる。また事業費を含めたコンバインドレシオで実行してみることも必要であろう。

次は、カタストロフリスクの計測である。公表された統計データのみで計測することはおそらく不可能であるが、火災と海上において有意に異なる年度があることからより超長期のデータを使うことにより、ストレスシナリオの作成は可能であるかもしれない。他に興味があるのは、損害保険業界の再編が最近、急速に進んでいるが、損保引受リスクの合併によるリスク分散効果を計測することである。

また、ソルベンシー規制との関連では、最近改訂、厳格化した日本のソルベンシー規制および欧州のソルベンシーⅡとの比較を行うことがある。ソルベンシーⅡでは、いわゆる保険料リスクと準備金リスクを考えているが、今回は準備金リスクについて考慮していない。これも今後の課題である。

参考文献

[1] Glenn G.Meyers,Fredrick L.Klinker and ,David A.Lalonde, ”The Aggregation and Correlation of Insurance Exposure”,CAS Forum Summer,2003

[2] Philip E.Heckman and Glenn G.Meyers,” The Calculation of Aggregate Loss Distributions from Claim Severity Distributions and Claim Count Distributions”, PCAS LXX, (1983)

[3] Glenn Meyers,”Estimating Between Line Correlations Generated by Parameter Uncertainty,” CAS Forum, Summer (1999a)

[4] Glenn Meyers,”Discussion of Aggregation of Correlated Risk Portfolios by Shaun Wang,” PCAS LXXXVI, PCAS LXXXVI, (1999b)

[5] Stuart A.Klugman,Harry H.Panjer,and Gordon E.Willmot, ”Loss Models From Data to Decisions”, (1998),John Wiley & Sons

[6] International Association of Actuaries,”A Globel Framework for Insurer Solvency Assessment”,(2004)

[7] Bowers,Gerber,Jones,Hickman and Nesbitt,”Actuarial Mathematics”(1988),Society of Actuaries

[8] インシュアランス統計号（損害保険版）2001 年度〜2010 年度

[9] 損害保険ファクトブック, 各年度版

On a generalization from ruin to default in Lévy insurance risks

Yasutaka Shimizu

Graduate School of Engineering Science
Osaka University

“経済リスクの統計学の新展開：稀な事象と再起的事象”

2013 年 12 月 26 日

(Joint work with R.Feng, University of Illinois)

Contents

- Insurance ruin theory and the recent developments
- "Ruin" to "Default": Lévy risk models and default-related quantities
- Some representations for "default-related risks"
- (Connection to Lévy fluctuation theory)
- Concluding remarks

Part I

Ruin theory and recent developments

Insurance Risk Models

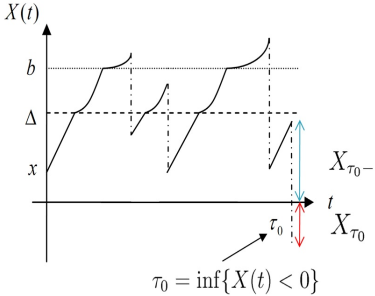
- **Objective:** quantify and analyze the risk of insolvency associated with insurance business.
- **Model setup:** $X_t = A_t - L_t$ (risk process)
  - **Assets** $A = (A_t)_{t\geq 0}$ (incoming cash flows):
    Initial surplus, premium income, investment income, ...
  - **Liabilities** $L = (L_t)_{t\geq 0}$ (outgoing cash flow):
    Claim payments, dividends, expenses, ...

- **Risk measures:**
  - Ruin probability: $\mathbb{P}^x(X_0 = x) = 1$,

$$\psi(x) = \mathbb{P}^x\left(\tau_0 < \infty\right).$$

  - Distribution of "severity" of ruin:

$$\phi(x; u, v) = \mathbb{P}^x\left(X_{\tau_0 -} \leq u, |X_{\tau_0}| \leq v, \tau_0 < \infty\right).$$

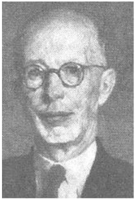


Classical ruin theory

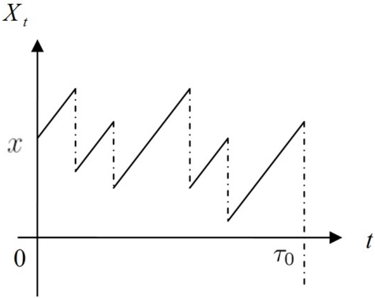
Cramér-Lundberg model: classical model; Lundberg (1903, Ph.D Thesis):

H.Cramér

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i,$$



P.Lundberg



- $x > 0$: initial surplus.
- $c > 0$: premium rate.
- $N_t \sim Po(\lambda t)$: a number of claims.
- $Y_i \in Q$: IID; ith claim size.
- Main concern:

$$\psi(x) = \mathbb{P}^x(\tau_0 < \infty).$$

Classical tools for analysis of "ruin"

Let $T > 0$ and distinguish the following 3 cases:

1. No claim in $(0, T)$ (not ruin): probability $e^{-\lambda T}$.
2. First claim occurs in $[t, t + dt)$ with $t < T$: probability $\lambda e^{-\lambda t}\,dt$
   its amount is $y < x + cT$: (not ruin)
3. First claim occurs in $[t, t + dt)$ with $t < T$: probability $\lambda e^{-\lambda t}\,dt$
   its amount is $y > x + cT$: (ruin)

- Distinguish the cases: for any $T > 0$,

$$\begin{aligned}\psi(x) &= e^{-\lambda T}\psi(x+cT) \ \text{(1. no claim)} \\ &+ \int_0^T \lambda e^{-\lambda t}\,dt \int_0^{x+ct} \psi(x+ct-y)\,Q(dy) \\ &\qquad \text{(2. first claim } t < T,\ y < u+ct\text{)} \\ &+ \int_0^T \lambda e^{-\lambda t}\,dt \int_{x+ct}^{\infty} Q(dy) \\ &\qquad \text{(3. first claim } t < T\text{, but } y > u+ct\text{)}\end{aligned}$$

- Take $\frac{d}{dT}$ on both sides, and set $T = 0$: *Integro-differential equation*,

$$c\psi'(x) + \lambda \int_0^x [\psi(x-y) - \psi(x)]\,Q(dy) + \lambda\overline{Q}(x) = 0.$$

Making some computations, we have a **renewal-type equation**:

$$\psi(x) = \frac{\lambda}{c}[\psi * \mathcal{T}_\rho Q'](x) + \frac{\lambda}{c}\mathcal{T}_\rho \overline{Q}(x),$$

where:

- $\mathcal{T}_s f(x) = e^{sx} \int_x^\infty e^{-sy} f(y)\, dy$: *Dickson-Hipp operator*,
- $\rho > 0$: Solution to the *Lundberg fundamental equation*:

$$\log \mathbb{E}^x[e^{\rho(X_1 - x)}] = c\rho + \lambda \int_0^\infty (e^{-\rho z} - 1)\, Q(dz) = 0.$$

Recent development of ruin theory

Expected discounted penalty function (Gerber and Shiu, 1998, NAAJ):

$$\phi(x) = \mathbb{E}^x\left[e^{-\delta\tau_0} w(X_{\tau_0 -}, |X_{\tau_0}|)\mathbf{1}_{\{\tau_0<\infty\}}\right],$$

where:

- $\tau_0 := \inf\{t > 0 | X_t < 0\}$: Time of ruin.
- $w : \mathbb{R}^2 \to \mathbb{R}$: “Penalty” depending on surplus prior to ruin and deficit at ruin.
- Expected Present Value (EPV) of risk at ruin.
- *Gerber-Shiu function*.

Example (Gerber-Shiu functions)

- $\delta = 0$; $w \equiv 1$: $\phi(x) = \mathbb{P}^x(\tau_0 < \infty)$.
- $\delta = 0$; $w(x, y) = \mathbf{1}_{\{(x \le u, y \le v)\}}$: a (defective) density of $(X_{\tau_0 -}, |X_{\tau_0}|)$:

$$\phi(x; du, dv) = \mathbb{P}^x\left(X_{\tau_0} \in du, |X_{\tau_0}| \in dv, \tau_0 < \infty\right).$$

- $\delta \ge 0$; $w = (\alpha x + \beta y)^k$: the kth-order (discounted) moment of a claim causing ruin: $\alpha, \beta \in \mathbb{R}$,

$$\phi(x; \alpha, \beta) = \mathbb{E}^x\left[e^{-\delta\tau_0}(\alpha X_{\tau_0 -} + \beta|X_{\tau_0}|)^k \mathbf{1}(\tau_0 < \infty)\right].$$

- $\delta \ge 0$; $w(x, y) = e^{-\xi x - \eta y}$: moment generating function of $(\tau_0, X_{\tau_0 -}, |X_{\tau_0}|)$:

$$\phi(x; \delta, \xi, \eta) = \mathbb{E}^x\left[e^{-(\delta\tau_0 + \xi X_{\tau_0 -} + \eta|X_{\tau_0}|)}\mathbf{1}(\tau_0 < \infty)\right] \quad (\delta, \xi, \eta \ge 0).$$

- e.g., option pricing, dividend strategy, capital injection, ...; Gerber and Shiu (1998a,b), Cai et al. (2009a,b,c), Eisenberg and Schmidli (2011), etc.

Risk measures by Gerber-Shiu function

- VaR-type risk measure due to G-S risks:

$$V_\epsilon := \inf\{x > 0 \,|\, \phi(x) < \epsilon\},$$

the minimum requirement of G-S risk does not exceed the level $\epsilon > 0$.

- e.g., $\delta = 0$, $w(x, y) = \mathbf{1}(y > z)$ in ϕ,

$$\phi(x; z) = \mathbb{P}^x\left(|X_{\tau_0}| > z,\, \tau_0 < \infty\right),$$

the tail function of the *Deficit at Ruin*:

$$DaR_\alpha(x) := \inf\{z > 0 | \phi(x; z) \leq 1 - \alpha\}.$$

so-called "VaR at ruin" when the initial capital is $x > 0$.

- Solve $x_\alpha = DaR_\alpha(x_\alpha)$, then

x_α: Surplus level to cover the deficit at ruin with $100\alpha\%$

Part II

"Ruin" to "Default"

Generalized risk process

Lévy insurance risk: spectrally negative Lévy process

$$X_t = x + ct + \sigma W_t - S_t,$$

- W: Wiener process, ($\sigma \geq 0$: const.)
  - Uncertainty of income process; Dufresne and Gerber (1991),
- S: Subordinator,
  - Pure jump Lévy process with increasing path, possibly infinite activity, representing frequent "small" claims, costs, etc; Huzack et al. (2004), Biffis and Morales (2011), etc.
  - Lévy characteristics:

$$\mathbb{E}[e^{iuS_t}] = \exp\left(t \int_0^\infty (e^{iu} - 1)\nu(z)\,dz \right),$$

where ν: Lévy density with $\int_0^\infty z\nu(z)\,dz < \infty$.

- **Time of Ruin**:

$$\tau_0 := \inf\{t > 0 | X_t < 0\}.$$

Connection to *Credit Risk Modeling*

Firm's asset value (or stock price) process:

$$V_t := V_0 exp\left(ct + \sigma W_t - S_t\right),$$

"Geometric" spectrally negative Lévy process,

- Madan and Schoutens (2008; J. of Credit Risk): It reasonably includes jumps and incorporates skewness in the underlying return distribution. A firm's asset value is exposed to shocks (represented by negative jumps), which is the main concern in risk management practice.
- Carr et al. (2002; J. of Business): risk-neutral processes for equity prices should be processes of infinite activity and finite variation.
- **Time of Default** (structural model):

$$\tau_d := \inf\{t > 0 | X_t < d\}, \quad d \in \mathbb{R},$$

where $X_t := \log V_t$, $x := \log V_0$.

"Ruin" to "Default"

Definition

Time of default:

$$\tau_d := \inf\{t > 0 | X_t < d\}, \quad d \in \mathbb{R},$$

"Default-related" quantities:

$$H_d(x) = \mathbb{E}^x \left[\int_0^{\tau_d} e^{-\delta t} l(X_t)\, dt \right] \mathbf{1}_{\{x \geq d\}},$$

a *"path-dependent penalty up to default"*.

- G-S function due to default:

$$\phi_d(x) = \mathbb{E}^x \left[e^{-\delta \tau_d} w(X_{\tau_d -}, |X_{\tau_d}|) \mathbf{1}_{\{\tau_d < \infty\}} \right], \quad x > d,$$

  is given by $l(x) = w(0,0)\epsilon_d(x) + \int_x^\infty w(x, z - x)\nu(z)\, dz$.
- As $w \equiv 1, \delta = 0$: probability of default

$$\psi_d(x) = \mathbb{P}^x(\tau_d < \infty), \quad x > d.$$

- Hence $\psi_d \subset \phi_d \subset H_d$.

Example (Total costs due to claims up to default)

- Suppose that dealing with a claim at time t costs $C(X_{t-}, X_t)$, where C is a positive cost function.
- EPV of total costs up to time T is given by

$$H_d(x;T) = \mathbb{E}^x\left[\sum_{0\le t\le \tau_d\wedge T} e^{-\delta t}C(X_{t-},X_t)\mathbf{1}_{\{\Delta X_t>0\}}\right]$$
$$= \mathbb{E}^x\left[\int_0^{\tau_d\wedge T}\int_0^\infty e^{-\delta t}C(X_{t-},X_{t-}-z)\nu(z)\,dzdt\right].$$

- As $T\to\infty$,

$$H_d(x) := \lim_{T\to\infty} H_d(u;T) = \mathbb{E}^x\left[\int_0^{\tau_d} e^{-\delta t}l(X_t)\,dt\right],$$

where $l(x) = \int_0^\infty C(x,x-z)\nu(z)\,dz$.

Part III

Some representations for “Default-related quantities”

Integro-differential equation

Theorem (Integro-differential Eq.)

Suppose that l is continuous on (d, ∞) except for a countable set of discontinuities D such that

$$\mathbb{E}^x \left[\int_0^{\tau_d} e^{-\delta t} |l(X_t)| \, dt \right] < \infty \quad \text{for all } x > d,$$

and that H_d has the bounded second derivative on $(d, \infty) \cap D^c$. Then H_d is the solution to the following integro-differential equation:

$$(\mathcal{A} - \delta) H_d(x) = -l(x), \quad x \in (d, \infty) \cap D^c,$$

where

$$\mathcal{A} f(x) = c f'(x) + \frac{\sigma^2}{2} f''(x) + \int_0^\infty [f(x - z) - f(x) + z f'(x)] \nu(z) \, dz.$$

Renewal type equation

Let

$$\mathcal{E}_\beta f(x) = e^{-\beta x}\int_0^x e^{\beta y} f(y)\,dy, \quad \mathcal{T}_\rho f(x) = e^{\rho x}\int_x^\infty e^{-\rho y} f(y)\,dy.$$

Theorem (Defective renewal Eq.)

Suppose the same assumptions as in the previous, and the net profit condition: $c > \int_0^\infty z\nu(z)\,dz$, and $\sigma > 0$. Then H_d satisfies the following (defective) renewal equation:

$$H_d(x) = h_l(x) + \int_0^{x-d} H_d(x-y)g(y)\,dy, \quad x \ge d,$$

where

$$h_l(x) = \frac{2}{\sigma^2}\mathcal{E}_\beta\mathcal{T}_\rho l(x) + \left[H_d(d) - \frac{2}{\sigma^2}\mathcal{E}_\beta\mathcal{T}_\rho l(d)\right] e^{-\beta(x-d)},$$

$$g(x) = \frac{2}{\sigma^2}\mathcal{E}_\beta\mathcal{T}_\rho \nu(x),$$

$\beta = 2c/\sigma^2 + \rho$ *and* ρ *is a solution to the Lundberg fundamental equation:*

$$\Psi(\rho) := \log \mathbb{E}^x[e^{\rho(X_1 - x)}] = \delta.$$

Remarks I

- As $\sigma^2 > 0$:
  - $H_d(d) = 0$ if $l(0) < \infty$.
  - $H_d(d) = w(0,0)$ in G-S cases.
- As $\sigma^2 = 0$, $H_d(d)$ is not clear (depending on the case): e.g., $H_d(x) = \mathbb{P}^x(\tau_d < \infty)$,

$$H_d(d) = \frac{1}{c}\int_0^\infty \nu(z,\infty)\,dz$$

- Case of $\sigma^2 = 0$ is obtained by $\sigma \to 0$:

$$\phi(x) = \frac{1}{c}[\phi * \mathcal{T}_\rho \nu](x) + \frac{1}{c}\mathcal{T}_\rho l(x),$$

since $\frac{2}{\sigma^2}\mathcal{E}_\beta f(x) \to \frac{1}{c} f(x)$.

Remarks II

- **Series representation**:

$$
\begin{aligned}
H_d &= h_l + g * H_d \\
&= h_l + g * [h_l + g * H_d] = h_l + g * h_l + g^{*2} * H_d \\
&= \ldots \\
&= h_l * \sum_{n=0}^{\infty} g^{*n} \\
&= \frac{1}{1-p} \int_0^{x-d} h_l(x-y) G_\delta(dy),
\end{aligned}
$$

where G_δ: *Compound Geometric Distribution* s.t.

$$
G_\delta(x) = (1-p) + \sum_{k=1}^{\infty} (1-p)p^k \int_0^x q_\delta^{*k}(x)\,dx,
$$

$$
q_\delta(x) = p^{-1} g(x), \quad p = \int_0^\infty g(x)\,dx < \infty.
$$

Corollary (Compound geometric representation)

$$H_d(x) = \frac{1}{1-p}\mathbb{E}[h_l(x-Y)], \quad x > d,$$

where $Y = U_1 + \cdots + U_N$; $U_i \sim q_\delta(x)\,dx$ *(iid)*, $N \sim Geo(p)$.

Corollary (Fourier transform)

$$\mathcal{F}H_d(s) = \frac{\mathcal{F}l(s) - \mathcal{L}l(\rho) + H_d(d)(\rho + is)\sigma^2/2}{\delta - \Psi(is)}, \quad s \in \mathbb{R},$$

where

$$\mathcal{F}f(s) = \int_{\mathbb{R}} f(x)e^{isx}\,dx, \quad \mathcal{L}f(s) = \int_0^\infty f(x)e^{-sx}\,dx,$$

$$\Psi(is) = isc + \frac{\sigma^2}{2}s^2 - \int_0^\infty (e^{isx} - 1)\,\nu(dx).$$

Corollary (Large initial capital)

Suppose that, for $R_0 > 0$,

$$\int_0^\infty e^{R_0 u}\nu(z)\,dz < \infty, \quad (light - tailed)$$

and that there exists the negative solution $-R \in (-R_0, 0)$ to the Lundberg fundamental equation:

$$\Psi(-R) = -cR + \frac{\sigma^2}{2}R^2 + \int_0^\infty (e^{Ru} - 1)\nu(z)\,dz = \delta,$$

and Then we have

$$H_d(x) \sim \frac{\int_0^\infty A(u)\,du}{\int_0^\infty B(u)\,du} e^{-R(x-d)}, \quad x \to \infty,$$

where

$$A(u) = e^{Ru}\left\{\frac{2}{\sigma^2}\mathcal{E}_\beta\mathcal{T}_\rho l(u) + \left[H_d(d) + \frac{2}{\sigma^2}\mathcal{E}_\beta\mathcal{T}_\rho l(d)\right]e^{-\beta u}\right\},$$

$$B(u) = ue^{Ru}\frac{2}{\sigma^2}\mathcal{E}_\beta\mathcal{T}_\rho\nu(u).$$

Part IV

Connection to Lévy fluctuation theory

Scale function

- First passage problem for Lévy process: e.g., Kyprianou (2006).
- *δ-scale function* is defined via

$$\int_0^\infty e^{-sx} W^{(\delta)}(x)\,dx = \frac{1}{\Psi(s)-\delta}, \quad s > \rho,$$

  where $\rho > 0$: $\Psi(\rho) = \delta$ *(Lundberg fundamental equation)*.
- Linking to *"Potential Measure"*: $U^{(\delta)}(dx) = \mathbb{E}\left(\int_0^\infty e^{-\delta t}\mathbf{1}_{(X_t \in dx)}\,dt\right)$.
- Our target H_d has a representation via the scale function.
- Recently, scale functions for various *spectrally negative Lévy processes* are explicitly known; see, e.g., Hubalek and Kyprianou (2011).

Example (Scale functions)

- *BM*: $X_t = \mu t + \sigma W_t$,

$$W^{(\delta)}(x) = \frac{2}{\sqrt{2\delta\sigma^2 + \mu}} e^{-\mu x/\sigma^2} \sinh\left(\frac{x}{\sigma^2}\sqrt{2\delta\sigma^2 + \mu}\right), \quad x \geq 0.$$

- *CPP with Exp.Jumps*: $X_t = ct - \sum_{i=1}^{N_t} Y_i$ with $Y_i \sim Exp(1/\mu)$, $N_t \sim Po(\lambda t)$

$$W^{(\delta)}(x) = \frac{(\mu + \rho^+)e^{\rho^+ x} - (\mu + \rho^-)e^{\rho^- x}}{\sqrt{r^2 + 4c\mu\delta}}, \quad x \geq 0,$$

where $\rho^\pm := (r \pm \sqrt{r^2 + 4c\mu\delta})/2c$, $r = \lambda + \delta - c\mu$.

- s.n. β-stable process with $\beta \in (1,2)$: $\log \mathbb{E}[e^{\theta X_t}] = e^{t\theta^\beta}$,

$$W^{(\delta)}(x) = \beta x^{\beta-1} E'_\beta(\delta x^\beta), \quad x \geq 0,$$

where $E_\beta(z) = \sum_{k\geq 0} z^k/\Gamma(1 + \beta k)$: the *Mittag-Leffler function*.

Fluctuation-theoretic solution

Theorem (via the scale function)

Let $W^{(\delta)}$ be the scale function of the risk process $X = (X_t)_{t \geq 0}$, and suppose that the corresponding δ-scale function $W^{(\delta)}$ is differentiable. Then

$$H_d(x) = \int_0^{x-d} \mathcal{T}_\rho l(x-y)\, K(dx) + \frac{\sigma^2}{2} K[0, x-d), \quad x \geq d$$

where

$$K(dx) = \left[W^{(\delta)}(0)\epsilon_0(x) + \frac{d}{dx} W^{(\delta)}(x) + \rho W^{(\delta)}(x) \right] dx.$$

- It is well known that the ruin probability

$$\psi(x) := \mathbb{P}^x(\tau_0 < \infty) = 1 - \Psi'(0+) W^{(0)}(x).$$

- For the classical G-S function: see, e.g., Biffis and Kyprianou (2011).

Example (Total costs due to claims up to default)

- Total costs up to τ_d: for $l(x) = \int_0^\infty C(x, x-z)\nu(z)\,dz$,

$$H_d(x) := \mathbb{E}^x\left[\int_0^{\tau_d} e^{-\delta t} l(X_t)\,dt\right].$$

- Suppose, e.g., $C(x,y) = \alpha(x-y)$ for $\alpha \in (0,1)$ (100α%-cost for each claim):

$$l(x) \equiv \alpha \int_0^\infty z\nu(z)\,dz =: \mu_\alpha,$$

- Then

$$H_d(x) = \frac{\mu_\alpha}{\delta}\left(1 - \mathbb{E}^{x-d}\left[e^{-\delta\tau_0}\mathbf{1}_{\{\tau_0<\infty\}}\right]\right),$$

  the last term is the "Gerber-Shiu function" $\phi(x-d)$, which is easily computed.

- In particular,

$$H_d(x) = \mu_\alpha\left(\frac{1}{\rho}W^{(\delta)}(x-d) - \int_0^{x-d} W^{(\delta)}(y)\,dy\right).$$

Part V

Concluding remarks

Concluding remarks

- We consider a generalized insurance risk process using spectrally negative Lévy processes.
- Beyond the classical insurance ruin theory, we consider "default-related quantities", which is EPVs of (path-dependent) default risks under the general s.n. Lévy risks.
- We obtain several representations for quantities of interest: from classical analytical ways to modern fluctuation-theoretic argument.
- Those results are possibly applied to the "credit risk" problems.
- Statistical inference and more reasonable approximations for those quantities are future issues.

Thank you!

先物市場の高頻度データ〜立会時間延長の影響分析 *

川崎能典 †　吉田靖 ‡

概要

グローバルな取引市場間の競争が強まる中，投資家の利便性向上を目的として，特にわが国ではデリバティブ市場において，立会時間の延長が実施されている．本報告では，東京商品取引所に上場されている金先物および大阪証券取引所に上場されている日経平均先物の高頻度データを利用し，立会時間の制度変更に照らして取引発生頻度の経時的変化を点過程モデルのあてはめを通じて観察する．特に，従来の国内の取引所は株式市場・商品市場共に，海外の取引所にはない昼休みが存在することが，特徴であった．この制度により，国内の高頻度データを使用した日内季節変動の分析では，昼休み前後の変動パターンが海外の実証結果と大きく異なる原因となっていた．本報告では，取引時間の延長により日中の変動パターンが変化していく様子を点過程モデルにより検証する．

キーワード：点過程，非定常ポアソン過程，金先物，日経平均先物

1　はじめに

1990 年に大阪証券取引所の日経平均先物が取引代金ベースで CME (Chicago Mercantile Exchange) の S&P500 先物を上回った翌年，Miller(1991) は米国の先物取引の国際競争力について分析している．その後も，取引所間の競争はデリバティブに限らず，2007 年のニューヨーク証券取引所とユーロネクストの合併に代表されるように，クロスボーダー取引の発達によって，激しさを増している．国内でもかつては最大の先物取引所であった東京穀物取引所が 2013 年 2 月にすべての取引を終了し，東京工業品取引所は農産物が移管され，東京商品取引所に商号変更し，商品取引所は大阪堂島商品取引所との 2 取引所のみとなった．また，東京証券取引所と大阪証券取引所は 2013 年 1 月に経営統合するなど，統廃合が進んでいる．

取引所は，単に流動性を提供するだけではなく，価格の発見機能を通じて，効率的な資源配分に寄与し，また，金融センターの中心となる社会的インフラでもあり，国全体の競争力向上にとっても重要な存在である．さらに，投資家のニーズは，高度化・複雑化が著しく，取引システムの安定性，高速性，取引時間の延長から，決済制度の改善まで多岐に亘っている．具体的には，東京工業品取引所（当時）は，2009 年 5 月から新取引システムを稼働させ，立会時間の延長や，処理速度の向上を実施し，東京証券取引所は，2010 年 1 月から arrowhead を稼働させ高速注文への対応や，呼値の刻みの縮小など，高速性，信頼性，拡張性を向上させ，大阪証券取引所では，デリバティブ市場において 2011 年 7 月から早朝 3 時までのナイトセッションを開設するなど，積極的な利便性の向上が実施されている．

一方で学術的には，高頻度データによる分析が進展しており，特にリアライズド・ボラティリティの推計に代表される理論と実証については，Takahashi et al.(2009) などの多くの研究が発表され，生方・渡部（2011）にあるようにリスク管理などの分野でも大きな貢献が期待されている．また，Barclay,Hendershott(2003) や Cooper et al.(2008) のように，夜間取引の分析をおこなっている論文も多数ある．しかしながら，前述の取引システム改善の効果の学術的な検証は必ずしも多くはなく，宇野・柴田（2012）や太田 (2013) など

*2013 年 12 月 26 日、科学研究費課題「経済リスクの統計学の新展開：稀な事象と再起的事象」(基盤研究 (A) 研究代表者：国友直人 東京大学・経済学部) に基づく研究集会。

†統計数理研究所・モデリング研究系：〒 190-8562　東京都立川市緑町 10-3, e-mail address: kawasaki@ism.ac.jp
‡東京経済大学経営学部〒 185-8502　東京都国分寺市南町 1-7-34, e-mail address: yyoshida@tku.ac.jp

の東証における取引高速化の分析に限定されており，同時に進められている取引時間変更の評価を行っている論文は非常に少ない．

このような背景の下，本研究は，大阪証券取引所の日経平均先物と東京商品取引所の金先物のティックデータを用いて，立会時間の延長が日中取引数量の推移パターンに与えた影響を検証し，新システムの導入が市場に与えた影響を評価することを目的とする．

従来，国内の取引所は株式市場・商品市場共に，海外の取引所にはない昼休みが存在することが特徴であった．この制度により，渡部 (2004) が指摘するように国内の高頻度データを使用した日内季節変動の分析では，特に昼休み前後の変動パターンが海外の実証結果と大きく異なっていた．また，そのため渡部 (2010) のようなリアライズド・ボラティリティの調整なども必要になっている．株式現物市場においても昼休みの廃止も議論されているものの，短縮が実施されたのみで現時点で昼休みは存在している．しかし，東京商品取引所と大阪証券取引所のデリバティブ市場においては，以前は昼休みが存在したが，現在は廃止されている．また，夜間の立会は，東京工業品取引所は午前 4 時まで，大阪証券取引所のデリバティブ市場は午前 3 時までと延長されている．現在では両取引所の分析をおこなうためのデータ蓄積も十分であり，さらに相互の比較も可能であるため，日内季節性の変動分析にとって，意義のある研究が可能である．

したがって，本研究は，大阪証券取引所および東京商品取引所の先物のティックデータを用いて，立会時間の延長が日中の価格変動性や取引数量に与えた影響を検証し，新制度の導入が市場の流動性に与えた影響を評価することを目的とし，取引システムの今後の改善やリスク管理の精度向上の基礎となることを目的とする．

つづく第 2 節では，分析対象である東京商品取引所の金先物と大阪証券取引所の日経平均先物の取引制度の変遷と分析に用いたデータについて述べる．第 3 節では，本稿で推計に用いた点過程モデルの推計について概説する．第 4 節は実証分析の結果である．最後に第 5 節として，まとめと今後の課題を述べる．

2　分析対象商品とデータ

東京商品取引所での分析対象は，商品先物の中で，最も流動性が高く，またグローバルな銘柄で，各地の取引所を利用することにより 24 時間の取引が可能でもあり，さらに ETF（上場投資信託）の形での取引も多いことで証券市場との関連も高い金先物とする．東京商品取引所の制度では，金先物が 2 か月毎に 6 限月あるが，芹田他 (2005) と同様にその中で最も取引量の多い先限の約定データを対象とする．分析にあたっては，約定データを 1 秒毎のデータに集約する．すなわち，取引量（枚数）は同一秒の中で合計し，価格は同一秒の中で最も遅い取引価格を採用する．

大阪証券取引所での分析対象は，最も取引量が多い日経平均先物とする．日経平均先物は，3 か月毎に限月が存在するが，最も取引量の多い期近物とする．ただし，SQ 前日において次の限月の取引量が多くなるのが通例であることから，SQ 前日午前零時をもって，限月交代するものとして取り扱う．約定データを使用することと，秒データへの集約方法は，金先物と同様である．

季節的変動パターンモデルの構築にあたっては，既存研究では森保 (2008) のようにスプライン関数を使用したモデルが多いが，周期性が残存している可能性があり，TIMSAC84(Akaike et al. 1985) の EPTREN を用いて条件付き強度関数によるモデリングを行ない比較する．

3　点過程モデルによる周期性解析

本節ではまず，さまざまな点過程データを解析するための一般的な枠組みを，条件付き強度関数[1](conditional intensity function) に基づき説明する．主として統計地震学を念頭に置いたものではあるが，点過程モデリングの優れた展望論文として Ogata (1999) を挙げておく．

[1] Intensity の訳語として強度よりは生起度のほうが適切であるとする書もある．マクニール他 (2008) を参照．

3.1 点過程モデルの最尤推定

正の実時間軸 $(0,\infty)$ 上でランダムに生起する系列事象 $\{t_i; 0 < t_1 < t_2 < \cdots\}$ を考える．系列の記述にあたっては，まず隣り合う事象に関して生起時刻の差 $X_i = t_i - t_{i-1}$ を考えると，基本的に正の値を取る確率過程を考えることになる．(後に尤度関数の定義で問題が生じないように，生起時刻の差は厳密に正であるとし，もし現実のデータに tie が存在する場合は，観測精度未満の時刻分だけいずれかをずらす操作を人為的に行う．) この生起間隔が独立同一分布に従う場合はよく知られた再生過程 (renewal process) であり，その周辺分布が指数分布であれば，系列事象は定常ポアソン過程に従う．

$N(a,b)$ で，区間 (a,b) に入る点 (事象) の個数を表そう．N は 0 個であるかもしれないが負にはならず，非負整数値確率変数といえる．このとき，微小な時間区間内に生起する事象の個数を予測する問題を考えよう．正の実時間軸 $(0,\infty)$ 上の点過程を想定し，それらを幅 δ の微少な区間に分割する．このとき，$\xi_k = N[(k-1)\delta, k\delta)$ を分割後の区間 $[(k-1)\delta, k\delta)$ 上の k 番目の確率変数とするとき，$\{\xi_k\}$ はひとつの確率過程になっている．もし δ が十分小さければ，$\{\xi_k\}$ が二値確率過程 (binary process) であると見なして良い．もしいま考えている点過程が定常ポアソン過程なのであれば，$\{xi_k\}$ 独立同一の分布に従うベルヌーイ系列である．しかし一般的には，系列事象の同時確率は条件付き確率 $P\{\xi_k = 1|\xi_1, \ldots, \xi_{k-1}\}$, $k = 1, 2, \ldots$ の列，すなわち過去の履歴に依存してきまると考えるべきである．

このとき条件付き確率を時間に関して微分して，条件付き強度関数 $\lambda(t|\mathcal{F}_t)$ を

$$P\{N(t, t+\delta) = 1|\mathcal{F}_t\} = \lambda(t|\mathcal{F}_t)\delta + o(\delta),$$

あるいは

$$\lambda(t|\mathcal{F}_t) = \lim_{\Delta\to 0} P\{\text{an event in } [t, t+\Delta]|\mathcal{F}_t\}/\Delta \tag{1}$$

で定義する．ここで $\mathcal{F}_t$ は時刻 t までの観測値の集合が生成する情報セットであり，その中には事象生起時刻の履歴 $\mathcal{H}_t = \{t_i; t_i < t\}$ も含む．条件付き強度関数の定式化によって，対応する点過程は完全に特徴付けられることが知られている (Liptzer and Shiryayev, 1978) ．条件付き強度関数が定数であれば，それは明らかに定常ポアソン過程に帰着する．条件付き強度関数が過去の履歴とは独立だが生起時刻 t には依存している場合は，t に関する任意の非負関数 $\nu(t)$ を使って $\lambda(t|\mathcal{F}_t) = \nu(t)$ と書ける場合だが，この場合は非定常ポアソン過程と呼ばれる．本報告で用いるのはこのタイプのモデルである．

他にも興味深い点過程のクラスと，それに対応する条件付き強度関数がある．Hawkes の自己励起過程 (self-exciting process)

$$\lambda(t|\mathcal{F}_t) = \mu + \int_0^t g(t-s)dN_s = \mu + \sum_{t_i<t} g(t-t_i)$$

はその一例である (Hawkes, 1971, Hawkes and Oakes, 1974)．この関数形は，時系列解析における自己回帰モデルを想起させる．このモデルでは，事象が起きるかどうかの期待値が過去の生起事象の線形結合になっており，いわゆるインパルス応答関数 $g(\cdot)$ がその重みを与える．

区間 $[0,T]$ 上で事象の生起時刻が $t_1, t_2, \ldots, t_n$ と観測されたとして，そのときパラメトリックに表現された条件付き強度関数 $\lambda_\theta(t|\mathcal{F}_t)$ が与えられたとしよう．このとき尤度関数は以下の形を取る．

$$L_T(\theta|t_1, t_2, \ldots, t_n; 0, T) = \left\{\prod_{i=1}^n \lambda_\theta(t_i|\mathcal{F}_{t_i})\right\} \exp\left\{-\int_0^T \lambda_\theta(t|\mathcal{F}_t)dt\right\} \tag{2}$$

パラメータ θ の最尤推定値は，尤度関数の対数値

$$\log L_T(\theta|t_1, t_2, \ldots, t_n; 0, T) = \sum_{i=1}^n \log \lambda_\theta(t_i|\mathcal{F}_{t_i}) - \int_0^T \lambda_\theta(t|\mathcal{F}_t)dt \tag{3}$$

を最大化することで得られる．

(3) 式右辺第 2 項が θ に関して解析的に表現されていれば，対数尤度関数の勾配微分は容易に得られる．そのような場合には，対数尤度関数の最大化は，標準的な非線形最適化の方法で実行できる．

複数の競合するモデルからひとつに絞らなければならないときには，赤池情報量規準 (Akaike, 1974)

$$\text{AIC} = -2\times(\text{maximum log-likelihood})+2\times\ (\text{number of parameters})$$

を最小化するモデルを選択する．

3.2　点過程の周期性モデリング

非定常ポアソン過程を想定し，以下のように条件付き強度関数を特定化する．

$$\lambda_\theta(t|\mathcal{F}_t) = a_0 + P_J(t) + C_K(t). \tag{4}$$

(4) 式の第 2 項は発展型のトレンドモデルとして

$$P_J(t) = \sum_{j=1}^{J} a_j \phi_j(t/T),\ 0 < t < T \tag{5}$$

という定式化を与える．ここで T は観測の全区間であり，$\phi_j(\cdot)$ は j 次の多項式である．(4) 式第 3 項は周期性をモデル化したものであり，フーリエ級数により

$$C_K(t) = \sum_{k=1}^{K} \left\{ b_{2k-1}\cos(2k\pi t/T_0) + b_{2k}\sin(2k\pi t/T_0) \right\}, \tag{6}$$

という定式化を与える．ここで T_0 は分析に先立って特定化する，現象の周期であり，例えば一日の長さに相当しているとすれば，この周期性モデルは日内周期性を表現していることになる．

(4) 式の代わりに，これらのモデルを指数の肩に乗せたモデルを考えることがしばしば有益である．

$$\lambda_\theta(t|\mathcal{F}_t) = \exp\{a_0 + P_J(t) + C_K(t)\} \tag{7}$$

これにより，条件付き強度関数の正値性が保証される．条件付き強度関数を対数線形型にモデリングしていると考えて良い．この場合は，積分項 ((3) 式の右辺第 2 項) の解析的評価は，特殊な場合を除いて不可能である．例えば Lewis (1970) を参照．一般にはこの項を数値積分で評価することになる．非定常ポアソンで強度関数が例えば指数レートでしか緩慢変動しない場合には最尤推定は実行可能であることが知られており (MacLean 1974)，ソフトウェア実装としては Ogata and Katsura (1985) の EPTREN がある．

4　実証分析

4.1　金先物の結果

分析の対象期間は，2006 年 10 月 2 日から 2011 年 4 月 28 日までとし，これを立会時間の違いにより 4 分割し，

- 期間 A：2006 年 10 月 2 日から 2008 年 1 月 4 日まで
  - 立会時間は 9:00-11:00 と 12:30-15:30
- 期間 B: 2008 年 1 月 7 日から 2009 年 5 月 1 日まで
  - 立会時間は 9:00-11:00 と 12:30-17:30
- 期間 C：2009 年 5 月 7 日から 2010 年 9 月 17 日まで
  - 立会時間は 9:00-15:30 と 17:00-23:00
- 期間 D: 2010 年 9 月 21 日から 2011 年 4 月 28 日まで

– 立会時間は 9:00-15:30 と 17:00-翌日 4:00

とする．期間 A と期間 B の違いを EPTREN による推計結果 (図 1 のパネル A と B) を用いて比較すると，前場の寄り付きが最も強度の高い時間帯となり，次いで後場の引けとなっている．そして多くの先行研究で指摘されているように，昼休みの前後でも強度が高くなり，強度のグラフは W 字型になっている．後場の推移をより詳細に見ると，期間 A では 15 時前後が日中で最も強度の弱い時間帯となってから急速に強度が引けまで上昇するが，期間 B では強度の最も低くなる時間帯はやや早まり，引けにかけて緩やかに上昇するパターンとなっている．

期間 C(図 1 のパネル C) では，それ以前と比べて昼休みがなくなるという大きな制度変更がなされているが，その結果，正午前後と夜間の 20 時前後に強度が低い時間帯が現れ，それまでと比べて，横長の W 字型となっている．そして，日中立会の寄り付きが最も強度が高い時間帯となることは変わらず，その次に強度が高くなる時間帯が日中の引けである点も同様であり，夜間立会は，全般的に日中立会よりも強度が低いので，W 字型よりも，Vv 型という表現がより正確である．

最後に期間 D(図 1 のパネル D) の推定結果を見ると，強度が最大となる点が 15:30 の日中立会の引けの時刻となる点が，それ以前の A から C の全期間と大きく異なる点である．夜間立会に関しては，20 時前後の強度が最も低くなる点は期間 C と同様で，その後 23 時頃まで強度が上昇する点も同様であるが，それ以降は 4 時にかけて徐々に強度が低くなる点も，それまで引けにかけては強度が上昇していたことと比べて大きな違いがある．

以上のように，日内季節変動のパターンは，特に期間 D において大きく変動したことがわかった．

なお，ここでは推定された強度の縦軸は基準化していない．従って，強度関数の面積は当該標本期間に生起した事象の総数に等しい．グラフは制度改変に対応して 4 つの期間に分けられているが，それぞれ事象総数は A:743,447，B:1,257,597，C:2,032,120，D:987,151 である．

4.2 日経平均先物の結果

日経平均先物に関しては，立会時間の延長 (の有無) によって以下の 6 期間に分けて分析している．金先物同様，強度を表す縦軸は事象数などで基準化はしていない．n は各期間での事象数である．

- 期間 A: 2006 年 9 月 7 日〜2007 年 9 月 14 日
  - イブニングセッション導入以前, $n = 1,168,040$
- 期間 B: 2007 年 9 月 18 日〜2008 年 10 月 10 日
  - 16:30–19:00 のイブニングセッション始まる, $n = 1,560,363$
- 期間 C: 2008 年 10 月 14 日〜2010 年 7 月 16 日
  - 16:30–20:00 にイブニングセッション延長, $n = 2,008,698$
- 期間 D: 2010 年 7 月 20 日〜2011 年 2 月 10 日
  - 16:30–23:30 にイブニングセッション延長, $n = 487,672$
- 期間 E: 2011 年 2 月 14 日〜2011 年 7 月 15 日
  - 日中立会昼休み廃止, $n = 399,835$
- 期間 F: 2011 年 7 月 19 日〜2012 年 12 月 12 日
  - 16:30–03:00 のナイトセッションに, $n = 1,006,938$

結果は図 2 の 6 枚のパネルに示されている．期間 4 の取引が前場の終わりに集中しており，ここだけ突出して強度が高い．

5　まとめ

実証分析の結果，条件付き強度関数のモデリングである EPTREN を用いて，先物市場での取引頻度の日中の推移を検証できることが確認された．夜間取引そのものの取引量は多くないが朝方の寄り付きパターンには影響を与えている可能性があることと，昼休みの廃止が大きく影響していることが示唆された．リアライズド・ボラティリティの影響などは今後の課題である．

参考文献

[1] Akaike, H., A new look at the statistical model identification, *IEEE Trans. Autom. Control*, **19**, 716–723.

[2] Akaike, H. et al., TIMSAC-84 part 2, *Computer Science Monographs No. 23*, The institute of Statistical Mathematics, 1985.

[3] Barclay Michael J.,and Hendershott,Terrence,Price Discovery and Trading After Hours, *Review of Financial Studies*,16(4),1041–1071,2003.

[4] Breedon, Francis and Ranaldo, Angelo, Intraday Patterns in FX Returns and Order Flow (April 3, 2012). Queen Mary, University of London, School of Economics and Finance WP 694. Available at SSRN: `http://ssrn.com/abstract=2099321` or `http://dx.doi.org/10.2139/ssrn.2099321`

[5] Cooper, Michael J., Cliff, Michael T. and Gulen, Huseyin, Return Differences between Trading and Non-Trading Hours: Like Night and Day (September 26, 2008). Available at SSRN: `http://ssrn.com/abstract=1004081` or `http://dx.doi.org/10.2139/ssrn.1004081`

[6] Hawkes, A. G., Point spectra of some mutually exciting point processes, *J. Roy. Statist. Soc. (B)*, **33**, 438–443, 1971.

[7] Hawkes, A. G. and D. A. Oakes, A cluster process representations of self-exciting process, *J. Appl. Probab.*, **11**, 493–503, 1974.

[8] Lewis, P. A. W., Remarks on the theory, computation and application of the spectral analysis of series of events, *J. Sound. Vib.*, **12**, 353–375, 1970.

[9] Liptzer, R. S. and A. N. Shiryayev, *Statistics of Random Processes II : Applications*, Springer-Verlag, New York, 1978.

[10] Lopez de Prado, Marcos, Advances in High Frequency Strategies (December 1, 2011). Doctoral Dissertation, Complutense University, Madrid, 2011. Available at SSRN: `http://ssrn.com/abstract=2106117` or `http://dx.doi.org/10.2139/ssrn.2106117`

[11] MacLean, C. J., Estimation and testing of an exponential polynomial rate function within the non-stationary Poisson process, *Biometrika*, **61**, pp. 81–86, 1974.

[12] マクニール・A, フライ・R, エンブレヒツ・P,「定量的リスク管理–基礎概念と数理技法–」(塚原英敦・小林俊・三浦良造・川崎能典・中川秀敏 訳), 共立出版,2008.

[13] Merton H. Miller, International Competitiveness of U.S. Futures Exchanges, Journal of Applied Corporate Finance, Volume 3, Issue 4, Winter 1991, Pages: 6–19, DOI: 10.1111/j.1745-6622.1991.tb00559.x

[14] 森保洋, 金先物市場のマイクロストラクチャー–ティック・データを利用した実証分析–, 日本商品先物振興協会　先物取引研究, 第 11 巻第 1 号 No.15,pp. 1–19,2008.

[15] Ogata, Y. and K. Katsura, EPTREN in TIMSAC-84 part 2 (Akaike et al. eds.), *Computer Science Monographs No. 23*, The institute of Statistical Mathematics, pp. 187–197, 1985.

[16] Ogata, Y., Seismicity analysis through point-process modeling: a review, *Pure and Applied Geophysics*, **155**, 471–507, 1999.

[17] 太田亘, 取引システム高速化の流動性に対する長期的影響，日本ファイナンス学会　第 21 回大会予稿集,2013.

[18] 芹田敏夫, 濱田隆道, 荒木浩介, 坂本智幸, 金先物市場の日中の価格形成と流動性：取引データに基づく実証研究，東京工業品取引所 市場構造研究所ワーキングペーパーシリーズ,05(1),2005.

[19] Takahashi, Makoto, Yasuhiro Omori, and Toshiaki Watanabe, Estimating Stochastic Volatility Models Using Daily Returns and Realized Volatility Simultaneously, *Computational Statistics and Data Analysis*, 53, 2404–2426,2009.

[20] Treepongkaruna, Sirimon, Brooks, Robert Darren and Gray, Stephen, Do Trading Hours Affect Volatility Links in the Foreign Exchange Market? (April 4, 2012). Australian Journal of Management, Vol. 37, No. 1, 2012. Available at SSRN: http://ssrn.com/abstract=2050440

[21] 生方雅人, 渡部敏明, 実現ボラティリティーボラティリティの計測方法の発展とリスクマネジメントへの応用可能性ー，証券アナリストジャーナル，49(8)，16–26,2011.

[22] 宇野淳, 柴田舞, 取引の高速化と流動性へのインパクト: 東証アローヘッドのケース. 現代ファイナンス 31, 87–107,2012.

[23] 渡部敏明, 日経 225 先物の価格および取引高の日中の変動パターン，先物・オプションレポート，大阪証券取引所，16(12),2004.

[24] 渡部敏明, 日経 225 の Realized Volatility ーマイクロストラクチャ・ノイズと夜間・昼休みの調整ー，先物・オプションレポート，大阪証券取引所，22(2),2010.

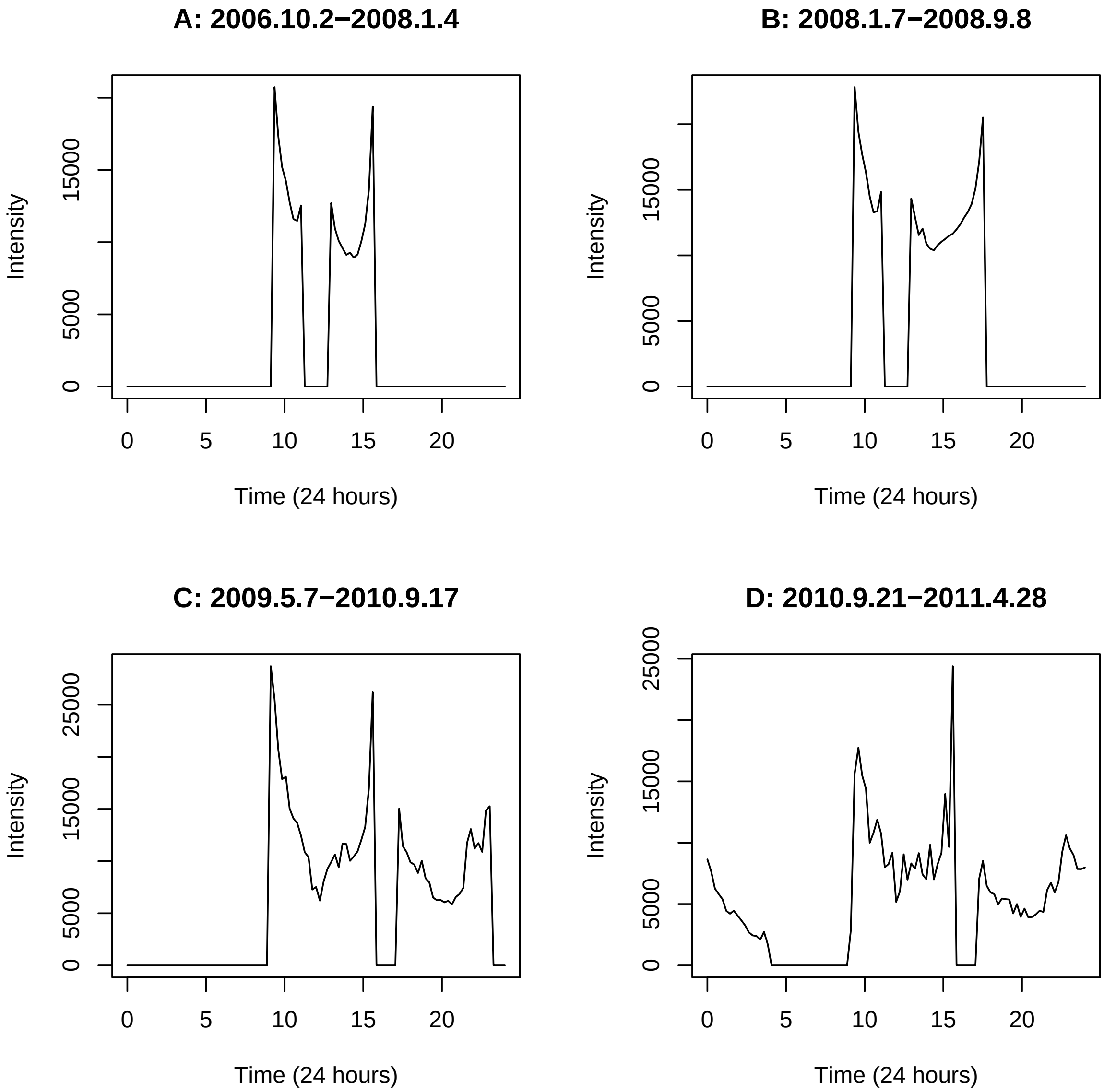


図 1: 点過程モデルから推定した金先物の取引強度

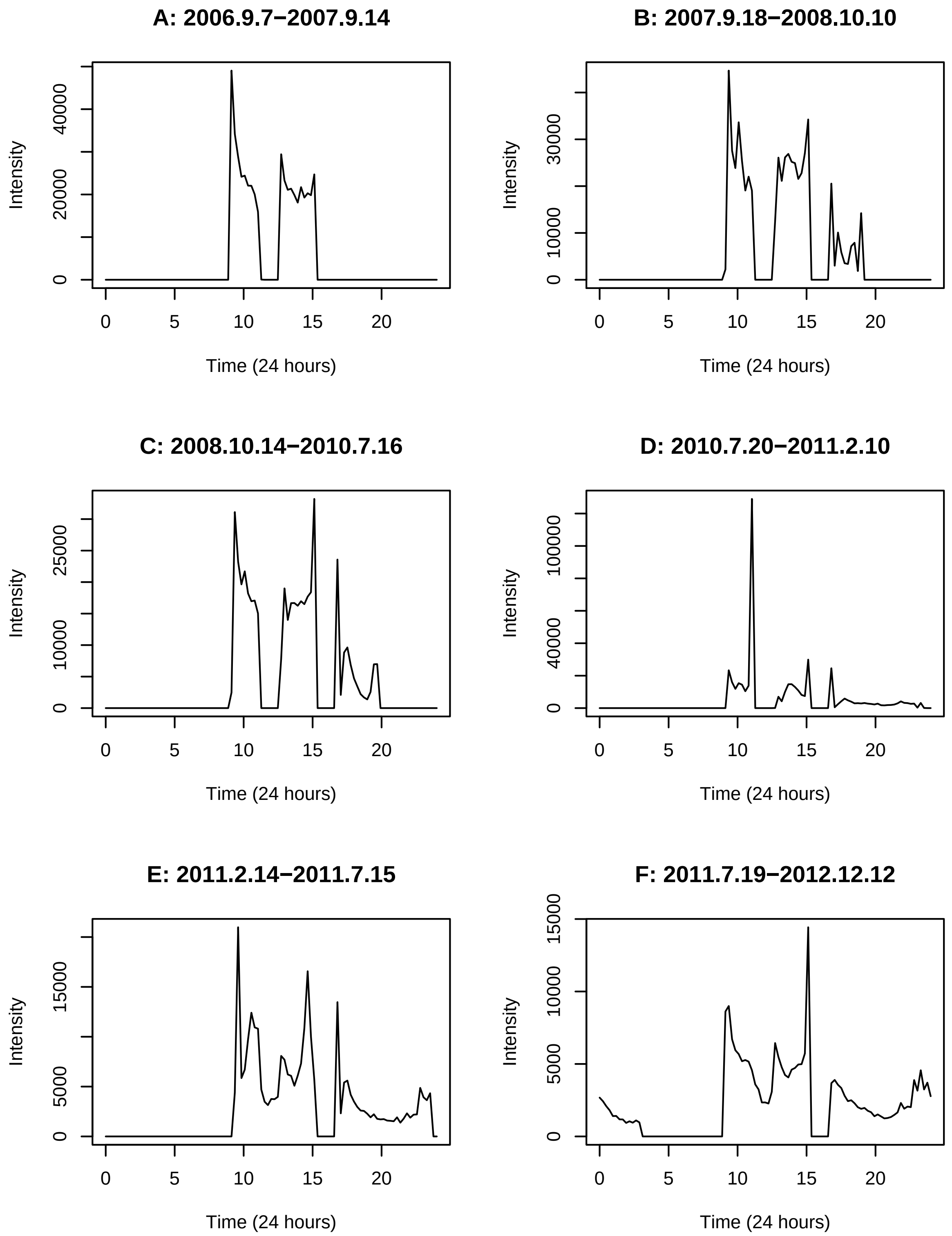


図 2: 点過程モデルから推定した日経平均先物の取引強度

Improving the Separating Information Maximum Likelihood Method for Continuous Diffusion Processes with Micro-Market Noise *

Naoto Kunitomo †

December 10, 2013

Abstract

For estimating the integrated covariances of continuous time diffusion process with micro-market noise, Kunitomo and Sato (2008, 2013) have proposed the Separating Information Maximum Likelihood (SIML) method by using high frequency financial data. We can improve the SIML method such that the modified SIML (MSIML) method is asymptotically optimal in a sense while it has the asymptotic robustness when the sample size is large. We investigate the effects of market adjustments (autocorrelated noises), round-off errors, and random sampling. We find that the MSIML estimator has reasonable finite sample properties and thus it would be useful for practice.

Key Words

Continuous Diffusion Process, Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML), Modified SIML (MSIML), Optimal Rate.

*This is a very preliminary manuscript. I thank Seisho Sato for providing some numerical examples.

†Graduate School of Economics, University of Tokyo, Bunkyo-ku, Hongo 7-3-1, Tokyo 113-0033, JAPAN, kunitomo@e.u-tokyo.ac.jp, phone: +81-3-5841-5614.

1. Introduction

Recently a considerable interest has been paid on the estimation problem of the continuous time diffusion processes and their relationships. It is partly because there are many theoretical use of diffusion processes in the area of mathematical finance. Since it is possible to use a large number of high-frequency data in financial markets, the estimation of the continuous time diffusion processes and their relationships have potentially many applications in practice. Although there were some discussion on the estimation of continuous stochastic processes in the statistical literature, the earlier studies often had ignored the presence of micro-market noises in financial markets when they tried to estimate the underlying stochastic processes. Because there are several reasons why the micro-market noises are important in high-frequency financial data both in economic theory and in statistical measurement, several new statistical estimation methods have been developed.

The main purpose of this paper is to propose a way to improve the Separating Information Maximum Likelihood (SIML) method for estimating the continuous time diffusion process by using high frequency data in the presence of possible micro-market noise. We shall call the resulting one as the MSIML (modified SIML) method because the SIML method was originally proposed by Kunitomo and Sato (2008, 2011) and its properties has been investigated by Kunitomo and Sato

(2013), Kunitomo and Misaki (2013). In this paper we shall show that the MSIML method estimator has reasonable asymptotic properties in the sense that it is asymptotically optimal when the sample size is large and it improves the finite sample properties of the SIML method under quite general situations.

The main motivation of our study is two-fold. First, the SIML method has reasonable asymptotic properties, but generally it does not attain the optimal rate in the ideal situation. Hence it may be important to improve the SIML and also understand the underlying main reason why the original SIML estimator does not attain the optimal convergence rate. Secondly, the variance-covariances of micro-market noise are also important because they cause important effects and thus the estimation of their magnitude gives key information on the underlying process. Because it is difficult to observe the micro-market noises, the assumption of i.i.d. random variables may be often too strong in the view of micro-market structure. In this paper we shall investigate the effects for estimating the variance and covariances of noises when they are weakly auto-correlated.

2. The MSIML estimation of the diffusion process with micro-market noise

2.1 The statistical models in continuous time and discrete time

Let $y_s(t_i^s)$ be the $i-$th observation of the (log-) price of the first asset at t_i^s for $0 = t_0^s < t_1^s < \cdots < t_{n_s^*}^s \leq t_{n_s}^s = 1$ and $y_f(t_j^f)$ be the $j-$th observation of the (log-) price of the second asset at t_j^f for $0 = t_0^f < t_1^f < \cdots < t_{n_f^*}^f \leq t_{n_f}^f = 1$. Let $t_{n_s^*}^s = \max_{i \geq 1, t_i^s \leq 1} t_i^s$, $t_{n_f^*}^f = \max_{i \geq 1, t_i^f \leq 1} t_i^f$ and we denote n as a constant index and n^* as a stochastic index. We consider the situation that the high-frequency data are observed at random times t_i^a $(a = s$ or $f)$ under some conditions on random sampling.

Assumption 2.1 : There exist positive constants c_a $(a = s$ or $f)$ such that

$$t_n^a \longrightarrow 1 \ , \ \frac{n_a^*}{n} \xrightarrow{p} c_a \tag{2.1}$$

and

$$\mathcal{E}\left[|t_i^a - t_{i-1}^a|\right] = O(n^{-1}) \tag{2.2}$$

as $n \to \infty$, where $a = s$ or f.

These conditions imply that n^{-1} corresponds to the average duration of observations of the intervals in $[0, 1]$ when n is relatively large. Without loss of generality we take $c_s = c_f = 1$.

A typical example is the Poisson Process Sampling on t_i^s and t_i^f with the intensity functions $\lambda_n^{(s)} = nc_s$ and $\lambda_n^{(f)} = nc_f$. In this case the sequence of random variables τ_i^a $(a = s, f)$ are exponentially distributed with $\mathcal{E}(\tau_i^a) = 1/n$ $(\tau_i^a = t_i^a - t_{i-1}^a)$ if we take $c_s = c_f = 1$. In this paper we make a further assumption on the independence of $X(t)$ and t_i^a $(i \geq 1)$.

Assumption 2.2 : The stochastic process $X(t)$ $(0 \leq t \leq 1)$ is independent of the random sequences t_i^s and t_j^f $(i, j \geq 1)$.

The underlying two-dimensional continuous process $\mathbf{X}(t) = (X_s(t), X_f(t))')$ $(0 \leq t \leq 1)$ is not necessarily the same as the observed (log-)price at t_i^s and t_j^f $(i, j \geq 1)$ and

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{C}_x(s) d\mathbf{B}(s) \quad (0 \leq t \leq 1), \tag{2.3}$$

where $\mathbf{B}(s)$ is the two-dimensional Brownian motion, $\mathbf{C}_x(s)$ is the 2×2 instantaneous volatility matrix adapted to the $\sigma-$field $\mathcal{F}(\mathbf{x}(r), \mathbf{B}(r), r \leq s)$. The main statistical objective is to estimate the quadratic variation or the integrated volatility matrix

$$\mathbf{\Sigma}_x = \int_0^1 \mathbf{\Sigma}_x(s) ds = \begin{pmatrix} \sigma_{ss}^{(x)} & \sigma_{sf}^{(x)} \\ \sigma_{sf}^{(x)} & \sigma_{ff}^{(x)} \end{pmatrix} \tag{2.4}$$

$(\mathbf{\Sigma}_x(s) = \mathbf{C}_x(s)\mathbf{C}_x^{'}(s))$ of the underlying continuous process $\mathbf{X}(t)$ $(0 \leq t \leq 1)$ from the set of discrete observations on $(y_s(t_i^s), y_f(t_j^f))$ with the condition that $\mathbf{\Sigma}_x(s)$ is a progressively measurable matrix and $\sup_{0 \leq s \leq 1} \mathbf{\Sigma}_x(s) < \infty$ $(a.s.)$.

We also consider the situation that the observed (log-)prices $y_s(t_i^s)$ and $y_f(t_j^f)$ are the sequence of discrete stochastic processes generated by

$$y_s(t_i^s) = h_s\left(\mathbf{X}(t_i^s), y_s(t_{i-1}^s), u_s(t_i^s)\right) \tag{2.5}$$

and

$$(2.6) \qquad y_f(t_j^f) = h_f\left(\mathbf{X}(t_j^f), y_f(t_{j-1}^f), u_f(t_j^f)\right) ,$$

where $h_s(\cdot)$ and $h_f(\cdot)$ are measurable functions, the (unobservable) continuous martingale process $\mathbf{X}(t)$ $(0 \leq t \leq 1)$ is defined by (2.3) and the micro-market noises $u_s(t_i^s)$ and $u_f(t_j^f)$ are the discrete stochastic processes. In particular, we assume that $u_s(t_i^s)$ and $u_f(t_j^f)$ are a sequence of independently and identically distributed random variables with $\mathcal{E}(u_s(t_i^s)) = 0, \mathcal{E}(u_f(t_j^f)) = 0$ and $\mathcal{E}(u_s(t_i^s)^2) = \sigma_{ss}^{(u)}, \mathcal{E}(u_f(t_j^f)^2) = \sigma_{ff}^{(u)}, \mathcal{E}(u_s(t_i^s)u_f(t_j^f)) = \delta(t_i^s, t_j^f)\sigma_{sf}^{(u)}$.

There are special cases of (2.3), (2.5) and (2.6), which reflect the important aspects on modeling financial markets and the high frequency financial data. The basic (high-frequency) financial model with micro market noises can be represented by

$$(2.7) \qquad y_s(t_i^s) = X_s(t_i^s) + u_s(t_i^s) \ , \ y_f(t_j^f) = X_f(t_j^f) + u_f(t_j^f) \ ,$$

where the underlying process $\mathbf{X}(t) = (X_s(t), X_f(t))'$ is given by (2.3). The synchronous sampling means $t_i^s = t_i^f$ and the fixed grid observation means $t_i^a - t_{i-1}^a = n^{-1}$. We shall consider the more general situations, that is, we have the non-synchronous observations as well as the random sampling.

The most important statistical aspect of (2.7) is the fact that it is an additive (signal-plus-noise) measurement error model. However, there are some reasons why the standard situation as (2.7) is not enough for applications. For instance, the high frequency financial models for micro-market price adjustments and the round-off-errors models for financial prices are not in the form of (2.7), but they can be represented as special cases of (2.3), (2.5) and (2.6). Sato and Kunitomo (2012) have discussed several important examples of (2.5) and (2.6) when the state variable is one dimension.

More generally, it is straightforward to extend our analysis to the cases when the observatios are the p-dimensional vector value process $y_j(t_i^{(j)}), j = 1, \cdots, p$. The model we have discussed has been the case when $p = 2$.

2.2 The MSIML estimation

We consider the situation when $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are independent with $\mathbf{\Sigma}_x(s) = \mathbf{\Sigma}_x$ $(0 \leq s \leq 1)$, and $\mathbf{v}_i$ are independently, identically and normally distributed as $N_p(\mathbf{0}, \mathbf{\Sigma}_v)$. We use an $n \times p$ matrix $\mathbf{Y} = (\mathbf{y}_i^{'})$ and consider the distribution of $np \times 1$ random vector $(\mathbf{y}_1^{'}, \cdots, \mathbf{y}_n^{'})^{'}$. Given the initial condition $\mathbf{y}_0$, we have

$$\mathbf{Y}_n \sim N_{n \times p} \left(\mathbf{1}_n \cdot \mathbf{y}_0^{'}, \mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n \mathbf{C}_n^{'} \otimes h_n \mathbf{\Sigma}_x \right) \ , \tag{2.8}$$

where $\mathbf{1}_n^{'} = (1, \cdots, 1)$, $h_n = 1/n \ (= t_i^n - t_{i-1}^n)$ and

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} . \tag{2.9}$$

We transform $\mathbf{Y}_n$ to $\mathbf{Z}_n \ (= (\mathbf{z}_k^{'}))$ by

$$\mathbf{Z}_n = h_n^{-1/2} \mathbf{P}_n \mathbf{C}_n^{-1} \left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0 \right) \tag{2.10}$$

where

$$\mathbf{C}_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} , \tag{2.11}$$

$$\mathbf{P}_n = (p_{jk}) \ , \ p_{jk} = \sqrt{\frac{2}{n+\frac{1}{2}}} \cos \left[\frac{2\pi}{2n+1} (k - \frac{1}{2})(j - \frac{1}{2}) \right] \tag{2.12}$$

and $\bar{\mathbf{Y}}_0 = \mathbf{1}_n \cdot \mathbf{y}_0^{'}$. We have the spectral decomposition $\mathbf{C}_n^{-1} \mathbf{C}_n^{'-1} = \mathbf{P}_n \mathbf{D}_n \mathbf{P}_n^{'} = 2\mathbf{I}_n - 2\mathbf{A}_n$ and $\mathbf{D}_n$ is a diagonal matrix with the k-th element

$$a_{kn} = 2[1 - \cos(\pi(\frac{2k-1}{2n+1}))] = 4n \sin^2 \left[\frac{\pi}{2} \left(\frac{2k-1}{2n+1} \right) \right] . \tag{2.13}$$

Then given the initial condition $\mathbf{y}_0$ the likelihood function can be defined as

$$L_n = \sum_{k=1}^{n} \log |a_{kn} \mathbf{\Sigma}_v + \mathbf{\Sigma}_x|^{-1/2} - \frac{1}{2} \sum_{k=1}^{n} \mathbf{z}_k^{'} [a_{kn} \mathbf{\Sigma}_v + \mathbf{\Sigma}_x]^{-1} \mathbf{z}_k . \tag{2.14}$$

Since $a_{k_n,n} \to 0$ as $n \to \infty$ when $k_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $a_{n+1-l_n,n} = O(n)$ when $l_n = O(n^{\beta})$ $(0 < \beta < 1)$. Then we may approximate $2 \times L_n$ by $2 \times L_{m_n}$ and the separating information maximum likelihood (SIML) estimator of $\hat{\mathbf{\Sigma}}_x$ is defined by

$$\hat{\mathbf{\Sigma}}_x = \frac{1}{m_n} \sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'} \tag{2.15}$$

and the SIML estimator of $\mathbf{\Sigma}_v$ is defined by

$$\hat{\mathbf{\Sigma}}_v = \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} \mathbf{z}_k \mathbf{z}_k^{'} \ . \tag{2.16}$$

For both $\hat{\mathbf{\Sigma}}_v$ and $\hat{\mathbf{S}}_x$, the number of terms m_n and l_n are dependent on n and we have the order requirements that $m_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $l_n = O(n^{\beta})$ $(0 < \beta < 1)$ for $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$, respectively.

In order to improve the SIML method, we notice that the (1st order) asymptotic bias term of $\hat{\mathbf{\Sigma}}_x$ is given by

$$\text{ABIAS} = (\frac{1}{m_n} \sum_{k=1}^{m_n} a_{kn}) \mathbf{\Sigma}_v + o_p(\frac{m_n^2}{n}) \ . \tag{2.17}$$

In this paper we propose to modify the SIML method and to use the MSIML estimator of $\mathbf{\Sigma}_v$ by

$$\hat{\mathbf{\Sigma}}_v = \frac{1}{l_n} \sum_{k=n_1+1}^{n_2} \left[a_{kn}^{-1} \mathbf{z}_k \mathbf{z}_k^{'} \right] \tag{2.18}$$

where $l_n = n_2 - n_1$ and $n_1 = n^{\beta_1}, n_2 = n^{\beta_2}$ $(\frac{1}{2} < \beta_1 < \beta_2 < 1)$.
Then given (2.14) the modified SIML (MSIML) estimator of $\hat{\mathbf{\Sigma}}_x$ is defined by

$$\hat{\mathbf{\Sigma}}_{x,m} = \left[\frac{1}{m_n} \sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'} \right] - \left[\frac{1}{m_n} \sum_{k=1}^{m_n} a_{kn} \right] \hat{\mathbf{\Sigma}}_v \ . \tag{2.19}$$

2.3 Asymptotic Properties of the MSIML estimator

The asymptotic properties of the SIML estimator have been investigated by Kunitomo and Sato (2008, 2011) on the estimation problem of the integrated volatility and integrated covariances. For the simplicity, we take the case of $p = 1$ and we consider the estimation problem of the variance of micro-market noise. It seems that there have not been any clear statement on the asymptotic properties of alternative estimators for the noise variance.

We have the following result.

Theorem 2.1 : We assume that $X(t_i^n)$ and $v_i = v(t_i^n)$ $(i = 1, \cdots, n^*)$ are given by (2.1) and (2.2) with $\sup_{0 \le s \le 1} \sigma_x^2(s) < \infty$.
For $1/2 < \beta_1, \beta_2 < 1$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$,

$$\sqrt{l_n}\left[\hat{\sigma}_v^2 - \sigma_v^2\right] \xrightarrow{d} N\left[0, V\right] \ , \tag{2.20}$$

where

$$V = 2\sigma^4 \ . \tag{2.21}$$

For the deterministic time varying volatility case the asymptotic properties of the MSIML estimator can be summarized as the next proposition.

Theorem 2.2 : We assume that $X(t_i^n)$ and $v_i = v(t_i^n)$ $(i = 1, \cdots, n^*)$ are given by (2.1) and (2.2) with $\sup_{0 \le s \le 1} \sigma_x^2(s) < \infty$ and $\sigma_x^2 = \int_0^1 \sigma_x^2(s) ds$ is a positive constant (or deterministic). Define the MSIML estimator of σ_x^2 by (2.19).

For $m_n = n^\alpha$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}^2_{x,m} - \sigma^2_x\right] \xrightarrow{d} N\left[0, V\right] \ , \tag{2.22}$$

where

$$V = 2\int_0^1 \left[\sigma^2_x(s)\right]^2 ds \ . \tag{2.23}$$

When σ^2_x is a random variable, we need the concept of *stable convergence in law* because the limiting distribution of the SIML estimator is the mixed-Gaussian distribution. In order to discuss the stable convergence in law we extend the probability space $(\Omega, \mathcal{F}, P)$ to the extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ as explained by Chapter VIII of Jacod and Syriyaev (2003) or Jacod (2007). We say that a sequence of random variables Z_n with an index n stably converges in law if

$$\mathbf{E}[Y f(Z_n)] \longrightarrow \tilde{\mathbf{E}}[Y f(Z)] \tag{2.24}$$

for all bounded continuous functions $f(\cdot)$ and all bounded random variables Y, and $\tilde{\mathbf{E}}[\cdot]$ is the expectation operator with respect to the extended probability space. We write this convergence as

$$Z_n \xrightarrow{\mathcal{L}-s} Z \ . \tag{2.25}$$

(See Jacod and Shriyaev (2003) and Jacod (2007) for the details.)
As a typical stochastic volatility case in the continuous time, we consider that the volatility function $\sigma_x(t)$ is a strong solution of the stochastic differential equation (SDE)

$$\begin{aligned}\sigma_x(t) &= \sigma_x(0) + \int_0^t \mu_\sigma(s, \sigma_x(s))ds + + \int_0^t \gamma_\sigma(s, \sigma_x(s))dB(s) \\ &\quad + \int_0^t \gamma^*_\sigma(s, \sigma_x(s))dB^*(s) \ ,\end{aligned} \tag{2.26}$$

where the coefficients $\mu_\sigma(s), \gamma_\sigma(s)$ and $\gamma_\sigma^*(s)$ are in the class of $\mathcal{A}^1$ (extensively measurable, continuous and bounded), and $B^*(s)$ is a Brownian motion which is orthogonal to $B(s)$. Here we set $\mathbf{B}(t) = (B(t), B(t)^*)'$ as the vector of Brownian motions. Then there exists a strong solution such that $\sup_{0\leq s\leq 1} \mathcal{E}[\sigma_x^4(s)] < \infty$. (There can be weaker conditions on the coefficients which give the existence of a strong solution and the moment conditions. See Chapter III of Ikeda and Watanabe (1989) for the notations and the details.)

The asymptotic properties of the MSIML estimator in the stochastic volatility cases can be summarized as Theorem 2.3.

Theorem 2.3 : We assume that $X(t_i^n)$ and $v_i = v(t_i^n)$ $(i = 1, \cdots, n^*)$ are given by (2.1) and (2.2) with $\sup_{0\leq s\leq 1} \sigma_x^2(s) < \infty$ and $\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds$ (> 0) is finite (a.s.). We assume that $\mathcal{E}[v(t_i^n)^4] < \infty$. For $m_n = n^\alpha$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$ we have the weak convergence

(2.27) $$Z_{n^*} = \sqrt{m_n}\left[\hat{\sigma}_{x,m}^2 - \sigma_x^2\right] \xrightarrow{w} Z^* ,$$

where the characteristic function $g_n(t) = \mathcal{E}[\exp(itZ_{n^*})]$ of Z_{n^*} converges to the characteristic function of Z^*, which is written as

(2.28) $$g(t) = \mathcal{E}[e^{-\frac{Vt^2}{2}}] ,$$

where

(2.29) $$V = 2\int_0^1 \left[\sigma_x^2(s)\right]^2 ds .$$

2.4 Generalizations

It is straightforward to extend our analysis in the previous section to the p dimension cases ($p \geq 1$). Another direction to extend our analysis would be to assume that the $p \times 1$ vector noise process $\{v_i\}$ is stationary process which can be represented as

$$v_i = \sum_{s=-\infty}^{\infty} \gamma_s w_{i-s} \; (i = \cdots, -1, 0, 1, \cdots) \,, \tag{2.30}$$

where $w_j = (w_{ij})$ are the vector sequence of independent random variables with $\mathcal{E}(w_{ij}) = 0, \mathcal{E}(w_{ij}^2) = \sigma_{jj}, \mathcal{E}(w_{li}w_{kj}) = 0 \; (i \neq j), \mathcal{E}(w_{ij}^4) < \infty$ and $\sum_{-\infty}^{\infty} \|\gamma_s\|^2 < \infty$.

3. Simulations

We have investigated the robustness properties of the MSIML estimator for the integrated volatility based on a set of simulations and the number of replications is 1,000. We have taken the sample size $n = 20,000$, and we have chosen $\alpha = 0.5$ and $n_1 = n^{c_1}, n_2 = n - n^{c_2}$ ($c_1 = 0.85, c_2 = 0.66$). The other details of the simulation procedure are similar to the corresponding ones reported by Kunitomo and Sato (2008, 2011).

In our simulations we consider several cases when the observations are generated by (2.3) and (2.7) as the basic case. The volatility function ($\sigma_x^2(s)$) is given by

$$\sigma_x^2(s) = \sigma(0)^2 \left[a_0 + a_1 s + a_2 s^2\right] , \tag{3.1}$$

where a_i $(i = 0,1,2)$ are constants and we have some restrictions such that $\sigma_x(s)^2 > 0$ for $s \in [0,1]$. It is a typical time varying (but deterministic) case and the integrated volatility σ_x^2 is given by

$$\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right] . \tag{3.2}$$

In this example we have taken several intra-day volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.

In our Monte-Carlo simulations, we also investigate the situation that the observed (log-)price $y(t_i^n)$ is a sequence of discrete stochastic process generated by

$$y(t_i^n) = h\left(X(t_i^n), y(t_{i-1}^n), v(t_i^n)\right) , \tag{3.3}$$

where $h(\cdot)$ is a measurable function and the (unobservable) continuous martingale process $X(t)$ $(0 \le t \le 1)$ is defined by (2.3) and $v(t_i^n)$ is the micro-market noise process. In Appendix we give some results and each model corresponds to the cases when we take $h(\cdot,\cdot,\cdot)$ as

Model 1 $\quad h_1(x,y,u) = y + g(x-y) + u \ (g : \text{a constant})$,

Model 2 $\quad h_2(x,y,u) = y + g_\eta(x-y+u) \ \ (g_\eta(\cdot) \text{ is } (2.7))$,

Model 3 $\quad h_3(x,y,u) = y + g_\eta(x-y) + u \ \ (g_\eta(\cdot) \text{ is } (2.7))$,

Model 4 $\quad h_4(x,y,u) = y + u + \begin{cases} g_1(x-y) & \text{if } y \ge 0 \ (g_1 : \text{a constant}) \\ g_2(x-y) & \text{if } y < 0 \ (g_2 : \text{a constant}) \end{cases}$,

Model 5 $\quad h_5(x,y,u) = y + \left[g_1 + g_2 \exp(-\gamma|x-y|^2)\right](x-y)$

$(g_1, g_2 : \text{constants})$,

respectively.

Model 1 is the basic additive model when $g = 1$. When $0 < g < 2$, Model 1 corresponds to the linear price adjustment model with the micro-market noise. Model 2 and Model 3 are the micro-market models with the round-off errors. Model 2 is the basic round-off errors model and Model 3 has a more complicated nonlinearity. Model 4 and Model 5 are the SSAR model and the exponential AR model, which have been known as nonlinear (discrete) time series models.

For a comparison we have calculated the historical volatility (HI) estimates. Overall the estimates of the MSIML method are quite stable and robust against the possible values of the variance ratio even in the nonlinear transformations we have considered.

For Model-1, the estimates obtained by historical-volatility (H-vol) are badly-biased, which have been known in the analysis of high frequency financial data. Actually, the values of H-vol are badly-biased in all cases of our simulations.

By examining these results of our simulations we conclude that we can estimate the integrated volatility of the hidden martingale part reasonably by the MSIML estimation method despite of the possible non-linear transformations. It may be surprising to find that the MSIML method gives reasonable estimates even when we have non-linear transformations of the original unobservable security (intrinsic) values. We have conducted a number of further simulations, but the results are quite similar as we have reported in this section.

4. Concluding Remarks

In the present study we propose a way to improve the statistical estimation method of the integrated volatility and covariances in the presence micro-market noises. We extend the Separating Information Maximum Likelihood (SIML) method proposed by Kunitomo and Sato (2008, 2011). We have shown that the modified SIML (MSIML) method has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) and it is asymptotically optimal in a sense when the sample size is large and the data frequency interval is small under reasonable conditions. The MSIML estimator has reasonable finite sample properties and also it has the asymptotic robustness properties.

The MSIML estimator is so simple that it can be practically used not only for the integrated volatility but also the integrated covariances of the multivariate high frequency financial series and the hedging ratios. Further developments of applications will be discussed in other occasions.

References

[1] Gloter, A. and J. Jacod (2001), "Diffusions with Measurement Errors. II : Optimal Estimators," *ESAIM : Probability and Statistics*, Vol. 5, 243-260.

[2] Ikeda, N. and S. Watanabe (1989), *Stochastic Differential Equations and Diffusion Processes*, 2nd Edition, North-Holland.

[3] Jacod, J. (2007), "Statistics and high frequency data," Unpublished Manuscript, Paris.

[4] Jacod, J. and A. N. Shiryaev (2003), *Limit Theorems for Stochastic Processes*, Springer.

[5] Kunitomo, N. and S. Sato (2008), "Separating Information Maximum Likelihood Estimation of Realized Volatility and Covariance with Micro-Market Noise," Discussion Paper CIRJE-F-581, Graduate School of Economics, University of Tokyo, (http://www.e.u-tokyo.ac.jp/cirje/research/dp/2008), forthcoming in *North American Journal of Economics and Finance.*

[6] Kunitomo, N. and S. Sato (2011), "The SIML Estimation of Realized Volatility of Nikkei-225 Futures and Hedging Coefficient with Micro-Market Noise," *Mathematics and Computers in Simulation*, Vol.81, 1272-1289.

[7] Kunitomo, N. and H. Misaki (2013), "The SIML Estimation of Integrated Covariance and Hedging Coefficients with Micro-market noise and Random Sampling," Unpublished Manuscript.

[8] Sato, S. and N. Kunitomo (2012), "A Robust Estimation of Integrated Volatility and covariance with Micro-market Adjustments and Round-off Errors," Unpublished Manuscript.

APPENDIX : TABLES

In Tables the estimates of the variance (σ_x^2) are calculated by the MSIML method while H-vol are calculated by the historical volatility estimation. The true-val means the true parameter value in simulations and mean, SD and MSE correspond to the sample mean, the sample standard deviation and the sample mean squared error of each estimators, respectively.

B-1 : Estimation of integrated volatility (Model-1)
($a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 1.00\text{E} - 04, g = 0.2$)

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	1.00E-04	1	1
mean	1.000648	6.93E-05	2.333708	1.000422
SD	0.118420	9.85E-07	0.023389	0.118420
MSE	0.014024	9.44E-10	1.779325	0.014024

B-2 : Estimation of integrated volatility (Model-1)
($a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 0.0, g = 0.2$)

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	0.00E+00	1	1
mean	0.992748	1.16E-06	0.1110989	0.992744
SD	0.111878	2.38E-08	0.0024040	0.111878
MSE	0.012569	1.35E-12	0.7901510	0.012569

B-3 : Estimation of integrated volatility (Model-1)
($a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 0.0, g = 1.5$)

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	0.00E+00	1	1
mean	1.003343	6.53E-05	2.99923	1.00313
SD	0.12221	7.87E-07	0.036817	0.12221
MSE	0.014946	4.27E-09	3.998269	0.01494

B-4 : Estimation of integrated volatility (Model-1)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 1.00\text{E} - 05, g = 1.0)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	1.00E-05	1	1
mean	1.004956	3.84E-05	1.3997	1.00483
SD	0.121406	4.88E-07	0.013967	0.121406
MSE	0.014764	8.06E-10	0.159962	0.014763

B-5 : Estimation of integrated volatility (Model-1)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 1.00\text{E} - 06, g = 0.01)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	1.00E-06	1	1
mean	0.523326	5.77E-07	0.0250976	0.523324
SD	0.071447	8.16E-09	0.0005492	0.071447
MSE	0.232323	1.79E-13	0.9504351	0.232325

B-6 : Estimation of integrated volatility (Model-2)

$(a_0 = 7, a_1 = -12, a_2 = 6; \sigma_v^2 = 2.00\text{E} - 02, \eta = 0.5)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	45	0.02	45	45
mean	46.59003	0.004002	136.3733	46.57694
SD	6.399936	0.00019	6.166522	6.399906
MSE	43.48737	0.000256	8387.097	43.44555

B-7 : Estimation of integrated volatility (Model-3)

$(a_0 = 7, a_1 = -12, a_2 = 6; \sigma_v^2 = 1.00\text{E} - 02, \eta = 0.5)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	45	1.00E-02	45	45
mean	47.185	1.17E-02	394.7923	47.14678
SD	6.548271	2.29E-04	7.1138	6.548203
MSE	47.6541	2.91E-06	122405.23	47.48761

B-8 : Estimation of integrated volatility (Model-3)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 0.0, \eta = 0.005)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	0.00E+00	1	1
mean	1.002935	1.94E-05	0.68542	1.002872
SD	0.117878	3.32E-07	0.008776	0.117878
MSE	0.013904	3.78E-10	0.099037	0.013903

B-9 : Estimation of integrated volatility (Model-4)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 0.0, g_1 = 0.2, g_2 = 5)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	0.00E+00	1	1
mean	1.001114	6.76E-05	2.221652	1.000893
SD	0.11993	2.22E-06	0.068095	0.11993
MSE	0.014384	4.57E-09	1.497069	0.014384

B-10 : Estimation of integrated volatility (Model-4)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 1.00\text{E} - 03, g_1 = 0.2, g_2 = 5)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	1.00E-03	1	1
mean	1.029582	1.99E-03	66.6307	1.02307
SD	0.122565	4.86E-05	1.547753	0.122566
MSE	0.015897	9.86E-07	4309.795	0.015555

B-11 : Estimation of integrated volatility (Model-5)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_v^2 = 0.0, g_1 = 1.9, g_2 = -1.7, \gamma = 10000)$

n=20000	σ_x^2	σ_v^2	H-vol	$\sigma_{x,m}^2$
true-val	1	0.00E+00	1	1
mean	0.996518	9.72E-05	6.375448	0.9962
SD	0.121183	3.21E-06	0.367669	0.12118
MSE	0.014697	9.46E-09	29.03062	0.01470